On the numerical implementation of a finite strain anisotropic damage model based upon the logarithmic rate

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ABSTRACT. *We present a finite strain anisotropic damage model and show its numerical implementation. The framework of XIAO, BRUHNS & MEYERS [XIA 00] is adopted; i. e. strain and stress measures are, respectively, the logarithmic strain and its work-conjugate stress. Also, a combination of the additive decomposition of the stretching and the multiplicative decomposition of the deformation gradient is used, and the logarithmic rate is used for all rate constitutive laws. To model the change in material symmetry induced by damage, the damage parameter is regarded as an evolving structural tensor. On the irreversible thermodynamics side, it is treated as an internal state variable. Gurson's flow potential is used as a micromechanical basis.*

RÉSUMÉ. *On présente un modèle d'endommagement anisotrope en transformations finies et on discute son implémentation numérique. Le formalisme proposé par XIAO, BRUHNS & MEYERS [XIA 00] est utilisé: les mesures des déformations et des contraintes sont données respectivement, par le tenseur des deformations logarithmiques et sa force conjuguée. Une combination entre la décomposition additive, les dilatations pures et une décomposition multiplicative du gradient de la transformation est utilisée. Toutes les variables flux sont définies par une vitesse logarithmique. Pour modéliser le changement dans la symmétrie de la matière induite par l'endommagement anisotrope, la variable de l'endommagement est considérée comme un tenseur dévolution structurale. Sur le plan thermodynamique, cette variable est la variable interne associée à l'endommagement. Le potentiel d'écoulement de Gurson est utilisé comme base de la modélisation micromécanique*

KEYWORDS: *Anisotropic damage, Logarithmic strain and rate, Numerical implementation*

MOTS-CLÉS : *Endommagement anisotrope, déformation et vitesse logarithmiques, implémentation numérique*

REEF – 10/2001. NUMEDAM'00, pages 385 to 400

1. Introduction

Materials undergoing finite deformation suffer a loss in load carrying capacity caused by the nucleation, growth, and coalescence of microdefects which is generally referred to as damage. Save the simplest, but nontrivial, special case where the material is initially isotropic and the microdefects are voids, any damaged material is anisotropic. The first damage model was presented by KACHANOV [KAC 58]. This model is generally accepted and forms the fundamental basis of Continuum Damage Mechanics (CDM). Several generalizations of this one-dimensional model to the three-dimensional regime exist. Here, we mention isotropic damage models of the type described in [LEM 90] and anisotropic models based upon ideas introduced in [COR 79], and [MUR 88]. For an extensive bibliography, we refer to the review article [KRA 89] and the books [LEM 96, KRA 96] on CDM.

The isotropic damage model is limited to the case where the material is initially isotropic and all the defects are voids. Moreover, as shown by JU [JU 90], this model fails to predict the effects of damage on Poisson's ratio. Assuming isotropic conditions, one can easily show that the anisotropic damage model predicts only the change in a single material parameter. And, there is even a more fundamental problem with this model, since it is usually developed using a symmetric effective stress measure $\hat{\boldsymbol{\sigma}}$. Generally, $\hat{\boldsymbol{\sigma}}$ is derived by the symmetrization of a stress measure denoted in [MUR 88] by $\boldsymbol{\sigma}^*$, which is a function of the damage variable, and considered to be unsymmetric. To derive $\boldsymbol{\sigma}^*$, finite deformation kinematics is used. In fact, $\boldsymbol{\sigma}^*$ is a first Piola-Kirchhoff type stress measure, a two-point tensor, and represents a bilinear map between two different vector spaces – the current and the fictitious undamaged configuration. One can therefore not expect $\boldsymbol{\sigma}^*$ to be symmetric, since symmetry will require that we identify the two vector spaces. We mention here the fact that the nominal stress tensor, also a two-point tensor, plays a prominent role in non-linear elasticity and elastoplasticity [OGD 84, HIL 78]. We note also that using the work-conjugacy notion introduced by HILL [HIL 78], any stress measure (and its work-conjugate strain measure) can be used in formulating non linear constitutive equations. Moreover, note that one can derive, using finite deformation kinematics, a symmetric stress measure (a second Piola-Kirchhoff type tensor) that is defined in the fictitious undamaged configuration, and that none of the effective stresses employed in the literature is equal to this stress measure. The ad hoc symmetrization of $\boldsymbol{\sigma}^*$ and damage models based upon the effective stress $\hat{\boldsymbol{\sigma}}$ are therefore questionable.

Within a phenomenological framework, the anisotropy induced by damage is indistinguishable from the so-called microstructural or initial anisotropy. In general, microstructural anisotropy is modelled by the introduction of so-called structural or material tensors [DOY 56, BOE 87]. BOEHLER [BOE 87], for example, uses the following set as material tensors:

$$\boldsymbol{m}_{ij} = (\boldsymbol{n}_i \otimes \boldsymbol{n}_j), \quad (i,j = 1,2,3), \qquad [1]$$

where $(\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3)$ denote orthonormal vectors. Apparently, the set in Equation [1] can be used as a basis for the representation of any second order tensor.

In this paper, we present an anisotropic damage model and show its numerical implementation. Based on the above mentioned fact about structural tensors and anisotropy, we make use of an interpretation of the damage parameter as an evolving structural tensor to model damage-induced anisotropy. We assume the damage parameter to be a second order, symmetric, and positive semi-definite tensor. On the irreversible thermodynamics side, the damage parameter is treated as an internal state variable. With these interpretations, damage modelling effectively reduces to formulating thermodynamically consistent evolution equations, finding critical values for the damage variable and determining material constants. The framework of [XIA 00] is capable of effectively incorporating anisotropic material behaviour, and is therefore going to be adopted here as basis for the damage model; i. e. we use the spatial logarithmic strain and its work-conjugate stress measure. Also, we use a corotating frame and all our rate constitutive equations are formulated using the logarithmic rate.

An outline of this paper is as follows. In Section 2, we briefly review the kinematic foundations of finite deformation. Strain and stress measures are presented in Section 3. In Section 4, we describe briefly the logarithmic rate and the logarithmic spin, and discuss the advantages of the logarithmic rate over other objective corotational time derivatives. In Section 5, within the framework of [XIA 00], we specify our damage model. The Gibbs and dissipation potentials, and the evolution equations are specified. Using thermodynamics with internal state variables, restrictions on the material parameters are derived. From Gurson's flow potential (which serves here as a micromechanical basis), we determine the material parameters in our Hill-type yield condition. In Section 6, the material moduli for computing the stiffness matrix and a step-by-step summary of the integration of the constitutive laws are given. We also show that our model is kinematically consistent.

2. Basic kinematic quantities

Let X denote a particle of a body B undergoing finite deformation. Let X have position vectors $\boldsymbol{X}$ in the reference and undeformed configuration $\mathcal{B}_0$and $\boldsymbol{x}$ in the current configuration $\mathcal{B}$. The motion that takes $\boldsymbol{X}$ into $\boldsymbol{x}$ is denoted by $\boldsymbol{\chi}(\boldsymbol{X},t)$. The two-point tensor $\boldsymbol{F} = \mathrm{Grad}\boldsymbol{\chi}(\boldsymbol{X},t)$ with Jacobian $J = \det\boldsymbol{F} > 0$ is called the deformation gradient. From the polar decomposition theorem, we obtain

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}, \qquad [2]$$

where $\boldsymbol{R}$ is the rotation and $\boldsymbol{U}$ and $\boldsymbol{V}$ are the right and left stretch tensors, respectively. The tensors defined as

$$\boldsymbol{C} = \boldsymbol{U}^2 = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} \quad \text{and} \quad \boldsymbol{B} = \boldsymbol{V}^2 = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}} \qquad [3]$$

are referred to, respectively, as the right and left Cauchy-Green tensors. Let λ_σ ($\sigma = 1,\cdots,n$) denote the eigenvalues of $\boldsymbol{V}$, $\chi_\sigma = \lambda_\sigma^2$ those of $\boldsymbol{B}$, $\boldsymbol{B}_\sigma$ their corresponding

orthonormal eigenprojections, and n the number of distinct eigenvalues. Then, $\boldsymbol{B}$ has the spectral representation

$$\boldsymbol{B} = \sum_{\sigma=1}^{n} \chi_\sigma \boldsymbol{B}_\sigma, \qquad \boldsymbol{B}_\sigma = \delta_{n1}\mathbf{1} + \prod_{\tau\neq\sigma}^{n} \frac{\boldsymbol{B} - \chi_\tau \mathbf{1}}{\chi_\sigma - \chi_\tau}. \qquad [4]$$

Here, δ_{ij} is the Kronecker delta, and $\mathbf{1}$ denotes the second order identity tensor. Let $\boldsymbol{v}$ denote the velocity of $\boldsymbol{X}$. The spatial velocity gradient is given by $\boldsymbol{L} = \mathrm{grad}\boldsymbol{v} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$ and has the following unique additive decomposition:

$$\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W}, \quad \boldsymbol{D} = \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}), \quad \boldsymbol{W} = \frac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^{\mathrm{T}}). \qquad [5]$$

The symmetric tensor $\boldsymbol{D}$ is called the stretching, and the antisymmetric tensor $\boldsymbol{W}$ the vorticity.

3. Strain and stress measures

Following [XIA 00], we use the following form for Hill's generalized strain measures:

$$\boldsymbol{E}^{(m)} = \sum_{\sigma=1}^{n} f(\lambda_\sigma)\, \boldsymbol{C}_\sigma, \qquad \boldsymbol{e}^{(m)} = \sum_{\sigma=1}^{n} f(\lambda_\sigma)\, \boldsymbol{B}_\sigma, \qquad [6]$$

where $\boldsymbol{C}_\sigma = \boldsymbol{R}^{\mathrm{T}}\boldsymbol{B}_\sigma\boldsymbol{R}$. The scale function $f(\lambda_\sigma)$, a smooth monotonic increasing function with the property $f(1) = f'(1) - 1 = 0$, is given by

$$f(\lambda_\sigma) = \frac{1}{m}\,(\lambda_\sigma^m - 1)\,, \qquad [7]$$

where for $m = 0$ the limiting process is understood. Note that [6] include most commonly used strain measures; Hencky's logarithmic strain measures $\boldsymbol{H}$ and $\boldsymbol{h}$ are obtained by setting $m = 0$; i. e. $f = \ln\lambda$:

$$\boldsymbol{H} = \sum_{\sigma=1}^{n} \ln\lambda_\sigma \boldsymbol{C}_\sigma = \ln\boldsymbol{U}, \qquad \boldsymbol{h} = \sum_{\sigma=1}^{n} \ln\lambda_\sigma \boldsymbol{B}_\sigma = \ln\boldsymbol{V}. \qquad [8]$$

The spatial Hencky strain $\boldsymbol{h}$ is a forward rotation of the material Hencky strain $\boldsymbol{H}$, and vice versa:

$$\boldsymbol{h} = \boldsymbol{R} \star \boldsymbol{H} = \boldsymbol{R}\boldsymbol{H}\boldsymbol{R}^{\mathrm{T}}, \qquad \boldsymbol{H} = \boldsymbol{R}^{\mathrm{T}} \star \boldsymbol{h}. \qquad [9]$$

Let $\boldsymbol{\sigma}$ denote the Cauchy stress tensor, and $\boldsymbol{\tau}$ the Kirchhoff stress tensor. $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are related by $\boldsymbol{\tau} = J\boldsymbol{\sigma}$. Again, following [XIA 00], we denote the work-conjugate stress measure to $\boldsymbol{H}$ and $\boldsymbol{h}$ in the sense of HILL [HIL 78] independent of any material

symmetry by $\boldsymbol{\Pi}$ and $\boldsymbol{\pi}$, respectively; i. e. the stress power per unit reference volume can be written as

$$\dot{w} = \boldsymbol{\tau} : \boldsymbol{D} = \boldsymbol{\Pi} : \dot{\boldsymbol{H}} = \boldsymbol{\pi} : \overset{\circ}{\boldsymbol{h}}{}^{\mathrm{R}}, \qquad [10]$$

where $\overset{\circ}{(\bullet)}{}^{\mathrm{R}}$ denotes the Green-Naghdi rate of $(\bullet)$, see Section 4. The following expression for the Eulerian stress measure $\boldsymbol{\pi}$ is taken from [XIA 00]:

$$\boldsymbol{\pi} = \sum_{\alpha,\beta=1}^{n} \sqrt{\chi_\alpha^{-1}\chi_\beta^{-1}} \left(\frac{\chi_\alpha - \chi_\beta}{\ln \chi_\alpha - \ln \chi_\beta} \right) \boldsymbol{B}_\alpha \boldsymbol{\tau} \boldsymbol{B}_\beta. \qquad [11]$$

The Lagrangian stress measure $\boldsymbol{\Pi}$ is a back-rotation of $\boldsymbol{\pi}$, and vice versa.

4. Log-spin and Log-rate

Inelastic material behaviour has to be formulated in rate or incremental form. In a Eulerian setting, this requires the use of objective corotational time derivatives. The general form for an objective corotational time derivative is

$$\overset{\circ}{\boldsymbol{G}} = \dot{\boldsymbol{G}} + \boldsymbol{G}\boldsymbol{\Omega} - \boldsymbol{\Omega}\boldsymbol{G}, \qquad [12]$$

where $\boldsymbol{G}$ is a time-differentiable, objective, second order, Eulerian tensor, and $\boldsymbol{\Omega}$ is a time-dependent skew-symmetric, second order tensor chosen such that $\overset{\circ}{\boldsymbol{G}}$ is objective. For a general discussion of objective corotational rates and their defining spins, we refer to [XIA 98a] and the references cited therein. Well-known examples of objective corotational rates are the Zaremba-Jaumann rate $\overset{\circ}{\boldsymbol{G}}{}^{\mathrm{J}}$ for $\boldsymbol{\Omega} = \boldsymbol{W}$, and the Green-Naghdi or polar rate $\overset{\circ}{\boldsymbol{G}}{}^{\mathrm{R}}$ for $\boldsymbol{\Omega} = \boldsymbol{\Omega}^{\mathrm{R}} = \dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}}$. For the logarithmic rate (log-rate) $\overset{\circ}{\boldsymbol{G}}{}^{\mathrm{Log}}$ of $\boldsymbol{G}$, we use the the logarithmic spin (log-spin)

$$\boldsymbol{\Omega}^{\mathrm{Log}} = \boldsymbol{W} + \sum_{\sigma \neq \tau}^{n} \left[\frac{(1 + (\chi_\sigma/\chi_\tau))}{1 - (\chi_\sigma/\chi_\tau)} + \frac{2}{(\ln(\chi_\sigma/\chi_\tau))} \right] \boldsymbol{B}_\sigma \boldsymbol{D} \boldsymbol{B}_\tau \qquad [13]$$

in Equation [12]. The time-dependent rotation tensor $\boldsymbol{R}^{\mathrm{Log}}$ that defines the log-spin is referred to as the logarithmic rotation (log-rotation) and is the solution of the tensor differential equation

$$\dot{\boldsymbol{R}}^{\mathrm{Log}} = -\boldsymbol{R}^{\mathrm{Log}}\boldsymbol{\Omega}^{\mathrm{Log}}, \qquad \boldsymbol{R}^{\mathrm{Log}}|_{t=0} = \boldsymbol{1}. \qquad [14]$$

The log-rate was recently proposed by XIAO, BRUHNS & MEYERS [XIA 97, XIA 98a, XIA 98b] and has since been successfully used in formulating constitutive equations [BRU 99, XIA 00]. The log-rate has two major advantages over any other objective corotational rate. First, the log-rate of the spatial Hencky strain $\boldsymbol{h}$ is equal to the stretching $\boldsymbol{D}$:

$$\overset{\circ}{(\ln \boldsymbol{V})}{}^{\mathrm{Log}} = \boldsymbol{D}. \qquad [15]$$

Actually, the validity of Equation [15] was the prime motivation for the derivation of the log-spin and the log-rate. Second, the hypo-elastic model based upon the log-rate is self-consistent; i. e. it is exactly integrable to deliver an (hyper)elastic relation [BRU 99].

5. Constitutive equations

The damage variable $\mathcal{D}$ is assumed to be a second order, symmetric, and positive semi-definite tensor. It is a member of the set $\boldsymbol{a}$ of internal state variables that also includes the hardening variables κ and $\boldsymbol{\alpha}$:

$$\boldsymbol{a} = \{\kappa, \boldsymbol{\alpha}, \mathcal{D}\}. \qquad [16]$$

The Gibbs potential Φ is assumed to be a function of π and $\boldsymbol{a}$, and the stretching $\boldsymbol{D}$ to have the additive decomposition

$$\boldsymbol{D} = \boldsymbol{D}^{\mathrm{e}} + \boldsymbol{D}^{\mathrm{ei}}, \qquad [17]$$

where $\boldsymbol{D}^{\mathrm{e}}$ denotes the elastic part of $\boldsymbol{D}$ and $\boldsymbol{D}^{\mathrm{ei}}$ the elastic-inelastic part. Then, starting from the Clausius-Duhem inequality and using a standard procedure due to COLEMAN & NOLL [COL 63], we derive the (hyper)elastic constitutive equation

$$\boldsymbol{h}^{\mathrm{e}} = \rho_0(\partial \Phi / \partial \pi) \qquad [18]$$

and the dissipation inequality

$$\pi : \boldsymbol{D}^{\mathrm{ei}} + \rho_0(\partial \Phi / \partial \boldsymbol{a}) \cdot \overset{\circ}{\boldsymbol{a}}{}^{\mathrm{Log}} \geq 0, \qquad [19]$$

where ρ_0 is the density in the reference configuration $\mathcal{B}_0$. Using Equation [15], and introducing the complementary hyperelastic potential Σ as $\Sigma = \rho_0 \Phi$, we obtain the following exactly-integrable rate form of the elastic constitutive law:

$$\boldsymbol{D}^{\mathrm{e}} = \overset{\circ}{\overline{(\partial \Sigma / \partial \pi)}}{}^{\mathrm{R}} + (\partial \Sigma / \partial \pi)(\boldsymbol{\Omega}^{\mathrm{Log}} - \boldsymbol{\Omega}^{\mathrm{R}}) - (\boldsymbol{\Omega}^{\mathrm{Log}} - \boldsymbol{\Omega}^{\mathrm{R}})(\partial \Sigma / \partial \pi). \qquad [20]$$

To separate elastic from inelastic deformations, we use, following KRÖNER [KRO 60] and LEE [LEE 69], the multiplicative decomposition of the deformation gradient:

$$\boldsymbol{F} = \boldsymbol{F}^{\mathrm{e}} \boldsymbol{F}^{\mathrm{i}}, \qquad [21]$$

where $\boldsymbol{F}^{\mathrm{e}}$ denotes the elastic part and $\boldsymbol{F}^{\mathrm{i}}$ the damage-plastic or simply the inelastic part. Note that [21] introduces an intermediate configuration which is assumed to be stress free and is unique only to within an arbitrary rotation. To relate the two decompositions [17] and [21], we use

$$\boldsymbol{D}^{\mathrm{e}} = \mathrm{sym}(\dot{\boldsymbol{F}}^{\mathrm{e}} \boldsymbol{F}^{-\mathrm{e}}), \qquad \boldsymbol{D}^{\mathrm{ei}} = \mathrm{sym}(\boldsymbol{F}^{\mathrm{e}} \dot{\boldsymbol{F}}^{\mathrm{i}} \boldsymbol{F}^{-\mathrm{i}} \boldsymbol{F}^{-\mathrm{e}}), \qquad [22]$$

where here sym(•) denotes the symmetric part of (•). We note that [22] represents the only natural, and direct relation between the two decompositions [17] and [21] and that such a relation does not exist if we assume, as widely done, that $\boldsymbol{F}^{\mathrm{e}}$ is a symmetric tensor [XIA 00].

5.1. *Particular form of the Gibbs and dissipation potentials*

Following LEHMANN [LEH 89], we assume that the Gibbs potential takes the additive decomposition

$$\Phi = \Phi^{e}(\pi,\mathcal{D}) + \Phi^{i}(\kappa,\alpha), \qquad [23]$$

where the elastic part Φ^{e} is assumed to depend only on the stress, and the damage variable and the inelastic part Φ^{i} only on the hardening variables. Using the interpretation of $\mathcal{D}$ as a structural tensor, it follows that Φ^{e} is an isotropic tensor function of π and $\mathcal{D}$. Using the representation theorem of tensor functions, we postulate the following quadratic function in π for Σ:

$$2\,\Sigma = \pi_{ij}(\mathbb{D}^{0}_{ijkl} + \mathbb{D}^{d}_{ijkl})\pi_{kl} \qquad [24]$$

with

$$\mathbb{D}^{0} = \eta_1 \mathbf{1} \otimes \mathbf{1} + 2\eta_2 \mathbb{I} \qquad [25]$$

and

$$\mathbb{D}^{d}_{ijkl} = \eta_3(\delta_{ij}\mathcal{D}_{kl} + \delta_{kl}\mathcal{D}_{ij}) + \eta_4(\delta_{ik}\mathcal{D}_{jl} + \delta_{il}\mathcal{D}_{jk} + \delta_{jk}\mathcal{D}_{il} + \delta_{jl}\mathcal{D}_{ik}). \qquad [26]$$

Here, $\mathbb{I}$ denotes the symmetric fourth order identity tensor, and η_1–η_4 are assumed to be material constants. The additive split of the compliance tensor $\mathbb{D}$ into an isotropic or undamaged part $\mathbb{D}^{0}$ and a damage-induced part $\mathbb{D}^{d}$ is motivated by micromechanical results given in [BUD 76]. With [24] and for isotropic damage, the elastic law takes the form

$$\pi = \sum_{\sigma=1}^{n} \left[(k - \frac{2}{3}\mu) \ln J^{e} + \mu \ln \chi^{e}_{\sigma} \right] \boldsymbol{B}_{\sigma}, \qquad [27]$$

where k,μ depend on the damage variable. The elastic law in the form [27] is due to Hencky and is widely used in finite elastoplasticity [ANA 86, SIM 92, SCH 95].

For the inelastic part Φ^{i} of the Gibbs potential, we use

$$-\rho_0 \Phi^{i} = H_{kin} \frac{1}{2} \boldsymbol{\alpha} : \boldsymbol{\alpha} + K(\kappa), \qquad [28]$$

where H_{kin} is the kinematic hardening modulus. The conjugate forces to $\mathcal{D}$, $\boldsymbol{\alpha}$, and κ are, respectively,

$$\boldsymbol{Y} = \eta_3(\mathrm{tr}[\pi])\pi + 2\,\eta_4 \pi^2, \qquad [29]$$

$$\boldsymbol{X} = -H_{kin}\,\boldsymbol{\alpha} \quad \text{and} \qquad [30]$$

$$K' = \rho_0 \partial K(\kappa)/\partial\kappa. \qquad [31]$$

Let the function $F(\pi,\boldsymbol{a}) = 0$ denote the yield criterion in stress space. Then, using F, we define the elastic domain as the set $\mathbb{E}_{\sigma} := \{(\pi,\boldsymbol{a}) : F(\pi,\boldsymbol{a}) \leq 0\}$. Taking

the back stress tensor $\boldsymbol{X}$ as the centre of the yield surface ($F = 0$), and the damage parameter as an evolving structural tensor, we derive the following anisotropic or Hill-type yield condition:

$$F = \pi_{eq} - \pi_0 \cdot m - K'(\kappa), \qquad [32]$$

$$\pi_{eq} = \|\bar{\boldsymbol{\pi}}\|_{\mathbb{A}} = \sqrt{\bar{\boldsymbol{\pi}} : \mathbb{A} : \bar{\boldsymbol{\pi}}}, \quad \bar{\boldsymbol{\pi}} = \boldsymbol{\pi} - \boldsymbol{X}, \quad m = (b_0 + b_1 I_3)^{\frac{1}{2}}, \qquad [33]$$

$$\mathbb{A} = \mathbb{A}^0 + \mathbb{A}^{\mathrm{d}}, \qquad \mathbb{A}^0 = \frac{3}{2}(\mathbb{I} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1}), \qquad [34]$$

$$\begin{aligned} \mathbb{A}^{\mathrm{d}}_{ijkl} &= a_3(\delta_{ij}\mathcal{D}_{kl} + \delta_{kl}\mathcal{D}_{ij}) + a_5(\delta_{ik}\mathcal{D}_{jl} + \delta_{il}\mathcal{D}_{jk} + \delta_{jk}\mathcal{D}_{il} + \delta_{jl}\mathcal{D}_{ik}) \\ &+ a_7(\delta_{ij}\mathcal{D}^2_{kl} + \mathcal{D}^2_{ij}\delta_{kl}) + a_9(\delta_{jk}\mathcal{D}^2_{il} + \delta_{ik}\mathcal{D}^2_{jl} + \delta_{jl}\mathcal{D}^2_{ik} + \delta_{il}\mathcal{D}^2_{jk}). \end{aligned} \qquad [35]$$

Here a_3,a_5,a_7,a_9; b_0 and b_1; π_0(the initial yield stress) are material constants and I_3 denotes the determinant of $\boldsymbol{\mathcal{D}}$. The fourth order tensor $\mathbb{A}$ is called the anisotropy tensor, and its additive decomposition into an isotropic or undamaged part $\mathbb{A}^0$ and a damage-induced part $\mathbb{A}^{\mathrm{d}}$ is partly motivated by the additive decomposition of the compliance tensor. Note that $\mathbb{A}^0$ is an orthogonal projection that maps a second order tensor into 3/2-times its deviatoric part, and that without damage, Equation [32] reduces to the flow condition of classical J_2 isotropic plasticity.

Perhaps the most generally accepted and widely used isotropic damage model is the micromechanical model of GURSON [GUR 77]. The most important ingredient of this model is the flow condition

$$F_G = \frac{3}{2}\pi^{\mathrm{D}}_{ij}\pi^{\mathrm{D}}_{ij} + 2q_1\, f\, \pi_0^2 \cosh\left[\frac{q_2\, \pi_{kk}}{2\pi_0}\right] - \pi_0^2\left[1 + (q_3 f)^2\right], \qquad [36]$$

where $q_1 = q_3 = 1.5$ and $q_2 = 1$ are material parameters [TVE 82], and $(\bullet)^{\mathrm{D}}$ refers to the deviator of $(\bullet)$. In its original form, F_G is a function of $\boldsymbol{\sigma}$, so that Gurson's flow condtion as given here in [36] is a further modification. Gurson's model has been extensively and successfully used by many researchers to model, amongst many other things, localization, shear banding and macrocracking.

To put [32] on a sound micromechanical basis, its material parameters are therefore determined from [36]. To this end, we consider a material that is isotropically damaged under one-dimensional conditions; i. e. we take $\boldsymbol{\mathcal{D}} = f\,\mathbf{1}$, and assume a loading case where only one normal component, say, $\pi_{11} = \pi$ of the stress tensor is different from zero. In this case, f can be regarded as the void volume fraction. From [32] and [36], the yield stresses are, respectively,

$$\pi^2 = w_2(f)\,\pi_0^2 \quad \text{and} \quad \pi^2 = w_1(f)\,\pi_0^2 \qquad [37]$$

with the weighting functions w_2 and w_1 given by

$$w_2(f) = [b_0 + b_1 f^3][1 + (2\,a_3 + 4\,a_5)\,f + (2\,a_7 + 4\,a_9)\,f^2]^{-1}, \qquad [38]$$

$$w_1(f) = [1 - 2\,q_1 f + q_1^2 f^2][1 + 1/4 q_1 f]^{-1}. \qquad [39]$$

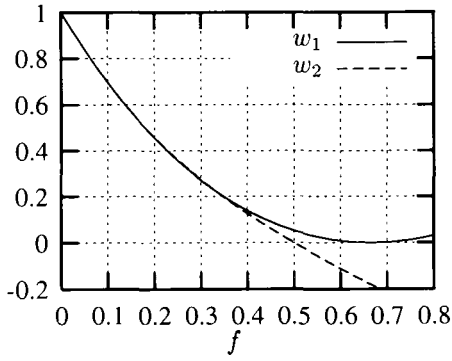

Figure 1. *Weighting functions w_1 and w_2*

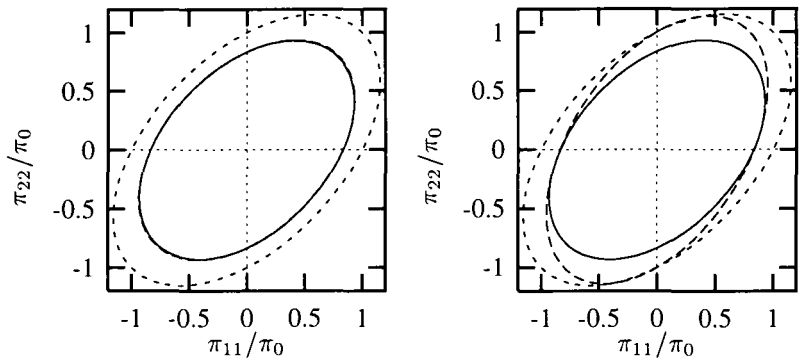

Figure 2. *Flow conditions F_G and F. Left: isotropic case with $f = 0.10$. Right: anisotropic case, where only the component $\mathcal{D}_{11} = 0.10$ is different from zero*

Considering [36] as exact, and taking the critical value of f as 0.5 [HIL 65], the material parameters

$$\begin{array}{lll} a_3 = -0.50 & a_5 = 1.08 & a_7 = -2.25 \\ a_9 = 3.51 & b_0 = +1.00 & b_1 = -8.00 \end{array} \qquad [40]$$

for [32] were chosen to get the best fit between w_1 and w_2 as shown in Figure 1. The two-dimensional case with the set of parameters from [40] is shown in Figure 2. For the isotropic case, there is virtually no difference between the two flow conditions. Both flow conditions are inside the initial or undamaged yield surface. For the simplest case of anisotropic damage also shown in Figure 2, the flow condition [32] lies between the initial yield surface and Gurson's yield surface and predicts, as expected, a decrease in flow stress in π_1-direction with virtually no change in the π_2-direction.

5.2. *Evolution equations and thermodynamic restrictions*

The evolution equations for the elastic-inelastic strain and the hardening variables are of the associative plasticity type:

$$\boldsymbol{D}^{\mathrm{ei}} = \lambda \frac{\partial F}{\partial \boldsymbol{\pi}} = \lambda \frac{\mathbb{A} : \bar{\boldsymbol{\pi}}}{\|\bar{\boldsymbol{\pi}}\|_{\mathbb{A}}} \quad [41]$$

$$\dot{\kappa} = \|\boldsymbol{D}^{\mathrm{ei}}\|_{\mathbb{A}^{-1}} = \lambda \quad [42]$$

$$\overset{\circ}{\boldsymbol{X}}{}^{\mathrm{Log}} = H_{kin}\, \boldsymbol{D}^{\mathrm{ei}}. \quad [43]$$

The plastic multiplier λ is determined from the consistency condition $\dot{F} = 0$ and satisfies the loading/unloading condition

$$\lambda \geq 0, \quad F \leq 0, \quad \lambda F = 0. \quad [44]$$

Based on the characteristics of the damage variable mentioned above, we postulate the following evolution equation:

$$\overset{\circ}{\boldsymbol{\mathcal{D}}}{}^{\mathrm{Log}} = \lambda \left[\beta_1 \mathbf{1} + \beta_2 \boldsymbol{Y}^{+}\right], \quad [45]$$

where $\beta_1, \beta_2 > 1$ are material parameters and $(\bullet)^{+}$ denotes the positive projection of $(\bullet)$. Substituting the evolution equations into the dissipation inequality, we obtain the following thermodynamic restrictions on the material constants η_3 and η_4 and the function m:

$$\eta_3 \geq 0, \quad \eta_4 \geq 0, \quad 0 \leq m \leq 1. \quad [46]$$

Note that [46] is consistent with the fact that for a given stress, the complementary hyperelastic potential Σ of a damaged material is greater than that of an undamaged material, and that these inequalities simply state (or ensure) that damage causes softening.

6. Numerical implementation

6.1. *Material moduli*

In standard displacement-type finite elements, the starting point of the solution process with the Newton-Raphson iteration scheme is the discretized form of the weak form of momentum balance. The Lie derivative

$$\overset{\circ}{\boldsymbol{\tau}}{}^{\mathrm{L}} = \dot{\boldsymbol{\tau}} - \boldsymbol{\tau}\boldsymbol{L}^{\mathrm{T}} - \boldsymbol{L}\boldsymbol{\tau} \quad [47]$$

of the Kirchhoff stress appears naturally in the rate form of the weak form of momentum balance [SIM 98, Equation (7.2.21)]. To obtain the material moduli $\mathbb{C}^{\mathrm{m}}$ which are

needed in computing the material part of the stiffness matrix, we proceed as follows. Using the basis-free expression for $\boldsymbol{\Omega}^{\mathrm{Log}}$ from [BRU 99], we obtain the following relation between $\overset{\circ}{\boldsymbol{\tau}}{}^{\mathrm{L}}$ and $\overset{\circ}{\boldsymbol{\tau}}{}^{\mathrm{Log}}$:

$$\overset{\circ}{\boldsymbol{\tau}}{}^{\mathrm{L}} = \overset{\circ}{\boldsymbol{\tau}}{}^{\mathrm{Log}} + \mathbb{G} : \boldsymbol{D}, \qquad [48]$$

$$\mathbb{G}_{ijkl} = \mathbb{B}_{irkl}\tau_{rj} - \mathbb{B}_{rjkl}\tau_{ri} - \delta_{ik}\tau_{jl} - \delta_{jl}\tau_{ik}. \qquad [49]$$

The fourth order tensor $\mathbb{B}$ is for $\chi_1 = \chi_2 = \chi_3$; i. e. $n = 1$ equal to zero. For $\chi_1 \neq \chi_2 = \chi_3$; i. e. $n = 2$

$$\mathbb{B}_{ijkl} = (B_{ik}\delta_{jl} - \delta_{ik}B_{jl})\frac{1}{\chi_1 - \chi_2}\left(\frac{1 + \chi_1/\chi_2}{1 - \chi_1/\chi_2} + \frac{2}{\ln(\chi_1/\chi_2)}\right) \qquad [50]$$

and for $\chi_1 \neq \chi_2 \neq \chi_3 \neq \chi_1$; i. e. $n = 3$

$$\mathbb{B}_{ijkl} = \nu_3(B^2_{ik}B_{jl} - B_{ik}B^2_{jl}) + \nu_2(B^2_{ik}\delta_{jl} - \delta_{ik}B^2_{jl}) + \nu_1(B_{ik}\delta_{jl} - \delta_{ik}B_{jl}), \quad [51]$$

$$\begin{cases} \nu_k = -\frac{1}{(\chi_1-\chi_2)(\chi_2-\chi_3)(\chi_3-\chi_1)}\sum_{i=1}^{3}(-\chi_i)^{3-k}\left(\frac{1+\epsilon_i}{1-\epsilon_i} + \frac{2}{\ln\epsilon_i}\right), \\ \epsilon_1 = \chi_2/\chi_3, \epsilon_2 = \chi_3/\chi_1, \epsilon_3 = \chi_1/\chi_2, \quad k = 1,2,3 \end{cases} \qquad [52]$$

Substituting Equation [48] into the rate form of the weak form of momentum balance, we obtain the material moduli $\mathbb{C}^{\mathrm{m}}$ as

$$J\mathbb{C}^{\mathrm{m}} = \mathbb{C}^{\mathrm{ep}} + \mathbb{G}, \qquad [53]$$

where $\mathbb{C}^{\mathrm{ep}}$ are the elastoplastic tangent moduli. SIMO & TAYLOR [SIM 85] showed that to retain the quadratic rate of asymptotic convergence of Newton's method, the elastoplastic tangent in Equation [53] should be replaced by the consistent tangent matrix. For our model, the consistent tangent matrix is currently computed numerically using the perturbation method as described, for example, in [KOJ 87].

6.2. *Integration of the constitutive equations*

To integrate the constitutive equations from Section 5, we use the operator split method [SIM 85, SIM 98]. Tables 1 and 2 contain a step-by-step summary of the integration algorithm. In the following, we explain our notation, and comment on some aspects of the integration. First, we note the fact that in the elastic predictor step, and consistent with the use of the log-rate, the tensor internal variables are forward-rotated with $\boldsymbol{R}_u^{\mathrm{Log}}$, the relative log-rotation:

$$(\boldsymbol{R}_u^{\mathrm{Log}})^{\mathrm{T}} = (\boldsymbol{R}_{n+1}^{\log})^{\mathrm{T}} \cdot \boldsymbol{R}_n^{\mathrm{Log}}, \qquad [54]$$

where $\boldsymbol{R}_{n+1}^{\log}$ and $\boldsymbol{R}_n^{\log}$ refer to the log-rotation at time t_{n+1} and t_n, respectively. Numerical solution of the tensor differential equation [14] yields an exponential map for

Step 1	Geometric update
	$\boldsymbol{\chi}_{n+1} = \boldsymbol{\chi}_n + \boldsymbol{u}$
	$\boldsymbol{F}_{n+1} = (\mathbf{1} + \nabla_n \boldsymbol{u})\boldsymbol{F}_n$
	$\boldsymbol{F}^{\mathrm{i}(0)}_{n+1} = \boldsymbol{F}^{\mathrm{i}}_n$
	$\boldsymbol{F}^{\mathrm{e}(0)}_{n+1} = \boldsymbol{F}_{n+1}(\boldsymbol{F}^{\mathrm{i}(0)}_{n+1})^{-1}$
	$\boldsymbol{B}^{\mathrm{e}(0)}_{n+1} = \boldsymbol{F}^{\mathrm{e}(0)}_{n+1}(\boldsymbol{F}^{\mathrm{e}(0)}_{n+1})^{\mathrm{T}}$
	$\boldsymbol{h}^{\mathrm{e}(0)}_{n+1} = \frac{1}{2}\ln \boldsymbol{B}^{\mathrm{e}(0)}_{n+1}$
Step 2	Elastic predictor
	$\mathbb{C}^{(0)}_{n+1} = (\boldsymbol{R}^{\mathrm{Log}}_u)^{\mathrm{T}} \star \mathbb{C}_n$
	$\boldsymbol{a}^{(0)}_{n+1} = (\boldsymbol{R}^{\mathrm{Log}}_u)^{\mathrm{T}} \star \boldsymbol{a}_n$
	$\boldsymbol{\pi}^{(0)}_{n+1} = \mathbb{C}^{(0)}_{n+1} : (\boldsymbol{h}^{\mathrm{e}(0)}_{n+1})$
GOTO Table **2**	
Step 6	Update intermediate configuration
	$\boldsymbol{V}^{\mathrm{e}}_{n+1} = \exp(\partial\Sigma/\partial\boldsymbol{\pi}_{n+1})$
	$\boldsymbol{R}^{\mathrm{e}}_{n+1} = \exp(\Delta t\boldsymbol{\Omega}^{\mathrm{e}})\boldsymbol{R}^{\mathrm{e}}_n$
	$\boldsymbol{F}^{\mathrm{e}}_{n+1} = \boldsymbol{V}^{\mathrm{e}}_{n+1}\boldsymbol{R}^{\mathrm{e}}_{n+1}$
	$\boldsymbol{F}^{\mathrm{i}}_{n+1} = \boldsymbol{F}^{-\mathrm{e}}_{n+1}\boldsymbol{F}_{n+1}$

Table 1. *Integration of the constitutive equations*

$\boldsymbol{R}^{\mathrm{Log}}_u$, and this is evaluated as described in [SIM 98]. Second, we mention the fact that for moderate strains the log-rotation $\boldsymbol{R}^{\mathrm{Log}}$ approximately equals the transpose of the rotation tensor $\boldsymbol{R}$ from the polar decomposition of $\boldsymbol{F}$. Third, note that with the help of Lemma A from [XIA 00] all quantities related to the decomposition [21] can be uniquely determined; from the stress update algorithm (Table 2), we obtain the stress $\boldsymbol{\pi}_{n+1}$. Then using [18], the hyperelastic form of the constitutive equation, we determine $\boldsymbol{V}^{\mathrm{e}}$:

$$\boldsymbol{V}^{\mathrm{e}}_{n+1} = \exp(\partial\Sigma/\partial\boldsymbol{\pi}_{n+1}) \qquad [55]$$

The elastic stretching $\boldsymbol{D}^{\mathrm{e}}$ is taken from the rate form of the elastic law. Then, with the elastic spin tensor given by

$$\boldsymbol{\Omega}^{\mathrm{e}} = \boldsymbol{\Omega}^{\log} - \sum_{\sigma\neq\tau}^{n}\left(\frac{2\,\lambda^{\mathrm{e}}_\sigma\lambda^{\mathrm{e}}_\tau}{{\lambda^{\mathrm{e}}_\tau}^2 - {\lambda^{\mathrm{e}}_\sigma}^2} + \frac{1}{\ln\lambda^{\mathrm{e}}_\sigma - \ln\lambda^{\mathrm{e}}_\tau}\right)\boldsymbol{B}^{\mathrm{e}}_\sigma\boldsymbol{D}^{\mathrm{e}}\boldsymbol{B}^{\mathrm{e}}_\sigma \qquad [56]$$

and from a tensorial differential equation similar to [14] for $\boldsymbol{R}^{\mathrm{e}}$, we determine the elastic rotation tensor $\boldsymbol{R}^{\mathrm{e}}$. From the polar decomposition and the multiplicative decomposition, $\boldsymbol{F}^{\mathrm{e}}$ and $\boldsymbol{F}^{\mathrm{i}}$ follow. The model presented is therefore regarded as kinematically consistent, since $\boldsymbol{F}^{\mathrm{e}}$ and $\boldsymbol{F}^{\mathrm{i}}$ and their related kinematic quantities can be

Step 3	Check for yielding
	$F_{n+1}^{(0)} = F(\boldsymbol{\pi}_{n+1}^{(0)}, \boldsymbol{a}_{n+1}^{(0)}) \leq 0?$
	YES $\boldsymbol{a}_{n+1} = \boldsymbol{a}_{n+1}^{(0)}$
	$\boldsymbol{\pi}_{n+1} = \boldsymbol{\pi}_{n+1}^{(0)}$
	RETURN
	NO
	$\Delta\lambda^{(0)} = 0, \quad i = 0$
	GOTO 4
Step 4	Return-Mapping
	$\Delta\lambda^2 = -\frac{F_{n+1}^i}{DF_{n+1}^{(i)}}$
	$\kappa_{n+1}^{(i+1)} = \kappa_{n+1}^{(i)} + \Delta\lambda^2$
	$\boldsymbol{\pi}_{n+1}^{(i+1)} = \boldsymbol{\pi}_{n+1}^{(i)} - \mathbb{C}_{n+1}^{(i)} : (\frac{\Delta\lambda^2}{\pi_{eq}}\mathbb{A} : \bar{\boldsymbol{\pi}}_{n+1}^{(i)}) - \mathbb{C}^{\mathrm{d}} : \boldsymbol{h}_{n+1}^{\mathrm{e(i)}}$
Step 5	Check for convergence
	$F\left(\boldsymbol{\pi}_{n+1}^{(i+1)}, \boldsymbol{a}_{n+1}^{(i+1)}\right) \leq \mathrm{TOL}?$
	YES
	RETURN
	NO, $\quad i \leftarrow i+1$
	GOTO 4

Table 2. *Integration of the constitutive equations*

uniquely determined without the use of ad hoc assumptions about $\boldsymbol{F}^{\mathrm{e}}$. Note also that with [56] our model satisfies the objectivity requirement in a general sense [NAG 90]. For details, we refer to [XIA 00].

6.3. *Example*

To show the performance of our model, we consider the plane-stress extension of a perforated strip. This is a classical example, and here the geometry, boundary conditions, and material parameters are chosen as in [SIM 98] page 189. For damage evolution, the material parameters are taken as $\beta_1 = 0.10$ and $\beta_2 = 0.0$; i.e. we are considering only isotropic damage. In Figure 3, we have plotted the reaction-displacement curves for the computation without damage, and with damage. Damage causes softening of the material. On the structural level, this softening is reflected in the reduction of the total reaction forces and is clearly, as shown in Figure 3, captured by the model. Qualitatively, these results are in a good agreement with other results reported in the literature, see for example [SOU 92].

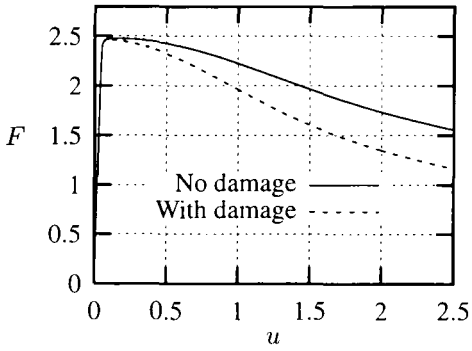

Figure 3. *Reaction versus displacement curves*

7. Summary and conclusions

A finite anisotropic damage model was presented. The elastoplasticity framework of [XIA 00] was used. Thus, the model can be regarded as kinematically, and self-consistent. It was shown that using a dual interpretation of the damage parameter as 1) an evolving structural tensor and 2) an internal variable, damage modelling reduces to formulating thermodynamically consistent evolution equations, finding critical values for the damage variable and determining material constants. We think that the former interpretation of the damage parameter, since it ties progress in damage modelling with that in the modelling of so-called strongly anisotropic materials, will prove useful in future. On the numerical implementation side, a closed form expression for the material moduli was given. Thus, the implementation of the model in existing finite element codes can be readily accomplished. More numerical examples are presented elsewhere.

Acknowledgements

We thank Dr. H. Xiao, Dr.-Ing. A. Meyers and Dipl.-Ing. H. Schütte for their interest and the fruitful discussions. This paper is abstracted from work conducted within the Collaborative Research Center SFB 398 at the Ruhr University Bochum. Financial support was provided by the Deutsche Forschungsgemeinschaft (DFG).

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