Fluid-Structure Interaction: A Theoretical Point of View

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ABSTRACT. *This paper deals with the mathematical and numerical analysis of fluid-structure interaction phenomena, and present some of the existence results that can be found on this subject. We explain the various approaches and review the technical tools required. In all cases we have a fluid interacting with a moving (rigid or deformable) structure. The fluid is supposed to be viscous (compressible or incompressible) and the fluid equations are set in an unknown, time dependent domain, determined by the structure deformations, itselves resulting from a stress applied by the fluid.*

RÉSUMÉ. *Nous présentons ici un panorama de résultats concernant l'analyse et l'analyse numérique de phénomènes d'interaction fluide-structure, nous donnons en particulier les résultats d'existence de solution à ces problèmes. Il s'agit dans tous les cas de l'interaction d'un fluide avec une structure rigide ou déformable. Le fluide est supposé visqueux (compressible ou incompressible) et modélisé par une équation posée dans un domaine inconnu, dépendant du temps et limité par la structure, elle-même soumise aux contraintes exercées par le fluide.*

KEYWORDS: *existence results, Navier-Stokes, rigid bodies, elasticity, fluid-structure interactions, time discretisation, space discretisation.*

MOTS-CLÉS : *existence de solutions, Navier-Stokes, corps rigides, élasticité, interactions fluide structure, discrétisation en temps et en espace.*

1. Introduction

This paper deals with the mathematical and numerical analysis of problems dealing with unsteady fluid-structure interaction phenomena. These phenomena are of major importance for aerospace, mechanical or biomedical applications, and thus have been studied by many authors over the past few years from different point of view (theory, numerical analysis and simulations). The problem is to describe the evolution of a viscous fluid coupled with a moving structure. The fluid can be compressible or incompressible, and the structure can be rigid or elastic. Several conditions traduce the coupling between the two media at their interface. First, the kinematic condition states that the fluid velocity and the structure velocity are equal. The second coupling condition traduces the action-reaction principle. The interaction is not reduced to these only transmission conditions since, in most interesting cases where the deformations of the structure are large enough, one can not neglect the variation of the fluid domain. We thus have to solve a problem defined (at least for the fluid part) over a time dependent domain. We focus here on the analysis and numerical analysis of the fluid-structure interaction problems in case where the deformation of the fluid domain is actually part of the unknown. We refer to [MOR 92] for a study of the vibrations of coupled problems in which the domain is fixed.

Section 2 is devoted to a general presentation of the problem. Energy estimates are formally derived. Then we expose some existence results: existence of weak solutions (section 3), existence of strong solutions (section 4). Finally in section 6, we give some results on the numerical analysis of the discretizations.

2. Standard mathematical formulation

In order to simplify the presentation we assume that the flow is viscous and homogeneous, incompressible or compressible and that its behaviour is described by the Navier-Stokes equations. We denote by ν its constant viscosity, ρ_f its density. For the structure, we can consider several cases: rigid bodies immersed in fluid, full $3D$ elasticity or hyperelastic models. But we can also consider asymptotic models such as plates, shells, beams, that are used when the thickness or the section of the elastic media is small with respect to its other dimensions. We can even consider that the displacement of the structure is a linear combinaison of a finite number of elastic eigenmodes. This latter modelization and the case of rigid bodies actually reduces the structure equations to ordinary differential equation (o.d.e). In the other cases, the partial differential equations (p.d.e) describing the structure are classically set in a fixed domain $\hat{\Omega}_S$. The domain $\hat{\Omega}_S$ is, in general, the reference configuration of the body (that will be assumed to coincide with its initial state for the sake of simplicity). The behaviour of the structure is described by the displacement $\mathbf{d}$ of each point $\mathbf{x}$ of the reference configuration. Consequently, each point $\mathbf{x} \in \hat{\Omega}_S$ is, at time t, at the position $\mathbf{x}(t) = \mathbf{x} + \mathbf{d}(\mathbf{x}, t)$. On the contrary, the fluid equations are set in Eulerian coordinates and are thus defined in an unknown domain $\Omega_f(t)$ depending on the structure displacement $\mathbf{d}$. All the unknowns linked to the fluid part are thus evaluated at each point of the physical domain, at time t. The incompressible Navier-Stokes equations are: *Find* $(\mathbf{u}, p)$ *in* $\Omega_f(t)$ *such that*

$$\rho_f \frac{\partial \mathbf{u}}{\partial t} + \rho_f \mathbf{u}.\nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \rho_f \mathbf{f}_f \quad \text{in } \Omega_f(t) \qquad [1]$$

$$\text{div}\, \mathbf{u} = 0 \quad \text{in } \Omega_f(t), \qquad [2]$$

where $\mathbf{u}$ denotes the fluid velocity, p its pressure and $\mathbf{f}_f$ a given exterior force. These equations are completed by initial data

$$\mathbf{u}(t=0,.) = \mathbf{u}_0(.) \quad \text{in } \Omega_f(0) \qquad [3]$$

and by boundary conditions. These boundary conditions are of two different types. We have to distinguish the part of $\partial\Omega_f(t)$ which is not in contact with the structure from the fluid-structure interface that we will denote by $\Gamma(t)$. On $\Gamma_f = \partial\Omega_f(t) \setminus \Gamma(t)$, we consider standard boundary conditions and in order to simplify the presentation, we will suppose that they are of Dirichlet type:

$$\mathbf{u}(t,.) = \mathbf{0} \quad \text{on } \Gamma_f. \qquad [4]$$

On $\Gamma(t)$, coupling conditions have to be considered and will be detailed later on.

The compressible isentropic Navier Stokes model is: *Find* $(\mathbf{u}, \rho_f)$ *in* $\Omega_f(t)$ *such that*

$$\frac{\partial \rho_f \mathbf{u}}{\partial t} + \text{div}\, (\rho_f \mathbf{u} \otimes \mathbf{u}) - \nu \Delta \mathbf{u} - (\lambda + \nu) \nabla \text{div}\, \mathbf{u} + \nabla \rho_f^{\gamma} = \rho_f \mathbf{f}_f \text{ in } \Omega_f(t), [5]$$

$$\frac{\partial \rho_f}{\partial t} + \text{div}\, (\rho_f \mathbf{u}) = 0 \quad \text{in } \Omega_f(t), \; [6]$$

where γ is a real number ≥ 2 is the parameter of the pressure law. Again, these equations are completed by initial data

$$\rho_f(t=0,.) = \rho_{f,0}, \quad (\rho_f \mathbf{u})(t=0,.) = \mathbf{m}_0(.) \quad \text{in } \Omega_f(0) \qquad [7]$$

and by boundary conditions.

Before precising the proper boundary conditions over the interface $\Gamma(t)$ for both modelizations, we have to specify the different models for the structure we consider: $3D$ linearised elasticity, asymptotic model (many situations can be thought about, like beams, plate or shells equations...) or the equations of rigid body motion or even finite dimensional modal decomposition.

First, we consider the $3D$ linearised elasticity. The constitutive law of the elastic media is supposed to be the Hooke law. The Lamé constants of the media are λ_s and μ_s and ρ_s is its density, $\mathbf{f}_s$ denotes the volumic force applied on the structure. The equations can be written as follows

$$\rho_s \frac{\partial^2 \mathbf{d}}{\partial t^2} - \text{div}\sigma(\mathbf{d}) = \rho_s \mathbf{f}_s \quad \text{in } \hat{\Omega}_s, \qquad [8]$$

where the stress tensor σ is given by

$$\sigma(\mathbf{d}) = \lambda_s \mathrm{tr}\varepsilon(\mathbf{d})\mathbf{Id} + 2\mu_s\varepsilon(\mathbf{d}), \qquad [9]$$

and $\varepsilon(\mathbf{d})$ represents the linearised strain tensor and is equal to

$$\varepsilon(\mathbf{d}) = \frac{1}{2}(\nabla\mathbf{d} + \nabla\mathbf{d}^T).$$

Initial conditions have to be added, for instance

$$\mathbf{d}(t=0) = 0, \quad \frac{\partial \mathbf{d}}{\partial t}(t=0) = \mathbf{v}_0. \qquad [10]$$

Boundary conditions on the part of the structure Γ_s that is not in contact with the flow have also to be added (for instance homogeneous Dirichlet boundary conditions). Next boundary conditions on the interface have to be precised. Denoting by Γ the part of the boundary of $\hat{\Omega}_s$ which corresponds to the fluid-structure interface, we write that Γ and $\Gamma(t)$ represent the same entity.

$$\forall t,\ \forall \mathbf{x} \in \Gamma, \quad \mathbf{x} + \mathbf{d}(t,\mathbf{x}) \in \Gamma(t), \qquad [11]$$

and

$$\forall t,\ \forall \mathbf{y} \in \Gamma(t), \exists \mathbf{x} \in \Gamma, \quad \mathbf{x} + \mathbf{d}(t,\mathbf{x}) = \mathbf{y}. \qquad [12]$$

Next, we traduce the kinematic condition on the interface: on $\Gamma(t)$ the fluid sticks to the structure, and thus the velocities of the fluid part and of the structure part are equal. Since the fluid is supposed to be viscous (in the case of an invicid flow only the continuity of the normal component of the velocities is required):

$$\forall t,\ \forall \mathbf{x} \in \Gamma, \quad \mathbf{u}(t, \mathbf{x} + \mathbf{d}(t,\mathbf{x})) = \frac{\partial \mathbf{d}}{\partial t}(t,\mathbf{x}). \qquad [13]$$

The other boundary condition corresponds to the action-reaction principle and can be written as follows:

$$\forall t,\ \forall \mathbf{x} \in \Gamma, \quad T_F(\mathbf{u},p)(t,\mathbf{x}) = \sigma(\mathbf{d}).\mathbf{n}(t,\mathbf{x}), \qquad [14]$$

where $T_F(\mathbf{u},p)$ stands for the expression of the normal component of the fluid stress tensor σ_f ($\sigma_f = 2\nu D(\mathbf{u}) - p\mathbf{I}$ in the incompressible case and $\sigma_f = 2\nu D(\mathbf{u}) + (\lambda\ \mathrm{div}\mathbf{u} - \rho_f)\mathbf{I}$ in the compressible case with $D(\mathbf{u})$ denoting the symmetric part of $\nabla\mathbf{u}$) written in the reference configuration.

If the structure under consideration possesses a small section or thickness, then, asymptotic models can be proposed to describe its behaviour. The coupling conditions are expressed again by the equality of the velocities at the interface and, present in the right hand side of the plate, beam or shell equations, since a forcing term appears that comes from the stress applied by the fluid to the structure. Let us consider, for instance, a plate of thickness $2e$ and of average surface ω. The elastic media is supposed to be isotropic and homogeneous. E denotes its Young modulus, $\underline{\nu}$ its Poisson ratio and ρ_s its volumic mass. With the previous notations we have

$$\hat{\Omega}_s = \omega \times] - e, e[.$$

We only present here the equations satisfied by the transverse displacement of the plate $d_3 = d_3(x_1, x_2)$. The equations can be written as follows using the Einstein convention, and considering that the Greek indices belong to $\{1, 2\}$:

$$\begin{aligned} 2\rho_s e \frac{\partial^2 d_3}{\partial t^2} + \frac{2Ee^3}{3(1-\underline{\nu}^2)} \Delta^2_{x_1,x_2} d_3 = \; & g_3^+ + g_3^- + \int_{-e}^{e} f_{s3} \\ & + e\frac{\partial (g_\alpha^+ - g_\alpha^-)}{\partial x_\alpha} + \int_{-e}^{e} x_3 \frac{\partial f_{s\alpha}}{\partial x_\alpha}, \text{ in } \omega, \end{aligned} \quad [15]$$

where $\mathbf{f}_s$ denotes a volumic force and $\mathbf{g}^+$, $\mathbf{g}^-$ surfacic forces applied respectively on $\omega \times \{e\}$ and $\omega \times \{-e\}$. The plate is moreover supposed to be clamped on $\partial\omega \times [-e, +e]$. The longitudinal displacement is given by

$$d_\alpha(x_1, x_2, x_3) = -x_3 \frac{\partial d_3}{\partial x_\alpha}(x_1, x_2).$$

Let us express the coupling conditions between this plate and the viscous flow on the interface which is supposed to be $\Gamma = \omega \times \{-e\}$. We have, for all $\mathbf{x} \in \Gamma$,

$$\begin{aligned} u_3(\mathbf{x} + \mathbf{d}(t, \mathbf{x})) &= \frac{\partial d_3}{\partial t}(t, x_1, x_2), \\ u_\alpha(\mathbf{x} + \mathbf{d}(t, \mathbf{x})) &= e\frac{\partial^2 d_3}{\partial t \partial x_\alpha}(t, x_1, x_2). \end{aligned}$$

Furthermore, the structure is submitted to a surfacing force coming from the fluid, and thus $\mathbf{g}^-$ depends on the fluid stress tensor written in the reference configuration. More details about plate, beam, shell equations can be found in [CIA 90], [DES 86], [BER 94].

The third case that has been considered is the rigid body motion or the reduced basis motion where the structure is deformable with displacements that are written as a linear combinaison of a finite number of modal functions associated to the continuous elastic operator. The first coupling condition is the equality of the velocities at the interface. The second coupling condition appears in the right hand side of the structure equations which are now o.d.e with respect to the position of the center of gravity $\mathbf{x}_G$ and the rotation angle vector θ and the coefficients of the modal decomposition. Considering a solid sphere immersed in a fluid, we have the Newton equations

$$m\frac{d^2\mathbf{x}_G}{dt^2} = \int_{\partial\Omega_s(t)} \sigma_f.\mathbf{n} + \int_{\Omega_s(t)} \rho_s \mathbf{f}_s, \quad [16]$$

$$J\frac{d^2\theta}{dt^2} = \int_{\partial\Omega_s(t)} (\mathbf{x} - \mathbf{x}_G(t)) \wedge (\sigma_f.\mathbf{n}) + \int_{\Omega_s(t)} \rho_s(\mathbf{x} - \mathbf{x}_G(t)) \wedge \mathbf{f}_s, \qquad [17]$$

where $\Omega_s(t)$ is the rigid region at time t and m and J denote respectively the mass of the rigid body and its inertia momentum. The displacement of the structure can be recovered, writing

$$\mathbf{d}(t, \mathbf{x}) = \mathbf{x}_G(t) + \exp((\theta(t).B).(\mathbf{x} - \mathbf{x}_G(0))) - \mathbf{x}, \qquad [18]$$

where $\theta.B = \sum_{i=1}^{3} \theta_i.B_i$

and $B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $B_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

2.1. *Energy estimates*

First, we write the coupled problem in a variational formulation, assuming that the solution of the problem exists and is sufficiently smooth to justify all the manipulations. Multiplying the fluid equations [1] by a divergence free function $\mathbf{v}$, the trace of which over $\partial\Omega_f(t) \setminus \Gamma(t)$ is equal to zero, and then integrating over $\Omega_f(t)$, it comes, after integration by parts

$$\int_{\Omega_f(t)} \rho_f \frac{\partial \mathbf{u}}{\partial t}\mathbf{v} + 2\nu \int_{\Omega_f(t)} D(\mathbf{u}) : D(\mathbf{v}) + \int_{\Omega_f(t)} \rho_f (\mathbf{u}.\nabla)\mathbf{u}.\mathbf{v}$$
$$+ \int_{\Gamma(t)} (p\mathbf{n} - \nu(2D(\mathbf{u}).\mathbf{n})\mathbf{v} = \int_{\Omega_f(t)} \rho_f \mathbf{f}_f \mathbf{v}. \qquad [19]$$

For the structure part – for instance in the case of $3D$ linearised elasticity – we also multiply equation [8] by a test function $\mathbf{b}$, satisfying homogeneous Dirichlet boundary conditions over $\partial\hat{\Omega}_s \setminus \Gamma$, then integrate over $\hat{\Omega}_s$. After integrating by parts, we obtain

$$\int_{\hat{\Omega}_s} \rho_s \frac{\partial^2 \mathbf{d}}{\partial t^2}\mathbf{b} + a(\mathbf{d}, \mathbf{b}) = \int_{\Gamma} \sigma(\mathbf{d}).\mathbf{n}\mathbf{b} + \int_{\hat{\Omega}_s} \rho_s \mathbf{f}_s \mathbf{b}, \qquad [20]$$

with

$$a(\mathbf{d}, \mathbf{b}) = \int_{\hat{\Omega}_s} \lambda \varepsilon_{kk}(\mathbf{d}) \varepsilon_{\ell\ell}(\mathbf{b}) + 2\mu \varepsilon_{ij}(\mathbf{d}) \varepsilon_{ij}(\mathbf{b}).$$

If we choose the test functions such as

$$\mathbf{v}(t, \mathbf{x} + \mathbf{d}(t + \mathbf{x})) = \mathbf{b}(t, \mathbf{x}), \quad \forall\, \mathbf{x} \in \Gamma \qquad [21]$$

then adding [20] to [19] and taking into account the coupling condition [14], we have for any $\mathbf{b}$ and $\mathbf{v}$ satisfying [21]

$$\int_{\Omega_f(t)} \rho_f \frac{\partial \mathbf{u}}{\partial t}\mathbf{v} + 2\nu \int_{\Omega_f(t)} D(\mathbf{u}) : D(\mathbf{v}) + \int_{\Omega_f(t)} \rho_f \mathbf{u}.\nabla\mathbf{u}.\mathbf{v}$$
$$+ \int_{\hat{\Omega}_s} \rho_s \frac{\partial^2 \mathbf{d}}{\partial t^2}\mathbf{b} + a(\mathbf{d}, \mathbf{b}) = \int_{\Omega_f(t)} \rho_f \mathbf{f}_f \mathbf{v} + \int_{\hat{\Omega}_s} \rho_s \mathbf{f}_s \mathbf{b}. \qquad [22]$$

Similarly, starting from equations [5], [6], we deduce for any $\mathbf{b}$ and $\mathbf{v}$ satisfying [21]

$$\int_{\Omega_f(t)} \frac{\partial \rho_f \mathbf{u}}{\partial t}\mathbf{v} + 2\nu \int_{\Omega_f(t)} D(\mathbf{u}) : D(\mathbf{v}) + \int_{\Omega_f(t)} \rho_f \mathbf{u} \otimes \mathbf{u} : D(\mathbf{v})$$

$$+ \int_{\Omega_f(t)} (\lambda \text{div } \mathbf{u} - \rho^\gamma)\text{div } \mathbf{u}$$

$$+ \int_{\hat{\Omega}_s} \rho_s \frac{\partial^2 \mathbf{d}}{\partial t^2}\mathbf{b} + a(\mathbf{d}, \mathbf{b}) = \int_{\Omega_f(t)} \rho_f \mathbf{f}_f \mathbf{v} + \int_{\hat{\Omega}_s} \rho_s \mathbf{f}_s \mathbf{b}. \qquad [23]$$

REMARK. — The same kind of variational formulation can be obtained for the plate equations. Concerning the rigid body it is slightly different and we can obtain a global weak formulation where the unknown is the global velocity, obtained by extending the velocity of the fluid by a rigid body velocity in the domain occupied by the structure at time t.

REMARK. — In all the cases, the test functions of the coupled problem depend on time (the problem is set in a non cylindrical domain). Moreover the test functions depend on the solution of the problem, which is not standard.

Let us derive energy estimates. We choose $(\mathbf{u}, \frac{\partial \mathbf{d}}{\partial t})$ as test functions in [22]. Note that they are admissible test functions, in particular they satisfy [21]. This leads to

$$\int_{\Omega_f(t)} \frac{1}{2}\rho_f \frac{\partial \mathbf{u}^2}{\partial t} + 2\nu \int_{\Omega_f(t)} D(\mathbf{u}) : D(\mathbf{u}) + \int_{\Omega_f(t)} \rho_f \mathbf{u}.\nabla \mathbf{u}.\mathbf{u}$$

$$+\frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}_s} \left(\frac{\partial \mathbf{d}}{\partial t}\right)^2 + \frac{1}{2}\frac{d}{dt}a(\mathbf{d}, \mathbf{d}) = \int_{\Omega_f(t)} \rho_f \mathbf{f}_f \mathbf{u} + \int_{\hat{\Omega}_s} \rho_s \mathbf{f}_s \frac{\partial \mathbf{d}}{\partial t}, \qquad [24]$$

We recall the Reynolds formula

$$\frac{d}{dt}\int_{\Omega_f(t)} \phi(x,t)dx = \int_{\Omega_f(t)} \frac{\partial \phi(x,t)}{\partial t}dx + \int_{\Gamma(t)} \phi \mathbf{w}.\mathbf{n},$$

where $\mathbf{n}$ is the outer unit normal vector of $\Omega_f(t)$ and $\mathbf{w}$ is the velocity of each point of the interface $\Gamma(t)$. Using it with $\phi = \frac{\mathbf{u}^2}{2}$ we derive (we recall that ρ_f is a constant — the fluid is homogeneous—)

$$\int_{\Omega_f(t)} \frac{1}{2}\rho_f \frac{\partial \mathbf{u}^2}{\partial t} = \frac{1}{2}\frac{d}{dt}\int_{\Omega_f(t)} \rho_f \mathbf{u}^2 - \frac{1}{2}\int_{\Gamma(t)} \rho_f \mathbf{u}^2 \mathbf{u}.\mathbf{n}.$$

Considering the incompressibility [2], the convection term becomes, after integration by parts,

$$\int_{\Omega_f(t)} \rho_f (\mathbf{u}.\nabla)\mathbf{u}.\mathbf{u} = \frac{1}{2}\int_{\Gamma(t)} \rho_f \mathbf{u}^2 \mathbf{u}.\mathbf{n}.$$

Thus taking into account these two equalities, [24] yields

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_f(t)} \rho_f \mathbf{u}^2 + 2\nu \int_{\Omega_f(t)} D(\mathbf{u}) : D(\mathbf{u}) + \frac{1}{2}\frac{d}{dt}\int_{\hat{\Omega}_s} \rho_s \left(\frac{\partial \mathbf{d}}{\partial t}\right)^2 + \frac{1}{2}\frac{d}{dt} a(\mathbf{d},\mathbf{d}) = \int_{\Omega_f(t)} \rho_f \mathbf{f}_f \mathbf{u} + \int_{\hat{\Omega}_s} \rho_s \mathbf{f}_s \frac{\partial \mathbf{d}}{\partial t}, \quad [25]$$

recalling the ellipticity of the bilinear form a over $H^1_{0,\Gamma_s}(\Omega_s)$ (c.f [CIA 86]), an energy estimate can then be deduced if the solution of the coupled system [1], [2], [8], [13], [14] exists:

$$||\mathbf{u}||_{L^\infty(0,T;L^2(\Omega_f(t))} + ||\mathbf{u}||_{L^2(0,T;H^1(\Omega_f(t))} + ||\mathbf{d}||_{W^{1,\infty}(0,T;L^2(\hat{\Omega}_s))} + ||\mathbf{d}||_{L^\infty(0,T;H^1_{0,\Gamma_s}(\hat{\Omega}_s))} \leq C(T, \mathbf{f}_f, \mathbf{f}_s, \mathbf{v}_0, \mathbf{u}_0). \quad [26]$$

In case of compressible flows, we obtain, in a similar way from [23], if the solution of the coupled system [5], [6], [8], [13], [14] exists:

$$||\rho_f||_{L^\infty(0,T;L^\gamma(\Omega_f(t))} + ||\sqrt{\rho_f}\mathbf{u}||_{L^\infty(0,T;L^2(\Omega_f(t))} + ||\mathbf{u}||_{L^2(0,T;H^1(\Omega_f(t))} + ||\mathbf{d}||_{W^{1,\infty}(0,T;L^2(\hat{\Omega}_s))} + ||\mathbf{d}||_{L^\infty(0,T;H^1_{0,\Gamma_s}(\hat{\Omega}_s))} \leq C(T, \mathbf{f}_f, \mathbf{f}_s, \mathbf{v}_0, \mathbf{m}_0, \rho_{f,0}). \quad [27]$$

The same type of energy estimate can also be obtained (at least formally) for the two other models presented before. Such estimates are the first step to prove the existence of weak solutions.
REMARK. — As mentioned in [ERR 94], the convection term seems to be necessary, in most cases, in order to obtain energy estimates — without imposing supplementary conditions on the data (small enough data, small time....) — when dealing with unknown time dependent domains. Besides the theoretical analysis, this fact is important to point out, especially for the numerical simulation of fluid-structure interaction. Indeed, as a starting point, the Stokes problem is often the first step of the implementation of the fluid discrete problem:the coupling of the Stokes model with the structure interaction may be unstable regardless of the presence of bugs, indeed the corresponding continuous problem may not be stable either, to start with, as was exhibited in [ERR 94]. To simulate the coupling it is thus important to consider the full, nonlinear, Navier-Stokes problem.

Let us now present some of the theoretical results that can be found on such models. In a first section we review the results dealing with the existence of weak solutions, then look at the question of existence of strong solutions.

3. Weak solution

3.1. *Elasticity*

For the time being, as far as we are aware of, no result is available on the full interaction problem: Navier-Stokes coupled with the $3D$ elasticity in time dependent domain. Nevertheless, we refer to [GRA 00a] for an existence result on a similar but steady state problem.

For the time being, the rigid body case seems to be the most accessible one.

Note however that, in the case of weak perturbations of the interface, one can assume that the fluid occupies a fixed domain: $\Omega_f(t) \equiv \Omega_f$. Numerically, when this assumption is made, other interface boundary conditions are often considered in order to take into account the interface motion: these are the transpiration techniques (see [BAR 94], [FAN 00]). From a theoretical point of view, J.L. Lions [LIO 69] proves the existence of a unique weak solution "à la Leray" using the Galerkin method for the coupled problem: Navier-Stokes coupled with the linearised elasticity. Since $\Omega_f(t) = \Omega_f$, the convection term does not disappear when the energy estimates are derived. Consequently, the interface condition [13] is modified in order to obtain a bound of the solution. If we want to keep [13], existence of weak solutions can be proven considering the Stokes equations instead of the Navier-Stokes equations.

3.2. *Rigid Body*

Several papers treat of the model of rigid bodies immersed in a viscous, incompressible or compressible flow from the theoretical point of view and answer the question of the existence of weak solutions. We tell about three of them, where quite the same weak formulation is used but where the techniques to prove the solvability are different. The unknown is the **global velocity** $\mathbf{u}$ equal to the fluid velocity in $\Omega_f(t)$ and to the rigid bodies velocity in $\Omega_s(t)$ $(\frac{d\mathbf{x}_G}{dt} + \frac{d\theta}{dt} \wedge (\mathbf{x} - \mathbf{x}_G))$. Let us denote by ρ the **global density**: $\rho = \rho_f 1_{\Omega_f(t)} + \rho_s 1_{\Omega_s(t)}$ where 1_E denotes the characteristic function of a given part E. In view of the conservation of mass [2] or [6], ρ satisfies a linear transport equation

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0 \text{ in } \Omega. \qquad [28]$$

Now, we consider a test function $\mathbf{w}$ that is rigid in $\Omega_s(t)$ and divergence free in the incompressible case, which is the case exposed below. The fact that $\mathbf{w}$ is rigid in $\Omega_s(t)$ can be written as follows $1_{\Omega_s(t)} D(\mathbf{w}) = 0$. Let us introduce the space of test functions

$$\mathcal{V} = \left\{ \mathbf{w} \in H^1([0,T] \times \Omega), 1_{\Omega_s(t)} D(\mathbf{w}) = 0, \mathrm{div}\, \mathbf{w} = 0 \right\}.$$

Note that the space of test functions depends on the continuous solutions. Starting from equations [2], [16], [17], one can derive the following variational formulation *for all* $\mathbf{w} \in \mathcal{V}$ *and a.e.* t,

$$\int_\Omega \rho \mathbf{u}(t)\mathbf{w}(t) - \int_0^t \int_\Omega \rho \mathbf{u} \partial_t \mathbf{w} - \int_0^t \int_\Omega \rho \mathbf{u} \otimes \mathbf{u} : D(\mathbf{w})$$

$$+2\nu \int_0^t \int_\Omega D(\mathbf{u}) : D(\mathbf{w}) = \int_\Omega \rho_0 \mathbf{u}_0 \mathbf{w}(0) + \int_0^t \int_\Omega \rho \mathbf{f} \mathbf{w}, \qquad [29]$$

where $\mathbf{f} = \mathbf{f}_s$ in the structure part and $\mathbf{f} = \mathbf{f}_f$ in the fluid part.

REMARK. — Such a global formulation can be used for the numerical simulation of particles in flow, and enables one to employ fixed mesh. The fictitious domain method relies also on such a formulation. We refer to [GLO 94a], [GLO 94b] for more details on the fictitious domain method.

The fact that this problem is a weak form of the original coupled frame is standard in regard to the fluid domain, indeed, we recover [1] by using a test function $\mathbf{w} = 0$ on $\Omega_s(t)$. Let us assume now enough regularity on the solutions, then we first observe that over $\Omega_s(t)$, the functions $\mathbf{u}$ and $\mathbf{w}$ correspond to a rigid motion and thus have the form

$$\mathbf{u}(x,t) = \mathbf{a}(t) + \omega(t) \wedge (\mathbf{x} - \mathbf{x}_G(t))$$

$$\mathbf{w}(x,t) = \mathbf{b}(t) + \zeta(t)J \wedge (\mathbf{x} - \mathbf{x}_G(t))$$

where $\mathbf{b}$ and ζ are arbitrary functions of t and $\mathbf{a}(t) = \frac{d\mathbf{x}_G(t)}{dt}$ using first $\zeta = 0$ then $\mathbf{b} = 0$, we obtain [16] and [17].

In [DES 99], [DES 00a], [CON 99] and [HOF 99] this weak formulation (or a quite similar one) is introduced. There are mainly three non standard features in those type of problems:

1. The Navier-Stokes equations are set in a non cylindrical domain, so classical Galerkin method does not apply.
2. The test functions depend on the solution.
3. The convection term requires compactness results on the velocity.

In [DES 99], [DES 00a] the existence of weak solutions for all $T > 0$ is proven, assuming that there is no collision (i.e. the body does not touch the exterior boundary $\partial\Omega$ or if there are several particles they do not enter in collision). In the incompressible case, the velocity belongs to $L^2((0,T), H_0^1(\Omega)) \times L^\infty((0,T), L^2(\Omega))$ and the density belongs to $L^\infty((0,T), L^\infty(\Omega))$, provided that $\mathbf{u}_0 \in L^2(\Omega)$, $\mathbf{f} \in L^2((0,T) \times \Omega)$ and $\rho_0 \in L^\infty(\Omega)$. In the compressible isentropic case the velocity is such that $\sqrt{\rho}\mathbf{u} \in L^\infty((0,T), L^2(\Omega))$ and $\mathbf{u} \in L^2((0,T), H_0^1(\Omega))$ and the density belongs to $L^\infty((0,T), L^\gamma(\Omega))$, provided suitable assumptions on the data. The proof is based on Schauder fixed point theorem and compactness results.

Step 1

The motion of the rigid body is supposed to be given, together with a global velocity $\mathbf{v}$ satisfying a nonlinear constraint (that we denote by $(*)$) stating that $\mathbf{v}$ corresponds to the rigid body motion in the given rigid body region. This velocity is regularized in space and time through an operator R_ε that preserves the constraint $(*)$. Then the original problem is linearised using this regularized velocity $\mathbf{v}_\varepsilon = R_\varepsilon(\mathbf{v})$. For ε fixed, the problem is now linear and the test functions depends on $\mathbf{v}_\varepsilon$. The equations are next written, thanks to a change of variables, in Lagrangian coordinates. A Galerkin method is then used to solve them. The basis functions considered are of two types: first the eigenfunctions of a Stokes-like problem defined in the initial fluid domain, where the coefficients depend on $\mathbf{v}_\varepsilon$, and a basis of the set of rigid motion extended in the fluid part. At this step, the assumption that there are no collision is required.

Step 2

Knowing that the previous linearized problem is well posed, we obtain a new velocity $\mathbf{u}_\varepsilon$, that gives us a new rigid body motion. The Schauder fixed point theorem is applied on the mapping S that associates with $\mathbf{v}$, $S(\mathbf{v}) = \mathbf{u}_\varepsilon$ (which has been actually slightly modified in order to satisfy the nonlinear constraint $(*)$). The compactness is obtained because of regularity results on $\mathbf{u}_\varepsilon$.

Step 3

Next, one has to pass to the limit when ε goes to zero, using an *a priori* energy bound that states that $\sqrt{\rho_\varepsilon}\mathbf{u}_\varepsilon \in L^\infty((0,T), L^2(\Omega))$ and $\mathbf{u}_\varepsilon \in L^2((0,T), H_0^1(\Omega))$ (and also in the compressible case that $\rho_\varepsilon \in L^\infty((0,T), L^\gamma(\Omega))$). One can then pass to the limit in the mass equation thanks to Di Perna-Lions compactness results for linear transport equations [DIP 89] (the case of isentropic compressible flow is slightly more complicated). Next, one has to study the momentum equation. At this step, compactness results on the velocity are required: it is shown that the velocity is compact in $L^2((0,T) \times \Omega)$, thanks to Riesz-Fréchet-Kolmogorov compactness theorem in L^p (see [BRE 83], p. 72). In order to pass to the limit in the weak equation, considering $\mathbf{w} \in \mathcal{V}$, a test function $\mathbf{w}_\varepsilon \in \mathcal{V}_\varepsilon$ is built such as $\mathbf{w}_\varepsilon$ converges in the good spaces though $\mathbf{w}$. At this step the assumption that no collision occurs is needed again.

Note that more recently, M. Tucknak has proven in [TUC 00] a compactness result stating that the weak limit of any weakly convergent sequence of solution of a fluid-structure interaction problem is still a solution of such a problem using basic theory on Navier-Stokes, thus simplifying the approach based on the use of renormalized solutions involved in the previous approach.

These types of methods can be extended to tackle a structure described by a finite number of modal functions (see [DES 00b]). It seems difficult to apply them to more general structure models since one needs space and time regularity of the fluid-structure interface, and $3D$ elastic models do not provide the required regularity.

The next articles deal with the incompressible model.

In [CON 99], the same result is proven but for only *one* rigid body. The problem is written in a new system of coordinates moving together with the body (thus making difficult the generalization of the approach to more than one rigid body). This tools had been already used by D. Serre in [SER 87] where he studies a ball in a viscous fluid that occupies the whole space $\mathbb{R}^3$. So the ball is then fixed inside a domain $\tilde{\Omega}(t)$ whose exterior boundary is moving (and over which a homogeneous Dirichlet boundary condition is set). They embed this moving domain inside a fixed larger one E and extend the functions defined over $\tilde{\Omega}(t)$ by 0 over the complementary set noted as $\tilde{\Omega}^c(t)$. As in [FUJ 70] a penalizing term is added to the translated-rotated equation [29] corresponding to a mass term on the velocity over $\tilde{\Omega}^c(t)$, so that, in the limit, the velocity will be zero over $\tilde{\Omega}^c(t)$. This allow to suppress the difficulty induced by the time dependent domain. A standard Faedo-Galerkin method is used to prove that the penalized problem is well posed and the *a priori* bounds on the solution are then used to pass to the limit following the same compactness arguments as the ones used in [FUJ 70].

In [HOF 99], the problem of the motion of a solid body $\Omega_s(t)$ in a bounded domain filled with a viscous incompressible fluid is adressed. The approach is again to write the problem over the whole domain but the equation over ρ is replaced by an equation over $\phi = 1_{\Omega_s(t)}$ that belongs to the space $Char(Q)$ the class of characteristic functions of subsets of Q. The problem reads as follows: *find* $(\mathbf{u}, \phi)$ *with*

$$v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\mathcal{V}) \qquad [30]$$

$$\phi \in Char(Q) \cap C^{1/p}(0,T;L^p(\Omega)), 1 \leq p < \infty \qquad [31]$$

such that [29] is satisfied together with

$$\int_Q \phi(\eta_t + \mathbf{u}.\nabla\eta)dxdt = 0 \qquad [32]$$

holds for any $\eta \in C^1(Q), \eta(T) = 0$.

The analysis of problem [29], [32] then proceeds by using a penalized method that consists in adding the term $\frac{1}{\varepsilon}D(\mathbf{v})D(\psi)$ in [29], since the rigid body motion belongs to the kernel of $D(.)$. An additional ingredient is required that consists in regularizing the equations by adding the term $\delta\nabla\Delta\mathbf{v}\nabla\Delta\psi$. It is not hard to check that this resulting problem is well posed, providing a solution $\mathbf{v}_{\varepsilon,\delta}$ and that the following a priori estimate holds

$$\int_\Omega |\mathbf{v}_{\varepsilon,\delta}(x,t)|^2 + \int_0^t \int_\Omega |\nabla\mathbf{v}_{\varepsilon,\delta}(x,t)|^2 + \frac{1}{\varepsilon}\phi|D(\mathbf{v}_{\varepsilon,\delta})(x,s)|^2$$
$$+\delta|\nabla\Delta\mathbf{v}_{\varepsilon,\delta}(x,s)|^2 dxds \leq C. \qquad [33]$$

The proof proceeds by passing to the limit first in ε second in δ using a generalized Aubin compactness theorem. In their paper, the authors also investigate the behaviour of the body near the wall and prove that if the body comes to the wall, its speed must vanish, which proves the limit of the modelization of the phenomenon in this extreme situation by the Navier-Stokes equation.

3.3. *Asymptotic models*

By asymptotic models we mean models of plates, beams or shells. In [FLO 00], the authors study a three-dimensional problem where a plate interacts with a linear compressible fluid. The plate equations are the ones described previously, with an additional term which represents the rotation inertia ($= \Delta \frac{\partial^2 d_3}{\partial t^2}$). This term regularizes the plate equation. The existence of weak solutions (in the energy spaces) is obtained thanks to a fixed point procedure (Kakutani fixed point theorem), provided that the data are small enough. The fluid equations are studied for a given geometry of the fluid domain. In order to deal with the non cylindrical domain an elliptic regularization of the equation is performed. These tools have been used in [SAL 85], where Navier-Stokes equations defined in a given time dependent geometry were considered. The structure equations are studied for a given forcing (coming from the regularized fluid equations). These are linear equations defined in a fixed domain, so there is no particular difficulty. Then the problem is recoupled thanks to a fixed point procedure. The final step is the convergence of the regularized problem through the real one. We refer also to [FLO 99], where the same problem in $2D$ is treated, and where the existence of a smooth solution is proven.

4. Strong solution

In [GRA 00b], we study the existence of strong solutions for rigid bodies immersed in a viscous incompressible fluid contained in a bounded domain. To handle the problem of the time dependent domain the Navier-Stokes equations are written in Lagrangian coordinates, and thus the new unknowns are the lagrangian velocity and pressure (still denoted by $\mathbf{u}$ and p). The space $K_T^r(\Omega)$ in which we search the solution is defined as follows: we set for $T > 0$,

$$K_T^r(\Omega) = L^2(0,T;H^r(\Omega)) \cap H^{r/2}(0,T;L^2(\Omega)).$$

The main result is the following:

Let r be a real number, $1 < r < 3/2$. We assume that $\mathbf{u}_0 \in H^{r+1}(\Omega_f(0))$, $\mathbf{f}_f$ and $\mathbf{f}_s$ are sufficiently smooth and that the mass and the momentum of inertia of the body is sufficiently large, then there exists a time $T_1 > 0$ depending on the data ($\Omega_f(0)$, $\|\mathbf{u}_0\|_{H^{r+1}(\Omega_f((0)))}$, $\mathbf{f}_f$...) such that the problem has a unique solution with $\mathbf{u} \in K_{T_1}^{r+2}(\Omega_f((0))$, $\nabla p \in K_{T_1}^r(\Omega_f((0))$, $\mathbf{x}_G \in H^{r/2+2}(0,T_1)$ and $\theta \in H^{r/2+2}(0,T_1)$.

The proof is based on several fixed point procedures. We study, in a first step, a fluid problem with a given velocity over $\partial\hat{\Omega}_s$. For such equations we prove that there exists a smooth solution with the help of a fixed point theorem (contraction mapping principle). The ideas are the same that one can find in the papers [ALL 83], [ALL 87], [BEA 81], [SOL 77], [SOL 88a] where the authors have studied the solvability of the Navier-Stokes equations with free surface in bounded or unbounded domains. The approach is the following: the equations are rewritten in Lagrangian coordinates (this change of variable enables to solve the difficulties linked to the time dependent domain) and it is shown that solutions for the initial value problem exist locally in time, in smooth functions spaces $K_T^r(\Omega)$. This first step enables us, for a given velocity of the rigid body, to define a fluid velocity and a fluid pressure. This gives us the fluid constraints. The next step is to solve the structure equations for given exterior forces (coming from the fluid constraints). A new velocity of the interface is derived. The problem is recoupled thanks to a fixed point procedure. It is at this step that we need to add the constraint on the size of the mass and inertia momentum of the rigid body. Nevertheless, this additional condition that seems unnatural, allows to obtain an uniqueness statement on the solution of the coupled problem.

Theses results can also be extended to a deformable structure whose displacement is a linear combinaison of a finite number of modal functions associated to the continuous elastic operator. The equations that described the evolution of the structure are o.d.e. and admit sufficiently smooth solutions in order to apply the same techniques described above (see [GRA 00c]).

5. Summary

Let's summarize the different techniques and strategies used in those type of problems. The question of time dependent domain can be solve as follows:

– Considering the fluid equations written in Lagrangian coordinates;

– Penalization techniques;

– Elliptic regularization of the equations.

We have also seen that one can try to find a global formulation of the fluid-structure interaction problem or look at the fluid equations and the structure equations separately and used a fixed point procedure to recoupled the problem.

Numerically, the dependence in time of the fluid domain requires also special procedures and can be treated thanks to different techniques. One has to deal with moving mesh or to develop special formulations in order to work on a fixed mesh. Nevertheless, in the latter case one has to find a way to track the moving interface. For flows in time dependent domains, the ALE (Arbitrary Lagrangian Eulerian) formulation is often used. It consists in working on a moving mesh whose motion is determined by a mesh velocity whose only constraint is to be equal to the fluid velocity at the interface. We refer to [HUG 81], [DOE 82] for more details. Other approaches have also been proposed among which we note the space-time formulations [TEZ 92].

We have already mentioned the fictitious domain techniques that has been used in the case of a fluid interacting with rigid bodies (to simulate sedimentation phenomena). The mesh is fixed and a global formulation on the global velocity is used for the fluid-structure problem. A Lagrange multiplier is introduced in order to enforce the rigid motion in the rigid region. One can also refer to the level set techniques used in a different context: modelization of the evolution of nonmiscible flows, where a global formulation is used and solved on a fixed mesh and where the interface is tracked thanks to a level set function satisfying a transport equation [CHA 96].

Nevertheless, in most cases, one has a fluid code able to solve the fluid equations (on a moving mesh) and a different structure code that treats the structure part. The question is then how to couple these two codes in order to obtain efficient algorithms, the advantage is that the fluid or structure models can be easily modified in such a "black-box" approach. In the next section, we focus on the numerical problems encountered for fluid-structure interaction simulation and more particularly: the time discretization and the spatial discretization.

6. Numerical analysis

6.1. *Time discretization*

In order to propose a time discretization of the model, we have to choose between different strategies for the treatment of the interaction. Many different schemes from fully implicit to fully explicit can be investigated. When the structure and the fluid are represented by models having the same dimension, there are numerous strategies that can be though about.

The first one is a complete implicit treatment where at each time step, the geometry, the interaction and the fluid and structural data (velocities and stresses) are all balanced. This highly nonlinear algorithm has to be solved iteratively. It is naturally stable but may require a special procedure (relaxation) to ensure the convergence at each iteration [MOU 96]. In this article, the authors prove –on a linear fluid-structure interaction problem– that an iterative process, based on the separated resolution of a fluid problem and a structure problem, converges through the coupled problem, provided a relaxation procedure is used.

A less implicit scheme is the one where we treat explicitly the behaviour of the geometry while retaining the coupling implicit. Thus, knowing the approximations of velocity, displacement and geometry at the time $n\Delta t$, we begin to extrapolate the geometry at the time $(n+1)\Delta t$. We then have to discretize the fluid part and the structure part and we can propose to solve (iteratively again) the fluid and the structure equations so that the interface conditions (velocities and stresses) are balanced at time $(n+1)\Delta t$. Another possibility is to make the coupling more explicit by solving at each time step the fluid and the structure parts only once and independently. This leads to the so-called staggered or partitioned strategies. We can compute the fluid equation with Dirichlet boundary conditions (provided from the previous time steps) and then the structure equation with Neumann boundary conditions (obtained from the recently computed fluid motion) or the opposite, first the fluid with Neumann boundary conditions and then structure with Dirichlet boundary conditions or first structure with either Dirichlet or Neumann boundary conditions. We can also compute first the structure part and the fluid part with the same possibilities for the interface conditions. In the case of a 3D fluid interacting with a plate, a shell or a structure modelled by modal functions then the only possible staggered strategy is to treat the fluid part with Dirichlet boundary conditions and compute the displacement knowing the applied fluid efforts. There are numerous variations for all those strategies depending on the choice of the time integrators for the fluid and the structure part, on the choice of the evaluation of the load applied by the fluid to the structure.... One can also think of prediction-correction procedures. Here, we review few articles that discuss the efficiency and stability properties of some of those strategies.

In [PIP 95], the authors introduce a criterion that ensures the stability of the numerical solution. This criterion expresses the energy balance at the fluid-structure interface. Next, they built and study few staggered procedures applied to a 1D linearized fluid-structure interaction problem (an Euler compressible flow interacting with a piston). The basic and popular staggered algorithm denoted by CSS (Conventional Serial Staggered) they consider is the following: (1) predict the motion of the interface, (2) update the fluid mesh, (3) compute the fluid part with a given velocity, (4) compute the new force applied to the structure, (5) advance the structural system. In particular, they prove that the basic CSS method completed with a well-chosen correction procedure provides an unconditionally stable and time-accurate scheme. Their theoretical linear analysis is confirmed by numerical simulations on a two-dimensional aeroelastic problem.

In the part II of the previous article [PIP 99], the authors develop a new criterion that predicts the performance of the considered partitioned procedure. They consider a three field formulation – the fluid, the structure and the dynamic of the mesh– to describe the fluid-structure interaction problem. They estimate the energy that is introduced at the interface of the two media by the various staggered schemes, assuming that the structure and the pressure induced by the flow are vibrating with constant amplitudes at the same frequency but assuming that they are not in phase. This gives an estimate of the created energy with respect to the time step. They validated their approach on two-dimensional and three-dimensional aeroelastic applications (supersonic and transonic flow/ panel and wing).

In [GRA 98b], three different strategies are studied, applied on a one-dimensional nonlinear problem where a modified Bürger equation in an unknown time dependent domain (written in ALE formulation) is coupled with a wave equation. In all those strategies the geometry is predicted. In the first one the interface conditions are treated implicitly, in the two others the treatment of the interface conditions is explicit. The first explicit strategy consists in solving the fluid part with Neumann boundary condition and then the structure part with Dirichlet boundary condition, and the second one corresponds to the so-called CSS procedure. The considered schemes are first order in time. We prove, provided that the time step and the data are sufficiently small, that the implicit strategy is stable and convergent with a rate of convergence of $\Delta t^{3/4}$. Concerning the explicit schemes, the first one is stable with a stability constant that can explode as time increases. Moreover if the equations are also discretized in space using a finite element discretization, then the scheme is stable under CFL-like conditions. Under more constraining CFL-like conditions, stability estimates are derived for the second explicit scheme but we are not able to derive stability without considering space discretization. These latter results seem to confirm the well-known limitation of CSS procedure.

6.2. *Spatial discretization*

We now consider the question of spatial discretization of the fluid-structure interaction problem [22]. In most of the applications and because one has –in most of the cases– at its disposal a fluid code and a structure code built independently, the fluid mesh and the structure mesh may be non-matching or incompatible (even if the meshes match, the partial differential equations describing the two media may require different types of discrete functions). In this framework, the question is how to couple these two codes, how to transfer the information at the interface of the two media, in a reliable and energy-consistant way. In most of the cases the structure solver is based on a finite element discretization, and we will denote by h_s the associated spatial mesh size. Concerning the fluid one can consider either finite element discretization or finite volume discretization. At the fluid-structure interface one has to traduce the equality of the velocities and the load balance. Nevertheless, at the discrete level, when the discretizations are incompatible, this can not be done in a strong way. The weak equality (all the quantities are now discrete ones) can be written as follows:

a) $\mathbf{u} = Q(\partial_t \mathbf{d})$,

b) $\sigma_s.\mathbf{n} = P(\sigma_f.\mathbf{n})$,

where Q and P are two linear operators (possibly depending on time if dealing with moving boundary) and σ_s (resp. σ_f) represents the numerical structure (resp. fluid) stress tensor. The question is how to define Q and P in order to be conservative?

If Q (depending for instance on the fluid discretization) is fixed the balance of the fluid and structure virtual works requires that $P = Q^T$ (see [GRA 98c] and [GRA 98a]. This is what is underlined in [FAR 98] where the authors present a conservative algorithm where the operator Q is the fluid interpolation operator and compare the choice $P = Q^T$ to the non conservative interpolation based methods where Q and P are both interpolation operators. Thanks to some numerical simulations, they show that these non-conforming methods can be accurate, in some cases (when the fluid and the structure interface share the same geometrical support) and that the conforming methods are accurate in all the considered cases. Considering finite element discretizations for the structure part they also discuss the accuracy of the mortar element approach introduce by Bernadi, Maday, Patera [BE 90]. This method is conservative and (as we shall see) mathematically optimal, in the sense that the error estimate for the fluid-structure interaction problem obtained when considering incompatible meshes is the same that the one obtained when the meshes match. Nevertheless, it can be noted that, considering P_1 finite element discretization for the fluid part, and if the fluid mesh size h_f is chosen of the order of h_s^2 then the conservative interpolation method is also accurate.

For a numerical analysis, we refer to [GRA 98a], where error estimates are derived for a steady-state limit model interaction problem. On the one hand, for the fluid part, a two-dimensional second-order equation is considered discretized with a P_k finite element discretization. On the other hand, for the structure part, several cases can be considered: a $2D$ structure, a beam (modelized by two decoupled one-dimensional equations: a fourth-order equation for the transverse displacement and a second-order equation for the longitudinal displacement), or a structure modelized by a finite number of modal functions. In this article, two different types of matching are considered: a pointwise matching, i.e. Q is equal to the P_k finite element interpolation operator associated to the fluid part, and an integral matching, i.e. the mortar element method. In all the cases, the error estimate will be of the form $O(h_f^\alpha) + O(h_s^\beta)$, where β is optimal in regard to the discretization associated to the structure part. The possible lack of optimality will come from α. The following results hold (with no assumption made on the relative size of h_f with respect to h_s)

1. For $2D$ structure modelized by second order elliptic equation then the standard conclusion holds (see [BE 90]) i.e.:

(a) the mortar method is optimal, $\forall k$ ($\alpha = k$),

(b) the pointwise matching is not optimal ($\alpha = 1/2$).

2. For a $1D$ fourth-order equation :

(a) the mortar method is optimal, $\forall k$ ($\alpha = k$),

(b) the pointwise matching is optimal for $k \leq 2$ ($\alpha = max(k, 2)$).

3. For a $1D$ second-order equation :

(a) the mortar method is optimal, $\forall k$ ($\alpha = k$),

(b) the pointwise matching is optimal for $k \leq 1$ ($\alpha = max(k, 1)$).

4. For a finite number of model functions (then the motion of the structure is modelized by an o.d.e)

(a) the mortar method is optimal, $\forall k$ ($\alpha = k$),

(b) the pointwise matching is also optimal, $\forall k$.

As we see the optimality of the pointwise matching is linked to the regularity of the displacement at the interface of the two media.

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