A new reduced basis method for non-linear problems

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RÉSUMÉ. *Une nouvelle technique de base réduite est proposée afin de résoudre une large classe de problèmes non linéaires. L'idée principale est de réduire les problèmes linéaires obtenus par une technique de perturbation et non pas le problème non linéaire initial. On a examiné en détail l'efficacité numérique de cette nouvelle méthode qui s'avère être très attrayante pour des problèmes de grande taille. Une analyse détaillée des algorithmes classiques de base réduite est également présentée.*

ABSTRACT. *An alternative reduced basis technique is proposed to solve a large class of non-linear problems. The basic idea is to reduce the linear problems obtained by perturbation technique and not the initial non-linear problem. The numerical efficiency of the new method is discussed in details and it turns out to be very attractive for large scale problems. A detailed analysis of classical reduced basis algorithms is also presented.*

MOTS-CLÉS : *Méthodes Asymptotiques Numériques, technique de perturbation, technique de base réduite, élasticité non linéaire, coques minces, Approximants de Padé.*

KEY WORDS : *Asymptotic Numerical Methods, perturbation technique, reduced basis technique, non-linear elasticity, thin shells, Padé Approximants.*

1. Introduction

The application of the reduced basis method to non-linear problems was proposed about twenty years ago. The employed basis resulted from a linear buckling analysis [BES 74], from phases of correction [ALM 78] or from a perturbation technique. This latter turned out to be the most efficient. Only some vectors were used in the applications (from 2 to 10, generally 6 or 7) since from a certain order, the computed vectors are less and less linearly independant. The proposed computation algorithm could be summed up in four steps: step 1: computation of the basis vectors with a perturbation technique; step 2: computation of the non-linear reduced problem; step 3: resolution of the non-linear reduced problem; step 4: correction phase and beginning of a new step. The gain obtained with this process is from Riks' point of view [RIK 84], counterbalanced by the excessive cost of steps 2 and 4. We will analyse once more this algorithm, in particular the essential point which is the cost of the computing time to get the reduced problem. This computation becomes too expensive for high orders (15 to 30) but it is different for the asymptotic numerical method (ANM) that uses only a representation in series of the solution branch in the following form:

$$U(a) = U_0 + \sum_{i=1}^{p} a^i U_i \qquad \textit{and} \qquad \lambda(a) = \lambda_0 + \sum_{i=1}^{p} a^i \lambda_i \qquad [1]$$

where U is the unknown vector and λ a scalar parameter. The computation of the series (1) needs only the above first step, and thus we can obtain dozens of computation steps without any correction and for a reasonable cost [COC 941]. It has been established that it is interesting to compute a large number of terms of the series, although the new generated vectors are nearly a linear combination of the previous ones.

It was also proposed to transform the polynomial approximation (1) by an asymptotically equivalent rational approximation under the form:

$$U(a) = U_0 + \sum_{i=1}^{p} f_i(a) U_i \qquad \textit{and} \qquad \lambda = \lambda_0 + \sum_{i=1}^{p} f_i(a) \lambda_i \qquad [2]$$

where the $f_i(a)$ are rational fractions called Padé Approximants [AZR 92], [COC 943]. As compared to the polynomial approximation, the additional computing time is negligible. When it is coupled with a continuation technique, it allows us to divide approximatively by two the number of steps necessary to solve a given problem [AEH 00], [BRU 99].

The reduced basis method has been compared with these two kinds of ANM in a recent work [NAJ 98]. It has been shown that the step lengths obtained by the rational representation [2] were close to the ones obtained by the reduced basis technique while the latter needs more computing time because of the step 2. Furthermore, the step length with the representation [1] and [2] for high orders (example order 16) were much larger than those obtained with the reduced basis method in low orders (example order 8). It appears that the reduced basis technique is not efficient compared to the two others, unless a mean to decrease the computing time associated to the step 2 is found. This second step is then a critical point for a bigger utilization of the reduced basis technique in the resolution of non-linear problems.

One shall mention that it is not obvious to treat numerically a large number of vectors U_i obtained by perturbation technique. Especially the orthogonalisation process and the calculus of the coefficients of the rational fractions involve numerical instabilities [NAJ 98], [CHA 97]. Fortunately, this does not affect the efficiency of the Asymptotic Numerical Methods for large truncation orders p.

The aim of this study is to present a new way to apply the reduced basis technique which allows one to use a basis with higher dimensions, for a moderate computing time. The objective is to avoid the reduction of the initial non-linear problem and to reduce the linear problems obtained by the perturbation technique. This allows to avoid the second step and the factorisation of the global tangent matrix, that requires a too expensive computing time. An efficient application of the reduced basis technique supposes good strategies for the choice of the basis (step 1) and a continuation method, but herein we shall not try to establish definitive settlements about this choice. Most of the applications presented in the literature deal with non-linear elasticity. Quite the reverse, the proposed procedure can be applied to every problem where we are able to apply an efficient perturbation technique. This was established in viscous fluid mechanics [CAD 00], for the unilateral contact [AEH 98], in plasticity [BRA 97], [ZAH 98] and in viscoplasticity [BRU 99].

This article is organized in the following way. First,we recall two presentations of the reduced basis technique written in the case of non-linear elasticity. Next, we analyse carefully the computational cost of the aforementioned techniques to determine their range of applicability. Finally, the new version of the reduced basis technique is presented and we analyse it in the same way.

2. The classical reduced basis methods

This method has been proposed and tested by Almroth [ALM 78], Noor and Peters [NOO 83], [NOO 80], [NOO 81]. It consists in using vector fields $u_1, u_2, u_3 ..., u_p$ as a basis in a Rayleigh-Ritz approximation. The unknown u representing the displacement is then searched in the following way:

$$u = r_1 u_1 + r_2 u_2 + r_3 u_3 + ... + r_p u_p, \qquad [3]$$

where the coefficients r_i are the new unknowns to be determined. In order to illustrate the method, let us consider a problem involving the elastic behaviour with geometric non-linearities. The equilibrium of a solid body occupying a region Ω_o, in a reference configuration with boundary $\partial\Omega_o$ can be expressed by the equilibrium equation and the stress-strain relation :

$$\int_{\Omega_o} S : \gamma^l(\delta u)dv + \int_{\Omega_o} S : 2\gamma^{nl}(u,\delta u)dv - \lambda P_e(\delta u) = 0 \qquad [4]$$

$$S = D : (\gamma^l(u) + \gamma^{nl}(u,u)) \qquad [5]$$

where $\gamma = \gamma^l + \gamma^{nl}$ is the Green-Lagrange strain tensor, S the second Piola-Kirchhoff stress tensor, λ is a load parameter and (S_o,λ_o,u_o) is a solution state chosen as $(0,0,0)$ for the sake of simplification :

$$\text{with} \quad 2\gamma^l(u) = \nabla u +^t \nabla u \quad \text{and} \quad 4\gamma^{nl}(u,\delta u) =^t \nabla u.\nabla\delta u +^t \nabla\delta u.\nabla u \qquad [6]$$

In what follows, we shall present and test two ways to apply this reduced basis technique.

2.1. *Classical reduced basis method*

According to the expression (4), the equilibrium equation is quadratic regarding to the variables u and S. A displacement approach is carried out replacing the stress-strain relation (5) into (4). One then obtains a cubic expression regarding to the variable u:

$$\begin{array}{l} \int_{\Omega_o}(\gamma^l(u) + \gamma^{nl}(u,u)) : D : \gamma^l(\delta u)dv \\ + \int_{\Omega_o}(\gamma^l(u) + \gamma^{nl}(u,u)) : D : 2\gamma^{nl}(u,\delta u)dv - \lambda P_e(\delta u) = 0 \end{array} \qquad [7]$$

The principle of the reduced basis technique is to approach the displacement u and the virtual displacement δu in the following way :

$$u = \sum_{i=1}^{p} r_i u_i \qquad [8]$$

$$\delta u = \sum_{i=1}^{p} \delta r_i u_i \qquad [9]$$

where r_i and δr_i are real and r_i become the new unknowns.
Reporting equations (8) and (9) into the cubic expression (7) leads to a cubic expression regarding to the variables r :

$$l^*_{ij} r_j + q^*_{ijk} r_j r_k + c^*_{ijkl} r_j r_k r_l - \lambda f^*_i = 0 \quad i,j,k,l = 1,p \qquad [10]$$

The operators l^*, q^* and c^* are defined in Appendix A.

In order to solve these equations, the coefficients l^*_{ij}, q^*_{ijk} and c^*_{ijkl} have to be computed. For instance with a basis of 3 vectors, 78 coefficients have to be computed and for a basis of 10 vectors, 6105 coefficients. As we can see, the number of coefficients hugely increases with the size of the basis. A reduction of the number of coefficients to be computed would allow us to reduce the computing time of this method. That is what is tempted in the next section.

2.2. *A variant of the classical reduced basis method*

In order to reduce the computing time to get the reduced problem, an alternative method has been proposed in the literature [NAJ 98]. This consists in obtaining a quadratic problem using a mixed variable (u, S). Then with additional variables, this leads to a reduced quadratic problem. The basic lines are mentioned here and for further details refer to [NAJ 98]. The principle is to start from the mixed formulation (4) and (5) which is quadratic regarding to the displacement-stress variables (u, S). The equilibrium equations are approached choosing the vectors fields δu under the form (9) and searching the displacement fields under the same form (8). As for the stress-strain relation (5), one requires it to be satisfied exactly. Reporting (9) and (8) into the stress-strain relation (5) leads to:

$$S(u) = D : (\sum_{i=1}^{p} r_i(\gamma^l(u_i)) + \sum_{i=1}^{p}\sum_{j=1}^{p} r_i r_j \gamma^{nl}(u_i, u_j)) \qquad [11]$$

This can be rewritten:

$$S(u) = \sum_{i=1}^{p} r_i S_i \quad + \quad \sum_{k=p+1}^{N} r_k S_k \;\; with \;\; N = p + \frac{p(p+1)}{2} \qquad [12]$$

These notations have been introduced:

$$S_i = D : \gamma^l(u_i) \quad \text{for } 1 \le i \le p \qquad [13]$$

$$S_k = D : 2\gamma^{nl}(u_i, u_j) \qquad \text{for } i \neq j \qquad [14]$$

$$\text{or} \quad S_k = D : \gamma^{nl}(u_i, u_i) \qquad \text{for } i = j (p + 1 \le k \le N) \qquad [15]$$

Thus the key point is the introduction of $p(p+1)/2$ additional variables to represent the non-linear dependance of the stress with respect to the reduced basis variables r_i, $1 \le i \le p$. These additional variables are defined by a quadratic relation:

$$r_k = r_i r_j \quad \text{with} \quad k = ip + j - \frac{i(i-1)}{2} \qquad [16]$$

With this choice of the unknowns $r = ^t\!< r_1, r_2, ..., r_p, r_{p+1}, ..., r_N >$, the reduced equation obtained from the principle of virtual work is only quadratic:

$$l_{ij} r_j + q_{ijk} r_j r_k - \lambda f_i = 0 \quad i = 1,p \;\; j = 1,N \;\; \text{and} \;\; k = 1,p \qquad [17]$$

The operators l and q are defined in Appendix A. As in the previous section, in order to solve these equations, the coefficients l_{ij} and q_{ijk} have to be computed. For instance with a basis of 3 vectors, 60 coefficients have to be computed and for a basis of 10 vectors, 3630 coefficients. As we can see, the number of coefficients is smaller than the ones obtained previously, nevertheless it remains too large to achieve an acceptable computation cost.

Note that both the problems (10) and (17) represent rigorously the same problem, but they are not expressed regarding to the same variables r. It is then possible to apply an asymptotic numerical method (representation with the series or the Padé Approximants) to solve them as presented in the next section. These methods are not expensive in computing time; however, they may present numerical instabilities.

3. Numerical evaluation of the reduced basis methods

3.1. *Three variants of the ANM*

The asymptotic numerical methods permit to determine non linear equilibrium pathes by means of asymptotic expansions. Different methods have been proposed: the direct computation of series, where the unknowns are represented by power series as follows:

$$\left\{ \begin{array}{c} u - u_0 \\ S - S_0 \\ \lambda - \lambda_0 \end{array} \right\} = \sum_{i=1}^{p} a^i \left\{ \begin{array}{c} u_{(i)} \\ S_{(i)} \\ \lambda_{(i)} \end{array} \right\} \qquad [18]$$

p is the truncation order and a is a parameter which can be defined as $a = \{< u - u_0, u_1 > +(\lambda - \lambda_0)\lambda_1\}$. Introducing the expansions (18) into equations (4) and (5) and identifying like powers of a leads to a set of linear problems. A discretisation by the finite element method leads at each order to:

$$[K_t]\{u_i\} = \lambda_i\{f\} + \{f_i^{NL}\} \qquad [19]$$

where f_i^{NL} depends only on the u_i and S_i coming from previous orders (see [COC 942] for further details).

The transformation of the polynomial approximation into asymptotically equivalent rational fractions called Padé Approximants can improve the range of validity. From the vectors u_i an orthogonalised basis u_i^* has to be first computed for example by the Gram-Schmidt procedure. u and λ are then searched as follows, [NAJ 98]:

$$u(a) = \sum_{i=0}^{p} f_i(a) u_i^* \qquad \text{and} \qquad \lambda(a) = \sum_{i=0}^{p} f_i(a)\lambda_i.$$

Another way to extend the range of validity of these representations is to apply the reduced basis technique presented and tested by Noor and Peters [NOO 83], that has been recalled in Part 2. The principle is to apply a Rayleigh-Ritz reduction technique to the original problem, using the first vectors u_i or u_i^* of the series as a basis (these two basis should lead theoretically to the same results). u is then searched in the following way : $u = \sum_{i=1}^{p} r_i u_i$.

The computing time to find the solution depends on the method used to solve the problem. Indeed, for the direct computation of series, the computing time needed is governed by the computation of K_t, the decomposition of K_t (note that the method employed in programming is the Crout method), and the computation of the right hand sides f_i^{NL} and their associated u_i. The computing time of these terms depends on the number of d.o.f. of the problem.

In order to define the time needed by the representation with the Padé Approximants, one has to add to the time used by the direct computation of series the time to orthogonalise the basis u_i as well as the time to compute the Padé coefficients, but these additional times are not expensive.

Concerning the reduced basis techniques, the time needed to obtain the solution is ruled by the computation of the coefficients to perform the reduced system, if the basis is supposed to be already computed. Indeed, the number of c_{ijkl}^* (classical reduced basis technique) and q_{ijk} (variant of the reduced basis technique) increases a lot with the dimension of the basis.

3.2. *Two numerical benchmarks*

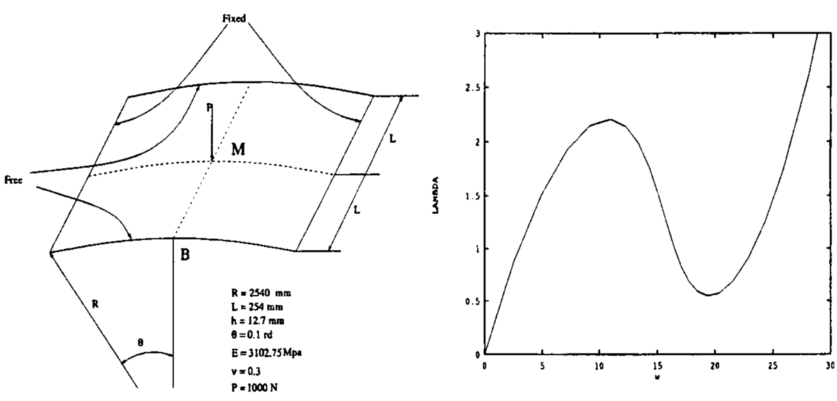

FIG. 1 – *a) Elastic cylindrical shallow shell loaded by a concentated force (roof problem). b) Load as a function of the displacement at the center M. Reference curve*

Two examples have been studied. The first one is the classical cylindrical shallow shell loaded by a single force. Hereinunder, it is referred to as the roof problem. A

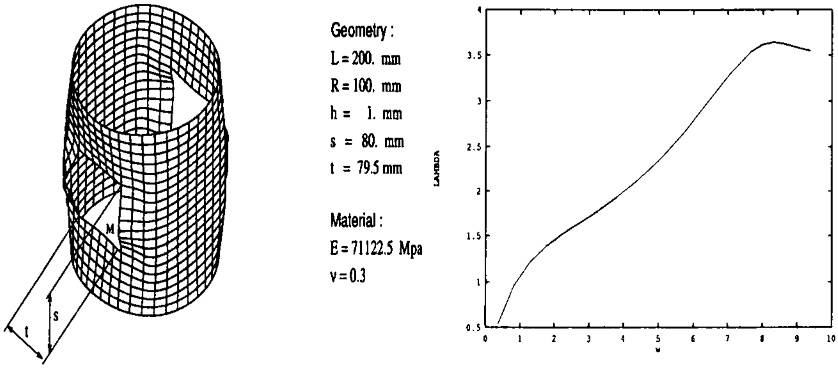

FIG. 2 – *a) Cylindrical shell with two diametrically opposite cut outs and loaded by a uniform axial compression. b) Load as a function of the radial displacement at point M. Reference curve*

mesh for a quarter of the shell, and 200 triangular DKT18 elements [BAT 90] have been used. The total number of d.o.f. is 726. The geometry and the material description are given in figure 1a. The reference solution obtained with a continuation method is represented in figure 1b.

The second example is a classical test problem already discussed in [RIK 84] and [NOO 81]. It is a cylindrical shell with two diametrically opposite cut outs and loaded by a uniform axial compression. The geometry and the material characteristics are given in figure 2a and the reference solution obtained with a continuation method is represented in figure 2b. For symmetry reasons, one eight of the shell has been discretised with a regular mesh involving 1608 triangular DKT18 elements. The total number of degrees of freedom is 5190.

Order	calculus of [Kt]	[Kt] decomposition	Fnl	Total (series)
5	2.1	0.7	1.6	4.4
10	2.1	0.7	3.4	6.2
15	2.1	0.7	5.1	7.9
20	2.1	0.7	7	9.8

TAB. 1 – *Roof (726 d.o.f.): Computing time for direct computation of series at order 5,10,15 and 20*

		Reduced basis (quadratic)		Reduced basis (cubic)	
Nb vectors	Basis vectors	reduced system	Total	reduced system	Total
5	4.4	1.9	6.3	3.4	7.8
10	6.2	15.5	21.7	64	70.2
15	7.9	65	72.9	177	184.9
20	9.8	183	192.8	559	568.8

TAB. 2 – *Roof (726 d.o.f.): Computing time for the classical reduced basis at order 5,10,15 and 20*

3.3. *Numerical results and discussion*

We first discuss the size of the step lengths obtained according to the truncation order and the employed method: the direct computation of series, the rational fractions, or one of the two reduced basis techniques. In accordance with what was expected, the two reduced basis techniques introduced in section 2 give exactly the same solution: they only differ in the way of computation.

We propose to analyse the step length from the residual curves. They represent the logarithm of the residual norm versus the displacement w. Generally, the quality of the solution is supposed to be good when the residual norm is less than 10^{-3} and this is what we keep in our discussion. In the case of the roof (figure 3), the best quality of the solution is obtained with the subspace method, the worse by the series, in accordance with what was expected. We also verify that the truncation order has a great influence on the quality of the solution and that the series at order 15 give better results than the reduced basis method at order 5. As already underlined by [NAJ 98], the best computational strategy is obtained by choosing large truncation orders. If a criterion less than 10^{-3} for the residual is considered, the obtained step lengths at order 5 are: $w_s = 4.2$ (series), $w_p = 5.5$ (Padé) and $w_r = 6.5$ (reduced basis); at order 10 : $w_s = 9$,$w_p = 11$ and $w_r = 12$, at order 15 : $w_s = 11$,$w_p = 14$ and $w_r = 15$. If the introduction of the reduced basis or of the Padé Approximants allows us to increase the range of validity, this improvement is not considerable. The reduced basis technique would only be interesting if the additional computing cost was relatively cheap.

Order	calculus of [Kt]	[Kt] decomposition	Fnl	Total(series)
5	17	27	15	59
10	17	27	30	74
15	17	27	46	90
20	17	27	63	107

TAB. 3 – *Cylinder (5190 d.o.f.): Computing time for direct computation of series at order 5,10,15 and 20.*

		Reduced basis (quadratic)		Reduced basis (cubic)	
Nb vectors	Basis vectors	reduced system	Total	reduced system	Total
5	59	15	74	27	86
10	74	125	199	301	375
15	90	522	612	1415	1506
20	107	1474	1581	4501	4608

TAB. 4 – *Cylinder (5190 d.o.f.): computing time for the classical reduced basis at order 5,10,15 and 20*

The reduced basis used herein is the basis u_i^* obtained by perturbation then orthonormalisation. We also tried to carry out this calculation with the basis u_i obtained from the perturbation, but without the orthonormalisation. The quality of the solution was always inadequate (residual greater than 10^{-3}), even for very small values of the displacement: the range of validity of this approximated solution was then nonexistent. We think it is due to an ill conditioning of the matrix l_{ij} and of the tangent matrix l_{ij}^t which occurs in the solution of the reduced problem (17). Even with the orthonormalised basis, the quality of the solution obtained by the reduced basis technique at order 15 for small values of the displacement (residual almost equal to 10^{-7}) is worse than the one obtained by the two direct asymptotic representations (residual almost equal to 10^{-12}). Likely, this unexpected behaviour is due to some numerical instabilities that are also connected to the ill conditioning of the matrices l_{ij}^t (see figure(3)).

The numerical tests for the cylinder lead to the same conclusions. The largest step length is obtained by the reduced basis technique and the smallest with the series, but the step length is not the only significant computational parameter. In figure (4), we have reported three load-deflection curves at order 5 and at order 15 with the Padé Approximants. Once more, the rational approximation at order 15 gives much better results than the reduced basis at order 5. Furthermore, despite many efforts, we have not been able to reach a satisfactory residual with the reduced basis technique, what is probably due to numerical instabilities.

Let us now discuss the computing times that are necessary to get all these asymptotic approximations. On tables 1 and 3, we reported the computing times needed to compute the series up to orders 5,10,15 and 20. The total computing time has been splitted into three parts : first the time to compute the tangent matrix, then the one to decompose it and finally, the one to compute the right hand sides F_i^{nl} and the u_i. The last time is the only one to depend on the truncation order and we notice that it is more or less proportional to this order. Note that the time distribution is very different in these two tests : with a small number of degrees of freedom (d.o.f.), the matrix decomposition is not time consuming, but it becomes very significant in the second test

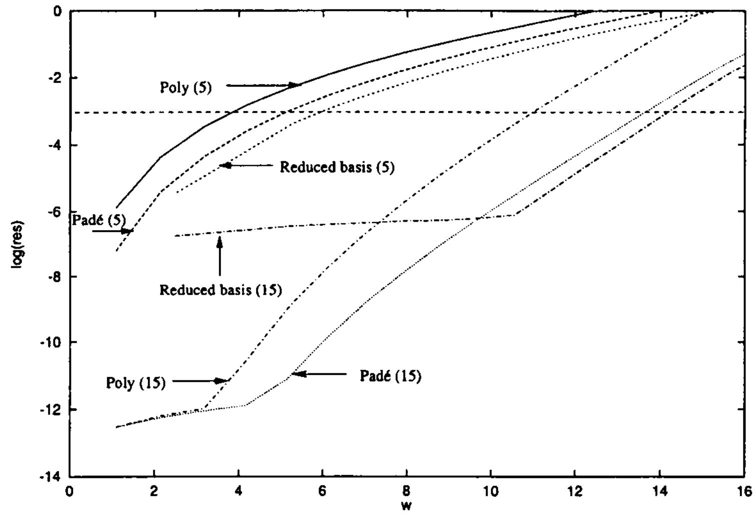

FIG. 3 – *Roof, (726 d.o.f.): representation of the residual obtained by the three methods.The series are truncated at order 5 and 15*

(5190 d.o.f.) and it will be prominent for large scale problems because it is known that the decomposition time increases as the cube of the number of unknowns.

The computing time to get the orthogonalised vectors u_i^* within the classical Gram-Schmidt orthogonalisation and to get the rational fractions is negligible (less than 1% for the orders 5 or 10, 2% for the order 15 and 2.5% for the order 20). That is why the polynomial and the Padé Approximants methods have nearly the same computing time. Because the range of validity of the rational approximations is larger than the polynomial one for about the same computational amount, the first method is better than the second one and we consider it as a reference in what follows.

Let us now broach the main point to be discussed in this section. On tables 2 and 4, we have presented the computing time needed by the two types of reduced basis method, by comparison with the computing time of the basis, that is about the one to build up the rational approximations. In these tables, we have only included the time needed to get the coefficients of the quadratic system (17) or of the cubic system (10) by getting rid of the time to solve this reduced system; in fact, the last time is not very small, but we have neglected it in this presentation, because it depends on the computational strategy and the ours could perhaps be improved. Whatever the reduction is, the computing time of the reduced system is huge so long the size of the basis is larger than 10. Thus the reduced basis techniques presented in part 2 can not be practically applied for an order beyond 7. Furthermore we notice the same computing time for the rational approximation at order 15 and the reduced basis at

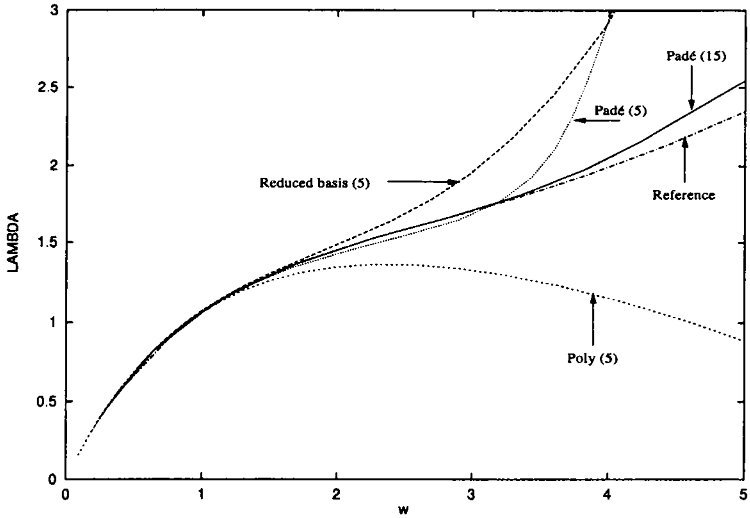

FIG. 4 – *Cylinder, (5190 d.o.f.) : displacement of the point M with the three methods. The series are truncated at order 5*

order 5, but we have already seen that the range of validity is much larger in the first case. From this table, it appears that the quadratic reduced basis method is better than the most classical cubic one, but this improvement is not sufficient to lead to an efficient numerical method for large truncation orders.

3.4. *Conclusion*

Clearly, the reduced basis technique is not an efficient numerical method as compared to the simple perturbation technique, especially when the latter is improved by the introduction of the Padé Approximants. The key point is the need of computing a large number of coefficients of a reduced system, what prevents to apply this technique with a large number of basis vectors. We recommand not to use more than 5 or 7 vectors for problems involving from 1000 to 10000 d.o.f. There are two more difficulties with this classical reduced basis technique. On the one hand, it is difficult to withdraw some numerical instabilities. On the other hand, it can only be applied in the case of very simple equations like non linear elasticity or Navier-Stokes equations. That is why we try to propose in the next section another way to apply the reduced basis technique, that will allow us to avoid all these drawbacks.

4. A modified reduced basis technique

In spite of the pessimistic conclusions of the previous section, we shall propose a variant of the reduced basis technique that remains efficient with a rather large basis. Because the main difficulty lies in the computation of the coefficients of the reduced system, we no longer compute these coefficients. We suggest first to apply the perturbation technique that transforms the non-linear problems into a sequence of linear ones.Secondly, to use the reduced basis technique to solve these linear problems. Thus, the perturbation is applied before the reduction, contrary to what is performed within the classical reduced basis techniques.

4.1. *Presentation of the method*

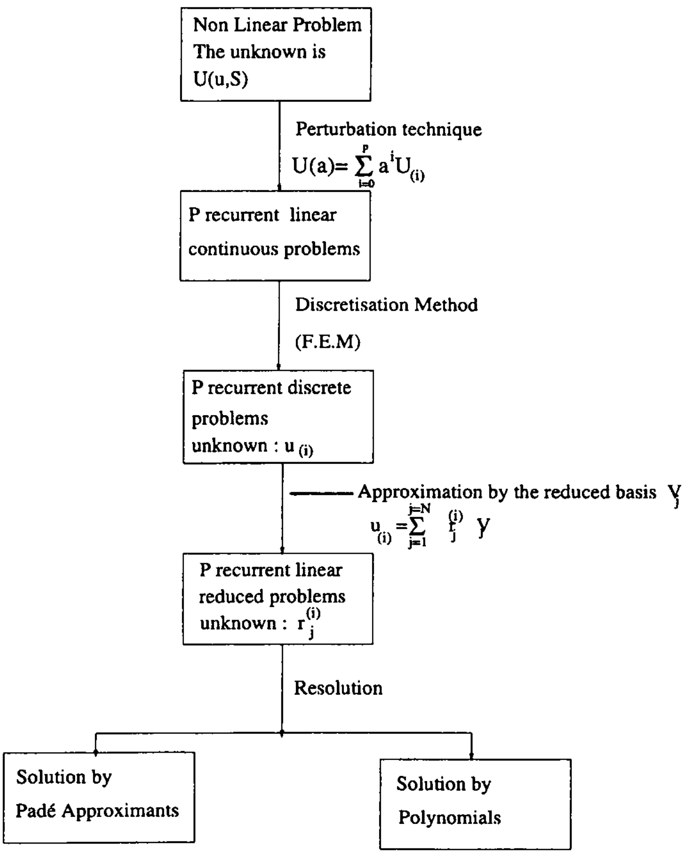

FIG. 5 – *Reduced basis technique's algorithm*

The first steps of the new method are exactly the same as in the classical asymptotic numerical method [COC 942]. First, we expand the variables into power series with respect to a parameter a. This leads to a sequence of linear problems. In the applications, we shall consider the same expansion parameter as previously. Then these problems are discretized by the finite element method, this leads to discrete linear equations as in equation (19). All these equations have the same form with various right hand sides:

$$[K_t]\{u\} = \{g^{nl}\} \qquad [20]$$

Supposing that we have chosen a basis of N vectors e_n, $1 \leq n \leq N$. The discrete unknowns u_i are sought as linear combinations of these vectors and an approximated solution of the linear equations (19) is deduced by the standard Galerkin procedure. Then, we transform these polynomials into rational fractions as explained previously. For the sake of simplicity, we have achieved the finite element discretisation before the reduction, but these two operations could be switched without any change in the final approximated equation. Consequently, the coefficients of the matrix k_{mn} should be about the same as the l^*_{ij} appearing in the system (10). It is clear that the quantity of the real numbers k_{mn} to be computed is exactly $N(N+1)/2$, which is much lower than the one required in the classical reduction method. Thus, in this way, the reduced basis technique can be efficiently applied with a rather large basis. This algorithm is summarised in figure (5).

4.2. *Numerical results and discussion*

Basis vectors	[Kt]	Fnl	Reduced matrix	Ortho-vec	Total
15	2.1	4.7	0.6	0.2	7.6
20	2.1	4.7	0.8	0.3	7.9
25	2.1	4.7	1	0.4	8.3
30	2.1	4.8	1.3	0.6	8.9

TAB. 5 – *Roof (726 d.o.f.): Computing time for the alternative Reduced basis using the polynomial approximations truncated at order 15*

Basis vectors	[Kt]	Fnl	Reduced matrix	Ortho-vec	Total
15	17	37	10	1	65
20	17	37	14	2	70
25	17	38	18	3	76
30	17	38	21	4	80

TAB. 6 – *Cylinder (5190 d.o.f.): Computing time for the alternative Reduced basis using the polynomial approximations truncated at order 15*

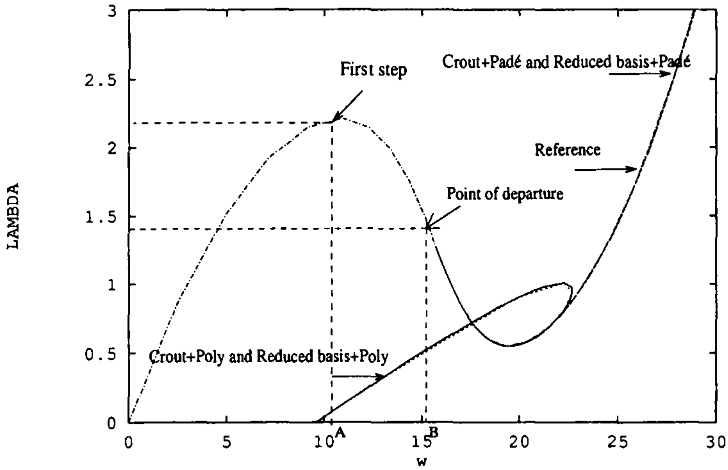

FIG. 6 – *Roof (726 d.o.f.) : The loading parameter λ as a function of the displacement at the point M.* For the sake of completeness, the starting points of the steps correspond to wA=10,1 and wB=15,06

The same two points as in the previous section have to be discussed to assess the reliability of this new numerical technique. First, we have to verify if the reduction of the linear problems does not affect the step length with a proper choice of the basis. Then, we have to analyse the computing times. In this study, we do not try to define the selection of the basis that will be efficient in many situations. We limit ourselves to establish that one can define such a basis in order to get both a good step length and a moderate computation cost.

For the numerical tests, we will proceed in the following way : two steps are going to be carried out using the direct computation of series truncated at order 15. This then defines 30 vectors that we transform into an orthonormal basis. This orthonor-

	calculus of [Kt]	treatment of [Kt]	Fnl	Ortho-vect	Total
Crout decomposition	32	2075	202	weak	2310
Reduced basis (30 vectors)	32	170	72	9.5	284

TAB. 7 – *Cylinder (39756 d.o.f.) : Computing time of one step with the standard A.N.M. (Crout, Padé) and with the new reduced basis technique (reduced basis, Padé). Order 15*

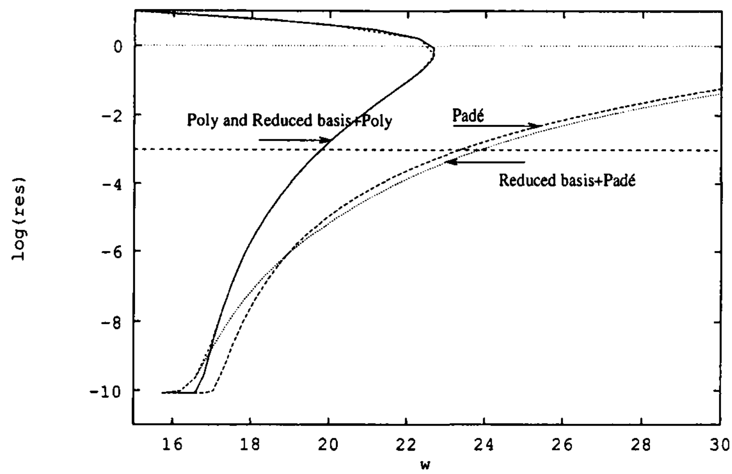

FIG. 7 – *Roof (726 d.o.f.), third step : residual curve according to the representation (polynomial or fractions) and to the solving technique of the linear problems (Crout or reduced basis). Truncation order : 15. (Point M)*

malisation has been achieved in different ways without any significant change of the result. This test deals with the third step of the calculus. It has been carried out at the end of the range of validity of the second series. We have then determined the response curves and the residual curves in four different ways : we started with two representations using the series, the coefficients u_i beeing computed either by a complete decomposition of the tangent matrix (noted Crout+Poly), or by the reduced basis technique presented in section 4 (noted Reduced basis+Poly). We have also two rational representations that differ in the resolution of the linear problems (noted Crout+Padé, Reduced basis+Padé). The results are presented on figures (6) and (7) in the case of the roof problem and on figures (8) and (9) for the cylinder problem. It clearly appears that the approached resolution by the reduced basis technique has a weak effect on the quality of the obtained results. It even seems that in one case of the rational representation (figure 7), the resolution with the reduced basis technique is preferable to the exact resolution using the direct method : this paradoxal result is probably linked to inherent numerical instabilities in the calculus of the Padé Approximants with high orders [NAJ 98], (Part 3.2). In the case of the cylinder and the rational approximations, the approached resolution slightly reduces the step length (with a maximum admissible residual of 10^{-3}, the limit of validity is $w = 5.1$ with a resolution by reduced basis and $w = 5.4$ with a resolution by the Crout method).

We have also analysed the step length obtained with the same subspace but without any orthogonalisation : the quality of the solution obtained this way was unacceptable.

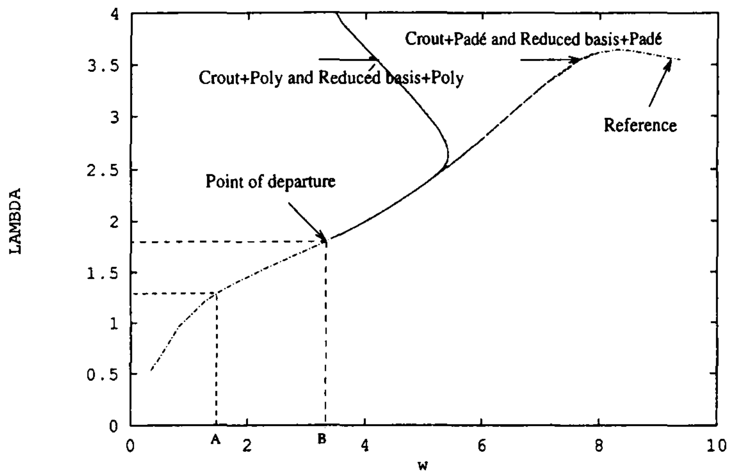

FIG. 8 – *Cylinder (5190 d.o.f.): Representation of the loading parameter λ as a function of the displacement at the point M.For the sake of completeness, the starting points of the steps correspond to wA=1,3 and wB=3,3*

The obtained residuals were greater than 10^{-3} even at the beginning of the step. So, the proposed reduced basis technique works well with a basis with a big size, but a procedure of normalisation is necessary in order to avoid numerical instabilities.

We may wonder what is the necessary dimension of the basis to obtain a satisfactory approximative solution. Such an analysis is presented on figure (10) where the number of vectors varies from 15 to 30. In the case of 15 vectors, they have been obtained by perturbation from the second starting point (point A). In the case of 20 vectors, the five first vectors obtained by perturbation from the initial point 0 have been added and so on. From the figure (10) one sees that at least 25 vectors are necessary to get a sufficiently large step length, and the approximation is not satisfactory below 20 vectors. The same results hold again for the roof problem.

Three numerical problems are considered to analyse the effective computation times : the roof with 726 d.o.f., the cylindrical shell with 5190 d.o.f. and the same cylindrical shell without account of the symetries. In the third test, the whole cylinder has been meshed with 12856 triangular DKT18 elements. This leads to a total number of d.o.f. equal to 39756. Note that the biggest problem has been tested on a HP C360 computer, it is approximatively 10 times faster than the HP 712/60 computer used for the other numerical tests. For the reduced basis method, we have used different basis, the dimension of the subspaces are between 15 and 30 vectors. In any case, the truncation order has always been chosen equalling to 15.

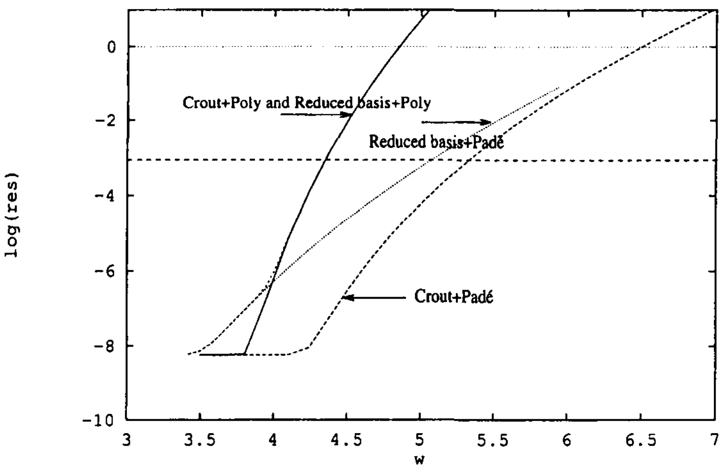

FIG. 9 – *Cylinder (5190 d.o.f.), third step : residual curve according to the representation (polynomial or fractions) and to the solving technique of the linear problems (Crout or reduced basis). Truncation order : 15. (Point M)*

For the two smallest problems (tables 5 and 6), the computing time is of the same order as the reference calculation (i.e., solving linear problems by Crout decomposition, order 15, rational approximation). The computing time increases with the dimension of the subspace, but not too much. Therefore, it is possible to work with a rather high dimension and it is recommanded to choose a higher dimension than 20, provided that such a basis is available. For the medium problem (cylinder, 5190 d.o.f.), the computation cost of the new method is 11% lower than the reference method (MAN+Padé), while the step lengths are about the same. The interest of the reduced basis technique is obvious from the results concerning the large scale problem (cylinder, 39756 d.o.f.). Indeed, the reduced basis technique permits to decrease the total CPU time of 87.5% as compared with the standard MAN-Padé algorithm while the two methods lead to about the same step length. In this case, most of the CPU time with the standard algorithm is spent by the matrix decomposition: this is consistent with a number of operations that is proportional to the cube of the number of d.o.f. Because the CPU time to generate the reduced matrix does not increase in the same manner, a time reduction from 2075 to 170 is not surprising. An additional time reduction is obtained in the calculation of the right hand sides f_i^{nl} and the corresponding displacement vectors u_i. In any case, the orthogonalisation process and the getting of the Padé Approximants have not a significant cost.

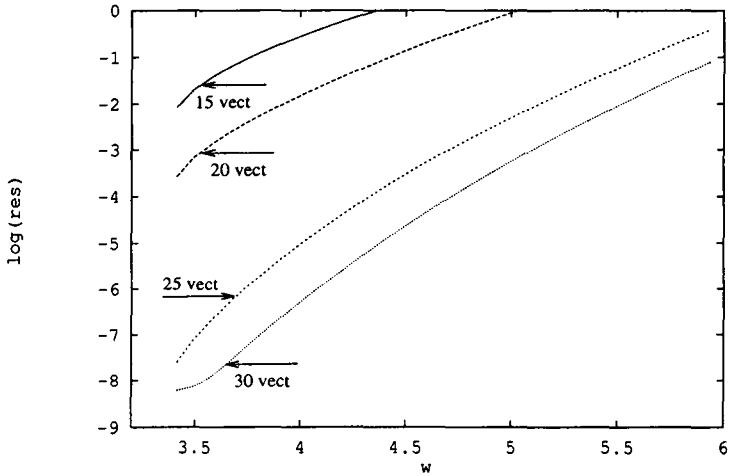

FIG. 10 – *Cylinder (5190 d.o.f.): Logarithm of the residual at order 15 for different basis vectors. Reduced basis technique combined with the rational fractions*

5. Conclusion

In this study a new reduced basis technique has been proposed and discussed to solve non-linear problems. Contrary to the classical reduced basis method [NOO 83], the perturbation is applied before the reduction. This avoids the calculation of many coefficients of a reduced non-linear problem. As compared to the classical asymptotic numerical method [COC 941] it differs only by the solving process of the linear problems that is performed by a reduced basis technique and not by a direct decomposition of the tangent matrix. It can also be strongly and cheaply improved by the introduction of Padé Approximants.

The presented numerical tests have clearly shown that the new method is very attractive for large scale problems. For instance, the computing time of one step of a problem having 39756 d.o.f. can be divided by 8. It has also been shown that the technique works well even with a rather large dimension of the subspace.

Likely, the new numerical technique could be useful in the prediction phase of prediction-correction algorithms applied to large scale problems. Indeed, many vectors can be computed using this type of numerical processes. This could help to define subspaces for the reduced basis technique. A complete discussion of this point is beyond the scope of the presented paper.

The application of the ANM to large scale problems (bigger than 10^5 d.o.f.) is a challenge. Finally, a direct linear solver would imply a too big computation time. An iterative method such as the conjugate gradient would allow a faster resolution, but the computation time would then be proportional to the truncation order. To account for previous resolutions with the same matrix, some techniques are available [REY 96]. Despite some attempts [MOK 99], the latter technique does not yet provide very good results within the ANM. The coupling of the ANM with the domain decomposition method is a more promising way [GAL 00]. The results of the present paper suggest that the reduced basis technique could also make easy the solving of large scale problems by the ANM.

The classical reduced basis method for non-linear problems has been revisited. Clearly, it is not recommanded when the dimension of the subspace is larger than five, because of the CPU time that hugely increases with the order. Furthermore and contrary to the new technique, it presents other drawbacks: numerical instabilities, application field restricted to quadratic or cubic systems.

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Appendix A

The operators l^*, q^* and c^* are defined as follows:

$$\begin{aligned}
l^*_{ij} &= \int_{\Omega_o} (\gamma^l(u_j) : D : \gamma^l(u_i))dv \\
q^*_{ijk} &= \int_{\Omega_o} (\gamma^{nl}(u_j, u_k) : D : \gamma^l(u_i))dv \\
&\quad + \int_{\Omega_o} (\gamma^l(u_j) : D : 2\gamma^{nl}(u_k, u_i))dv \\
c^*_{ijkl} &= \int_{\Omega_o} (\gamma^{nl}(u_j, u_k) : D : 2\gamma^{nl}(u_l, u_i))dv \\
and \quad f^*_i &= P_e(u_i)
\end{aligned} \qquad [21]$$

The operators l and q are defined as follows :

$$\begin{array}{l} l_{ij} = \int_{\Omega_o} S_j : \gamma^l(u_i)dv \\ \quad \text{for} \quad i = 1, p \quad \text{and} \quad j = 1, N \\ \\ q_{ijk} = \int_{\Omega_o} S_j : (\gamma^l(u_i) + 2\gamma^{nl}(u_k, u_i))dv \\ \quad \text{for} \quad i = 1, p \quad j = 1, N \quad \text{and} \quad k = 1, p \\ \\ and \quad f_i = P_e(u_i) \quad \text{for} \quad i = 1, p \end{array} \qquad [22]$$