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New wave solutions for the fractional-order biological population model, time fractional burgers, Drinfel'd–Sokolov–Wilson and system of shallow water wave equations and their applications

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ABSTRACT
In this research, by applying the improved $\left(\frac{G'}{G}\right)$-expansion method, we have found the travelling and solitary wave solutions of the fractional-order biological population model, time fractional Burgers equation, the Drinfel'd–Sokolov–Wilson equation and the system of shallow water wave equations. The advantage of this method is providing a new and more general travelling wave solutions for many non-linear evolution equations, it supply three different kind of solutions in the form (the hyperbolic functions, the trigonometric functions and the rational functions). This method included the extended $\left(\frac{G'}{G}\right)$-expansion method when $\sigma = 0$ and the $\left(\frac{G'}{G}\right)$-expansion method when N takes only positive value and zero. All of these merits help us in survey of the physical meaning of each models mentioned above for investigating stability of these models. Rapprochement between our results and the previous renowned outcome presented.

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1. Introduction

The second half of the twentieth century saw a second revolution in studying the classical physics. A wave equation having soliton solutions has both non-linearity and dispersive property. Scientists showed the effects of non-linearity in the dynamical equations. They showed the non-linearity has two interesting manifestations of opposite property. One of this property is chaos and the second is soliton. Both of these properties developed into paradigms with solid mathematical background and physical observation.

Over the years, many methods were developed to evaluate the exact and solitary wave solutions of physical, biological, plasma, fluid mechanics, optical fibers, solid state physics, chemical kinetics and geochemistry . . . etc. models. For

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examples Wazwaz (2004), Khater (2015), Zhang (2008), Seadawy and Dianchen (2016), Kim and Sakthivel (2012), Vitanov (2010), Xie, Yan, and Zhang (2001), Seadawy (2015), Ma and Fuchssteiner (1996), Hu and Zhang (2001), Seadawy (2012), Wedin & Kerswell (2004) and so on.

The technique adopted is to expand a solution to higher-dimensional equations around a solution to an ODE. This basic idea presented and analysed in the reference Ma and Fuchssteiner (1996) as well, and a more systematical approach to travelling waves was emerged into a so-called transformed rational function method (Ma & Lee, 2009). The current situation in the manuscript had an application of such explored idea or method, since an expansion in (8) is a rational function of the $\left(\frac{G'}{G}\right)$ function, whose Equation (9) implies that $\left(\frac{G'}{G}\right)$ satisfies a Riccati equation. The general solution to a general Riccati equation has been given by (40)–(42) in Ma and Fuchssteiner (1996). More generally, multiple wave solutions could be presented by the multiple exp-function method (Ma & Zhu, 2012), and this is the unique way to go with, following Fourier theory. Another interesting question is how to transform a fractional PDE into a non-fractional ODE. Is a dimensional reduction sufficient. The current manuscript deals with a few interesting questions and the computations are carefully made. The new method for finding the exact solutions of some time-fractional Kortewegde Vries (KdV) type equations appearing in shallow water waves are obtained. The new method for time-fractional equations viz. time-fractional KdV-Burgers and KdV-mKdV equations for finding the exact solutions is employed. The fractional complex transform accompanied by properties of local fractional calculus for reduction of fractional partial differential equations to ordinary differential equations are used Sahoo and Saha (2016). The fractional complex transform method for wave equations on Cantor sets within the local differential fractional operators are obtained. The proposed method was efficient to handle differential equations on Cantor sets (Yang, Baleanu, & He, 2013; Zhao, Cai, & Yang, 2016).

The basic point of this research is to stratify the improved $\left(\frac{G'}{G}\right)$-expansion method for obtaining the travelling and solitary wave solution of fractional-order biological population model, time fractional burgers equation, the Drinfel'd–Sokolov–Wilson equation and the system of shallow water wave equations (Arshad, Seadawy, Lu, & Wang, 2016; Khalfallah, 2009; Lu, Seadawy, & Arshad, 2017; Selima, Seadawy, & Yao, 2016; Seadawy, 2017; Seadawy, Arshad, & Lu, 2017; Zahran & Khater, 2014).

The remnant of this paper is systematised as follows: In Section 2, we give the description of the improved $\left(\frac{G'}{G}\right)$-expansion method. In Section 3, we use this method to get the exact solutions of (NLPDEs.) pointed out above. In Section 5, conclusions are given.

2. Basic steps of technique

2.1. Conformable fractional derivative and its properties

The conformable fractional derivative of order α is defined by the following expression (Anderson & Ulness, 2015; Hosseini, Bekir, & Ansari, 2017; Khalil et al., 2014)

$$D_\alpha f(x) = \lim_{\tau \to 0} \frac{f(t + \tau\, t^{1-\alpha}) - f(t)}{\tau} = t^{1-\alpha} f'(t), \tag{1}$$

where $(0 < \alpha < 1)$. We list some important properties for the conformable fractional derivative as follows:

$$D_t^\alpha t^r = r\, t^{r-\alpha}, \tag{2}$$

$$D_t^\alpha (f(t) g(t)) = g(t) D_t^\alpha f(t) + f(t) D_t^\alpha g(t), \tag{3}$$

$$D_t^\alpha \frac{f(t)}{g(t)} = \frac{g(t)\, D_t^\alpha\, f(t) - f(t)\, D_t^\alpha\, g(t)}{g^2(t)}. \tag{4}$$

2.2. The improved $\left(\frac{G'}{G}\right)$-expansion method

Consider the following non-linear evolution equations

$$\begin{cases} (u, u_x, u_y, u_t, u_{xy}, u_{xt}, \ldots) & = 0, \\ F(u, D_x^\alpha u, D_y^\alpha u, D_t^\alpha u, D_x^\alpha D_y^\alpha u, D_x^\alpha D_t^\alpha u, \ldots) & = 0. \end{cases} \tag{5}$$

where F is a polynomial in $u(x, t)$ and its partial derivatives in which the highest order derivatives and non-linear terms are involved. In the following, we give the main steps of this method:

Step 1. We use the wave transformation

$$\begin{cases} (x, y, t) = u(\xi), & \xi = x + y - c\, t, \\ u(x, y, t) = u(\xi), & \xi = x + y + \frac{c\, t^\alpha}{\alpha}. \end{cases} \tag{6}$$

where c is a constant, to reduce Equation (5) to the following ODE:

$$P(u, u', u'', u''', \ldots) = 0, \tag{7}$$

where P is a polynomial in $u(\xi)$ and its total derivatives.

Step 2. Suppose the solution of Equation (7) has the form:

$$u(\xi) = \sum_{i=-N}^{N} a_i \left(\frac{\left(\frac{G'}{G}\right)}{\left(1 + \sigma \left(\frac{G'}{G}\right)\right)} \right)^i, \tag{8}$$

where a_i, is constants to be determined, such that $a_N \neq 0$ and $G = G(\xi)$ satisfies the following second-order linear ordinary differential equation(LODE):

$$G'' + \mu G = 0, \tag{9}$$

Step 3. Determine the positive integer N in Equation (8) by balancing the highest order derivatives and the non-linear terms.

Step 4. Substitute Equation (8) along Equation (9) into Equation (7) and collecting all the terms of the same power $\left(\frac{G'}{G}\right)^i$ where ($i = N, N-1, \ldots, 1-N$, $-N$) and equating them to zero, we obtain a system of algebraic equations, which can be solved by Maple or Mathematica to get the values of a_i and σ.

Step 5. Substituting these values and the solutions of Equation (9) into Equation (8), we obtain the exact solutions of Equation (5).

3. Application

Here, we apply the improved $\left(\frac{G'}{G}\right)$-expansion method described in Section 2 to find the exact travelling wave solutions and the solitary wave solutions of fractional-order biological population model, time fractional burgers equation, the Drinfel'd–Sokolov–Wilson equation and the system of shallow water wave equations.

3.1. *Fractional-order biological population model*

Consider a time fractional biological population model (Bekir, Guner, & Cevikel, 2013; Lu, 2012; Rida, Arafa, & Mohamed, 2011; Zhang & Zhang, 2011):

$$\frac{\partial^2}{\partial x^2}v^2 + \frac{\partial^2}{\partial y^2}v^2 + h(v^2 - r) - \frac{\partial^\alpha u}{\partial t^\alpha} = 0, \tag{10}$$

where h and r are arbitrary constants. Using the wave travelling transformation

$$v(\xi) = v(x,t) \text{ where } \left(\xi = kx + iky + \frac{\omega t^\alpha}{\alpha}\right),$$

carries the partial differential equation (PDE.) (10) into the ordinary differential equation (ODE.):

$$v^2 - \delta v' - r = 0. \tag{11}$$

Where $\delta = \frac{\omega}{h}$. Balancing between the highest order derivatives and nonlinear terms appearing in Equation (11) $\Rightarrow \left(v' \,\&\, v^2\right)$ we obtain ($N = 1$). So that, by using Equation (8) we get the formal solution of Equation (11):

$$v = a_{-1}\frac{1 + \sigma\left(\frac{G'}{G}\right)}{\left(\frac{G'}{G}\right)} + a_0 + a_1\frac{\left(\frac{G'}{G}\right)}{1 + \sigma\left(\frac{G'}{G}\right)}. \tag{12}$$

Substituting Equation (12) and its derivative into Equation (11) and collecting all term with the same power of $\left(\frac{G'}{G}\right)^j$ where $\left(j = 4, 3, 2, 1, 0\right)$ we get algebraic system. By solving it by any computer program like maple, mathematica, . . . etc, we secure:

$$\mu = 0, \sigma = \sigma, a_{-1} = 0, a_0 = -\delta, a_1 = \delta\sigma.$$

So that, the travelling wave solution:

$$v = \delta\left(-1 + \sigma\left(\frac{\left(\frac{G'}{G}\right)}{1 + \sigma\left(\frac{G'}{G}\right)}\right)\right), \tag{13}$$

then, we have only rational case solutions:

$$v = \delta\left(-1 + \frac{\sigma A_2}{A_1 + (\xi + \sigma)A_2}\right). \tag{14}$$

3.2. Time fractional Burgers equation

Consider the one-dimensional time fractional Burgers equation (Bekir et al., 2013; Inc, 2008):

$$\frac{\partial^\alpha v}{\partial t^\alpha} + \epsilon v v_x - \sigma v_{xx} = 0. \tag{15}$$

where ϵ and σ are constants. Using wave transformation

$$u(x, t) = u(\xi) \quad \xi = kx + \frac{\omega t^\alpha}{\alpha},$$

which carries PDE. (15) into ODE.:

$$v + \rho v^2 + \beta v' + c = 0. \tag{16}$$

Where $\rho = \frac{ck}{2\omega}$ and $\beta = -\frac{\sigma k^2}{\omega}$. Balancing between the highest order derivatives and non-linear terms appearing in (16) $\Rightarrow$ $\left(v' \,\&\, v^2\right)$ we obtain $\left(N = 1\right)$. So that, by using Equation (8) we get the same formal solution of Equation (11). Substituting Equation (12) and its derivative into Equation (16) and collecting all term with the same power of $\left(\frac{G'}{G}\right)^j$ where $\left(j = 4, 3, 2, 1, 0\right)$ we get algebraic system. By solving it by any computer program like maple, mathematica, . . . etc, we procure:

Case 1

$$\rho = 0, a_{-1} = -c, a_0 = 2c\sigma, a_1 = -c\sigma^2$$

So that, the exact travelling wave solution:

$$v = -c\frac{1 + \sigma\left(\frac{G'}{G}\right)}{\left(\frac{G'}{G}\right)} + 2c\sigma - c\sigma^2\frac{\left(\frac{G'}{G}\right)}{1 + \sigma\left(\frac{G'}{G}\right)}. \tag{17}$$

Then, the solitary wave solutions:
When ($\mu < 0$), we obtain the hyperbolic function solution:

$$v = -c\frac{\left(A_1 + A_2\sigma\sqrt{-\mu}\right)\cosh\left(\sqrt{-\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{-\mu}\right)\sinh\left(\sqrt{-\mu}\xi\right)}{\sqrt{-\mu}\left(A_1\sinh\left(\sqrt{-\mu}\xi\right) + A_2\cosh\left(\sqrt{-\mu}\xi\right)\right)} + 2c\sigma - c\sigma^2\frac{\sqrt{-\mu}\left(A_1\sinh\left(\sqrt{-\mu}\xi\right) + A_2\cosh\left(\sqrt{-\mu}\xi\right)\right)}{\left(A_1 + A_2\sigma\sqrt{-\mu}\right)\cosh\left(\sqrt{-\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{-\mu}\right)\sinh\left(\sqrt{-\mu}\xi\right)}. \quad (18)$$

When ($\mu > 0$), we obtain the trigonometric function solution:

$$v = -c\frac{\left(A_1 - A_2\sigma\sqrt{\mu}\right)\sin\left(\sqrt{\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{\mu}\right)\cos\left(\sqrt{\mu}\xi\right)}{\sqrt{\mu}\left(A_1\cos\left(\sqrt{\mu}\xi\right) - A_2\sin\left(\sqrt{\mu}\xi\right)\right)} + 2c\sigma - c\sigma^2\frac{\sqrt{\mu}\left(A_1\cos\left(\sqrt{\mu}\xi\right) - A_2\sin\left(\sqrt{\mu}\xi\right)\right)}{\left(A_1 - A_2\sigma\sqrt{\mu}\right)\sin\left(\sqrt{\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{\mu}\right)\cos\left(\sqrt{\mu}\xi\right)}. \quad (19)$$

When ($\mu = 0$), we obtain the rational function solution:

$$v = -c\frac{A_1 + (\xi + \sigma)A_2}{A_2} + 2c\sigma - c\sigma^2\frac{A_2}{A_1 + (\xi + \sigma)A_2}. \quad (20)$$

Case 2

$$\rho = -\frac{a_{-1} + c}{a_{-1}^2}, \sigma = 0, a_{-1} = a_{-1}, a_0 = a_1 = 0$$

So that, the exact travelling wave solution:

$$v = a_{-1}\frac{1 + \sigma\left(\frac{G'}{G}\right)}{\left(\frac{G'}{G}\right)}. \quad (21)$$

Then, the solitary wave solutions:
When ($\mu < 0$), we obtain the hyperbolic function solution:

$$v = a_{-1}\frac{\left(A_1 + A_2\sigma\sqrt{-\mu}\right)\cosh\left(\sqrt{-\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{-\mu}\right)\sinh\left(\sqrt{-\mu}\xi\right)}{\sqrt{-\mu}\left(A_1\sinh\left(\sqrt{-\mu}\xi\right) + A_2\cosh\left(\sqrt{-\mu}\xi\right)\right)}. \quad (22)$$

When ($\mu > 0$), we obtain the trigonometric function solution:

$$v = a_{-1}\frac{\left(A_1 - A_2\sigma\sqrt{\mu}\right)\sin\left(\sqrt{\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{\mu}\right)\cos\left(\sqrt{\mu}\xi\right)}{\sqrt{\mu}\left(A_1\cos\left(\sqrt{\mu}\xi\right) - A_2\sin\left(\sqrt{\mu}\xi\right)\right)}. \quad (23)$$

When ($\mu = 0$), we obtain the rational function solution:

$$v = a_{-1}\frac{A_1 + (\xi + \sigma)A_2}{A_2}. \quad (24)$$

Case 3

$$\mu = \frac{-1 + 4c\rho}{4\beta^2}, \sigma = 0, a_{-1} = -\frac{1}{2\alpha}, a_0 = \frac{\beta}{\rho}, a_1 = 0$$

So that, the exact travelling wave solution:

$$v = -\frac{1}{2\alpha}\frac{1+\sigma\left(\frac{G'}{G}\right)}{\left(\frac{G'}{G}\right)} + 2c\sigma - c\sigma^2\frac{\left(\frac{G'}{G}\right)}{1+\sigma\left(\frac{G'}{G}\right)} + \frac{\beta}{\rho}. \tag{25}$$

Then, the solitary wave solutions:
When ($\mu < 0$), we obtain the hyperbolic function solution:

$$v = -\frac{1}{2\alpha}\frac{\left(A_1 + A_2\sigma\sqrt{-\mu}\right)\cosh\left(\sqrt{-\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{-\mu}\right)\sinh\left(\sqrt{-\mu}\xi\right)}{\sqrt{-\mu}\left(A_1\sinh\left(\sqrt{-\mu}\xi\right) + A_2\cosh\left(\sqrt{-\mu}\xi\right)\right)} + \frac{\beta}{\rho}. \tag{26}$$

When ($\mu > 0$), we obtain the trigonometric function solution:

$$v = -\frac{1}{2\alpha}\frac{\left(A_1 - A_2\sigma\sqrt{\mu}\right)\sin\left(\sqrt{\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{\mu}\right)\cos\left(\sqrt{\mu}\xi\right)}{\sqrt{\mu}\left(A_1\cos\left(\sqrt{\mu}\xi\right) - A_2\sin\left(\sqrt{\mu}\xi\right)\right)} + \frac{\beta}{\rho}. \tag{27}$$

When ($\mu = 0$), we obtain the rational function solution:

$$v = -\frac{1}{2\alpha}\frac{A_1 + \left(\xi + \sigma\right)A_2}{A_2} + \frac{\beta}{\rho}. \tag{28}$$

Case 4

$$\mu = -\frac{a_{-1}\left(-2a_{-1}^2 + a_{-1}a_0 + 2ca_0\right)}{\left(2a_{-1} + a_0\right)\left(8a_{-1}^2 + 4a_{-1}a_0 + a_0^2\right)}, \rho = -\frac{2a_{-1} + a_0}{2a_{-1}a_0},$$
$$\beta = -\frac{8a_{-1}^2 + 4a_{-1}a_0 + a_0^2}{2a_{-1}a_0}, \sigma = 1, a_{-1}, a_0 = a_0, a_1 = -a_{-1} - a_0.$$

So that, the exact travelling wave solution:

$$v = a_{-1}\frac{1+\sigma\left(\frac{G'}{G}\right)}{\left(\frac{G'}{G}\right)} + a_0 - a_{-1} - a_0\frac{\left(\frac{G'}{G}\right)}{1+\sigma\left(\frac{G'}{G}\right)}. \tag{29}$$

Then, the solitary wave solutions:
When ($\mu < 0$), we obtain the hyperbolic function solution:

$$v = a_{-1}\frac{\left(A_1 + A_2\sigma\sqrt{-\mu}\right)\cosh\left(\sqrt{-\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{-\mu}\right)\sinh\left(\sqrt{-\mu}\xi\right)}{\sqrt{-\mu}\left(A_1\sinh\left(\sqrt{-\mu}\xi\right) + A_2\cosh\left(\sqrt{-\mu}\xi\right)\right)} + a_0 - a_{-1} - a_0\frac{\sqrt{-\mu}\left(A_1\sinh\left(\sqrt{-\mu}\xi\right) + A_2\cosh\left(\sqrt{-\mu}\xi\right)\right)}{\left(A_1 + A_2\sigma\sqrt{-\mu}\right)\cosh\left(\sqrt{-\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{-\mu}\right)\sinh\left(\sqrt{-\mu}\xi\right)}. \tag{30}$$

When ($\mu > 0$), we obtain the trigonometric function solution:

$$\begin{aligned} v =& a_{-1}\frac{\left(A_1 - A_2\sigma\sqrt{\mu}\right)\sin\left(\sqrt{\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{\mu}\right)\cos\left(\sqrt{\mu}\xi\right)}{\sqrt{\mu}\left(A_1\cos\left(\sqrt{\mu}\xi\right) - A_2\sin\left(\sqrt{\mu}\xi\right)\right)} \\ &+ a_0 - a_{-1} - a_0\frac{\sqrt{\mu}\left(A_1\cos\left(\sqrt{\mu}\xi\right) - A_2\sin\left(\sqrt{\mu}\xi\right)\right)}{\left(A_1 - A_2\sigma\sqrt{\mu}\right)\sin\left(\sqrt{\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{\mu}\right)\cos\left(\sqrt{\mu}\xi\right)}. \end{aligned} \tag{31}$$

When ($\mu = 0$), we obtain the rational function solution:

$$v = a_{-1}\frac{A_1 + \left(\xi + \sigma\right)A_2}{A_2} + a_0 - a_{-1} - a_0\frac{A_2}{A_1 + \left(\xi + \sigma\right)A_2}. \tag{32}$$

Case 5

$$\mu = -\frac{a_{-1}\left(a_{-1}a_0 + 2a_{-1}^2 + 2ca_0\right)}{a_0\left(-a_0 + 4a_{-1}\right)\left(-a_0 + 2a_{-1}\right)}, \rho = \frac{-a_0 + 2a_{-1}}{2a_{-1}a_0}, \beta = \frac{-a_0 + 4a_{-1}}{2a_{-1}},$$

$$\sigma = -1, a_1 = a_1, a_0 = a_0, a_1 = -a_{-1} + a_0.$$

So that, the exact travelling wave solution:

$$v = a_{-1}\frac{1 + \sigma\left(\frac{G'}{G}\right)}{\left(\frac{G'}{G}\right)} + a_0 - a_{-1} + a_0\frac{\left(\frac{G'}{G}\right)}{1 + \sigma\left(\frac{G'}{G}\right)}. \tag{33}$$

Then, the solitary wave solutions:
When ($\mu < 0$), we obtain the hyperbolic function solution:

$$\begin{aligned} v =& a_{-1}\frac{\left(A_1 + A_2\sigma\sqrt{-\mu}\right)\cosh\left(\sqrt{-\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{-\mu}\right)\sinh\left(\sqrt{-\mu}\xi\right)}{\sqrt{-\mu}\left(A_1\sinh\left(\sqrt{-\mu}\xi\right) + A_2\cosh\left(\sqrt{-\mu}\xi\right)\right)} \\ &+ a_0 - a_{-1} + a_0\frac{\sqrt{-\mu}\left(A_1\sinh\left(\sqrt{-\mu}\xi\right) + A_2\cosh\left(\sqrt{-\mu}\xi\right)\right)}{\left(A_1 + A_2\sigma\sqrt{-\mu}\right)\cosh\left(\sqrt{-\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{-\mu}\right)\sinh\left(\sqrt{-\mu}\xi\right)}. \end{aligned} \tag{34}$$

When ($\mu > 0$), we obtain the trigonometric function solution:

$$\begin{aligned} v =& a_{-1}\frac{\left(A_1 - A_2\sigma\sqrt{\mu}\right)\sin\left(\sqrt{\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{\mu}\right)\cos\left(\sqrt{\mu}\xi\right)}{\sqrt{\mu}\left(A_1\cos\left(\sqrt{\mu}\xi\right) - A_2\sin\left(\sqrt{\mu}\xi\right)\right)} \\ &+ a_0 - a_{-1} + a_0\frac{\sqrt{\mu}\left(A_1\cos\left(\sqrt{\mu}\xi\right) - A_2\sin\left(\sqrt{\mu}\xi\right)\right)}{\left(A_1 - A_2\sigma\sqrt{\mu}\right)\sin\left(\sqrt{\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{\mu}\right)\cos\left(\sqrt{\mu}\xi\right)}. \end{aligned} \tag{35}$$

When ($\mu = 0$), we obtain the rational function solution:

$$v = a_{-1}\frac{A_1 + \left(\xi + \sigma\right)A_2}{A_2} + a_0 + a_{-1} - a_0\frac{A_2}{A_1 + \left(\xi + \sigma\right)A_2}. \tag{36}$$

Case 6

$$\rho = \frac{3}{16c}, \beta = 0, \sigma = -1, a_{-1} = -\frac{4c}{3}, a_0 = -\frac{16c}{3}, a_1 = -4c$$

So that, the exact travelling wave solution:

$$v = -\frac{4c}{3}\frac{1+\sigma\left(\frac{G'}{G}\right)}{\left(\frac{G'}{G}\right)} - \frac{16c}{3} - 4c\frac{\left(\frac{G'}{G}\right)}{1+\sigma\left(\frac{G'}{G}\right)}. \tag{37}$$

Then, the solitary wave solutions:
When $(\mu < 0)$, we obtain the hyperbolic function solution:

$$\begin{aligned} v = &-\frac{4c}{3}\frac{\left(A_1 + A_2\sigma\sqrt{-\mu}\right)\cosh\left(\sqrt{-\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{-\mu}\right)\sinh\left(\sqrt{-\mu}\xi\right)}{\sqrt{-\mu}\left(A_1\sinh\left(\sqrt{-\mu}\xi\right) + A_2\cosh\left(\sqrt{-\mu}\xi\right)\right)} \\ &- \frac{16c}{3} - 4c\frac{\sqrt{-\mu}\left(A_1\sinh\left(\sqrt{-\mu}\xi\right) + A_2\cosh\left(\sqrt{-\mu}\xi\right)\right)}{\left(A_1 + A_2\sigma\sqrt{-\mu}\right)\cosh\left(\sqrt{-\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{-\mu}\right)\sinh\left(\sqrt{-\mu}\xi\right)}. \end{aligned} \tag{38}$$

When $(\mu > 0)$, we obtain the trigonometric function solution:

$$\begin{aligned} v = &-\frac{4c}{3}\frac{\left(A_1 - A_2\sigma\sqrt{\mu}\right)\sin\left(\sqrt{\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{\mu}\right)\cos\left(\sqrt{\mu}\xi\right)}{\sqrt{\mu}\left(A_1\cos\left(\sqrt{\mu}\xi\right) - A_2\sin\left(\sqrt{\mu}\xi\right)\right)} \\ &- \frac{16c}{3} - 4c\frac{\sqrt{\mu}\left(A_1\cos\left(\sqrt{\mu}\xi\right) - A_2\sin\left(\sqrt{\mu}\xi\right)\right)}{\left(A_1 - A_2\sigma\sqrt{\mu}\right)\sin\left(\sqrt{\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{\mu}\right)\cos\left(\sqrt{\mu}\xi\right)}. \end{aligned} \tag{39}$$

When $(\mu = 0)$, we obtain the rational function solution:

$$v = -\frac{4c}{3}\frac{A_1 + (\xi + \sigma)A_2}{A_2} - \frac{16c}{3} - 4c\frac{A_2}{A_1 + (\xi+\sigma)A_2}. \tag{40}$$

3.3. The Drinfel'd–Sokolov–Wilson equation

Consider the Drinfel'd–Sokolov–Wilson equation (Liu & Liu, 2002; Seadawy, 2017; Xue-Qin & Hong-Yan, 2008)

$$\begin{cases} u_t + pvv_x & = 0, \\ v_t + ruv_x + su_xv + qv_{xxx} & = 0. \end{cases} \tag{41}$$

Using the wave transformation

$$u(x,t) = u(\xi),\ v(x,t) = v(\xi), \quad \xi = k(x - ct),$$

which carries PDE. (41) into ODE.:

$$\begin{cases} -kcu' + pkvv' & = 0, \\ -kcv' + rkuv' + sku'v + qk^3v''' & = 0. \end{cases} \tag{42}$$

By integrating the first equation in Equations (42), we obtain

$$u = \frac{p}{2c} v^2. \tag{43}$$

Substituting Equation (43) into the second equation in Equation (41) and by integrating once the result equation, we obtain:

$$6c^2 v + p(r + 2s)v^3 + 6qck^2 v'' = 0. \tag{44}$$

Balancing between the highest order derivatives and non-linear terms appearing in (44) $\Rightarrow \left(v'' \;\&\; v^3\right)$ we obtain $\left(N = 1\right)$. So that, by using Equation (8) we get the same formal solution of Equation (11). Substituting Equation (12) and its derivative into Equation (44) and collecting all term with the same power of $\left(\frac{G'}{G}\right)^j$ where $\left(j = 6, 5, 4, 3, 2, 1, 0\right)$ we get algebraic system. By solving it by any computer program like maple, mathematica, . . . etc, we gain:

$$c = 2qk^2\mu, a_{-1} = 0, a_0 = \pm\sqrt{\frac{24q^2k^2\mu}{rp(12\sigma^2 - 1) + 24ps\sigma^2 - 2ps}}, a_1 = -a_0\sigma$$

So that, the exact travelling wave solution:

$$v = \pm\sqrt{\frac{24q^2k^2\mu}{rp(12\sigma^2 - 1) + 24ps\sigma^2 - 2ps}} - a_0\sigma \frac{\left(\frac{G'}{G}\right)}{1 + \sigma\left(\frac{G'}{G}\right)}. \tag{45}$$

Then, the solitary wave solutions:

When ($\mu < 0$), we obtain the hyperbolic function solution:

$$v = \pm\sqrt{\frac{24q^2k^2\mu}{rp(12\sigma^2 - 1) + 24ps\sigma^2 - 2ps}} - a_0\sigma \frac{\sqrt{-\mu}\left(A_1 \sinh\left(\sqrt{-\mu}\xi\right) + A_2 \cosh\left(\sqrt{-\mu}\xi\right)\right)}{\left(A_1 + A_2\sigma\sqrt{-\mu}\right)\cosh\left(\sqrt{-\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{-\mu}\right)\sinh\left(\sqrt{-\mu}\xi\right)}. \tag{46}$$

When ($\mu > 0$), we obtain the trigonometric function solution:

$$v = \pm\sqrt{\frac{24q^2k^2\mu}{rp(12\sigma^2 - 1) + 24ps\sigma^2 - 2ps}} - a_0\sigma \frac{\sqrt{\mu}\left(A_1 \cos\left(\sqrt{\mu}\xi\right) - A_2 \sin\left(\sqrt{\mu}\xi\right)\right)}{\left(A_1 - A_2\sigma\sqrt{\mu}\right)\sin\left(\sqrt{\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{\mu}\right)\cos\left(\sqrt{\mu}\xi\right)}. \tag{47}$$

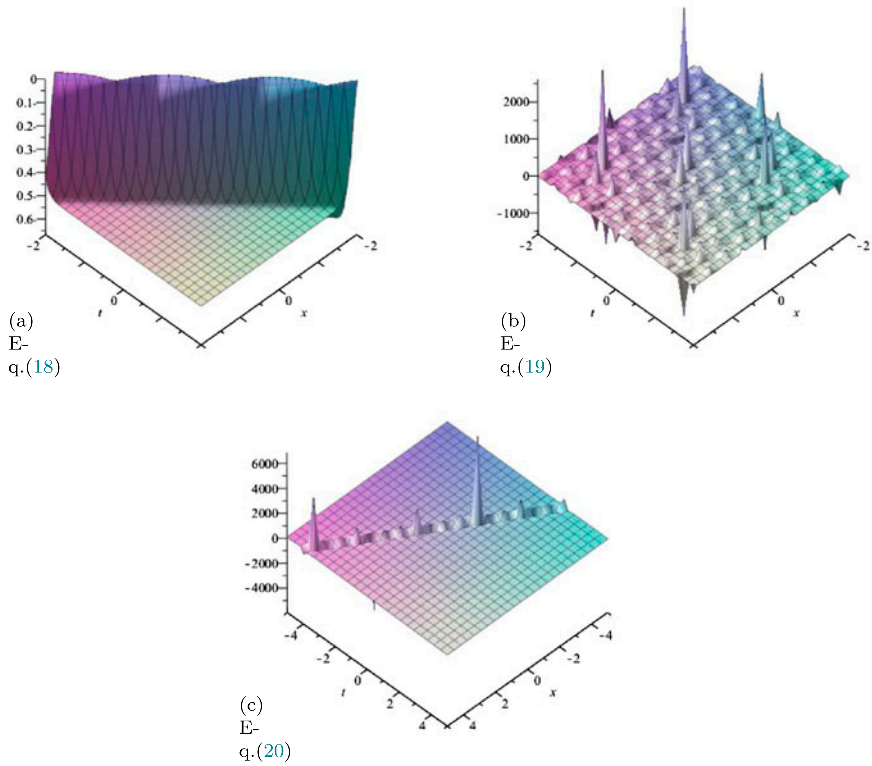

Figure 1. The solitary wave sketch of solution [Periodic singular soliton solution of Equation (18) & Singular kink soliton solution of Equations (19), (20)], when ($k = 5, \omega = 5, \alpha = .5, c = 4, \sigma = 6, A_1 = 2, A_2 = 3$)

When ($\mu = 0$), we obtain the rational function solution:

$$v = -\frac{a_0 \sigma A_2}{A_1 + \left(\xi + \sigma\right) A_2}. \tag{48}$$

3.4. *The system of shallow water wave equations*

Consider the system of the shallow water wave equation (Dolapci & Yildrim, 2013; Yang, Machado, & Baleanu, 2017):

$$\begin{cases} u_t + (uv)_x + v_{xxx} &= 0, \\ v_t + u_x + v v_x &= 0. \end{cases} \tag{49}$$

Using the wave transformation

$$u(x,t) = u(\xi) \quad \xi = (x - ct),$$

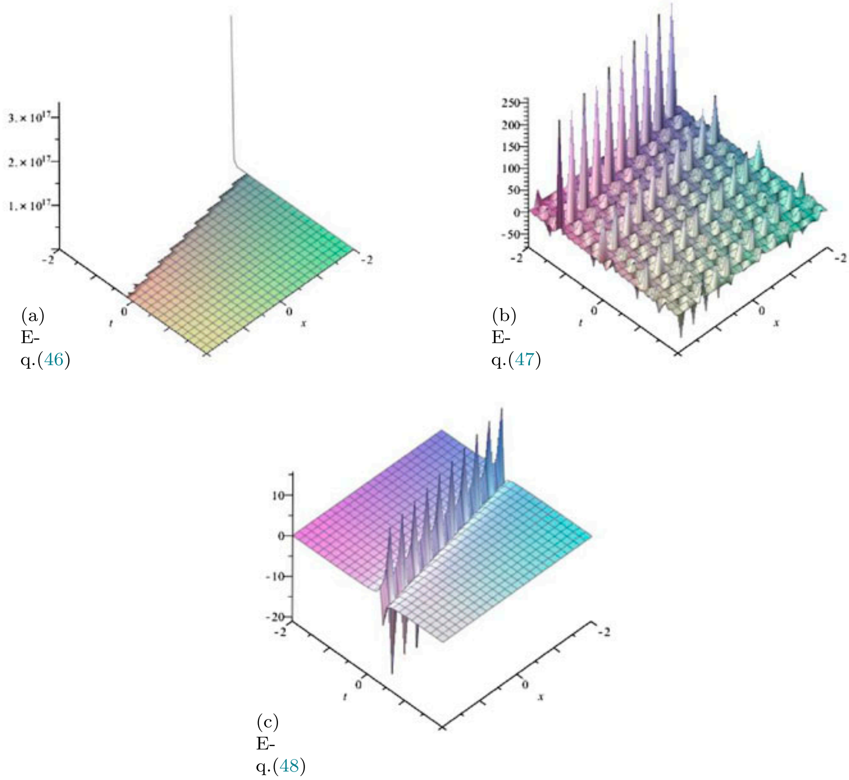

Figure 2. The solitary wave sketch of solution [Periodic Dark soliton solution of Equation (46) & Singular kink soliton solution of Equations (47),(48)], when ($k = 5, c = 5, q = 1, r = -2, p = 3, s = -2, a_0 = -3, \alpha = .5, c = 4, \sigma = -1, A_1 = 6, A_2 = 7$)

which carries PDE. (49) into ODE.:

$$\begin{cases} -cu' + vu' + uv' + v''' & = 0, \\ u' - cv' + vv' & = 0, \end{cases} \tag{50}$$

Integrating once the second equation with zero constant of integration, we get

$$u = cv - \frac{v^2}{2}. \tag{51}$$

substituting Equation (51) into the first equation of Equation (50) we obtain

$$v''' + (3cv - \frac{3v^2}{2} - c^2)v' = 0. \tag{52}$$

Integrating Equation (52) with zero constant of integration, we find

$$v'' + \frac{3}{2}cv^2 - \frac{1}{2}v^3 - c^2 v = 0. \tag{53}$$

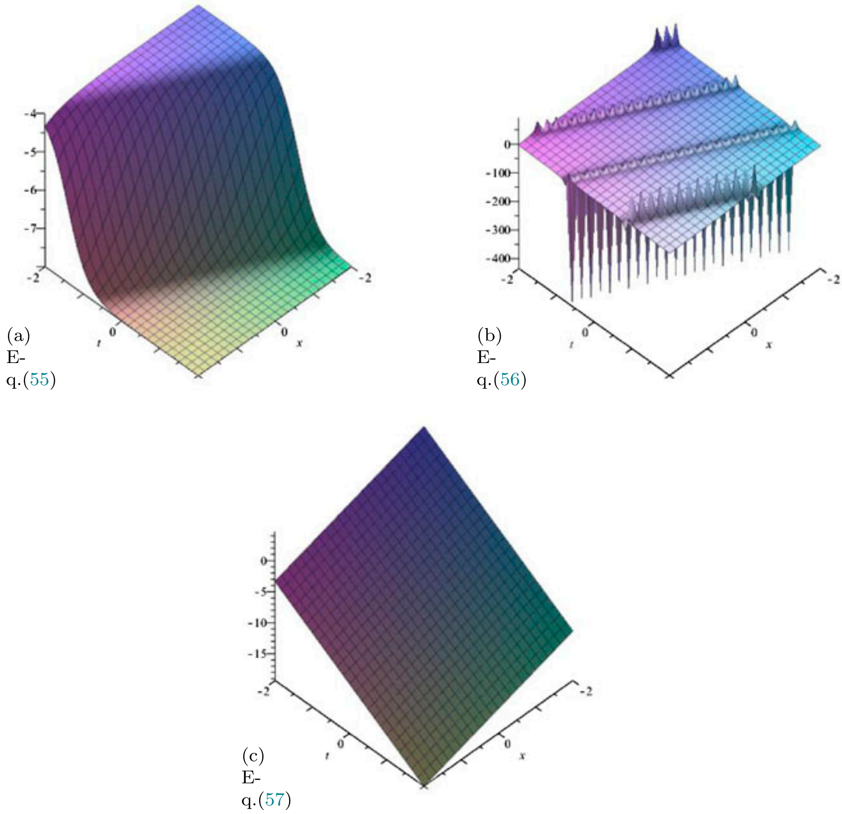

Figure 3. The solitary wave sketch of solution [Kink soliton solution of Equation (55) & Singular kink soliton solution of Equation (56) & Soliton wave profile of Equation (57)], when $(k = 5, \omega = 5, \alpha = .5, c = 4, \sigma = 6, A_1 = 2, A_2 = 3)$

Balancing between the highest order derivatives and non-linear terms appearing in (44) $\Rightarrow \left(v'' \,\&\, v^3\right)$ we obtain $\left(N = 1\right)$. So that, by using Equation (8) we get the same formal solution of Equation (11). Substituting Equation (12) and its derivative into Equation (53) and collecting all term with the same power of $\left(\frac{G'}{G}\right)^j$ where $\left(j = 6, 5, 4, 3, 2, 1, 0\right)$ we get algebraic system. By solving it by any computer program like maple, mathematica, . . . etc, we obtain:

$$\mu = -\frac{1}{4}c^2, \sigma = 0, a_{-1} = c, a_0 = \pm 2, a_1 = 0$$

So that, the exact travelling wave solution:

$$v = c\frac{1 + \sigma\left(\frac{G'}{G}\right)}{\left(\frac{G'}{G}\right)} \pm 2. \tag{54}$$

Then, the solitary wave solutions:

When ($\mu < 0$), we obtain the hyperbolic function solution:

$$v = c\frac{\left(A_1 + A_2\sigma\sqrt{-\mu}\right)\cosh\left(\sqrt{-\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{-\mu}\right)\sinh\left(\sqrt{-\mu}\xi\right)}{\sqrt{-\mu}\left(A_1\sinh\left(\sqrt{-\mu}\xi\right) + A_2\cosh\left(\sqrt{-\mu}\xi\right)\right)} \pm 2. \quad (55)$$

When ($\mu > 0$), we obtain the trigonometric function solution:

$$v = c\frac{\left(A_1 - A_2\sigma\sqrt{\mu}\right)\sin\left(\sqrt{\mu}\xi\right) + \left(A_2 + A_1\sigma\sqrt{\mu}\right)\cos\left(\sqrt{\mu}\xi\right)}{\sqrt{\mu}\left(A_1\cos\left(\sqrt{\mu}\xi\right) - A_2\sin\left(\sqrt{\mu}\xi\right)\right)} \pm 2. \quad (56)$$

When ($\mu = 0$), we obtain the rational function solution:

$$v = c\frac{A_1 + \left(\xi + \sigma\right)A_2}{A_2} \pm 2. \quad (57)$$

Note that:
All the obtained results have been checked with Maple 16 by putting them back into the original equation and found correct.

4. Discuss the results

In this research, we showed a good comparison between our results and that obtained by other researchers using the different methods. We sorted our comparison for each models in the main following steps:

Firstly: Solutions of fractional-order biological population model:

Equation (14) is new form of solution for the model from that obtained in Lu (2012) who used Backlund transformation of fractional Riccati equation.

Secondly: Solutions of time fractional burgers equation:

All our solutions for this model are new and more general solution from that obtained in Inc (2008) which use the variational iteration method.

Thirdly: Solutions of the Drinfel'd–Sokolov–Wilson equation:

Equations (46), (47) are similar to Equations (3.20), (3.21), (3.23), (3.25), (3.26) in Seadawy (2017) when $\left(\mu = -4q^2A^2\right)$ and $\left(\omega^2 = \frac{24q^2k^2\mu(r+2s)}{6(r(12\sigma^2-1)+24s\sigma^2-2s)},\right.$ $\left.A_1 = \frac{2\omega B+qA-\sigma\sqrt{-\mu}(2\omega B-qA)}{1+\sigma^2\mu^2}\right)$.

Equation (48) is a new form of solution for Equation (3.35) in Yang et al. (2017) who used modified simple equation method.

Fourthly: Solutions of the system of shallow water wave equations:

Equations (55), (56) are similar to Equations (21), (29) in Dolapci and Yildrim (2013) when $\left(c = 2 \pm \lambda, A_1 = 0, \sigma = 0, \lambda^2 = 1 + 4\mu\right)$.

Equation (57) is new form solutions from that in Dolapci and Yildrim (2013) who sue exp $-\phi(\xi)$-expansion function method. So that, it is shown that the improved $\left(\frac{G'}{G}\right)$-expansion method provides an effective and a more powerful mathematical tool for solving non-linear evolution equations in mathematical physics. It is one of the general method which depended on the auxiliary equation but also contain two methods which make this method an effective tool for obtaining exact analytical solutions for many of non-linear partial differential evolutions equations with integer and fraction order and it a second advantage of it.

5. Conclusion

In this research, we succeed to apply the improved $\left(\frac{G'}{G}\right)$-expansion method for finding new and more general solutions (exact and solitary wave solutions) of fractional-order biological population model, time fractional burgers equation, the Drinfel'd–Sokolov–Wilson equation and the system of shallow water wave equations. We discussed the travelling and solitary solutions of two non-linear partial differential equations with fraction order and two other non-linear partial differential equations with integer order. Figures 1–3 show different kinds of solitary wave solutions for these models. We believe that our results of these models will be useful for young researchers who are going to study the exact solutions of non-linear partial differential equations (NLPDEs.). We also hope that our results will be interesting for some referees. The improved $\left(\frac{G'}{G}\right)$-expansion method is very simple, direct, effective and powerful method to apply it for many non-linear evolution equations.

Author's contributions

All parts contained in the research were carried out by the authors through hard work and a review of the various references and contributions in the field of mathematics and the physical Applied.

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