Computational issues and applications for 3D anisotropic damage modelling: coupling effects of damage and frictional sliding

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ABSTRACT. *The inelastic response for quasi-brittle materials is due to generation and growth of oriented mesocracks. A 3D model taking into account most of effects induced by this kind of damage is outlined. The emphasis is put on the modelling of damage by mesocrack growth and frictional sliding on closed mesocrack lips. From a numerical point if view, a purely implicit local integration scheme has been chosen for both damage and sliding evolutions. This method has been found particularly adapted to the damage modelling at stake here. Although the proposed model couples two dissipative phenomena, the numerical integration is facilitated by a low degree of effective connection between the equations governing each evolution. Simulations of boundary-value problems illustrate the pertinence of the coupled model.*

RÉSUMÉ. *Le comportement inélastique des matériaux quasi-fragiles est dû à la création et à la croissance de mésofissures orientées. On décrit ici un modèle 3D prenant en compte la plupart des effets induits par ce type d'endommagement. On insiste sur la modélisation de l'endommagement par mésofissuration et du glissement avec frottement sur les lèvres des fissures fermées. D'un point de vue numérique, un schéma d'intégration purement implicite se trouve particulièrement adapté au traitement de l'évolution de l'endommagement. Bien que le modèle associe deux phénomènes dissipatifs, l'intégration numérique est facilitée par le faible degré de couplage entre les équations régissant les deux évolutions. Des simulations par éléments finis illustrent enfin la pertinence du modèle couplé.*

KEYWORDS: *model, quasi-brittle, damage, frictional sliding, coupling, numerical integration, simulation, engineering applications.*

MOTS-CLÉS : *modélisation, quasi-fragile, endommagement, glissement avec frottement, couplage, intégration numérique, simulation, applications.*

1. Introduction

This paper addresses some issues concerning the modelling of the behaviour of quasi-brittle materials, comprising some rocks, concrete, ceramics...These materials share the same damage process, namely the generation and growth of decohesion mesosurfaces (mesocracks). This phenomenon induces a degradation of the effective properties of the material. Besides, the generally oriented nature of flaws gives rise to a number of characteristic events such as induced anisotropy, volumetric dilatancy, irreversible stress/strain effects, dissymmetry between tension and compression, unilateral behaviour due to crack opening/closure, dissipative frictional sliding on closed mesocrack lips...The purpose of this paper is to summarize most salient features of a model capable of taking into account most of the above phenomena insisting specially on its numerical implementation and applications for a set of engineering problems concerning concrete structures.

The postulate of combining both physical pertinence and numerical simplicity led the authors to search a third way between micromechanical and phenomenological approaches: the former propose an accurate picture of the real mechanisms but their use is frequently limited to particular loading paths due to inherent complexities encountered; the latter are generally designed to be easily implanted in Finite Element codes but suffer from a lack of physical motivation. Section 2 of this paper describes a 3D damage model by mesocrack growth, originally proposed by Dragon [DRA 94], and recently developed by Halm and Dragon [HAL 96], [HAL 98]. Its particularity lies in its modular nature, with two main parts:

– A first step deals with the modelling of the mesocrack growth as well as with the moduli recovery phenomenon due to crack closure (unilateral effect). The emphasis is put on the stress continuity requirement when passing from open to closed cracks (and vice versa). Thus, f. ex., tension-compression cycles can be modelled.

– The second level couples damage with a second dissipative phenomenon, namely frictional sliding on closed mesocracks and allows to simulate more complex loading paths (torsion, f. ex.).

The purpose of the model depicted in Section 2 is to provide an efficient tool for resolving boundary-value problems involving non linear behaviour of quasi-brittle solids. Thus, a great care is taken of accuracy and simplicity of the numerical integration scheme related to both independent mechanisms as well as to the coupled model. It is worth noting that the use of an implicit integration scheme for damage leads to the resolution of a linear equation, while classical elastoplastic models require more complex numerical treatment. Moreover the low degree of coupling between the two equations governing respectively damage and sliding evolutions avoids to solve an intricate non linear system. Details are given in Section 3.

In order to illustrate the pertinence of the coupled model and the efficiency of the integration algorithm, the constitutive equations have been introduced in

Code_Aster, the Finite Element code developed by Electricité de France. Section 4 provides comments on some boundary-value problems underscoring the ability of the model for efficient structural analyses of concrete structures.

2. Anisotropic damage and sliding model

This section outlines the salient features of the anisotropic damage model by Dragon *et al.* [DRA 94], [HAL 96], [HAL 98]. The particularity of this model lies in its modular structure, each part dealing with a given dissipative mechanism: damage by mesocrack growth (with unilateral behaviour) and frictional sliding on closed mesocrack lips. The behaviour of the mesocracked material is assumed to be rate-independent, isothermal and restrained to small strain.

2.1. *Damage by mesocrack growth and unilateral behaviour*

The model at stake here aims at describing the progressive mesocrack-induced anisotropic degradation and related behaviour of elastic quasi-brittle solids. It is based on a series of assumptions combining micromechanical considerations and macroscopic formulation:

(i) Damage is described by a single internal variable, a second-order tensor **D** conveying information on crack orientation:

$$\mathbf{D} = \sum_{i} d^{i}(S)\mathbf{n}^{i} \otimes \mathbf{n}^{i} \qquad [1]$$

where $\mathbf{n}^i$ stands for the normal of the i-th set of parallel cracks and $d^{(i)}(S)$ is a dimensionless scalar function proportional to the extent S of decohesion. The form [1] derives from micromechanical considerations [KAC 92]. From a macroscopic point of view, Onat and Leckie [ONA 88] prove that **D** must be an even function of $\mathbf{n}^i$, and then at least quadratic. The spectral decomposition of **D** leads to:

$$\mathbf{D} = \sum_{k=1}^{3} D^{k}\mathbf{v}^{k} \otimes \mathbf{v}^{k} \qquad [2]$$

Expression [2] can be macroscopically interpreted as follows: any system of microcracks can be reduced to three equivalent orthogonal sets of cracks characterized by densities D^k and normal vectors $\mathbf{v}^k$.

REMARK.– Unlike the case of « 1–d » models (the value of d is then bounded by 0 and 1), values of the D_{ij}-components within the relative tensorial representation cannot be straightforwardly interpreted in the same simplistic manner. In fact, when considering the scalar dimensionless density function $d^i(S)$ as a part of the micromechanical interpretation of the damage tensor **D**, one can – for a particular nature of defects considered (e.g. penny-shaped microcracks) – interpret $d^i(S)$ in terms of the conventional crack density $r^c = \frac{\sum_i a_i^3}{V}$. In such a case, d(S) can theoretically vary within the interval [0,1]. So, one can state that D_{ij}-components values take their micromechanically licit values in the interval [0,1] while the effective control of the evolution equations (including their algorithmic management) and local instability phenomena generated by the CDM model put effective limits *well below* this conceptual absolute bound of unity. That is why such a damage model has to be associated with tools of detection of relevant local instabilities (*i.e.* localisation bifurcation) in the context of computational algorithms for efficient structural analysis. This association has been achieved for the first level of the model (frictionless damage model without unilateral behaviour), see [DRA 94]: it allows to correctly predict the incipience of localisation phenomena within 3D framework. The localisation detection is not treated in this paper.

(ii) Micromechanical studies [KAC 92] show that 3D damage configurations should be rigorously described not by the single variable **D** [1], but by two damage parameters, namely **D** and its extension to the fourth-order $\overline{\mathbf{D}}$:

$$\overline{\mathbf{D}} = \sum_i d^i(S)\mathbf{n}^i \otimes \mathbf{n}^i \otimes \mathbf{n}^i \otimes \mathbf{n}^i$$

However, when cracks are open, the influence of $\overline{\mathbf{D}}$ can be neglected and the single variable **D** appears sufficient to model the degradation of solids containing cracks. Under compressive loading, favourably oriented cracks may close, leading to an elastic moduli recovery phenomenon. In this case, the contribution of $\overline{\mathbf{D}}$ into the overall elastic properties can no longer be neglected. In order to maintain the macroscopic interpretation [2], the complementary fourth-order entity (named $\hat{\mathbf{D}}$) necessary to account for the unilateral effect is directly built with the eigenvalues and eigenvectors of **D** and slightly differs from $\overline{\mathbf{D}}$:

$$\hat{\mathbf{D}} = \sum_{k=1}^{3} D^k \mathbf{v}^k \otimes \mathbf{v}^k \otimes \mathbf{v}^k \otimes \mathbf{v}^k$$

Note that there is no new information in $\hat{\mathbf{D}}$ with respect to **D**, so $\hat{\mathbf{D}}$ is not considered as a new damage variable.

(iii) One assumes the existence of a thermodynamic potential (free energy per unit volume w), function of strain $\boldsymbol{\varepsilon}$, damage **D** and the fourth-order damage parameter $\hat{\mathbf{D}}$, and generating a form of elastic orthotropy for $\mathbf{D} \neq \mathbf{0}$, in connection with the three eigensystems [2]. Assuming linear elasticity and non interaction between cracks, the tensorial functions representation theory [BOE 78] gives the general form of the terms entering $w(\boldsymbol{\varepsilon}, \mathbf{D}, \hat{\mathbf{D}}(\mathbf{D}))$:

$$w(\varepsilon,\mathbf{D}) = \tfrac{1}{2}\lambda(\mathrm{tr}\,\varepsilon)^2 + \mu\,\mathrm{tr}(\varepsilon.\varepsilon) + g\,\mathrm{tr}(\varepsilon.\mathbf{D}) + \alpha\,\mathrm{tr}\,\varepsilon\,\mathrm{tr}(\varepsilon.\mathbf{D}) + 2\beta\,\mathrm{tr}(\varepsilon.\varepsilon.\mathbf{D})$$
$$-(\alpha+2\beta)\,\varepsilon:\left[\sum_{k=1}^{3} H(-v^k.\varepsilon.v^k)D^k v^k \otimes v^k \otimes v^k \otimes v^k\right]:\varepsilon \qquad [3]$$

H stands for the classical Heaviside function and activates or deactivates the $\hat{\mathbf{D}}$-term depending on whether the k-th equivalent set of mesocracks is open ($\mathbf{v}^k.\varepsilon.\,\mathbf{v}^k>0$) or closed ($\mathbf{v}^k.\varepsilon.\,\mathbf{v}^k\leq 0$). The proof for the form of the opening/closure criterion $\mathbf{v}^k.\varepsilon.\,\mathbf{v}^k=0$ can be found in [HAL 96]. λ and μ are the classical Lamé constants; α and β are material constants related to modified elastic moduli for a given damage state. The factor ($\alpha+2\beta$) in front of the $\hat{\mathbf{D}}$-term is obtained by assuming a total stiffness recovery in the direction normal to the closed crack. The linear term, reading g tr(ε.**D**), generates residual phenomena for $\mathbf{D}\neq\mathbf{0}$. The elastic stress $\boldsymbol{\sigma}$ and the damage thermodynamic force $\mathbf{F}^D$ are determined by partial derivation:

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \varepsilon} = \lambda\,(\mathrm{tr}\varepsilon)\,\mathbf{I} + 2\mu\,\varepsilon + g\mathbf{D} + \alpha\,[\mathrm{tr}(\varepsilon.\mathbf{D})\mathbf{I} + (\mathrm{tr}\varepsilon)\mathbf{D}] + 2\beta\,(\varepsilon.\mathbf{D} + \mathbf{D}.\varepsilon)$$
$$-2(\alpha+2\beta)\sum_{k=1}^{3} H(-v^k.\varepsilon.v^k)D^k(v^k.\varepsilon.v^k)v^k \otimes v^k$$

$$\mathbf{F}^D = -\frac{\partial w}{\partial \mathbf{D}} = -\,g\varepsilon - \alpha\,(\mathrm{tr}\varepsilon)\,\varepsilon - 2\beta\,\varepsilon.\varepsilon$$
$$+(\alpha+2\beta)\sum_{k=1}^{3} H(-v^k.\varepsilon.v^k)(v^k.\varepsilon.v^k)^2 v^k \otimes v^k$$

The forms of w, $\boldsymbol{\sigma}$ and $\mathbf{F}^D$ respect the continuity conditions for multilinear elasticity [CUR 95], so that these functions remain continuous despite the presence of H.

(iv) The evolution of **D**, corresponding to the brittle, splitting-like crack kinetics, has been found to follow the normality rule with respect to a criterion in the space of components of the proper thermodynamic force $\mathbf{F}^D$. The damage evolution is thus apparently following the principle of maximum dissipation and is related here to tensile (positive) straining ε^+ and to actual damage pattern. It should be stressed however that the particular damage criterion proposed in [DRA 94] $f(\mathbf{F}^D,\mathbf{D})\leq 0$ is explicitly dependent on the part $\mathbf{F}^{D1+} = -g\varepsilon^+ = \mathbf{F}^D - \mathbf{F}^{D2} - \mathbf{F}^{D1-}$ of the driving force $\mathbf{F}^D$. $\mathbf{F}^{D1}$ is the strain energy release rate term related to residual effects: $\mathbf{F}^{D1} = -g\varepsilon$, $\mathbf{F}^{D2}$ represents the remaining recoverable energy release rate. The former term is decomposed into the splitting part $\mathbf{F}^{D1+} = -g\varepsilon^+$, $\varepsilon^+ = \mathbf{P}^+:\varepsilon$, with $\mathbf{P}^+$ a positive fourth-order projection operator selecting positive eigenvalues from strain, and the non-splitting part $\mathbf{F}^{D1-} = -g(\varepsilon-\varepsilon^+)$. The damage criterion and rate-independent damage evolution law are thus as follows:

$$f(\mathbf{F}^D-\mathbf{F}^{D1-}-\mathbf{F}^{D2},\mathbf{D}) = \sqrt{\frac{1}{2}\mathrm{tr}\left[(\mathbf{F}^D-\mathbf{F}^{D1-}-\mathbf{F}^{D2}).(\mathbf{F}^D-\mathbf{F}^{D1-}-\mathbf{F}^{D2})\right]}$$

$$+ \mathrm{B}\ \mathrm{tr}\left[(\mathbf{F}^D-\mathbf{F}^{D1-}-\mathbf{F}^{D2}).\mathbf{D}\right] - (C_0 + C_1 \mathrm{tr}\mathbf{D}) \le 0$$

$$\dot{\mathbf{D}} = \Lambda_D \frac{\partial f}{\partial \mathbf{F}^D} = \Lambda_D \left(\frac{\boldsymbol{\varepsilon}^+}{\sqrt{2\,\mathrm{tr}(\boldsymbol{\varepsilon}^+.\boldsymbol{\varepsilon}^+)}} + \mathrm{B}\mathbf{D} \right) \quad , \quad \Lambda_D \ge 0 \qquad [4]$$

Note that the damage model including the unilateral effect necessitates the identification of eight material constants only, which can be relatively easily determined, see f. ex. [HAL 01].

2.2. *Frictional sliding on closed mesocrack lips*

Even if it takes into account the unilateral effect the previous model does not restore the shear moduli when cracks close, assuming thus that cracks are perfectly lubricated. Because of the roughness of the crack lips and the consecutive friction, this assumption appears too strong: experimental data involving loading-unloading cycles for specimens undergoing frictional sliding on the lips (torsional tests for example) exhibit a shear moduli recovery in the direction parallel to the crack plane, due to blocking of crack lips displacement. The work by Gambarotta and Lagomarsino [GAM 93] proposes a 3D micromechanical model for this phenomenon which constitutes a progress with respect to some earlier 2D attempts. This section provides a macroscopic formulation suitable for boundary-value problems involving frictional sliding. It is built within the same thermodynamic framework as for damage and is based on following hypotheses:

(i) Sliding occurs within the crack plane. A micromechanical study ([KAC 92], considering this time that crack displacement has no opening component) leads to the following possible expression for the sliding variable:

$$\gamma = \sum_i \frac{S^i \xi^i}{V} \mathrm{sym}(\mathbf{n} \otimes \mathbf{g})^i$$

S^i stands for the cracked surface of the i-th set of parallel mesocracks of normal $\mathbf{n}^i$, ξ^i the sliding following the direction $\mathbf{g}^i$, V the representative volume element. As the influence of **D** reduces to that of three equivalent sets according to [2], γ can be written in the analogous manner:

$$\gamma = \sum_{k=1}^{3} \frac{S^k \xi^{ki}}{V} \mathrm{sym}(\mathbf{v} \otimes \mathbf{g})^k = \sum_{k=1}^{3} \gamma^k$$

where $\mathbf{v}^k$, k = 1,2,3 are the **D**-eigenvectors.

(ii) Frictional blocking induces a macroscopic recovery of the shear moduli. In Expression [3], the degradation of the shear moduli is related to the β-term in the first line. Invariants involving γ will thus replace the previous β-term in the free energy w(ε,**D**,γ) taking into account this additional dissipative phenomenon. Due to the particular structure of **D** and γ and the fact that only simultaneous (γ,**D**)-invariants enter w, two additional invariants convey useful information: tr(ε.γ.**D**) and tr(γ.γ.**D**).

(iii) According to the points (i) and (ii), the following expression is proposed for the free energy of the solid containing sliding cracks:

$$w(\varepsilon,\mathbf{D},\gamma) = \tfrac{1}{2}\lambda\,(\mathrm{tr}\,\varepsilon)^2 + \mu\,\mathrm{tr}(\varepsilon.\varepsilon) + g\,\mathrm{tr}(\varepsilon.\mathbf{D}) + \alpha\,\mathrm{tr}\,\varepsilon\,\mathrm{tr}(\varepsilon.\mathbf{D}) + 2\beta\,\mathrm{tr}(\varepsilon.\varepsilon.\mathbf{D})$$
$$+\sum_{k=1}^{3} H(-\mathbf{v}^k.\varepsilon.\mathbf{v}^k)\Big[-\alpha\,\varepsilon:(D^k\mathbf{L}^k):\varepsilon - 2\beta\,\mathrm{tr}(\varepsilon.\varepsilon.\mathbf{D}^k)$$
$$+4\beta\,\mathrm{tr}(\varepsilon.\gamma^k.\mathbf{D}^k) - 2\beta\,\mathrm{tr}(\gamma^k.\gamma^k.\mathbf{D}^k)\Big]$$

with $\mathbf{L}^k = \mathbf{v}^k \otimes \mathbf{v}^k \otimes \mathbf{v}^k \otimes \mathbf{v}^k$ and $\mathbf{D}^k = D^k \mathbf{v}^k \otimes \mathbf{v}^k$. The coefficients 4β and −2β in the last line have been calculated by assuming: (1) the continuity between the expressions of w corresponding respectively to open and closed cracks, (2) sliding γ is equal to the strain ε in the crack plane at the very closure moment. The elastic stress as well as the thermodynamic force related to **D** contains the contribution of each equivalent set (open or closed, sliding or blocked):

$$\sigma = \lambda\,(\mathrm{tr}\varepsilon)\,\mathbf{I} + 2\mu\,\varepsilon + g\mathbf{D} + \alpha\,[\mathrm{tr}(\varepsilon.\mathbf{D})\mathbf{I} + (\mathrm{tr}\varepsilon)\mathbf{D}] + 2\beta\,(\varepsilon.\mathbf{D} + \mathbf{D}.\varepsilon)$$
$$+\sum_{k=1}^{3} H(-\mathbf{v}^k.\varepsilon.\mathbf{v}^k)\Big[-2\alpha\,D^k(\mathbf{v}^k.\varepsilon.\mathbf{v}^k)\mathbf{v}^k \otimes \mathbf{v}^k$$
$$-2\beta(\varepsilon.\mathbf{D}^k + \mathbf{D}^k.\varepsilon) + 2\beta(\gamma^k.\mathbf{D}^k + \mathbf{D}^k.\gamma^k)\Big] \qquad [5]$$

$$\mathbf{F}^D = -\,g\varepsilon - \alpha\,(\mathrm{tr}\varepsilon)\,\varepsilon - 2\beta\,\varepsilon.\varepsilon$$
$$+\sum_{k=1}^{3} H(-\mathbf{v}^k.\varepsilon.\mathbf{v}^k)\Big[\alpha\,(\mathbf{v}^k.\varepsilon.\mathbf{v}^k)^2\mathbf{v}^k \otimes \mathbf{v}^k + 2\beta\,\mathbf{L}^k:(\varepsilon.\varepsilon)$$
$$-4\beta\,\mathbf{L}^k:(\varepsilon.\gamma^k) + 2\beta\,\mathbf{L}^k:(\gamma^k.\gamma^k)\Big]$$

The thermodynamic force related to sliding concerning a particular equivalent set is:

$$\mathbf{F}^{\gamma k} = -\frac{\partial w}{\partial \gamma^k} = H(-\mathbf{v}^k.\varepsilon.\mathbf{v}^k)\Big[-2\beta(\varepsilon.\mathbf{D}^k + \mathbf{D}^k.\varepsilon) + 2\beta(\gamma^k.\mathbf{D}^k + \mathbf{D}^k.\gamma^k)\Big] \qquad [6]$$

(iv) The model considers frictional non-sliding/sliding phenomena on mesocrack lips on a macroscopic scale by an approach similar to that to damage. Although it is widely employed in many models [HOR 83], [GAM 93]..., the Coulomb's criterion is not suitable in this context because of its micromechanical formulation. The pertinent quantity governing sliding on an equivalent system k is the thermodynamic force $\mathbf{F}^{\gamma k}$ (which can be physically interpreted as the sliding energy release rate). One assumes that the sliding criterion explicitly depends on the norm of the tangential part $\mathbf{F}^{\gamma Tk}$ of the force $\mathbf{F}^{\gamma k}$ and on the normal strain $v^k.\varepsilon.v^k$. Unlike the Coulomb's law, the normality rule with respect to the function defining the reversibility domain has been found to keep a strong physical sense: it indicates a connection between γ and $\mathbf{F}^{\gamma Tk}$ indicating that sliding occurs in the crack plane (as long as damage axes do not rotate). The sliding convex reversibility domain h^k can be written as:

$$h^k(\mathbf{F}^{\gamma k}-\mathbf{F}^{\gamma Nk}, v^k.\varepsilon.v^k) = \sqrt{\tfrac{1}{2}\,\mathrm{tr}\left[(\mathbf{F}^{\gamma k}-\mathbf{F}^{\gamma Nk}).(\mathbf{F}^{\gamma k}-\mathbf{F}^{\gamma Nk})\right]} + \rho v^k.\varepsilon.v^k \leq 0 \quad \text{if } v^k.\varepsilon.v^k \leq 0$$

where ρ is a friction coefficient in the sense meant by the above thermodynamic force (tangential component) – normal strain relationship, and:

$$\mathbf{F}^{\gamma k} = \mathbf{F}^{\gamma Tk} + \mathbf{F}^{\gamma Nk} \quad ; \mathbf{F}^{\gamma Nk} = (v^k.\mathbf{F}^{\gamma k}.v^k)v^k \otimes v^k$$

The normality rule gives:

$$\dot{\gamma}^k = \Lambda_\gamma^k \frac{\partial h^k}{\partial \mathbf{F}^{\gamma k}} = \Lambda_\gamma^k \frac{\mathbf{F}^{\gamma Tk}}{\sqrt{2\,\mathrm{tr}\left(\mathbf{F}^{\gamma Tk}.\mathbf{F}^{\gamma Tk}\right)}}$$

2.3. *Damage and sliding coupling*

The both dissipative phenomena (damage and frictional sliding) described independently in the previous paragraphs may occur simultaneously under particular loading paths. One assumes that the splitting-like kinetics considered in Paragraph 2.1. is still valid for closed sliding cracks even when they branch: after a short transitional distance, cracks tend to grow perpendicularly to positive principal strain direction (see, f. ex., [BAR 97]). The sliding evolution law needs a rewriting, especially when the principal axes of $\mathbf{D}^k$ rotate (for $\mathbf{D}^k$-non-proportional loading paths): in this case, sliding tends to depart from the crack plane and thus the driving force for sliding has to incorporate not only the tangential part $\mathbf{F}^{\gamma Tk}$ of $\mathbf{F}^{\gamma k}$ but also a fraction of the normal part $\mathbf{F}^{\gamma Nk}$. Let be the following partition of $\mathbf{F}^{\gamma k}$:

$$\begin{aligned} \mathbf{F}^{\gamma k} = \mathbf{F}^{\gamma Tk} + \mathbf{F}^{\gamma Nk} &= \underbrace{\mathbf{F}^{\gamma Tk} + 4\beta(\gamma^k:\mathbf{D}^k)v^k \otimes v^k} - 4\beta(\varepsilon:\mathbf{D}^k)v^k \otimes v^k \\ &= \qquad\qquad \mathbf{F}^k \qquad\qquad - 4\beta(\varepsilon:\mathbf{D}^k)v^k \otimes v^k \end{aligned}$$

$\mathbf{F}^k$ is the appropriate part of $\mathbf{F}^{\gamma k}$ to enter the expression of the criterion h^k taking into account $\mathbf{D}^k$-axes rotation and avoiding discontinuities when cracks open [HAL 98]. Note that in the case of proportional loading paths (*i.e.* $\gamma^k:\mathbf{D}^k=0$), the term $\mathbf{F}^k$ reduces to $\mathbf{F}^{\gamma Tk}$.

$$h^k(\mathbf{F}^k, \mathbf{v}^k.\varepsilon.\mathbf{v}^k) = \sqrt{\tfrac{1}{2}\mathrm{tr}\left[\mathbf{F}^k.\mathbf{F}^k\right]} + \rho \mathbf{v}^k.\varepsilon.\mathbf{v}^k \leq 0 \text{ if } \mathbf{v}^k.\varepsilon.\mathbf{v}^k \leq 0$$

The normality assumption leads to:

$$\dot{\gamma}^k = \Lambda_\gamma^k \frac{\partial h^k}{\partial \mathbf{F}^{\gamma k}} = \Lambda_\gamma^k \frac{\mathbf{F}^k}{\sqrt{2\,\mathrm{tr}\left(\mathbf{F}^k.\mathbf{F}^k\right)}} \quad , \quad \Lambda_\gamma^k \geq 0 \qquad [7]$$

REMARK.– It was obvious for the present authors that controlling the following effects: (i) damage-induced anisotropy, (ii) damage related volumetric dilatancy, (iii) damage related residual effects, (iv) rigorous 3D treatment of the unilateral problem, (v) idem for the frictional resistance and sliding effect for closed microcracks involving the dissipative coupling with damage, should first have been embraced within the framework of classical local approach. The non locality of constitutive equations, which can be now postulated for particular purposeful aspects of the model, would allow enlarging its domain of pertinence by e.g. casting the underlying hypothesis of non interacting microcrack in the actual one and treat the problems of clustering and related enhancement *vs.* shielding microcracks interactions. This is planned as further work.

3. Numerical treatment

In order to treat complex boundary-value problems, an accurate numerical tool has to be associated with the previous model. The strong non linearity of the damage and frictional sliding mechanisms requires a time integration algorithm for the evolution of the damage variable **D** and of the sliding variable γ. This section summarises the local (*i.e.* for each integration point of a Finite Element discretization) integration scheme for both evolution laws [4] and [7], After dealing separately with damage and sliding mechanisms respectively, the coupling of the both ones is considered.

3.1. *Local integration for the damage model*

Let I be the time interval [0,T], with the partition $I = \bigcup_{r=1}^{N} [t_r, t_{r+1}]$. Given the mechanical state $\mathbf{q}_r = (\varepsilon_r, \mathbf{D}_r, \gamma^k{}_r, \sigma_r)$ at time t_r and the prescribed strain increment $\Delta\varepsilon$ (such as $\varepsilon_{r+1} = \varepsilon_r + \Delta\varepsilon$), the integration problem amounts to calculate the stat $\mathbf{q}_{r+1} = (\varepsilon_{r+1}, \mathbf{D}_{r+1}, \gamma^k{}_{r+1}, \sigma_{r+1})$. Since only damage evolution is concerned in this paragraph, γ^k is considered constant ($\gamma^k{}_r = \gamma^k{}_{r+1}$). The tensors ε_{r+1}, $\mathbf{D}_{r+1}$ and σ_{r+1} are determined by:

$$\varepsilon_{r+1} = \varepsilon_r + \Delta\varepsilon$$
$$\mathbf{D}_{r+1} = \mathbf{D}_r + \Delta\mathbf{D}$$

$$\boldsymbol{\sigma}_{r+1} = \mathbf{G}_\sigma(\boldsymbol{\varepsilon}_{r+1}, \mathbf{D}_{r+1}, \gamma^k{}_{r+1}) \qquad [8]$$

with $\mathbf{G}_\sigma$ standing for Relation [5]. This calculation comprises two steps.

(i) Elastic prediction: First, the increment is assumed elastic, *i.e.* $\Delta\mathbf{D} = \mathbf{0}$. One checks whether the mechanical state $(\boldsymbol{\varepsilon}_{r+1}, \mathbf{D}_r)$ meets the condition:

$$f(\boldsymbol{\varepsilon}_{r+1}, \mathbf{D}_r) = \sqrt{\frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\varepsilon}^+_{r+1}.\boldsymbol{\varepsilon}^+_{r+1}\right)} - Bg\,\mathrm{tr}\left(\boldsymbol{\varepsilon}^+_{r+1}.\mathbf{D}_r\right) - (C_0 + C_1\,\mathrm{tr}\,\mathbf{D}_r) \le 0 \qquad [9]$$

If [9] is satisfied, the elastic prediction coincides with the solution of the problem. Then,

$$\mathbf{D}_{r+1} = \mathbf{D}_r$$

and $\boldsymbol{\sigma}_{r+1}$ is calculated by [8]. Otherwise, if $f(\boldsymbol{\varepsilon}_{r+1}, \mathbf{D}_r) > 0$, the mechanical state has to be corrected in order to determine the increment $\Delta\mathbf{D}$.

(ii) Non linear correction: The incremental formulation leads to the following formulation for the damage evolution:

$$\Delta\mathbf{D} = \Delta\Lambda_D \mathbf{G}_D(\boldsymbol{\varepsilon}^+_{r+1}, \mathbf{D}_{r+1}) \quad \text{with} \quad \mathbf{G}_D(\boldsymbol{\varepsilon}^+_{r+1}, \mathbf{D}_{r+1}) = \frac{\boldsymbol{\varepsilon}^+_{r+1}}{\sqrt{2\,\mathrm{tr}\left(\boldsymbol{\varepsilon}^+_{r+1}.\boldsymbol{\varepsilon}^+_{r+1}\right)}} + B\,\mathbf{D}_{r+1}$$

The increment $\Delta\mathbf{D}$ depends on $\boldsymbol{\varepsilon}^+_{r+1}$ and $\mathbf{D}_{r+1}$, *i.e.* the value of $\boldsymbol{\varepsilon}^+$ and $\mathbf{D}$ at the end of the integration interval $[t_r, t_{r+1}]$. This assumption corresponds to a fully implicit integration scheme, which is known for its unconditional stability [ORT 85] whatever $\Delta\boldsymbol{\varepsilon}$ is. It is worth noting that unlike for most of elastoplastic models, the implicit scheme is well adapted to the damage model presented in Section 2.1: in fact, the damage multiplier increment $\Delta\Lambda_D$ is obtained by solving the criterion $f(\boldsymbol{\varepsilon}^+_{r+1}, \mathbf{D}_{r+1}) = 0$ which reduces to a linear equation whose solution is:

$$\Delta\Lambda_D = \frac{X_{r+1}}{BX_{r+1} + C_1 Y_{r+1} + BgZ_{r+1}}$$

$$\text{with: } X_{r+1} = \sqrt{\frac{g^2}{2}\,\mathrm{tr}\left(\boldsymbol{\varepsilon}^+_{r+1}.\boldsymbol{\varepsilon}^+_{r+1}\right)} - Bg\,\mathrm{tr}\left(\boldsymbol{\varepsilon}^+_{r+1}.\mathbf{D}_r\right) - \left(C_0 + C_1\,\mathrm{tr}\mathbf{D}_r\right)$$

$$Y_{r+1} = \frac{\mathrm{tr}\,\boldsymbol{\varepsilon}^+_{r+1}}{\sqrt{2\,\mathrm{tr}\left(\boldsymbol{\varepsilon}^+_{r+1}.\boldsymbol{\varepsilon}^+_{r+1}\right)}} + B\,\mathrm{tr}\mathbf{D}_r$$

$$Z_{r+1} = \sqrt{\frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\varepsilon}^+_{r+1}.\boldsymbol{\varepsilon}^+_{r+1}\right)} + B\,\mathrm{tr}\left(\boldsymbol{\varepsilon}^+_{r+1}.\mathbf{D}_r\right)$$

3.2. *Local integration for the sliding model*

In this paragraph, damage is assumed constant ($\mathbf{D}_{r+1} = \mathbf{D}_r$). The sliding integration follows the same scheme as for damage.

(i) Elastic prediction: The step is assumed elastic ($\gamma^k_{r+1} = \gamma^k_r$). The damage eigenvalues D^k_{r+1} and eigenvectors $\mathbf{v}^k_{r+1}$ are known, so the value of $\mathbf{F}^k_{r+1}$ entering h^k is given by:

$$\mathbf{F}^k_{r+1} = \mathbf{F}^{\gamma k}_{r+1} + 4\beta\, D^k_{r+1}\, (\mathbf{v}^k_{r+1}.\varepsilon_{r+1}.\mathbf{v}^k_{r+1})\, \mathbf{v}^k_{r+1}\otimes\mathbf{v}^k_{r+1}$$

Then the value of h^k is checked:

$$h^k_{r+1}(\mathbf{F}^k_{r+1},\mathbf{v}^k_{r+1}.\varepsilon_{r+1}.\mathbf{v}^k_{r+1}) = \sqrt{\frac{1}{2}\mathrm{tr}\left(\mathbf{F}^k_{r+1}.\mathbf{F}^k_{r+1}\right)} + \rho\mathbf{v}^k_{r+1}.\varepsilon_{r+1}.\mathbf{v}^k_{r+1}$$

If $h^k_{r+1}\leq 0$, $\gamma^k_{r+1} = \gamma^k_r$ and the mechanical state is fully determined. Otherwise (if $h^k_{r+1}>0$), the frictional sliding evolution undergoes the following correction.

(ii) Non linear correction: The sliding increment $\Delta\gamma^k$ is obtained by solving the following system:

$$\begin{cases} \sigma_{r+1} = \mathbf{G}_\sigma(\varepsilon_{r+1},\mathbf{D}_{r+1},\gamma^k_{r+1}) & \\ \mathbf{F}^{\gamma k}_{r+1} = \mathbf{G}_{F\gamma k}(\varepsilon_{r+1},\mathbf{D}_{r+1},\gamma^k_{r+1}) & [10] \\ \Delta\gamma^k = \Delta\Lambda^k_\gamma\, \mathbf{G}_\gamma(\mathbf{F}^k_{r+1}) & [11] \\ h^k_{r+1} = 0 & \end{cases}$$

Equations [10] and [11] stand respectively for Equations [6] and [7]. Again, this system corresponds to an implicit integration scheme. But while the damage integration reduced to a linear equation, the above system remains non-linear and its solving necessitates a Newton-Raphson algorithm.

3.3. *Fully coupled model*

The both above dissipative mechanisms (damage and frictional sliding) may occur simultaneously along particular loading paths. The problem is then to determine the coupled increments $\Delta\mathbf{D}$ and $\Delta\gamma^k$ simultaneously. The integration is here facilitated by the low degree of effective connection between f on one hand and h^k on the other: whereas h^k is a function of $\mathbf{D}$ and γ^k, f only depends on $\mathbf{D}$ and the equation f = 0 can be solved without explicit reference to sliding. The general algorithm is as follows:

(1) The value of $\mathbf{v}^k.\varepsilon.\mathbf{v}^k$ of the normal strain for each equivalent microcrack system is checked.

(2) If $\mathbf{v}^k.\varepsilon.\mathbf{v}^k>0$, the corresponding system is open; sliding does not occur and $\Delta\mathbf{D}$ is calculated as described in Paragraph 2.1.

(3) If $\mathbf{v}^k.\varepsilon.\mathbf{v}^k\leq 0$, the corresponding system is closed and may slide. Both criteria $f \leq 0$, $h^k\leq 0$ are checked; $\Delta\mathbf{D}$ and $\Delta\gamma^k$ are calculated, if necessary, by solving successively $f = 0$ for the damage evolution and later $h^k = 0$ for the sliding one.

REMARK.– The numerical algorithm is apparently standard and this constitutes paradoxically a non negligible contribution: the model deals with two strongly coupled dissipative phenomena (damage by mesocrack growth, frictional sliding) with only nine material constants. However, the particular structure of the equations governing the evolution of the internal variables, optimised somewhat by the modelling procedure, allows a classical backward difference time integration scheme to be efficient enough in spite of the complexity of the mechanisms at stake.

4. Numerical example

This section presents an application of the model for structural analysis. The three major phenomena, *i.e.* degradation, unilateral effect and frictional sliding are illustrated. A numerical simulation of boundary-value problem requires an efficient tool including a reliable model (in the case of damage model with unilateral effect, great care must be taken of the continuity of the response) and of an efficient integration scheme. With the implicit scheme used, the tangent operator has a great influence on the time needed by the simulation.

4.1. *Geometry and loads*

The numerical test is carried out on a slab with a symmetrical double edge notch. The material constants are given in Table 1. This set has been identified for a Fontainebleau sandstone. The geometry of the specimen is described in Figure 1. The structure is constrained against x- and y-displacements along the lower edge A_1-A_2 and the lower half A_2-B_2 of the right-hand side. Stress is applied on the top face A_3-A_4 and the upper half of the left-hand side A_4-B_4 via sheets considered to be infinitely rigid that are stuck to the test specimen. The upper face A_3-A_4 is maintained in the horizontal position and the left face A_4-B_4 in the vertical position. The test specimen is first subjected to a positive displacement of A_3-A_4, then a compressive force P_n is applied; finally, in addition to the compressive force, a shearing force P_s is applied. The slab is meshed by QUA4 elements under the hypothesis of plane stress. The concerned mesh appears in Figure 2: while relatively rough, it has been refined in the critical areas, *i.e.* in the central band and around the notches.

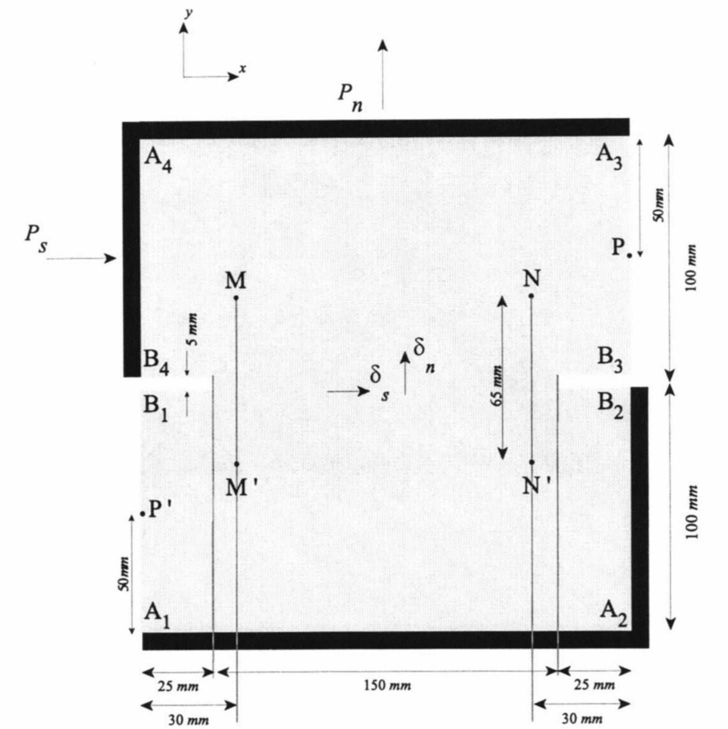

Figure 1. *Geometry of the test specimen*

λ (MPa)	μ (MPa)	α (MPa)	β (MPa)	g (MPa)	C_0 (MPa)	C_1 (MPa)	B (1)	ρ (MPa)
26 250	17 500	1 900	-20 400	-110	0.001	0.55	0	2 500

Table 1. *Constitutive parameters*

4.2. *Mesocrack growth*

Figure 2 shows a damage map during the first stage of the loading history, *i.e.* tension (by displacement imposed) on A_3-A_4. More precisely, the damage presented here is the component D_{yy} of the damage tensor. Due to the strongly brittle behaviour of sandstone, damage rapidly localizes around the notches for a low level of D_{yy} (maximum about 0.12) and becomes quasi-negligible in the central slab section.

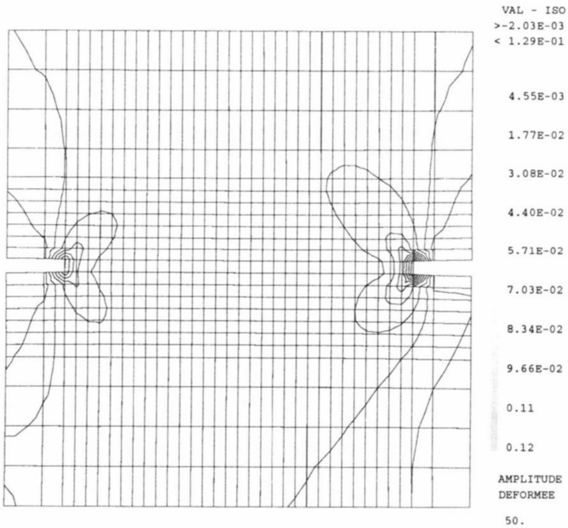

Figure 2. *D_{yy} map*

4.3. *Unilateral effect*

The unilateral effect is observed in Figure 3, (force P_n vs. difference of the vertical displacement of the two edge points in the right notch). After a degradation in the first stage of the loading history, the unloading stage exhibits two major effects: first the appearence of residual strain-like quantity δ for $P_n = 0$ and second the stiffening of the material caused by the mesocrack closure.

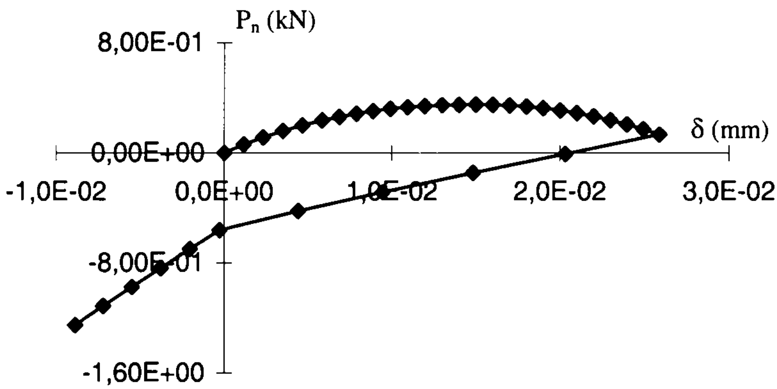

Figure 3. *P_n vs. δ (difference of the vertical displacement of the two edge points in the right notch)*

4.4. *Frictional sliding*

The first stage of the loading generates microcracks, principally concentrated around the notches. After crack closure (stage 2), a shear loading is applied. Figure 4a shows the intensity of the frictional sliding: even if numerous microcracks are located close to the notches, one observes a high density of sliding cracks within a band crossing the sample. Even the central zone, damaged to slighter degree than near-notch zones, is affected by this effect. However, due to the very low level of damage, the incipience of frictional sliding does not notably influence the distribution and the level of the stress (f.ex., Von Mises stress, Figure 4b): the difference between the values of Von Mises stress with or without frictional sliding does not exceed a few percent. The damage localization phenomenon acts here as an inhibitor for the sliding mechanism. This precocious influence of localization is corroborated by recent works [GIR 00]. Frictional sliding may influence more drastically the stress distribution for more ductile materials such as some concretes. Further work (some of which being under way) deals with this subject.

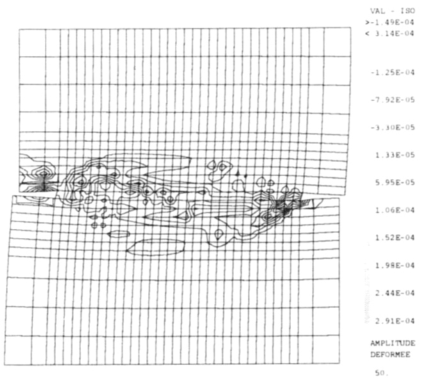

Figure 4a. *Sliding γ_{xy} map*

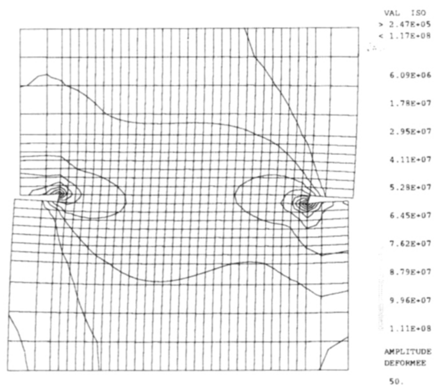

Figure 4b. *Von Mises stress map*

5. Conclusion

The vocation of the model depicted in this paper is to provide the engineer with an efficient while physically motivated tool for structural analysis: it seems that a reasonable compromise has been found between the pertinence of the 3D theoretical formalism and its applicability for industrial boundary-value problems. The constitutive equations and the required continuity of the stress-strain response stem from the tensor functions representation theory and the multilinear functions theory, the latter applied to managing unilateral effects linked to damage deactivation. Although two coupled dissipative phenomena – mesocrack growth and frictional sliding on closed mesocrack lips – are considered, the low degree of numerical coupling between the respective equations governing these two mechanisms allows a convenient algorithmic treatment and FE implementation; an example of application of the model shows its capacity to illustrate the mesocrack growth and the recovery of effective properties. Further works attempt to clarify whether the localization effects are premature compared to intrinsic material and structural response, *i.e.* whether they represent a specific excessive model tendency to be amended. It could be done, f. ex., by introducing some rate-dependance into the model which would in this manner account for genuine viscosity of engineering materials like concrete and would contribute as a regularizing factor for numerical calculations. Another axis of prospective short term research is to quantify on a broader basis the effects induced by frictional sliding.

6. References

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