Deriving Adequate Formulations for Fluid-Structure Interaction Problems: from ALE to Transpiration

Thierry Fanion* — **Miguel Fernández**** — **Patrick Le Tallec*****

** Dassault Aviation and INRIA, F-92214 Saint-Cloud*

thierry.fanion@dassault-aviation.fr

*** INRIA Rocquencourt, BP 105, F-78153 Le Chesnay Cedex*

miguel.fernandez@inria.fr

**** Ecole Polytechnique, F-91128 Palaiseau Cedex*

patrick.letallec@polytechnique.fr

ABSTRACT. *Most formulations describing low speed large displacements fluid-structure interaction problems use a totally lagrangian formulation for the structure, and an Arbitrary Euler Lagrange (ALE) formulation for the fluid. The purpose of the present paper is to review the derivation of such formulations, to describe different time discretisation strategies and to explain the type of numerical problems which arise when implementing these techniques. To overcome all technical difficulties arising when dealing with moving grids, we will also explain how an adequate asymptotic expansion can reduce the problem to a standard problem written on a fixed configuration, but using specific transpiration interface boundary conditions. This last formulation is rather popular in the aeronautical community, and will be illustrated by various numerical experiments.*

RÉSUMÉ. *La plupart des formulations de problèmes d'interaction fluide-structure en grands déplacements et faible vitesse utilisent une formulation lagrangienne totale pour la structure, et une formulation Arbitrairement Lagrangienne Eulerienne (ALE) pour le fluide. Le but de ce travail est de revoir la dérivation de telles formulations, de décrire les différentes stratégies de discrétisation en temps et d'éclaircir le type de problèmes numériques apparaissant dans l'implémentation de ces techniques. Pour surmonter les difficultés techniques provenant de l'utilisation de grilles mobiles, on montrera aussi comment un développement asymptotique adéquat permet de se ramener à un problème standard écrit en configuration fixe, mais avec des conditions de transpiration spécifiques à l'interface. Cette dernière formulation, plutôt en vogue dans le domaine de l'aéronautique, sera illustrée avec quelques expériences numériques.*

KEYWORDS: *fluid-structure interaction, large deformation, time discretisation, total energy conservation, linearisation, transpiration.*

MOTS-CLÉS : *interaction fluide-structure, grandes déformations, discrétisation en temps, conservation de l'énergie totale, linéarisation, transpiration.*

1. Introduction

Low speed large displacement problems where a flexible elastic structure interacts with the flow of an external or internal fluid occur frequently in practice, for example when studying hydraulic shock absorbers, biomedical flows in flexible pipes, aeroelastic instabilities of flexible aircrafts or tall bridges, or ocean flows around very long risers. The numerical challenge is to predict the longterm time evolution and stability of these coupled systems. It turns out that enforcing the kinematic compatibility at the fluid-structure interface and updating the geometry of the fluid domain requires a particular care, especially when this must be done within a numerical model which has been discretised in time and space.

The key is to properly respect mass and momentum conservation laws for the coupled fluid-structure system considered as a unique continuous medium sticking together because of a kinematic constraint mechanically imposed at the fluid-structure interface $\Gamma^s(t)$. These conservation laws when transported on a global fixed reference configuration define the mechanical problem to be solved (section 2). Consistent time discretisations can then be introduced (section 3). The problem is that, as observed in section 4, classical time integration schemes may loose their long term stability properties when used on moving domains, depending on the grid deformation smoothness and on the discretisation error in the equation of mass.

To overcome all technical difficulties arising when dealing with moving grids, we will then explain in section 5 how an adequate asymptotic expansion can reduce the problem to a standard problem written on a fixed configuration, but using specific transpiration interface boundary conditions. The efficiency of the resulting formulations will be illustrated by several numerical experiments in three dimensional aeroelasticity. Such transpirations boundary conditions formulations turn out to be quite popular in the engineering community, but up to now they were missing proper mathematical justifications and variational formulations.

2. Mechanical problem

The system under study occupies a moving domain $\Omega(t)$ in its present configuration. It is made of a deformable structure $\Omega^s(t)$ (aircraft, civil engineering structure) and of a surrounding fluid in motion in the complement $\Omega^f(t)$ of $\Omega^s(t)$ in $\Omega(t)$ (Figure 1). The problem consists in finding the time evolution of this configuration, of the velocity U and Cauchy stress tensor σ within the fluid and the structure, and to assess the long term stability of the system. The fluid may be inviscid which means that the normal component of the velocity field must be continuous at the interface, but that its tangential component may be discontinuous. Introducing the velocity field $U^s = U_{|\Omega^s}$ and $U^f = U_{|\Omega^f}$ within the structure and the fluid, and the unit normal vector $n(t)$ to the interface in its present deformed configuration (oriented towards the structure) this kinematic assumption takes the form

$$\mathrm{Tr}(U^s)_{|\Gamma^s} \cdot n(t) = \mathrm{Tr}(U^f)_{|\Gamma^s} \cdot n(t), \qquad [1]$$

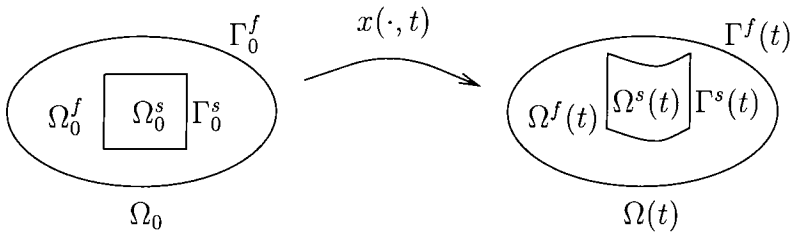

Figure 1. *Geometric configurations: the fixed reference configuration* Ω_0 *and the physical configuration* $\Omega(t)$

where Tr denotes the kinematic restriction (trace) of the different velocity fields on the interface.

The values of density, velocity and Cauchy stress tensor in the present configuration $\Omega(t)$ are governed by basic conservation and constitutive laws. Because of the large displacements which are involved, the configuration $\Omega(t)$ is time dependent. To overcome this difficulty, and to evaluate the strain field or write the elastic constitutive laws inside the structure, one can transport the conservation laws on a fixed reference configuration Ω_0, delimited for example by a given equilibrium configuration of the structure. For this purpose, one must introduce a continuous mapping

$$\begin{array}{rccc} x & : & \Omega_0 \times \mathbb{R}^+ & \longrightarrow & \mathbb{R}^3 \\ & & (x_0, t) & \longmapsto & x(x_0, t) \end{array} ,$$

which maps any point x_0 of the fixed configuration Ω_0 to its image $x(x_0,t)$ in the present configuration $\Omega(t)$. The choice of the configuration Ω_0 and of the map x may be arbitrary, hence the name of Arbitrary Lagrangian Eulerian (ALE) formulation which is given to the resulting equations. It is nevertheless more simple [LET 94], [LET 99b] to impose that on the structure Ω^s, the point $x(x_0,t)$ corresponds to the position $x^s(t)$ at time t of the material point which was located in x_0 at time t_0. This implies then that the configuration (or grid) velocity $U^G := \dfrac{\partial x}{\partial t}_{|x_0}$ (with $\dfrac{\partial x}{\partial t}_{|x_0}$ denoting the partial derivative of x with respect to time at a given fixed position x_0 in the reference configuration) is always equal to the real velocity U^s of the structure at any point x of Ω^s. On the fluid, the mapping x^f from Ω_0^f onto $\Omega^f(t)$ is characterised by its nodal values on the discretisation grid and can be any reasonable extension $x^f = \mathrm{Ext}(x^f_{|\Gamma_0^s})$ of the material interface deformation

$$x^f = \mathrm{Ext}(x^f_{|\Gamma_0^s}), \quad \frac{\partial x^f}{\partial t}_{|\Gamma_0^s} = \mathrm{Tr}(U^s)_{|\Gamma_0^s}.$$

The weak form of the conservation laws can now be directly transported on the fixed domain Ω_0 yielding

$$\int_{\Omega_0} \left\{ \frac{\partial J\rho}{\partial t}_{|x_0} + \mathrm{div}_0 \left[J\rho \left(U - U^G \right) F^{-T} \right] \right\} \hat{q} \, \mathrm{d}x_0 = 0, \quad \forall \hat{q} \in L^2(\Omega_0),$$

(Mass Conservation),

$$\int_{\Omega_0} \left\{ \frac{\partial J\rho U}{\partial t}_{|x_0} + \mathrm{div}_0 \left[J\rho U \otimes \left(U - U^G \right) F^{-T} \right] \right\} \cdot \hat{U} \, \mathrm{d}x_0$$

$$+ \int_{\Omega_0} J\sigma F^{-T} : \frac{\partial \hat{U}}{\partial x_0} \, \mathrm{d}x_0 + \int_{\Gamma_0^s} g_\Gamma \cdot \left(\hat{U}^s - \hat{U}^f \right) \frac{\mathrm{d}a}{\mathrm{d}a_0} \, \mathrm{d}a_0 = \int_{\Omega_0^s} f \cdot \hat{U} \, \mathrm{d}x_0$$

$$+ \int_{\partial\Omega} g \cdot \hat{U} \, \mathrm{d}a, \quad \forall \hat{U} = (\hat{U}^s, \hat{U}^f) \in V, \text{ (Momentum Conservation)},$$

under the notation

$$F = \frac{\partial x}{\partial x_0} = \nabla_0 x, \quad J = \det F, \quad n \, \mathrm{d}a = JF^{-T} n_0 \, \mathrm{d}a_0.$$

Above, the velocity test functions do not necessarily match at the interface, and are taken in the product space

$$V = \left\{ \hat{U} = (\hat{U}^s, \hat{U}^f) : \Omega_0 \longrightarrow \mathbb{R}^3; \quad \hat{U}^s \in H^1(\Omega_0^s), \quad \hat{U}^f \in H^1(\Omega_0^f) \right\}.$$

Moreover, the vectors f and g represent the external forces applied on the system (we assume for simplicity that there are no body forces on the fluid) and the vector g_Γ denotes the interface stress vector in the present configuration. For an inviscid fluid, the constitutive assumption imposes that the stress vector to be normal to the interface

$$g_\Gamma = -pn,$$

the interface pressure being the Lagrange multiplier of the kinematic interface continuity condition [1]. For a viscous fluid, we have $g_\Gamma = \sigma n$.

In the above variational formulation, our choice of reference configuration guarantees that we have $U = U^G$ on the structure. Hence the mass conservation equation reduces there to the identity $J\rho =$ constant, and can be omitted in all further calculations. In other words, the conservation of mass is automatically satisfied inside the structure, and must therefore be checked on the fluid domain only.

We must finally specify the different constitutive laws characterizing the materials under study. On the fluid, the constitutive law is simple when written in the present configuration $\Omega^f(t) = x^f(\Omega_0^f, t)$, where we have

$$\int_{\Omega_0^f} \mathrm{div}_0 \left[J\rho \left(U^f - U^G \right) F^{-T} \right] \hat{q} \, \mathrm{d}x_0 = \int_{\Omega^f(t)} \mathrm{div}_x \left[\rho \left(U^f - U^G \right) \right] \hat{q} \, \mathrm{d}x,$$

$$\int_{\Omega_0^f} \mathrm{div}_0 \left[J\rho U \otimes \left(U^f - U^G \right) F^{-T} \right] \cdot \hat{U} \, \mathrm{d}x_0$$

$$= \int_{\Omega^f(t)} \mathrm{div}_x \left[\rho U^f \otimes \left(U^f - U^G \right) \right] \cdot \hat{U} \, \mathrm{d}x,$$

$$\int_{\Omega_0^f} J\sigma F^{-T} : \frac{\partial \hat{U}}{\partial x_0} \, \mathrm{d}x_0 = \int_{\Omega^f(t)} \left\{ \mu \left[\nabla_x U^f + \left(\nabla_x U^f \right)^T \right] - p\,\mathrm{I} \right\} : \frac{\partial \hat{U}}{\partial x} \, \mathrm{d}x.$$

The viscosity coefficient μ is set to zero when dealing with inviscid fluids.

The integrals on the structure are evaluated on the reference material configuration Ω_0 and use objective (frame invariant) elastic constitutive laws. When dealing with three dimensional massive hyperelastic structures, the second Piola Kirchhoff stress tensor $S = JF^{-1}\sigma F^{-T}$ is obtained by differentiating the free energy function with respect to the Green Lagrange strain tensor $\underline{\underline{E}}$ [LET 94]

$$S(x_0) = \frac{\partial \psi}{\partial \underline{\underline{E}}}(x_0, \frac{\partial x}{\partial x_0}).$$

For such structures, the mass and stiffness integrals take the usual abstract form

$$\int_{\Omega_0^s} J\rho \ddot{x}^s \cdot \hat{U} \, \mathrm{d}x_0 := m^s(\ddot{x}^s, \hat{U}),$$

$$\int_{\Omega_0^s} FS : \frac{\partial \hat{U}}{\partial x_0} \, \mathrm{d}x_0 = \int_{\Omega_0^s} \frac{\partial \psi}{\partial \underline{\underline{E}}}(x_0, \frac{\partial x}{\partial x_0}) : \hat{\underline{\underline{E}}}(\hat{U}) \, \mathrm{d}x_0 := a^s(x^s, \hat{U}),$$

under the notation

$$\ddot{x}^s = \dot{U}^s = \frac{\partial^2 x^s}{\partial t^2}_{|x_0}, \quad \hat{\underline{\underline{E}}}(\hat{U}) = \frac{1}{2} \left(F^T \frac{\partial \hat{U}}{\partial x_0} + \frac{\partial \hat{U}}{\partial x_0}^T F \right).$$

Similar integrals appear when dealing with more general structures such as elastic beams or shells in large displacements [CAR 95]. Altogether, the conservation laws, kinematic constraints and constitutive laws governing the evolution of a fluid-structure system take the final form

$$\int_{\Omega_0^f} \frac{\partial J\rho}{\partial t}_{|x_0} \hat{q} \, \mathrm{d}x_0 + \int_{x(\Omega_0^f, t)} \mathrm{div}_x \left[\rho \left(U^f - U^G \right) \right] \hat{q} \, \mathrm{d}x = 0,$$

$$\forall \hat{q} : \Omega_0 \to \mathbb{R}, \quad \text{(Mass)}, \qquad [2]$$

$$\int_{\Omega_0^f} \frac{\partial J\rho U^f}{\partial t}_{|x_0} \cdot \hat{U} \, \mathrm{d}x_0 + \int_{x^f(\Omega_0^f, t)} \left\{ \mathrm{div}_x \left[\rho U^f \otimes \left(U^f - U^G \right) \right] \cdot \hat{U} \right.$$

$$\left. + \sigma : \nabla_x \hat{U} \right\} \mathrm{d}x + m^s(\ddot{x}^s, \hat{U}) + a^s(x^s, \hat{U})$$

$$- \int_{\Gamma_0^s} \left(J\sigma F^{-T} n_0 \right) \cdot \left[\mathrm{Tr}(\hat{U}^f)_{|\Gamma_0^s} - \mathrm{Tr}(\hat{U}^s)_{|\Gamma_0^s} \right] \mathrm{d}a_0$$

$$= \int_{\Omega_0^s} f \cdot \hat{U}\, \mathrm{d}x_0 + \int_{\partial\Omega(t)} g \cdot \hat{U}\, \mathrm{d}a, \quad \forall \hat{U} \in V, \quad \text{(Momentum)}, \qquad [3]$$

$$\left[\mathrm{Tr}(U^f)_{|\Gamma_0^s} - \mathrm{Tr}(U^s)_{|\Gamma_0^s}\right] \cdot JF^{-T} n_0 = 0, \quad \text{(kinematic continuity)}, \qquad [4]$$

$$x^f = \mathrm{Ext}(x^f_{|\Gamma_0^s}), \quad \frac{\partial x^f}{\partial t}_{|\Gamma_0^s} = \mathrm{Tr}(U^s)_{|\Gamma_0^s}, \quad U^G = \frac{\partial x^f}{\partial t}_{|x_0},$$

$$\text{(fluid configuration map)}. \qquad [5]$$

These equations completely characterize the evolution of the structural deformation $x^s \in V^s$, of the fluid density $\rho^f J$ in initial configuration, of the pressure $p \in Q = L^2(\Omega_0^f)$, of the fluid velocity $U^f \in V^f$, of the interface force $J\sigma F^{-T} n_0 \in W_\Gamma = (H^{1/2}(\Gamma_0^s))'$, and of the fluid configuration mapping $x^f \in V^f$ when complemented by a state law $p = g(\rho, T)$ relating the pressure p to the density ρ and temperature T inside the fluid, and by adequate initial and boundary conditions. Specific choices of state law or of boundary conditions to be imposed on the external boundary $\partial\Omega(t)$ will depend on the physical problem under consideration.

The above formulation is very general. It reduces the fluid-structure interaction to the kinematic condition [4] and to the associated kinetic Lagrange multiplier (interface force) $J\sigma F^{-T} n_0$, appearing in the global momentum conservation equation when using non kinematically admissible test functions.

REMARK. — The above formulation reduces in fact to three coupled subproblems, which are characteristic of fluid-structure interaction problems.

– Solving the mass conservation equation, and choosing $\hat{U}^s = 0$ and $\hat{U}^f$ arbitrary in the momentum conservation equation [3] while taking into account the kinematic interface boundary condition [4] as specified by the structural problem, we first obtain a standard fluid equation written in ALE form on the moving domain $\Omega^f(t)$. The corresponding solution U^f defines then the interface load $L_{interface}$ as the residual of these fluid equations on the interface

$$L_{interface}(\hat{U}_{|\Gamma^s}) = -\int_{\Gamma_0^s} \left(J\sigma F^{-T} n_0\right) \cdot \hat{U}\, \mathrm{d}a_0$$

$$= \int_{\partial\Omega(t)\cap\partial\Omega^f(t)} g \cdot \hat{U}^f\, \mathrm{d}a - \int_{\Omega_0^f} \frac{\partial J\rho U^f}{\partial t}_{|x_0} \cdot \hat{U}^f\, \mathrm{d}x_0$$

$$- \int_{x^f(\Omega_0^f,t)} \left\{ \mathrm{div}_x \left[\rho U^f \otimes \left(U^f - U^G\right)\right] \cdot \hat{U}^f + \sigma : \nabla_x \hat{U}^f \right\} \mathrm{d}x,$$

where $\hat{U}$ is any extension of $\hat{U}_{|\Gamma}$ defined inside $\Omega^f(t)$. This expression computing the interface load from the fluid equation residual has the major advantage of still making sense after finite element discretisation, and of leading to somewhat more stable numerical results [FAR 98a].

– Choosing in [3] $\hat{U}^f = 0$ and $\hat{U}^s$ arbitrary in V^s yields a standard structural problem with imposed traction forces $L_{interface}$ on the interface (specified by the fluid problem)

$$m^s(\ddot{x}^s, \hat{U}^s) + a^s(x^s, \hat{U}^s) = \int_{\Omega_0^s} f \cdot \hat{U}^s \, \mathrm{d}x_0 + \int_{\partial\Omega(t)\cap\partial\Omega^s(t)} g \cdot \hat{U}^s \, \mathrm{d}a + L_{interface}(\hat{U}^s_{|\Gamma^s}), \quad \forall \hat{U}^s \in V^s.$$

– The grid configuration map inside the fluid is finally defined by

$$x^f = \mathrm{Ext}(x^f_{|\Gamma_0^s}), \quad \frac{\partial x^f_{|\Gamma_0^s}}{\partial t} = (U^s)_{|\Gamma_0^s}, \quad U^G = \frac{\partial x^f}{\partial t}_{|x_0},$$

and is coupled to the other subproblems by the condition relating the fluid grid velocity on the interface to the local value of the structural velocity.

For compressible flows, the above conservation laws must be complemented by an energy equation expressing the conservation of the total energy of the fluid $E = \rho e + \frac{1}{2}\rho U^2$. Neglecting any external load acting on the fluid and any exchange of heat between the fluid and the structure, this equation takes the weak form

$$\int_{\Omega_0^f} \frac{\partial JE}{\partial t}_{|x_0} \hat{q} \, \mathrm{d}x_0 - \int_{x^f(\Omega_0^f, t)} \left[E\left(U^f - U^G\right) - \sigma U^f + \underline{q}\right] \cdot \frac{\partial \hat{q}}{\partial x} \, \mathrm{d}x + \int_{\partial\Omega(t)\cap\partial\Omega^f(t)} \left[E\left(U^f - U^G\right) - \sigma U^f + \underline{q}\right] \cdot n\hat{q} \, \mathrm{d}a - \int_{\Gamma_0^s} g_\Gamma \cdot \mathrm{Tr}(U^f)_{|\Gamma^s} \hat{q} \, \mathrm{d}a_0 = 0, \quad \forall \hat{q} : \Omega_0^f \to \mathbb{R}, \quad \text{(Energy)}. \qquad [6]$$

The finite volume approximation of this equation is then obtained by restricting the test functions $\hat{q}$ to be piecewise constant.

Total energy conservation is obtained either before or after discretisation by writing this energy equation with $\hat{q} = 1$, and by adding to it the structural equation multiplied by the structural velocity U^s. This cancels the action of the interface force g_Γ, and leads to an energy balance of the type

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\Omega_0^f} JE \, \mathrm{d}x_0 + \int_{\Omega_0^s} \psi(\underline{\underline{E}}) \, \mathrm{d}x_0 + \frac{1}{2} m^s(\dot{x}^s, \dot{x}^s)\right] = \int_{\Omega_0^s} f \cdot U^s \, \mathrm{d}x_0 + \int_{\partial\Omega(t)\cap\partial\Omega^s(t)} g \cdot U^s \, \mathrm{d}a + \int_{\partial\Omega(t)\cap\partial\Omega^f(t)} \left[E\left(U^f - U^G\right) - \sigma U^f + \underline{q}\right] \cdot n \, \mathrm{d}a.$$

3. Time discretisation

We now need to construct a time discretisation scheme respecting the compatibility condition [4] at the fluid-structure interface, and conserving energy. The simplest choice is to use a first order staggered time integration scheme where the fluid problem is first solved with the explicit kinematic structural interface boundary condition obtained at the previous structural iteration yielding new fluid velocity U^f_{n+1} and interface traction $L_{n+1}(\hat{U})$, and where the structural problem is then solved with this imposed traction L_{n+1} yielding the final prediction of the interface velocity $(U^s_{n+1})_{|\Gamma}$ and position x^f_{n+1}. In this choice, the works developed by the fluid to structure and structure to fluid interface traction forces during the present time step do not cancel because they do not act on the same velocity field. On the fluid side, they act on the structural velocity $U^f_{n+1} = U^s_n$ at time t_n; on the structural side, they act on the present structural velocity U^s_{n+1}. This error can be reduced to second order by replacing as in [PIP 95b], [PIP 95a], [FAR 98b] the predicted structural velocity $(\mathrm{Tr}\, U^s_{|\Gamma})_n$ by a higher order extrapolation.

An alternate way for getting a better energy conservation [LET 96, LET 99b] is to solve the full system (including the kinematic compatibility condition [4]) at a sequence of discrete times $t_n, n = 1, \ldots$, using independent finite difference approximations of the various time derivatives. Good accuracy and dissipation properties are obtained by approximating the structural acceleration by a generalised mid point rule (with governing unknown $x^s_{n+1/2}$) [SIM 92], [KUH 99]

$$x^s_n = \frac{x^s_{n+1/2} + x^s_{n-1/2}}{2},$$

$$F_n = \frac{\partial x^s_n}{\partial x_0}, \quad \underline{\underline{\hat{E}}}_n(\hat{U}) = \frac{1}{2}\left(F_n^T \frac{\partial \hat{U}}{\partial x_0} + \frac{\partial \hat{U}}{\partial x_0}^T F_n\right),$$

$$S^s_n = \frac{1}{2}(S^s(x^s_{n+1/2}) + S^s(x^s_{n-1/2})),$$

$$U^s_n = \frac{x^s_{n+1/2} - x^s_{n-1/2}}{\Delta t_n} = \frac{1}{2}(U^s_{n+1/2} + U^s_{n-1/2}),$$

$$(\ddot{x}^s)_n = \frac{U^s_{n+1/2} - U^s_{n-1/2}}{\Delta t_n}.$$

The acceleration of the fluid on the other hand can be approximated by a wide variety of discretisation schemes such as a standard first order backward Euler scheme

$$\left(\frac{\partial \rho J U^f}{\partial t}\right)_n = \frac{(\rho J U^f)_n - (\rho J U^f)_{n-1}}{\Delta t_n},$$

a second order Gear backward difference [MAR 96]

$$\left(\frac{\partial \rho J U^f}{\partial t}\right)_n = \frac{3}{2\Delta t}(\rho J U^f)_n - \frac{2}{\Delta t}(\rho J U^f)_{n-1} + \frac{1}{2\Delta t}(\rho J U^f)_{n-2},$$

or a second order Crank Nicolson formula

$$\left(\frac{\partial \rho J U^f}{\partial t}\right)_n = \frac{(\rho J U^f)_{n+1/2} - (\rho J U^f)_{n-1/2}}{\Delta t_n}.$$

The relevant unknown in this last choice is $U^f_{n+1/2}$.

4. Energy conservation

Energy conservation is a key point in studying fluid-structure interactions. In particular, the evolution of the kinetic energy must be carefully controlled. A time integration of the principle of conservation of momentum taking the real velocity field as test function indicates that the variation of the sum of the kinetic energy of the system and of the elastic energy of the structure must be equal to the difference between the energy introduced by the external boundary conditions and the energy dissipated by viscous effects inside the fluid or developed by the pressure field inside the fluid through compressibility effects. Respecting this energy principle is crucial for preserving stability, and for ensuring the long term accuracy of the numerical predictions. Moreover, this bound on the energy is *the* major tool in the theoretical and numerical analysis of the linearised version of the fluid structure interaction problem [LET 99a] following the steps of [DAU 84, chapter XVIII].

Most time integration schemes do violate this principle of energy conservation when dealing with deformable domains. More precisely, for fully coupled schemes using conservative formulations and non volume preserving grid configuration maps x^f, a small pollution term appears in the kinetic energy conservation principle, which may grow exponentially in time.

To study this energy conservation for the time discrete case, we multiply at each time t^n the variational equation [3] by U^f_n on the fluid, and by U^s_n on the structure. This choice cancels the action of the interface traction forces p_Γ because the imposed kinematic compatibility condition [4] is exactly satisfied at time t_n when using totally coupled schemes.

On the structure, the action of U^s_n on the inertia terms produces the correct variation of kinetic energy

$$\begin{aligned}\left(\frac{\partial U}{\partial t}\right)_n \cdot U^s_n &= \frac{U^s_{n+1/2} - U^s_{n-1/2}}{\Delta t_n} \cdot \frac{U^s_{n+1/2} + U^s_{n-1/2}}{2} \\ &= \frac{|U^s_{n+1/2}|^2 - |U^s_{n-1/2}|^2}{2\Delta t_n}.\end{aligned}$$

On stiffness terms, it produces the right variation of elastic energy

$$\frac{S^s(x^s_{n+1/2}) + S^s(x^s_{n-1/2})}{2} \hat{\underline{\underline{E}}}_n \left(\frac{U^s_{n+1/2} + U^s_{n-1/2}}{2}\right)$$

$$= \frac{S^s(x^s_{n+1/2}) + S^s(x^s_{n-1/2})}{2} \frac{\underline{\underline{E}}_{n+1/2} - \underline{\underline{E}}_{n-1/2}}{\Delta t_n}$$

$$= \frac{1}{\Delta t_n} \left[\psi(\underline{\underline{E}}_{n+1/2}) - \psi(\underline{\underline{E}}_{n-1/2}) + \frac{\partial^3 \psi}{\partial \underline{\underline{E}}^3}(\underline{\underline{E}}_*)(\underline{\underline{E}}_{n+1/2} - \underline{\underline{E}}_{n-1/2})^3 \right].$$

On the fluid, a direct integration of the inertia terms yields finally

$$I^f_n = \int_{\Omega^f_0} \left(\frac{\partial J \rho U^f}{\partial t} \right)_n \cdot U^f_n + \int_{x^f(\Omega_0, t_n)} \mathrm{div}_x \left[\rho U^f \otimes (U^f - U^G) \right]_n \cdot U^f_n.$$

Using direct algebraic manipulations and subtracting the weak equation of mass reduces this integral to

$$\begin{aligned} I^f_n &= \int_{\Omega^f_0} \frac{1}{2} \left(\frac{\partial J \rho |U^f|^2}{\partial t} \right)_n \\ &+ \int_{\Omega^f_0} \left[\left(\frac{\partial J \rho U^f}{\partial t} \right)_n \cdot U^f_n - \frac{1}{2} |U^f_n|^2 \left(\frac{\partial J \rho}{\partial t} \right)_n - \frac{1}{2} \left(\frac{\partial J \rho |U^f|^2}{\partial t} \right)_n \right] \\ &+ \int_{x^f(\Omega_0, t_n)} (\frac{1}{2} |U^f_n|^2 - \hat{q}) \left(\frac{1}{J_n} \left(\frac{\partial J \rho}{\partial t} \right)_n + \mathrm{div}_x [\rho (U^f - U^G)]_n \right). \end{aligned}$$

We do not recover here the exact variation of kinetic energy inside the fluid. Two error terms appear. The last line corresponds to a truncation error

$$e_h = \int_{\Omega^f(t_n)} \frac{\Delta t}{2} \inf_{q_h \in Q_h} (\frac{1}{2} |U^f_n|^2 - q_h) \left(\frac{1}{J_n} \left(\frac{\partial J \rho}{\partial t} \right)_n + \mathrm{div}_x \left[\rho (U^f - U_G) \right]_n \right),$$

which can be made very small by a careful choice of the space of pressure test functions Q_h. This error disappears for the space continuous problem, and for spatially uniform flows approximated by schemes satisfying the Discrete Geometric Conservation Law (that is exactly satisfying the local conservation of mass for spatially uniform fluids).

The second line is proportional to the truncation error induced by the time discretisation scheme, but the coefficient of proportionality depends on the regularity in time of the map ρJ , that is in particular on the time regularity of the grid configuration x^f. In other words, any abrupt changes of J can lead to large local errors. Actually, this second line can be studied in more details. For a backward Euler scheme, we have

$$\begin{aligned} &\left(\frac{\partial J \rho U^f}{\partial t} \right)_n \cdot U^f_n - \frac{1}{2} |U^f_n|^2 \left(\frac{\partial J \rho}{\partial t} \right)_n - \frac{1}{2} \left(\frac{\partial J \rho |U^f|^2}{\partial t} \right)_n \\ &= \frac{1}{2 \Delta t} (\rho J)_{n-1} |U^f_n - U^f_{n-1}|^2. \end{aligned}$$

This term is in fact positive. It can be considered as an additional numerical dissipation and will not affect the stability properties of the scheme. The situation is a bit different for a Crank Nicolson scheme, where the error

$$\left(\frac{\partial J\rho U^f}{\partial t}\right)_n \cdot U_n^f - \frac{1}{2}|U_n^f|^2 \left(\frac{\partial J\rho}{\partial t}\right)_n - \frac{1}{2}\left(\frac{\partial J\rho |U^f|^2}{\partial t}\right)_n$$
$$= -\frac{1}{8\Delta t}\left((\rho J)_{n+1/2} - (\rho J)_{n-1/2}\right)|U_{n+1/2}^f - U_{n-1}^f|^2.$$

is of smaller order, but where we can no longer control its sign.

REMARK. — The time regularity of the grid configuration x^f also appears in a standard truncation error analysis of the ALE formulation. The map x^f must satisfy the minimal regularity requirements needed to preserve the accuracy of the time integration scheme.

5. Transpiration

5.1. *Motivation*

The ALE formulation studied up to now has two practical drawbacks. First, at each time step, a new grid x^f must be built inside the fluid domain, and the associated grid velocity U^G must be computed. We have just seen that both fields must follow the deformation of the structure and be smooth in time and space. Second, the flux vectors $\phi(W, \sigma)$ (those appearing inside the divergence terms in the conservation laws) are modified by the ALE formulation, and thus the corresponding flow solvers must be changed in depth.

In order to overcome these drawbacks, and to be able to solve at low cost fluid structure interaction problems at moderate deformation, aeronautical engineers have developed transpiration techniques, from an idea of Lighthill [LIG 58]. These formulations do not require to update the computational grid or the flux solvers subroutines, but only involve modifications of the interface boundary conditions. They will now be derived and justified mathematically. The main mathematical principle is to write the fluid problem in variational form on the present configuration $x^f(x_0) = x_0 + \delta x(x_0)$, working with the fundamental unknown

$$\delta W(x_0) = W(x^f(x_0)) - W_0(x_0) - \nabla_0 W_0(x_0)\delta x(x_0), \qquad [7]$$

where W_0 represents the steady state reached by the fluid, when it flows around the structure at rest corresponding to a structural map given by $x(x_0) = x_0$. At first order with respect to the interface displacement, this new unknown describes the difference between the reference flow and the present flow at the *same physical point* $x^f(x_0)$. More precisely, we are now interested in the linearisation of the coupled problem of fluid-structure interaction introduced in section 2, around a steady state corresponding

to the reference structural configuration $x = I$ in Ω_0, taking as new unknowns the displacement δx of the domain and the fluctuation δW of the fluid state variables, as defined in [7].

In order to ensure that the reference configuration is at equilibrium under the action of the external fluid, we will assume that the residual stress σ_0^s (Piola-Kirchoff's first stress tensor) in this configuration equilibrates the steady state stress field on the interface.

In such a general setting, the unknowns x and W satisfy the conservation laws [2], [3] and [6]. Integrating the convective terms by parts, these laws reduce to the abstract variational form: find $W : \Omega_0^f \times \mathbb{R}^+ \to \mathbb{R}^5$ and $x : \Omega_0 \times \mathbb{R}^+ \to \mathbb{R}^3$ such that

$$
\begin{aligned}
& m^s(\ddot{x}^s, \hat{v}_2^s) + \int_{\Omega_0^f} \frac{\partial JW}{\partial t}_{|x_0} \cdot \hat{v}^f \, \mathrm{d}x_0 \\
& - \int_{\Omega_0^f} J \left[\phi(W,\sigma) - W \otimes U^G\right] F^{-T} : \nabla_0 \hat{v}^f \, \mathrm{d}x_0 + a^s(x^s, \hat{v}_2^s) \\
& - \int_{\Gamma_0^s} J\sigma F^{-T} n_0 \cdot (\hat{v}_2^f - \hat{v}_2^s) \, \mathrm{d}a_0 = \int_{\Omega_0^s} f \cdot \hat{v}_2^s \, \mathrm{d}x_0 \\
& \quad + \int_{\Gamma_0^s} U^G \cdot FSn_0 \hat{v}_3^f \, \mathrm{d}a_0, \quad \forall \hat{v} = (\hat{v}^s, \hat{v}^f) \in \mathcal{D}(\Omega_0)^5 \times \mathcal{D}(\Omega_0)^5, \qquad [8]
\end{aligned}
$$

with boundary condition at farfield

$$
\phi^E(W) JF^{-T} n_0 = \mathcal{F}(\bar{W}, JF^{-T} n_0, W_\infty), \quad \text{on} \quad \Gamma_0^f. \qquad [9]
$$

Here, we have used the notation

$$
W = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} \rho \\ \rho U^f \\ E \end{pmatrix}, \quad \hat{v} = \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \end{pmatrix} : \Omega_0 \longrightarrow \mathbb{R}^5 \times \mathbb{R}^5,
$$

$$
\phi(W,\sigma) = W \otimes U^f - \mathrm{I}_2 \, \sigma - \mathrm{I}_3 \otimes (\sigma U^f), \qquad [10]
$$

with

$$
\mathrm{I}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{I}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},
$$

and σ given by an adequate state and constitutive law. The flux $\phi^E(W)$ corresponds to the inviscid constitutive law $\sigma = -p\,\mathrm{I}$.

The flux $\mathcal{F}(\bar{W}, JF^{-T} n_0, W_\infty)$ at farfield is defined by flux vector splitting

$$
\mathcal{F}(\bar{W}, JF^{-T} n_0, W_\infty) = \underbrace{\sum_{\lambda_k > 0} \lambda_k R_k \otimes L_k \, W}_{A^+} + \underbrace{\sum_{\lambda_k < 0} \lambda_k R_k \otimes L_k \, W_\infty}_{A^-},
$$

with an outgoing flux A^+, governed by the local value W, and an ingoing flux A^-, governed by the farfield state W_∞. The vectors R_k and L_k are respectively the right and left eigenvectors, and λ_k the eigenvalues, of the inviscid jacobian matrix

$$\frac{\partial \phi^E}{\partial W}(\bar{W}) J F^{-T} n_0,$$

with $\bar{W} = W_\infty$ or $\bar{W} = W$ depending on the implementation.

By integration by parts of the variational problem used with

$$\hat{v}^f = \begin{pmatrix} \hat{v}_1^f \\ 0 \\ 0 \end{pmatrix},$$

and $\hat{v}^s = 0$, we get in particular that the unknown satisfies the kinematic boundary condition

$$W_2 \cdot J F^{-T} n_0 = \rho \dot{\delta x} \cdot J F^{-T} n_0, \quad \text{on} \quad \Gamma_0^s.$$

Recall that, in Ω_0^f, δx is arbitrary and can be any reasonable extension of the structural motion inside the fluid domain.

Now we focus on the linearisation of the above problem around the equilibrium steady state corresponding to $x = I$. The structure being at equilibrium in this steady state, we have

$$-\int_{\Omega_0^f} \phi(W_0, \sigma_0) : \nabla_0 \hat{v}^f \, \mathrm{d}x_0 + \int_{\Omega_0^s} \sigma_0^s : \nabla_0 \hat{v}_2^s \, \mathrm{d}x_0 - \int_{\Gamma_0^s} \sigma_0 n_0 \cdot (\hat{v}_2^f - \hat{v}_2^s) \, \mathrm{d}a_0$$

$$= \int_{\Omega_0^s} f \cdot \hat{v}_2^s \, \mathrm{d}x_0, \quad \forall \hat{v} = (\hat{v}^s, \hat{v}^f) \in \mathcal{D}(\Omega_0)^5 \times \mathcal{D}(\Omega_0)^5, \quad [11]$$

together with the kinematic boundary condition

$$\phi^E(W_0) n_0 = \mathcal{F}(\bar{W}, n_0, W_\infty), \quad \text{on} \quad \Gamma_0^f.$$

In particular, [11] implies

$$W_{0,2} \cdot n_0 = \rho_0 U_0^f \cdot n_0 = 0, \quad \text{on} \quad \Gamma_0^s. \qquad [12]$$

In this linearisation process, the unknowns are the fluctuations $(\delta W, \delta x)$ of the fluid and of the structure around the reference state (W_0, I), as induced by given small perturbations of data. Such fluctuations are defined as in [7] by

$$\begin{aligned} x = \quad & I + \delta x, && \text{in} \quad \Omega_0, \\ W(I + \delta x) = \quad & W_0 + \nabla_0 W_0 \delta x + \delta W, && \text{in} \quad \Omega_0^f. \end{aligned} \qquad [13]$$

and describe the variation of state variables taken at the same frozen physical point $x_0 + \delta x$, and hence at two different lagrangian points $(I + \delta x)^{-1}(x_0) \neq x_0$.

5.2. *The steady state problem after transport*

The variation being taken at lagrangian point $(I + \delta x)^{-1}(x_0)$, we first need to transport the steady state equation

$$\int_{\Omega_0^f} \phi(W_0, \sigma_0) : \nabla_0 w \, \mathrm{d}x_0 = 0, \quad \forall w \in \mathcal{D}(\Omega_0^f)^5, \qquad [14]$$

back to this auxiliary configuration.

This is the purpose of the next lemma.

Lemma 1 *For any smooth displacement* $\delta x \in \mathcal{C}^1(\overline{\Omega}_0^f)^3$ *and solution* $(W_0, \sigma_0) \in \mathcal{C}^1(\overline{\Omega}_0^f)^5 \times \mathcal{C}^1(\overline{\Omega}_0^f)^{3\times 3}$ *of the equilibrium steady state problem [14], we have*

$$\int_{\Omega_0^f} \underbrace{\left[\phi(W_0, \sigma_0)\left(\mathrm{I}\,\mathrm{div}_0\,\delta x - \nabla_0 \delta x^T\right) + \nabla_0 \phi(W_0, \sigma_0)\delta x\right]}_{G} : \nabla_0 w \, \mathrm{d}x_0 = 0,$$

$$\forall w \in \mathcal{D}(\Omega_0^f)^5. \qquad [15]$$

Proof: Consider a given test function $w \in \mathcal{D}(\Omega_0^f)^5$ with support $K = \mathrm{supp}\, w$. We can then construct a compact set $K_0 \subset \Omega_0^f$, and a bound ε_0, such that the map $x^\varepsilon = I + \varepsilon \delta x$ is one to one when ε is sufficiently small, and satisfies

$$Q^\varepsilon = (x^\varepsilon)^{-1}(K) \subset K_0, \quad x^\varepsilon(K_0) \subset \Omega_0^f, \quad \forall 0 < \varepsilon < \varepsilon_0.$$

By changing variables in the different integrals, we have from [14]

$$\begin{aligned}
0 &= \int_K \phi(W_0, \sigma_0) : \nabla_x w \, \mathrm{d}x \\
&= \int_{x^\varepsilon(Q^\varepsilon)} \phi(W_0, \sigma_0) : \nabla_x w \, \mathrm{d}x^\varepsilon \\
&= \int_{Q^\varepsilon} \phi\left(W_0(x^\varepsilon(x_0)), \sigma_0(x^\varepsilon(x_0))\right) : \nabla_x w(x^\varepsilon(x_0)) \frac{\mathrm{d}x^\varepsilon}{\mathrm{d}x_0} \, \mathrm{d}x_0 \\
&= \int_{Q^\varepsilon} \phi\left(W_0(x^\varepsilon(x_0)), \sigma_0(x^\varepsilon(x_0))\right) : \nabla_0 w(x^\varepsilon(x_0))(\nabla_0 x^\varepsilon)^{-1} \frac{\mathrm{d}x^\varepsilon}{\mathrm{d}x_0} \, \mathrm{d}x_0 \\
&= \int_{Q^\varepsilon} \phi\left(W_0(x^\varepsilon(x_0)), \sigma_0(x^\varepsilon(x_0))\right) (\nabla_0 x^\varepsilon)^{-T} : \nabla_0 w(x^\varepsilon(x_0)) \frac{\mathrm{d}x^\varepsilon}{\mathrm{d}x_0} \, \mathrm{d}x_0 \\
&= \int_{K_0} [\phi\left(W_0(I + \varepsilon\delta x), \sigma_0(I + \varepsilon\delta x)\right) \nabla_0 (I + \varepsilon\delta x)^{-T} : \nabla_0 w(I + \varepsilon\delta x) \\
&\qquad \det \nabla_0 (I + \varepsilon\delta x)] \, \mathrm{d}x_0,
\end{aligned}$$

the last equality coming from the fact that $w(x^\varepsilon(x_0))$ is identically equal to zero outside $(x^\varepsilon)^{-1}(K)$ since the support of w is included in K. At first order in ε, the above expression takes the form

$$\begin{aligned} 0 = & \int_{K_0} \phi(W_0,\sigma_0) : \nabla_0 w \, \mathrm{d}x_0 \\ & + \varepsilon \int_{K_0} \phi(W_0,\sigma_0) : \nabla_0(\nabla_0 w \delta x) \, \mathrm{d}x_0 \\ & + \varepsilon \int_{K_0} \left[\phi(W_0,\sigma_0) \left(\mathrm{I}\,\mathrm{div}_0\, \delta x - \nabla_0 \delta x^T \right) + \frac{\partial \phi}{\partial W}(W_0,\sigma_0)(\nabla_0 W_0 \delta x) \right. \\ & \qquad \left. + \frac{\partial \phi}{\partial \sigma}(W_0,\sigma_0)\,(\nabla_0 \sigma_0 \delta x) \right] : \nabla_0 w \, \mathrm{d}x_0 + o(\varepsilon). \end{aligned}$$

In this expression, the first and second terms correspond to [14] written with test functions w and $\nabla_0 w \delta x$ respectively, and therefore cancel. The lemma [15] then follows after division by ε and by making ε tend to zero. ■

After integration by parts, the above lemma can also be written under the strong form

$$\mathrm{div}_0\, G = 0, \quad \text{in} \quad \Omega_0^f.$$

By multiplying now this expression by $\hat{v}^f \in \mathcal{D}(\Omega_0)^5$ and by integrating by parts on Ω_0^f, we finally obtain

$$\int_{\Omega_0^f} G : \nabla_0 \hat{v}^f \, \mathrm{d}x_0 = \int_{\Gamma_0^s} G n_0 \cdot \hat{v}^f \, \mathrm{d}a_0, \quad \forall \hat{v}^f \in \mathcal{D}(\Omega_0)^5.$$

In other words, after transport, the solution of the steady state problem satisfies the linearised convected problem

$$\begin{aligned} & \int_{\Omega_0^f} \left[\phi(W_0,\sigma_0) \left(\mathrm{I}\,\mathrm{div}_0\, \delta x - \nabla_0 \delta x^T \right) + \nabla_0 \phi(W_0,\sigma_0) \delta x \right] : \nabla_0 \hat{v}^f \, \mathrm{d}x_0 \\ = & \int_{\Gamma_0^s} \left[\phi(W_0,\sigma_0) \left(\mathrm{I}\,\mathrm{div}_0\, \delta x - \nabla_0 \delta x^T \right) + \nabla_0 \phi(W_0,\sigma_0) \delta x \right] n_0 \cdot \hat{v}^f \, \mathrm{d}a_0 \\ = & - \int_{\Gamma_0^s} \left[\phi(W_0,\sigma_0) \eta(\delta x) - (\nabla_0 \phi(W_0,\sigma_0) \delta x)\, n_0 \right] \cdot \hat{v}^f \, \mathrm{d}a_0, \\ & \qquad \forall \hat{v}^f \in \mathcal{D}(\Omega_0)^5. \end{aligned} \tag{16}$$

Here $\eta(\delta x) = -\left(\mathrm{I}\,\mathrm{div}_0\, \delta x - \nabla_0 \delta x^T \right) n_0$ represents, at first order, the variation $\eta \, \mathrm{d}a_0 = -n \, \mathrm{d}a - n_0 \, \mathrm{d}a_0$ of surface vector $n_0^s \, \mathrm{d}a_0 = -n_0 \, \mathrm{d}a_0$ (where n_0^s denotes the unit normal vector to Γ_0^s, pointing towards the fluid domain). In two dimensions, we have simply

$$\eta(\delta x) = \begin{pmatrix} \partial_\tau \delta x_2 \\ -\partial_\tau \delta x_1 \end{pmatrix},$$

with ∂_τ representing the tangential derivative along Γ_0^s.

5.3. *Linearisation method*

The linearisation is performed with respect to the fluctuations [7] by subtracting the steady state problem [11] and the linearised convected problem [16] from the initial problem [8] and by neglecting high order terms,

$$[8] - [11] - [16].$$

Let us first consider the time derivatives. Taking into account the definition [7] of the fluctuations, we get

$$\int_{\Omega_0^f} \frac{\partial JW}{\partial t}_{|x_0} \cdot \hat{v}^f \, \mathrm{d}x_0 = \int_{\Omega_0^f} \frac{\partial J(W_0 + \nabla_0 W_0 \delta x + \delta W)}{\partial t} \cdot \hat{v}^f \, \mathrm{d}x_0.$$

At first order in δx, the jacobian J reduces to $J = 1 + \mathrm{div}_0 \, \delta x$, and the above expression becomes

$$\begin{aligned}
\int_{\Omega_0^f} \frac{\partial JW}{\partial t}_{|x_0} \cdot \hat{v}^f \, \mathrm{d}x_0 &= \int_{\Omega_0^f} \delta \dot{W} \cdot \hat{v}^f \, \mathrm{d}x_0 \\
&\quad + \int_{\Omega_0^f} \left(\mathrm{div}_0 \, \dot{\delta x} W_0 + \nabla_0 W_0 \dot{\delta x} \right) \cdot \hat{v}^f \, \mathrm{d}x_0 \\
&= \int_{\Omega_0^f} \delta \dot{W} \cdot \hat{v}^f \, \mathrm{d}x_0 + \int_{\Omega_0^f} \mathrm{div}_0 \left(W_0 \otimes \dot{\delta x} \right) \cdot \hat{v}^f \, \mathrm{d}x_0 \\
&= \int_{\Omega_0^f} \delta \dot{W} \cdot \hat{v}^f \, \mathrm{d}x_0 - \int_{\Omega_0^f} W_0 \otimes \dot{\delta x} : \nabla_0 \hat{v}^f \, \mathrm{d}x_0 \\
&\quad + \int_{\Gamma_0^s} W_0 \otimes \dot{\delta x} n_0 \cdot \hat{v}^f \, \mathrm{d}a_0.
\end{aligned}$$

Plugging this expression into the variational problem [8], and using the definition of $U^G = \dot{\delta x}$, we get at first order

$$\begin{aligned}
&\int_{\Omega_0^f} \delta \dot{W} \cdot \hat{v}^f \, \mathrm{d}x_0 + \int_{\Gamma_0^s} W_0 \otimes \dot{\delta x^s} n_0 \cdot \hat{v}^f \, \mathrm{d}a_0 + m^s(\ddot{\delta x^s}, \hat{v}_2^s) \\
&- \int_{\Omega_0^f} J \left[\phi(W_0, \sigma_0) + \nabla_0 \phi(W_0, \sigma_0) \delta x + \frac{\partial \phi}{\partial W}(W_0, \sigma_0) \delta W \right. \\
&\qquad \left. + \frac{\partial \phi}{\partial \sigma}(W_0, \sigma_0) \delta \sigma \right] F^{-T} : \nabla_0 \hat{v}^f \, \mathrm{d}x_0 + a^s(I + \delta x^s, \hat{v}_2^s) \\
- \int_{\Gamma_0^s} J \sigma F^{-T} n_0 \cdot (\hat{v}_2^f - \hat{v}_2^s) \, \mathrm{d}a_0 &= \int_{\Omega_0^s} f \cdot \hat{v}_2^s \, \mathrm{d}x_0 + \int_{\Gamma_0^s} \dot{\delta x^s} \cdot FSn_0 \hat{v}_3^f \, \mathrm{d}a_0, \\
&\qquad \forall \hat{v} = (\hat{v}^s, \hat{v}^f) \in \mathcal{D}(\Omega_0)^5 \times \mathcal{D}(\Omega_0)^5.
\end{aligned}$$

By subtracting the steady state problem [11] from this, and by developing the expressions for F and J, we get simply at first order

$$\int_{\Omega_0^f} \delta\dot{W} \cdot \hat{v}^f \, \mathrm{d}x_0 + m^s(\delta\ddot{x}^s, \hat{v}_2^s)$$

$$- \int_{\Omega_0^f} \left(\frac{\partial \phi}{\partial W}(W_0, \sigma_0)\delta W + \frac{\partial \phi}{\partial \sigma}(W_0, \sigma_0)\delta\sigma \right) : \nabla_0 \hat{v}^f \, \mathrm{d}x_0$$

$$- \int_{\Omega_0^f} \left[\phi(W_0, \sigma_0) \left(\mathrm{I}\,\mathrm{div}_0\, \delta x - \nabla_0 \delta x^T \right) + \nabla_0 \phi(W_0, \sigma_0)\delta x \right] : \nabla_0 \hat{v}^f \, \mathrm{d}x_0$$

$$- \int_{\Gamma_0^s} \left[\left(\nabla_0 \sigma_0 \delta x^s + \delta\sigma \right) n_0 - \sigma_0 \eta(\delta x^s) \right] \cdot (\hat{v}_2^f - \hat{v}_2^s) \, \mathrm{d}a_0$$

$$+ \int_{\Gamma_0^s} W_0 \otimes \delta\dot{x}^s n_0 \cdot \hat{v}^f \, \mathrm{d}a_0 + \delta a^s(\delta x^s, \hat{v}_2^s) = \int_{\Gamma_0^s} \delta\dot{x}^s \cdot \sigma_0^s n_0 \hat{v}_3^f \, \mathrm{d}a_0,$$

$$\forall \hat{v} = (\hat{v}^s, \hat{v}^f) \in \mathcal{D}(\Omega_0)^5 \times \mathcal{D}(\Omega_0)^5.$$

where

$$\delta a^s(\delta x^s, \hat{v}_2^s) = \int_{\Omega_0^s} \left(\frac{\partial FS}{\partial x}(I)\delta x^s \right) : \nabla_0 \hat{v}_2^s \, \mathrm{d}x_0,$$

corresponds to the linearisation of the structural elastic constitutive law.

Subtracting now the linearised convected problem [16] satisfied by the steady state W_0, and using the interface kinetic relation at equilibrium $\sigma_0^s n_0 = \sigma_0 n_0$, we obtain that the perturbation field $(\delta W, \delta x)$ satisfies the following variational problem:

$$\int_{\Omega_0^f} \delta\dot{W} \cdot \hat{v}^f \, \mathrm{d}x_0 + m^s(\delta\ddot{x}^s, \hat{v}_2^s)$$

$$- \int_{\Omega_0^f} \left(\frac{\partial \phi}{\partial W}(W_0, \sigma_0)\delta W + \frac{\partial \phi}{\partial \sigma}(W_0, \sigma_0)\delta\sigma \right) : \nabla_0 \hat{v}^f \, \mathrm{d}x_0$$

$$+ \delta a^s(\delta x^s, \hat{v}_2^s) + \int_{\Gamma_0^s} \Big\{ \delta\dot{x}^s \cdot n_0 W_0 + \left(\phi(W_0, \sigma_0) + \mathrm{I}_2\, \sigma_0 \right) \eta(\delta x^s)$$

$$- \left[\nabla_0 \left(\mathrm{I}_2\, \sigma_0 + \phi(W_0, \sigma_0) \right) \delta x^s \right] n_0 - \mathrm{I}_2\, \delta\sigma n_0 - \delta\dot{x}^s \cdot \sigma_0 n_0\, \mathrm{I}_3 \Big\} \cdot \hat{v}^f \, \mathrm{d}a_0$$

$$+ \int_{\Gamma_0^s} \left[\left(\nabla_0 \sigma_0 \delta x^s + \delta\sigma \right) n_0 - \sigma_0 \eta(\delta x^s) \right] \cdot \hat{v}_2^s = 0,$$

$$\forall \hat{v} = (\hat{v}^s, \hat{v}^f) \in \mathcal{D}(\Omega_0)^5 \times \mathcal{D}(\Omega_0)^5. \qquad [17]$$

In addition, the kinematic boundary condition [9], once written at first order in terms of δW and δx, reduce to

$$\frac{\partial \phi^E}{\partial W}(W_0)\delta W n_0 = \phi^E(W_0)\eta(\delta x) - \frac{\partial \mathcal{F}}{\partial n}(W_0, n_0, W_\infty)\eta(\delta x)$$

$$-\nabla_0\phi^E(W_0)\delta x n_0 + \frac{\partial \mathcal{F}}{\partial \bar{W}}(W_0, n_0, W_\infty)(\delta W + \nabla_0 W_0 \delta x), \quad \text{on} \quad \Gamma_0^f. \quad [18]$$

In the particular case where $\bar{W} = W_\infty$ and the extension map has no effect on the exterior edge (it means $\delta x = 0$ on Γ_0^f) the boundary condition [18] is more simple, in fact we get

$$\frac{\partial \phi^E}{\partial W}(W_0)\delta W n_0 = A^+ \delta W, \quad \text{on} \quad \Gamma_0^f,$$

in other words, there is no added incoming flux at infinity.

As before, an integration by parts of [17], written with

$$\hat{v}^f = \begin{pmatrix} \hat{v}_1^f \\ 0 \\ 0 \end{pmatrix},$$

and $\hat{v}^s = 0$, yields

$$\delta W_2 \cdot n_0 = \rho_0 \delta \dot{x}^s \cdot n_0 - \nabla_0 W_{0,2} \delta x^s \cdot n_0 + W_{0,2} \cdot \eta(\delta x^s), \quad \text{on} \quad \Gamma_0^s. \quad [19]$$

The kinematic condition [19] and the specific form [10] of the flux function enable us to greatly simplify the interface integrals in [17]. Indeed, from the kinematic condition at rest, $U_0^f \cdot n_0 = 0$, and [19], we first have

$$\begin{aligned} \rho_0 \delta U^f \cdot n_0 &= \delta(\rho U^f) \cdot n_0 - \delta\rho U_0^f \cdot n_0 \\ &= \delta W_2 \cdot n_0 \\ &= \rho_0 \delta \dot{x}^s \cdot n_0 - \nabla_0(\rho_0 U_0^f)\delta x^s \cdot n_0 + \rho_0 U_0^f \cdot \eta(\delta x^s) \\ &= \rho_0 \delta \dot{x}^s \cdot n_0 - \rho_0(\nabla_0 U_0^f)\delta x^s \cdot n_0 + \rho_0 U_0^f \cdot \eta(\delta x^s). \end{aligned}$$

The kinematic condition [19] therefore reduces to

$$\delta U^f \cdot n_0 = \delta \dot{x}^s \cdot n_0 - (\nabla_0 U_0^f)\delta x^s \cdot n_0 + U_0^f \cdot \eta(\delta x^s), \quad \text{on} \quad \Gamma_0^s. \quad [20]$$

For viscous fluids with no slip boundary conditions we have $U_0^f = 0$ and $\delta U^f + \nabla_0 U_0^f \delta x^s = \delta \dot{x}^s$ on Γ_0^s. We then deduce, from [10], [12] and [20], that on Γ_0^s

$$\begin{aligned} (\phi(W_0, \sigma_0) + \mathrm{I}_2\, \sigma_0)\, \eta(\delta x^s) - [\nabla_0 (\mathrm{I}_2\, \sigma_0 + \phi(W_0, \sigma_0))\, \delta x^s]\, n_0 \\ + \delta \dot{x}^s \cdot n_0 W_0 - \mathrm{I}_2\, \delta\sigma n_0 - \delta \dot{x}^s \cdot \sigma_0 n_0\, \mathrm{I}_3 \\ = \delta U^f \cdot n_0 W_0 - \mathrm{I}_2\, \delta\sigma n_0 - \sigma_0 \delta U^f \cdot n_0\, \mathrm{I}_3 \\ = \left(\frac{\partial \phi}{\partial W}(W_0, \sigma_0)\delta W + \frac{\partial \phi}{\partial \sigma}(W_0, \sigma_0)\delta\sigma \right) n_0. \end{aligned}$$

Therefore, the linearised fluid-structure interaction problem [17] finally reduces to the unique variational equation

$$\int_{\Omega_0^f} \delta\dot{W} \cdot \hat{v}_0^f \, \mathrm{d}x_0 + m^s(\delta\ddot{x}^s, \hat{v}_2^s)$$

$$- \int_{\Omega_0^f} \left(\frac{\partial \phi}{\partial W}(W_0, \sigma_0)\delta W + \frac{\partial \phi}{\partial \sigma}(W_0, \sigma_0)\delta\sigma \right) : \nabla_0 \hat{v}^f \, \mathrm{d}x_0$$

$$+ \int_{\Gamma_0^s} \left(\frac{\partial \phi}{\partial W}(W_0, \sigma_0)\delta W + \frac{\partial \phi}{\partial \sigma}(W_0, \sigma_0)\delta\sigma \right) n_0 \cdot \hat{v}^f \, \mathrm{d}a_0$$

$$+\delta a^s(\delta x^s, \hat{v}_2^s) + \int_{\Gamma_0^s} \left[(\nabla_0 \sigma_0 \delta x^s + \delta\sigma) \, n_0 - \sigma_0 \eta(\delta x^s) \right] \cdot \hat{v}_2^s = 0,$$

$$\forall \hat{v} = (\hat{v}^s, \hat{v}^f) \in \mathcal{D}(\Omega_0)^5 \times \mathcal{D}(\Omega_0)^5. \qquad [21]$$

complemented with the kinematic boundary conditions [18] and [20].

This linearisation leads to a coupled problem written on a fixed configuration Ω_0, using standard flux functions ϕ, and totally independent of the extension δx used inside the fluid domain Ω_0^f. Therefore, the problem obtained allows us to take into account the motion of the structure, while keeping a fixed fluid domain. This is achieved by using non-standard boundary conditions on the fixed interface Γ_0^s. On the one hand, the kinematic condition of continuity of the normal velocity [9] is replaced by a condition of transpiration [19], and on the other hand, the kinetic continuity of traction forces on the interface is modified by the introduction of a correcting term which appears in [16] as a surface integral.

Our method's underlying idea comes from the definition of fluctuations, [13], which leads to the transpiration condition [19], and from the transported problem [16], which enables us to transform the volume integral, with dependencies in δx, into a surface integral on the fixed interface.

5.4. *Coupled fluid and solid subproblems*

The variational formulation [21] is now equivalent to two subproblems coupled along the fixed interface, Γ_0^s (see remark of Section 2). If we take, in [21] $\hat{v}^s = 0$, we recover the standard linearised Euler equations for inviscid fluids, or the linearised Navier-Stokes equations for viscous flows,

$$\delta\dot{W} + \mathrm{div}_0 \left(\frac{\partial \phi}{\partial W}(W_0, \sigma_0)\delta W + \frac{\partial \phi}{\partial \sigma}(W_0, \sigma_0)\delta\sigma \right) = 0, \quad \text{in} \quad \Omega_0^f, \qquad [22]$$

completed with the kinematic condition of transpiration

$$\frac{\partial \phi^E}{\partial W}(W_0)\delta W n_0 = \phi^E(W_0)\eta(\delta x) - \frac{\partial \mathcal{F}}{\partial n}(W_0, n_0, W_\infty)\eta(\delta x)$$

$$-\nabla_0\phi^E(W_0)\delta x n_0 + \frac{\partial \mathcal{F}}{\partial \bar{W}}(W_0, n_0, W_\infty)(\delta W + \nabla_0 W_0 \delta x), \quad \text{on} \quad \Gamma_0^f,$$

$$\delta U^f \cdot n_0 = \delta \dot{x}^s \cdot n_0 - (\nabla_0 U_0^f)\delta x^s \cdot n_0 + U_0^f \cdot \eta(\delta x^s), \quad \text{on} \quad \Gamma_0^s.$$

Alternatively, and this will be the case in the section to come, the fluid subproblem can be replaced at first order by its non-linear equivalent by adding to [22] the equation, $\mathrm{div}_0\, \phi(W_0, \sigma_0) = 0$, satisfied in initial state, yielding

$$\dot{\mathcal{W}} + \mathrm{div}_0\, \phi(\mathcal{W}, \Sigma) = 0, \quad \text{in} \quad \Omega_0^f,$$

$$\phi^E(\mathcal{W})n_0 = \phi^E(\mathcal{W})\eta(\delta x) + \mathcal{F}(\bar{\mathcal{W}}, JF^{-T}n_0, W_\infty)$$

$$-\nabla_0\phi^E(\mathcal{W})\delta x n_0 + \frac{\partial \mathcal{F}}{\partial \bar{W}}(W_0, n_0, W_\infty)(\nabla_0 \mathcal{W}\delta x), \quad \text{on} \quad \Gamma_0^f,$$

$$\mathcal{W}_2 \cdot n_0 = \rho \dot{\delta x}^s \cdot n_0 - \nabla_0 \mathcal{W}_2 \delta x^s + \mathcal{W}_2 \cdot \eta(\delta x^s), \quad \text{on} \quad \Gamma_0^s.$$

The structural subproblem is simply obtained by taking $\hat{v}^f = 0$ in [21], yielding

$$m^s(\ddot{\delta x}^s, \hat{v}_2^s) + \delta a^s(\delta x^s, \hat{v}_2^s)$$

$$= \int_{\Gamma_0^s} [\sigma_0 \eta(\delta x^s) - \nabla_0 \sigma_0 \delta x^s n_0 - \delta \sigma n_0] \cdot \hat{v}_2^s \, \mathrm{d}a_0, \quad \forall \hat{v}_2^s \in \mathcal{D}(\Omega_0)^3.$$

Equivalently, after integration by parts, this structural problem can be written

$$\ddot{\delta x}^s - \mathrm{div}_0 \left(\frac{\partial FS}{\partial x}(I)\delta x^s \right) = 0, \quad \text{in} \quad \Omega_0^s,$$

$$\left(\frac{\partial FS}{\partial x}(I)\delta x^s \right) n_0^s = \delta \sigma n_0^s + \nabla_0 \sigma_0 \delta x^s n_0^s + \sigma_0 \eta(\delta x^s), \quad \text{on} \quad \Gamma_0^s.$$

The coupling with the fluid subproblem appears here on the interface by means of non-standard boundary conditions.

Finally, the configuration subproblem defining the extension of δx inside the fluid domain is no longer needed and therefore disappears from the problem.

6. Numerical tests

We have used the *Dassault Aviation* code "Eugenie" with its three-dimensionnal steady and unsteady capabilities. This industrial Euler code working on unstructured mono- or multi-domain meshes can deal with complex configurations, such as a complete aircraft with its engines. The finite volume cell vertex formulation (see [FEZ 89]) uses space-centered schemes. Two numerical fluxes are available: the first one is a predictor-corrector flux based on a Lax-Wendroff scheme (see [BAS 99]), and the second is a Peraire flux with second and fourth order artificial viscosity (see [SEL 89]). A dual time stepping technique allows unsteady computations, with a Gear

(second order) backward difference scheme for the physical time steps and an implicit strategy for the resolution of the local time-stepping unsteady problem.

Compared to its description in the former section, the practical implementation of the kinematic boundary condition is slightly relaxed in most Euler codes, including "Eugenie". More precisely, in the kinematic boundary condition [4], the velocity after transport $U^f(x^f(x_0))$ is approximated by its value before transport $U^f(x_0)$:

$$U^f(x^f(x_0)) \cdot n(x^f(x_0)) = U^s(x^s).n(x^s).$$

In other words, the gradient term $\nabla_0 U_0^f$ is neglected. Then, the transpiration kinematic boundary condition reduces to:

$$\begin{aligned} U^f(x^f(x_0)) \cdot n(x^f(x_0)) &= U^f(x_0) \cdot n_0(x_0) + U^f(x_0) \cdot (n(x^f(x_0)) - n_0(x_0)) \\ &= U^s(x^s(x_0)) \cdot n(x^s(x_0)) \end{aligned}$$

This means that in the numerical solver the usual boundary term

$$U^f(x_0) \cdot n_0(x_0) = g(x_0),$$

will be given by the modified expression

$$g(x_0) = U^f(x_0) \cdot n_0(x_0) - (U^f(x_0) - U^s(x^s)) \cdot n(x^f(x_0)),$$

prescribing weakly the interface transpiration boundary condition.

The Eugenie code has been linearised using the automatic differentiation tool *Odyssée* developed by INRIA (see [FAU 98]). Fortran routines corresponding to numerical flux and boundary conditions have been carefully differentiated with respect to the fluid state W and to a set of input conditions (for instance angle of attack or slip angles, motion of the body, ...) and were gathered to compute either steady or harmonic solutions. The resulting linear system is solved using iterative solvers such as preconditioned G.M.Res., without any local time stepping technique. This approach leads to smaller computation times but requires more memory, especially to store the preconditioner.

This linear code is interesting in many domains in aerodynamics such as:

– flight control and stabilisation methods, by predicting sensitivities of some coefficients such as (lift, ...) to variations of aerodynamic parameters (angle of attack, ...);

– aerodynamic shape design, by computing the sensitivity of a cost function to a given deformation of a body;

– stability analysis for flutter prediction, which needs generalised aerodynamic forces that can be computed by a harmonic linearised Euler code.

6.1. *Euler code, steady case*

The first test validates the transpiration technique in a transonic steady case. We will compare the pressure coefficients on a NACA64A010 wing (with a supposed infinite span) for a Mach number of 0.796. We compute these coefficients for an angle of attack of -0.21 degree (inflow condition) with a transpiration condition which corresponds to an angle of attack increase of 1 degree. The reference test is the computation of the same coefficients for an angle of attack angle of 0.79 degree (Figure 2). The same strategy is used for an angle of attack increase of 0.5 degree by transpiration around an angle of attack of -0.21 degree, compared to the results obtained with a 0.29 degree angle of attack (Figure 3).

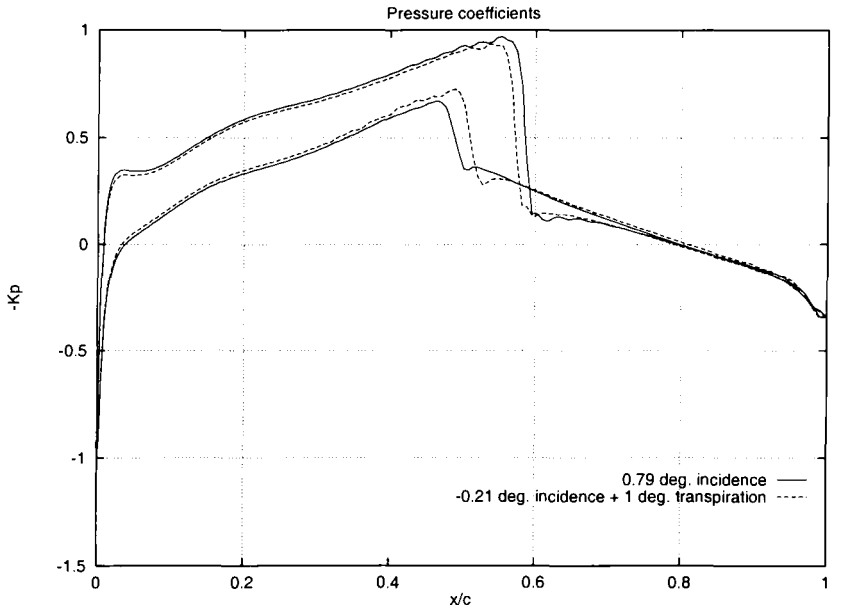

Figure 2. *1 degree by transpiration*

We can see that the results obtained by the transpiration method are very good for small variations of angle of attack, and deteriorate for larger ones.

6.2. *Euler code, unsteady case*

The second test validates both transpiration and ALE techniques. We consider an oscillatory pitch of the NACA64A010 wing in a transonic unsteady flow. We compare, in Figures 4 and 5, the real and imaginary parts of the pressure coefficients for different methods: transpiration, ALE with a solid rotation of the mesh and ALE with mesh deformation. Both ALE techniques were used with an second order geometric

conservation law (see [KOO 98]). For these computations, the oscillatory pitch amplitude chosen was 0.1 degree, the rotation axis was located at $x/c = 0.24$ (where c denotes the chord) and the frequency was 17.2 Hz. We have also plotted the experimental datas ([AGA 82]) to validate the computational results.

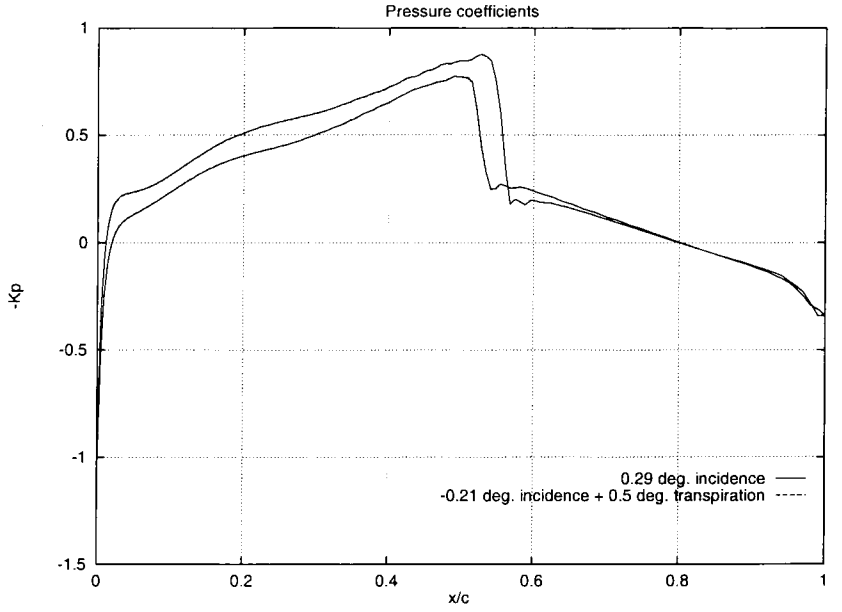

Figure 3. *0.5 degree by transpiration*

We can see that the three computational methods match quite well the experimental datas. Especially, the shock displacement is well located, even if the amplitudes are not the same.

From a CPU point of view, the ALE method with a global rotation is nearly thirty percent more expensive than the transpiration technique, while ALE with mesh deformation is much more expensive, due to the huge amount of time spent in the deformation processes.

6.3. *Linearised code, harmonic case*

The last computation validates the linearised harmonic code with transpiration conditions in the case of an imposed structural motion. We consider the three dimensional RAE wing with an oscillating flap (see Figure 6 and [AGA 82] for experimental datas) for a transonic Mach number of 0.9 and a flap frequency of 90 Hz. The mesh contains 228000 tetrahedra and 40000 nodes.

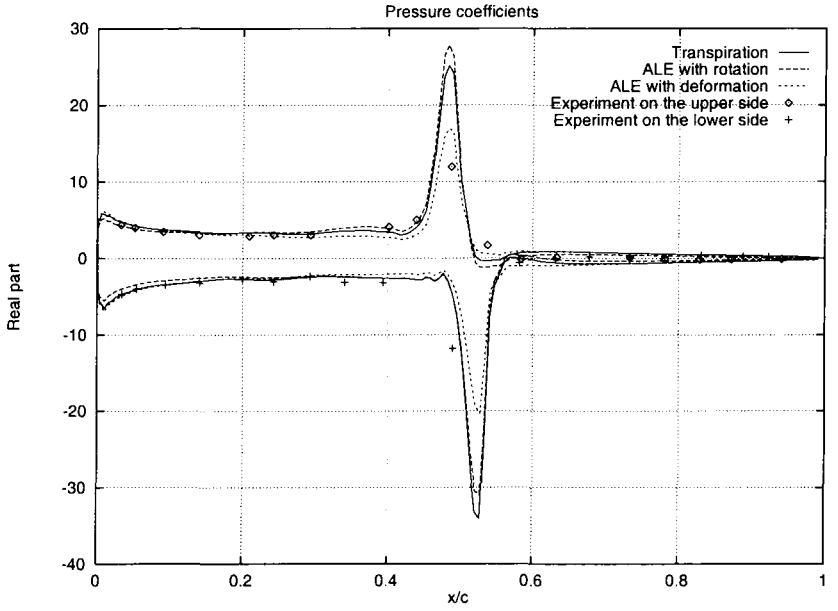

Figure 4. *Real part of the pressure coefficients*

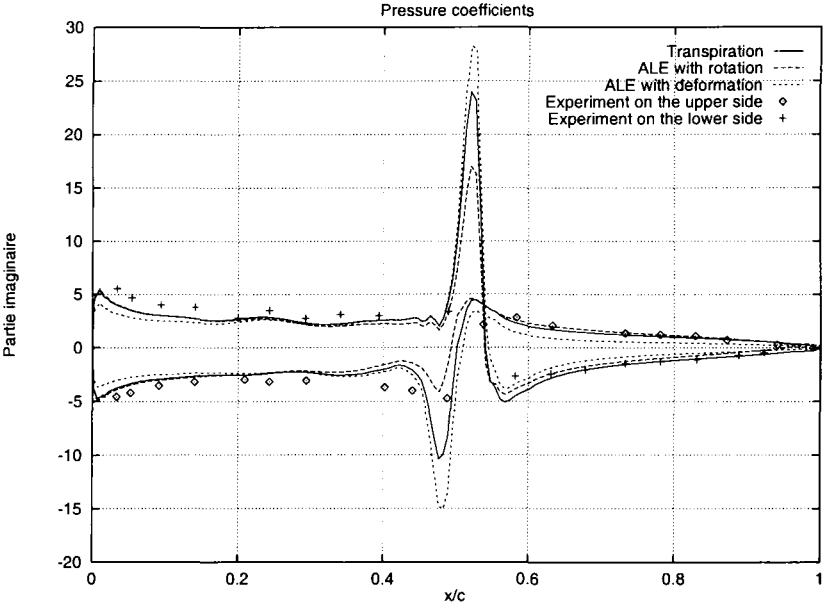

Figure 5. *Imaginary part of the pressure coefficients*

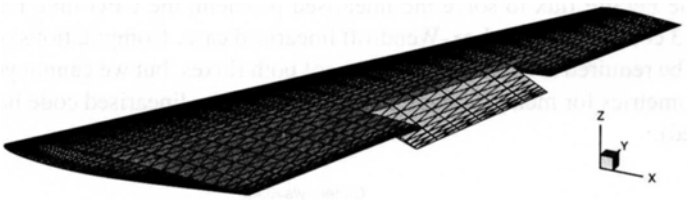

Figure 6. *RAE wing with its flap pulled down*

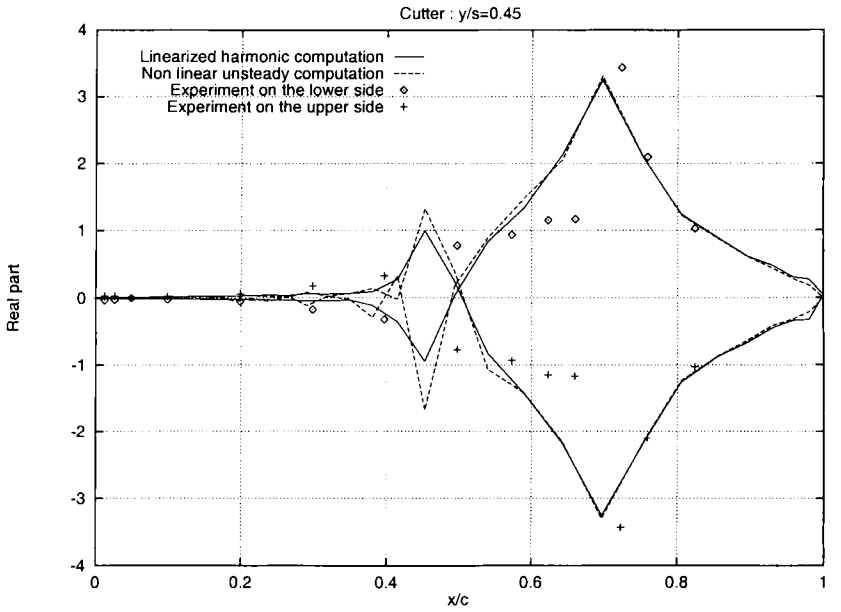

Figure 7. *Real part of the pressure coefficients*

We present on Figures 7 and 8 the real and imaginary parts of the first harmonic of the pressure coefficients on a cutter of the wing (at 45% of the span) obtained by the linearised Euler code, by the unsteady non-linear code and by the experiments.

We can see that the linearised harmonic results agree with the unsteady computation. Nevertheless, the mesh seems to be insufficiently refined to obtain a good comparison between computations and experiments.

The CPU gain of the linearised approach depends on the numerical flux used in the linearised and in the non-linear computation. The Lax-Wendroff flux was chosen for the non-linear cases, because of its reasonable cost and of the quality of its results. Using the same flux in the linearised approach leads to a gain of a factor 2. But if

we take the Peraire flux to solve the linearised problem, the CPU time reduces by a factor of 13 compared to the Lax-Wendroff linearised case. Computations on a refined mesh will be required to evaluate the quality of both fluxes, but we cannot yet consider bigger geometries for memory requirements reasons (the linearised code has not been parallelised).

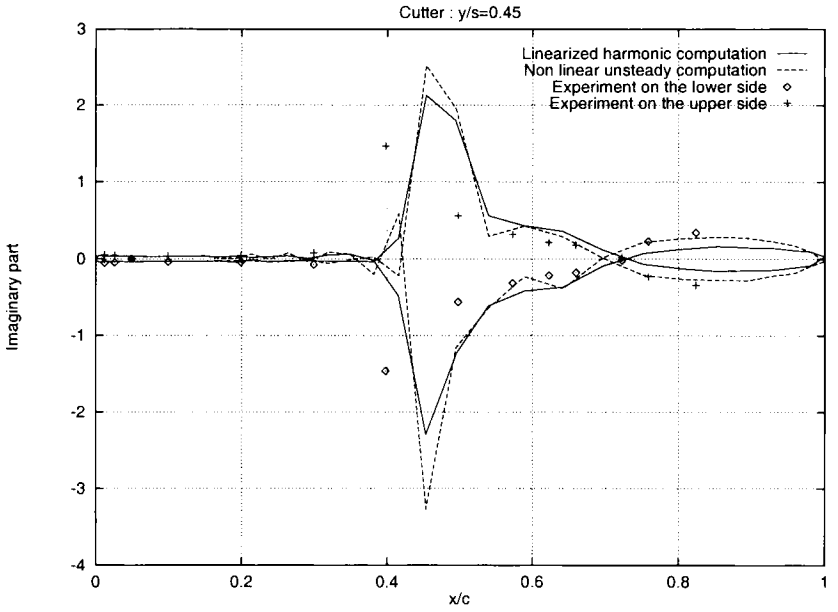

Figure 8. *Imaginary part of the pressure coefficients*

7. Conclusion

The analysis presented in this paper may give a better insight on the different formulations used in fluid-structure interaction, and on the stability properties of the different time integration schemes used in such problems. We have seen there the importance of using smooth grid deformation maps inside the fluid to preserve long term stability properties.

We also have proposed a mathematical derivation of the so called transpiration interface boundary conditions which seem to be good candidates for solving efficiently fluid-structure interaction problems while keeping a fixed grid and configuration on the fluid domain.

The real numerical issue is in any case to be able to obtain reliable numerical predictions of the physical stability of the coupled problem under study. This can either be carried out by a direct numerical integration in time of the full coupled problem us-

ing time accurate schemes with good energy conservation properties, or by computing the harmonic solutions of the linearised variational problem [21].

8. References

[AGA 82] AGARD, Ed., *Compendium of Unsteady Aerodynamic Measurements*, num. 702, Advisory Group for Aerospace Research and Development, North Atlantic Treaty Organization, 1982.

[BAS 99] BASTIN J., ROGE G., "A Multidimensional Fluctuation Splitting Scheme for the Three-Dimensional Euler Equations", *M2AN Math. Model. Numer. Anal.*, vol. 33, num. 6, 1999, pp. 1241-1259.

[CAR 95] CARRIVE-BÉDOUANI M., LE TALLEC P., MOURO J., "Approximations par éléments finis d'un modèle de coques minces géométriquement exact", *Rev. Européenne Élém. Finis*, vol. 4, num. 5-6, 1995, pp. 633-661.

[DAU 84] DAUTRAY R., LIONS J. L., *Analyse mathématique et calcul numérique pour les sciences et les techniques*, Collection du Commissariat à l'Energie Atomique, Masson, 1984.

[FAR 98a] FARHAT C., LESOINNE M., LE TALLEC P., "Load and Motion Transfer Algorithms for Fluid/Structure Interaction Problems with Non-Matching Discrete Interfaces: Momentum and Energy Conservation, Optimal Discretization and Application to Aeroelasticity", *Comput. Methods Appl. Mech. Engrg.*, vol. 157, num. 1-2, 1998, pp. 95-114.

[FAR 98b] FARHAT C., LESSOINNE M., "Higher-order Ataggered and Subiteration Free Algorithm for Coupled Dynamic Aeroelasticity Problems", *AIAA Paper 98-0516, 36th Aerospace Sciences Meeting and Exhibit*, Reno, Nevada, January 1998.

[FAU 98] FAURE C., PAPEGAY Y., "*Odyssée* User's Guide Version 1.7", report num. RT0224, 1998, INRIA.

[FEZ 89] FEZOUI L., STOUFFLET B., "A Class of Implicit Upwind Schemes for Euler Simulations with Unstructured Meshes", *J. Comput. Phys.*, vol. 84, num. 1, 1989, pp. 174-206.

[KOO 98] KOOBUS B., FARHAT C., "Second Order Implicit Schemes that Satisfy the GCL for Flow Computations on Dynamic Grids", *AIAA Paper 98-0113, 36th Aerospace Sciences Meeting and Exhibit*, Reno, Nevada, January 1998.

[KUH 99] KUHL D., RAMM E., "Generalized Energy-Momentum Method for Non-Linear Adaptive Shell Dynamics", *Comput. Methods Appl. Mech. Engrg.*, vol. 178, num. 3-4, 1999, pp. 343-366.

[LET 94] LE TALLEC P., *Numerical Methods for Non-Linear Three-Dimensional Elasticity*, vol. III of *Handbook of Numerical Analysis*, North Holland, 1994.

[LET 96] LE TALLEC P., MOURO J., "Structures en grands déplacements couplées à des fluides en mouvement", report num. RR2961, August 1996, INRIA.

[LET 99a] LE TALLEC P., MANI S., "Analysis of a Linearised Fluid Structure Interaction Problem", report num. 9917, May 1999, CEREMADE Université de Paris Dauphine, to appear in Numerische Mathematik.

[LET 99b] LE TALLEC P., MOURO J., "Fluid Structure Interaction with Large Structural Displacements", 1999, to appear in Computer Methods in Applied Mechanics and Engineering.

[LIG 58] LIGHTHILL M. J., "On Displacement Thickness", *J. Comput. Phys.*, vol. 4, 1958, pp. 383-392.

[MAR 96] MARTIN R., GUILLARD H., "Second-Order Defect-Correction Scheme for Unsteady Problems", *Computer Fluids*, vol. 25, num. 1, 1996, pp. 9-27.

[PIP 95a] PIPERNO S., "Simulation numérique de phénomènes d'interaction fluide structure", PhD thesis, Ecole Nationale des Ponts et Chaussées, 1995.

[PIP 95b] PIPERNO S., FARHAT C., LARROUTUROU B., "Partitioned Procedures for the Transient Solution of Coupled Aeroelastic Problems", *Comput. Methods Appl. Mech. Engrg.*, vol. 124, 1995, pp. 79-111.

[SEL 89] SELMIN V., "A Multistage Method for the Solution of the Euler Equations on Unstructured Grids", GRUBER R., PERIAUX J., SHAW R. P., Eds., *Proceedings of the 5th International Symposium on Numerical Methods in Engineering*, vol. 2, Springer-Verlag, 1989, pp. 449-454.

[SIM 92] SIMO J. C., TARNOW N., "The Discrete Energy-Momentum method. Conserving Algorithms for Non-Linear Elastodynamics", *Z. Angew. Math. Phys.*, vol. 43, num. 5, 1992, pp. 757-792.