On identification of small defects by vibration tests

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ABSTRACT. *This paper addresses the conceptual problem of whether there exist measurable quantities which are highly sensitive to small changes in the physical parameters of vibrating systems? We present an astonishing analytical example, showing that a small change in one of the springs of a large multi-degree-of-freedom mass-spring system can be identified by a special dynamic test. We then argue that despite this analytical example, which can be easily reproduced by all interested readers, the longstanding controversy whether a small damage in a realistic structure can be identified by a dynamic test is still open.*

RÉSUMÉ. *Ce papier s'intéresse à la question fondamentale qui est de savoir s'il existe des quantités mesurables hautement sensibles à de petites variations des paramètres caractéristiques de vibration d'un système. On présente un exemple analytique montrant qu'une petite variation dans la rigidité de l'un des ressorts d'un système masse-ressorts à grand nombre de degrés de liberté peut être révélée par un essai dynamique spécifique. On en conclu que, malgré cet exemple analytique simple, aisément vérifiable par le lecteur, le problème de mesure d'un faible endommagement dans une structure mécanique par des essais dynamiques demeure bien ouvert.*

KEYWORD: *Defects, damage characterization, vibrating systems, dynamic test, stiffness reduction, frequency response, crack identification.*

MOTS-CLÉS : *défauts, caractérisation du dommage, système vibrant, essais dynamiques, réduction de rigidité, réponse fréquencielle, identification de fissures*

1. Introduction

There is a wealth of literature associated with diagnostic criteria, procedures, and methods for damage detection in structures and systems. With technological progress, high-speed machinery and transport carriers are prone to fatigue and catastrophic failure. There is thus an increasing interest in developing reliable diagnostic methods allowing the detection of structural defects at an early stage. Following the early work of Cawley and Adams [CAW 79], a variety of methods for identifying the existence of damage have been proposed, see *e.g.* [GAS 98], [HEA 91], [LIA 92], [RAT 00], [REY 96], and [REY 00].

If the damage exceeds a certain significant level then it can be detected by changes in the spectral properties. It is a matter of debate, however, whether a damage can be identified at an early stage, at which the physical parameters of the system are only slightly altered from their nominal undamaged values. An inherent difficulty associated with repeatability of modal testing results conducted in independent laboratories prevents easy confirmation or rejection of published results and procedures associated with damage detection. It appears that the fundamental problem of determining whether there exist measurable quantities that are highly sensitive to small changes in the physical parameters, has not been appropriately addressed yet. In this context an analytical example demonstrating the possibility of identifying small damages in the theoretical model framework of vibrating systems may be of much significant importance than merely displaying experimental results. In this paper we furnished such an analytical example, and through it address the fundamental problem of identifying small defects in structures by means of vibration tests.

For simplicity consider a conservative vibrating system modelled by:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{o}, \qquad [1]$$

with symmetric positive definite mass matrix $\mathbf{M}$, and non-negative definite stiffness matrix $\mathbf{K}$. It is well known that the eigenvalues of the system [2] are continuous functions of the elements in the system matrices, see *e.g.* [PAR 80]. If the eigenvalues are distinct then the mode-shapes are continuous functions of the physical parameters as well. Changes in the physical parameters of the system are related by bounds to changes in the orientation of the eigenvectors, as shown in [RAM 93]. Hence small changes in the physical parameters may produce only small variation in the spectral data. The measurable data in modal test however are rational functions, *e.g.*,

$$H(j\omega) = \frac{P(j\omega)}{Q(j\omega)}, \quad j = \sqrt{-1},$$

where $H(j\omega)$ is the frequency response function and $P(j\omega)$ and $Q(j\omega)$ are functions depending on the physical properties of the system. Hence continuity of ω with respect to the coefficients of $P(j\omega)$ and $Q(j\omega)$ does no necessarily imply continuity of ω with respect to the elements of $H(j\omega)$.

We present in Section 2 an example demonstrating that a certain measured function is highly sensitive to changes in the physical parameters of the system. Physical interpretation of this result is given in Section 3. In Section 4, while applying the result to groove identification in a vibrating rod, we find that the frequency of excitation required determining the location of the groove is extremely high. We therefore conclude that the problem of whether a small damage in a realistic structure can be identified by a dynamic test is still subject to debate.

2. Damage in a discrete model of a uniform vibrating rod

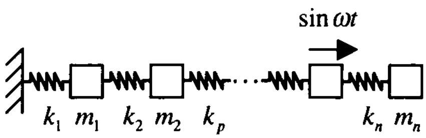


Figure 1. *Mass-spring system*

Consider the n-degree-of-freedom system shown in Figure 1, which consists of masses m_i and springs of constants k_i, $i = 1,2,\dots,n$. Such a system represents a discrete lumped-parameter-model, or finite difference model, of an axially vibrating rod. Suppose that the harmonic excitation $f(t) = \sin(\omega t)$ is applied to the j-th degree of freedom. Then, the motion of the system is described by the set of ordinary differential equations

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{e}_j \sin\omega t\,, \tag{2}$$

where:

$$\mathbf{M} = diag\{m_1, m_2, \dots, m_n\}, \tag{3}$$

$$\mathbf{K} = \begin{bmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1}+k_n & -k_n \\ & & & -k_n & k_n \end{bmatrix}, \tag{4}$$

and $\mathbf{e}_j$ is the j-th unit vector of dimension n. The system [2] has a particular solution of the form

$$\mathbf{x}_j(t) = \mathbf{h}_j \sin \omega t\,, \quad j = 1,2,\ldots,n\,, \qquad [5]$$

where $\mathbf{h}_j$ is a constant vector. Substituting [5] in [2] gives

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\mathbf{h}_j = \mathbf{e}_j\,, \quad j = 1,2,\ldots,n\,. \qquad [6]$$

The n-equations defined by [6] can be assembled as follows

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\mathbf{H} = \mathbf{I}\,, \qquad [7]$$

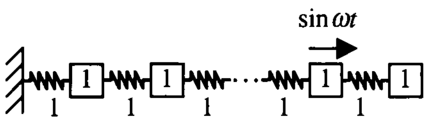
where $\mathbf{I}$ is the identity matrix and

$$\mathbf{H} = \left[\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_n\right]. \qquad [8]$$

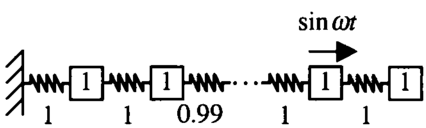
We thus have

$$\mathbf{H}(\omega) = \left(\mathbf{K} - \omega^2\mathbf{M}\right)^{-1}\,. \qquad [9]$$

The matrix $\mathbf{H}(\omega)$ is called the *Frequency Response Function* (FRF) matrix. Its element $h_{ij}(\omega)$ represents the steady state amplitude of the harmonic response at the i-th degree-of-freedom due to a unit sinusoidal excitation applied to the j-th degree-of-freedom. Hence, the elements of $\mathbf{H}(\omega)$ can be determined by simple vibration tests.



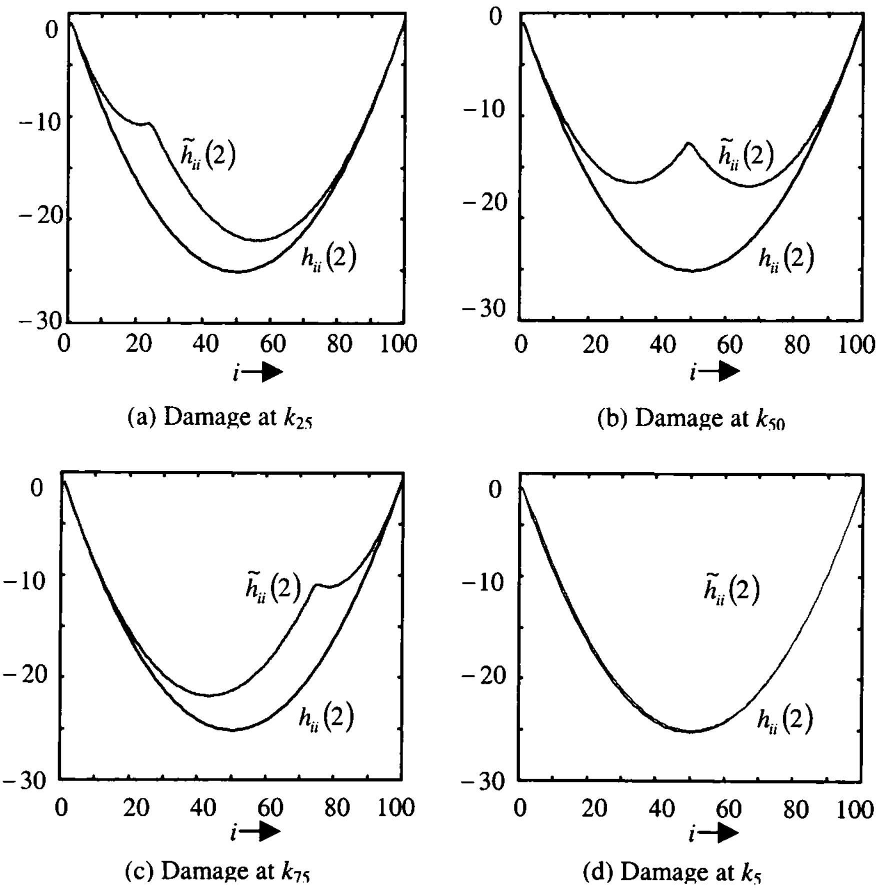
(a) Pure system



(b) Damaged system

Figure 2. *Systems description: (a) pure system, and (b) damaged system*

Consider now the uniform system of unit parameters $m_i = k_i = 1$, $1 = 1,2,\ldots,100$ of dimension $n = 100$, shown in Figure 2(a). A harmonic exciting force with frequency $\omega = 2$ is applied to the system at the j-th degree-of-freedom. Suppose that the constant of the p-th spring is reduced due to a damage to $k_p = 0.99$, as shown in Figure 2(b). Let $h_{ii}(2)$ and $\tilde{h}_{ii}(2)$ be the collocated frequency-response-functions at $i = 1,2,\ldots,n$ of the pure system and the damaged system, respectively. These functions are plotted in Figure 3(a) for the case where $p = 25$, i.e. the damage is applied to the 25-th spring. Similar graphs for the cases that the damage is applied to the 50-th and 75-th degrees of freedom are shown in Figures 3(b) and 3(c), respectively. It is apparent that the damage and its location is clearly observable by discontinuity in slope of $\tilde{h}_{ii}(2)$. In contrast, damaged applied to the 5-th spring cannot be unambiguously identified as shown in Figure 3(d). Such a result can be grasp by intuition. The fifth spring is too close to the support and hence cannot be excited easily by the harmonic force.



(a) Damage at k_{25}

(b) Damage at k_{50}

(c) Damage at k_{75}

(d) Damage at k_5

Figure 3. *Damage identification*

It should be note that other harmonic forces, of different frequencies, may not provide clear identification of the damage and its location. For example, Figures 4(a) and 4(b) display the functions $h_{ii}(1.5)$ and $\tilde{h}_{ii}(1.5)$ for the case where the damaged is located at the $p = 50$ spring and the exciting frequency is $\omega = 1.5$. The two functions associated with the pure and damaged systems look almost identical. Figures 4(c) and 4(d) display these frequency-response-functions for the case where the exciting frequency is $\omega = 2.5$. Here a variation between the two functions at the damage location $p = 50$ is observed. This variation is small, however, and cannot be considered as a reliable criterion for damage detection for realistic systems. These intriguing results deserve further considerations.

3. The interpretation

Denote the stiffness matrix of the undamaged system shown in Figure 2(a) by $\mathbf{K}$, and let $\tilde{\mathbf{K}}$ be the stiffness matrix of the damaged system (Figure 2(b)). Then the stiffness matrix of the undamaged system shown in Figure 2(a) is:

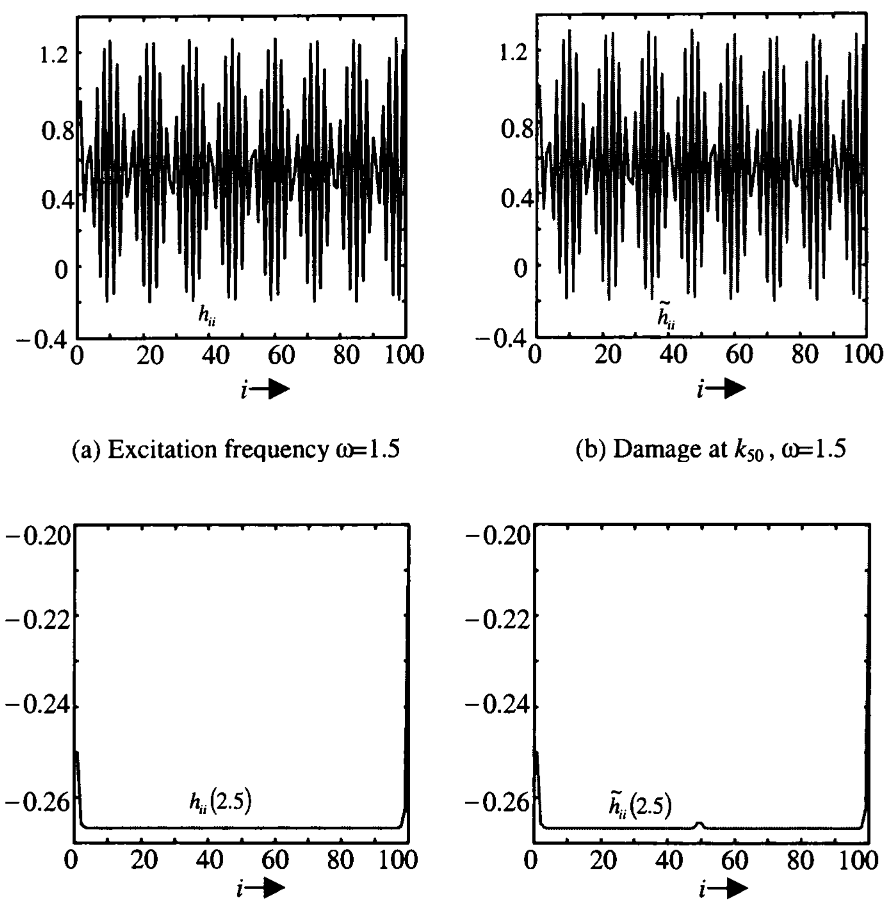
$$\mathbf{K} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}, \tag{10}$$

and the stiffness matrix of the damaged system (Figure 2(b)) is:

$$\tilde{\mathbf{K}} = \begin{array}{cc} \begin{array}{cccccccc} & & & & p\text{-th} & & & \\ & & & & \text{column} & & & \\ & & & & \downarrow & & & \end{array} & \\ \left[\begin{array}{cccccccc} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & \ddots & \ddots & \ddots & & & & \\ & & -1 & 2-\delta & -1+\delta & & & \\ & & & -1+\delta & 2-\delta & -1 & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 1 \end{array}\right] & \begin{array}{l} \\ \\ \\ \\ \leftarrow p\text{-th row} \\ \\ \\ \\ \end{array} \end{array} \tag{11}$$

Both systems share the same mass matrix $\mathbf{M} = \mathbf{I}$. Define a diagonal matrix:

$$\mathbf{D} = \begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}. \qquad [12]$$



(a) Excitation frequency ω=1.5

(b) Damage at k_{50}, ω=1.5

(c) Excitation frequency ω=2.5

(d) Damage at k_{50}, ω=2.5

Figure 4. *Excitations by various frequencies*

Then, since $\mathbf{D} = \mathbf{D}^{-1}$, we have by [6]

$$\mathbf{D}\left(\mathbf{K} - \omega^2\mathbf{M}\right)\mathbf{D}\mathbf{D}\mathbf{h}_j = \mathbf{D}\mathbf{e}_j, \qquad [13]$$

which implies that

$$-\mathbf{D}\left(\mathbf{K} - \omega^2\mathbf{M}\right)\mathbf{D}\mathbf{h}_j = -\mathbf{e}_j. \qquad [14]$$

For $\omega = 2$ equation [14] takes the following explicit form:

$$\begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -1 & 2 & -1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 3 \end{bmatrix} \begin{pmatrix} h_{1j} \\ h_{2j} \\ \vdots \\ h_{jj} \\ \vdots \\ h_{n-1,j} \\ h_{nj} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}. \qquad [15]$$

The physical interpretation of equation [15] is that h_{jj} is the static deflection of the j-th node due to a collocated unit static load applied to if, as shown in Figure 5(a). Note that the spring configuration of the system of Figure 5(a) is similar to that of the undamaged system shown in Figure 2(a), but with an additional spring of constant $k = 2$ attached between the n-th node and the ground.

Following a similar process we find that for the damaged system the equation

$$-\mathbf{D}\left(\tilde{\mathbf{K}} - \omega^2\mathbf{M}\right)\mathbf{D}\mathbf{h}_j = -\mathbf{e}_j \qquad [16]$$

holds, or explicitly:

$$\begin{bmatrix} 2 & -1 & & & & & & & & \\ -1 & 2 & -1 & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & & \\ & & -1 & 2 & -1 & & & & & \\ & & & \ddots & \ddots & \ddots & & & & \\ & & & & -1 & 2+\delta & -1+\delta & & & \\ & & & & & -1+\delta & 2+\delta & -1 & & \\ & & & & & & \ddots & \ddots & \ddots & \\ & & & & & & & -1 & 2 & -1 \\ & & & & & & & & -1 & 3 \end{bmatrix} \begin{pmatrix} h_{1j} \\ h_{2j} \\ \vdots \\ h_{jj} \\ \vdots \\ h_{p-1,j} \\ h_{pj} \\ \\ h_{n-1,j} \\ h_{nj} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \qquad [17]$$

The element h_{jj} in equation [15] represents the static deflection of the j-th node due to a collocated unit static load applied to if, as shown in Figure 5(b). When the applied force is in the close neighbourhood of the damaged the two springs of constant 2δ became dominant resulting with large change in the response.

The systems of Figure 5 can be represented by using equivalent springs as shown in Figure 6, where

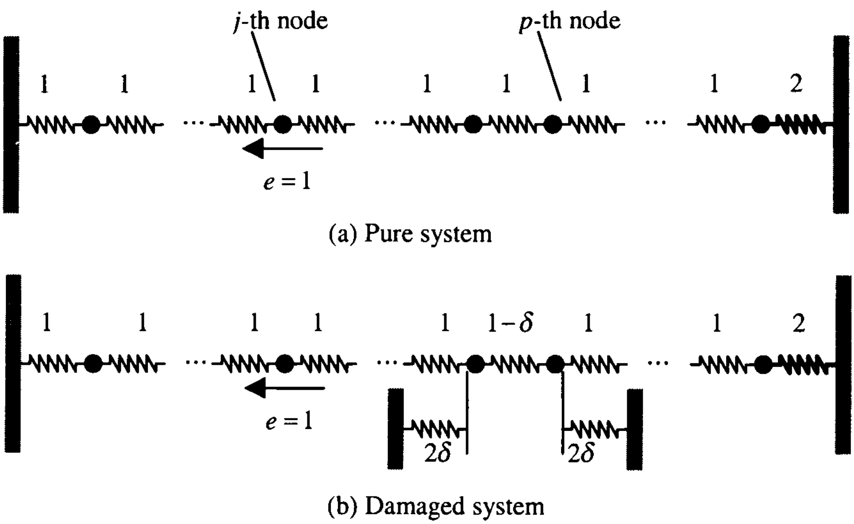


Figure 5. *Static models for (a) pure system, and (b) damaged system*

$$k_A = \frac{1}{j} \text{ and } k_B = \frac{1}{n - j + \frac{1}{2}} = \frac{2}{2n - 2j + 1}. \qquad [18]$$

It thus follows that

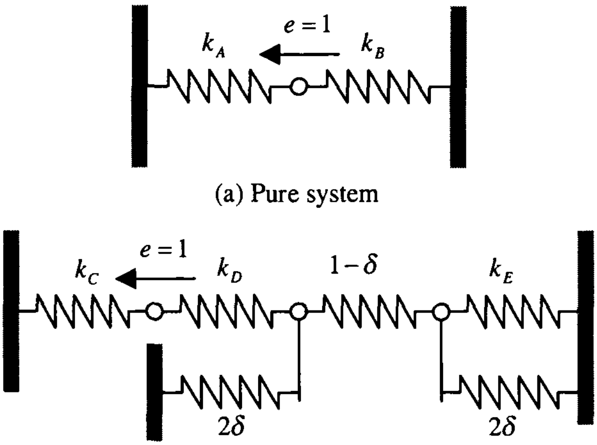
$$h_{jj}(2) = -\frac{1}{k_A + k_B} = -\frac{j(2n - 2j + 1)}{2n + 1}. \qquad [19]$$

In a similar manner, using:

$$k_C = \frac{1}{j},\ k_D = \frac{1}{p - j}, \text{ and } k_E = \frac{2}{2n - 2p - 1}, \qquad [20]$$

we obtain from Figure 6(b) that:

$$\tilde{h}_{jj}(2)=\begin{cases}\dfrac{j(2j-\delta(p(8n+8j+12-8p)-8jn-6n-6j-5)-1-2n)}{-8\delta p^2+8p\delta n+12\delta p+1-6\delta n-5\delta+2n} & j<p\\[2ex] \dfrac{(-1-2n+2j)(4j\delta p-4\delta p^2+4\delta p-3\delta j-d+j)}{-8\delta p^2+8p\delta n+12\delta p+1-6\delta n-5\delta+2n} & j\geq p\end{cases} \qquad [21]$$



(a) Pure system

(b) Damaged system

Figure 6. *Equivalent static models for (a) pure system, and (b) damaged system*

The response $h_{jj}(2)$ in Equation [19] describes a smooth function in j, while $\tilde{h}_{jj}(2)$ features derivative discontinuity at $j=p$, which allows identification of the damage and its location. In fact, the plots in Figures 3 and 4 are precisely $h_{jj}(2)$ and $\tilde{h}_{jj}(2)$ given by equations [19] and [21].

4. Damage identification in an axially vibrating rod

Consider an axially vibrating uniform rod of length L, modulus of elasticity E, density ρ and cross-sectional area A, which is fixed at one end, $x=0$, and free to oscillate at the other end, $x=L$. The axial vibrations of this rod are governed by the differential equation and boundary conditions

$$
\begin{aligned}
EA\frac{\partial^2 u}{\partial x^2} &= \rho A\frac{\partial^2 u}{\partial t^2} \\
u(0,t) &= \frac{\partial u}{\partial x}(L,t) = 0.
\end{aligned}
\tag{22}
$$

A discrete mass-spring system model of dimension n, with equal length elements of length $h = L/n$, leads to the eigenvalue problem:

$$(\mathbf{A} - \lambda\mathbf{B})\mathbf{u} = \mathbf{o} \tag{23}$$

where $\mathbf{A} = \frac{EA}{h}\mathbf{K}$, $\mathbf{B} = \rho A h \mathbf{I}_n$, and where $\mathbf{K}$ is given by [10].

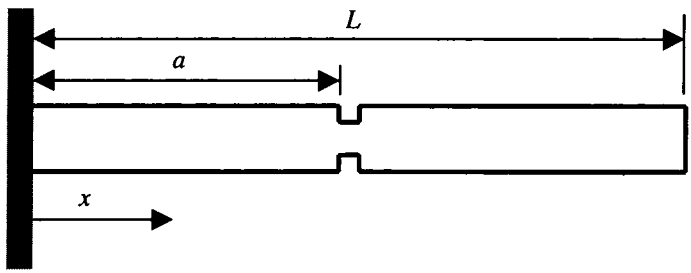


Figure 7. *An axially vibrating uniform rod with groove at* $x = a$

A corresponding damaged rod with groove at $x = a$ is shown in Figure 7. Let $a = (p-1)h$ for some integer $1 \le p \le n$. Let the groove width be w, and let $A_1 < A$ be the cross-sectional of the rod at the groove position. Then the eigenvalue problem associated with the damaged rod takes the form

$$(\tilde{\mathbf{A}} - \lambda\tilde{\mathbf{B}})\mathbf{u} = \mathbf{o} \tag{24}$$

where $\tilde{\mathbf{A}} = \frac{EA}{h}\tilde{\mathbf{K}}$, with $\tilde{\mathbf{K}}$ given by [11],

$$\delta = 1 - \frac{hA_1}{w(A - A_1) + hA_1}, \tag{25}$$

$$\tilde{\mathbf{B}} = \rho A h \begin{bmatrix} \mathbf{I}_{p-1} & & \\ & \eta & \\ & & \mathbf{I}_{n-p} \end{bmatrix}, \qquad [26]$$

and:

$$\eta = 1 - \frac{(A - A_1)w}{Ah}. \qquad [27]$$

In order to identify the location of the groove in this system using the method presented in Section 2 it is required to excite the system with the frequency

$$\omega = \frac{2}{h}\sqrt{\frac{E}{\rho}}. \qquad [28]$$

For steel rod $\rho = 7800$ kg/m^3, $E = 1.962 \times 10^{11}$ N/m^2, of length $L = 1$ m, with $A = 0.01$ m^2, $A_1 = 0.0095$ $w = 0.01$m, $n = 50$, and $p = 25$, we obtain: $\delta = 0.0256$, $\eta = 0.9750$, and $\omega = 5.0154 \times 10^5$ rad/s. The functions $h_{jj}(\omega)$ for the uniform rod, and $\tilde{h}_{jj}(\omega)$ associated with the damaged rod, are shown in Figure 8. Although the damaged position is observable, its effect is less drastic than that obtained in Section 2 for the chain of mass-spring system. The reason is that the groove in the rod affects both the stiffness and the mass of the p-th element. In practice it may be difficult to excite the rod with the high frequency ω required. With current progress in materials research, however, such excitations using piezoelectric materials are not entirely beyond futuristic expectations.

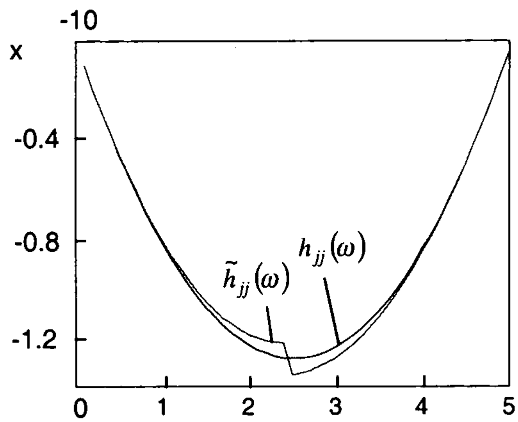


Figure 8. *Groove identification in an axially vibrating rod*

5. Conclusions

There are procedures in the literature for damage identification, accompanying with experimental results. The inherent difficulty associated with the repeatability of modal analysis testing forms a barrier in achieving a scientific consensus regarding the applicability of these procedures when slightly damaged systems are considered. In the theoretical arena the fundamental problem regarding the existence of measurable quantities, which are sensitive to small changes in the physical parameters of the system, is still open. In an effort to address this issue we have presented an analytical example showing that under certain circumstances small changes in the stiffness of a uniform chain of mass-spring system can be detected. The example was restricted by the need of imposing a special frequency of excitation. It is well known that the frequency response of a damped system is smoother than that associated with its conservative counterpart. Hence damage in damped system may less identifiable. In the context of realistic components, such as identification of a small groove in an axially vibrating steel rod, the method requires excitation with very high frequencies. We may thus argue that the fundamental question whether small defects in structures can be detected by vibration tests is still subject to debate.

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