Mesh adaptivity in finite elements using the mortar method

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ABSTRACT. *Relying on the mortar element technique, we propose an algorithm for mesh adaptivity in finite elements, where no conformity condition is enforced on the intersection of the triangles during the refinement process. We perform the numerical analysis of the final discretization.*

RÉSUMÉ. *À partir de la méthode d'éléments avec joint, nous proposons un algorithme d'adaptativité de maillages en éléments finis, où aucune condition de conformité n'est imposée sur l'intersection des triangles au cours du raffinement. Nous effectuons l'analyse numérique de la discrétisation finale.*

KEYWORDS: *Mesh adaptivity, finite elements, mortar method.*

MOTS-CLÉS : *Adaptation de maillage, éléments finis, méthode de joints.*

1. Introduction

Mesh adaptivity has become essential in the framework of finite element methods since it plays an important role for the efficiency of the discretization. It relies on the use of two different tools:

– the error indicators which allow for choosing the elements that must be cut up,

– an algorithm for cutting up (and also sometimes gluing back) the elements.

Now, some error indicators which satisfy optimal properties of coincidence with the local error have been exhibited (see for instance [BMV] and [VE1][VE2] for the numerical analysis of such indicators). But the procedure of cutting up the elements is not completely optimized and it is always expensive, specially for three-dimensional domains. Indeed, in order to take into account the standard assumptions of the finite element discretization, two criteria must be satisfied:

– the triangulation must be conforming, in the sense that the intersection of two different elements is either empty or a corner or a whole edge or a whole face of each of the two elements,

– the family of triangulations must be regular, which means that the largest ratio of the diameter of an element to the diameter of the inscribed circle or sphere in this element must be bounded independently of the triangulation.

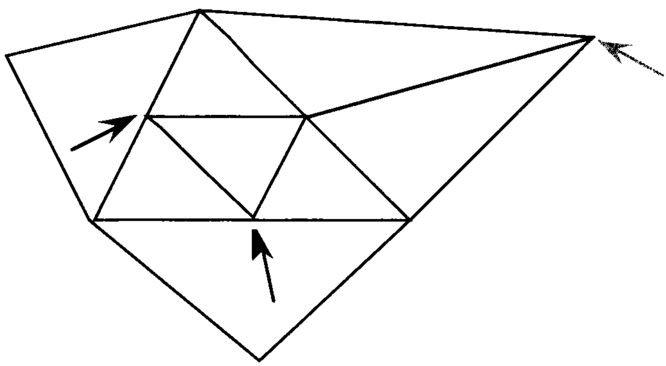

Figure 1. Conformity versus regularity

There is a contradiction between these two criteria: indeed, the first one does not allow for dividing a triangle into four elements (and iteratively into 2^{2k} for any positive integer k) by drawing lines between the middles of the edges, since this leads to a nonconformity between the cut-up triangles and their non-refined neighbours, while the second one does not allow for dividing the angles at each new triangulation.

These difficulties are illustrated in Figure 1 (with black arrows for the violation of conformity and a grey arrow for the violation of regularity). Of course, many correct (but not optimal) solutions have been proposed and successfully tested, see [GB] for a review of these techniques. However, we present in this paper a new algorithm for mesh refinement, which relies on the mortar method and allows for relaxing the first criterion.

Indeed, one feature of the mortar element method of [BMP1][BMP2] is the possibility of working with non-matching grids on different subdomains: then, the corresponding discrete functions are not globally continuous. The matching conditions between subdomains are only enforced in a weak way, by integral equations on the interfaces involving the so-called *mortar function*. So, the algorithm that we propose, in two-dimensional domains, is the following one:
1. define a coarse (and most often quasi-uniform) mesh;
2. iteratively,

– solve the problem on the previous mesh and compute the corresponding error indicators on each element (see e.g. [PS] and [WO] for the analysis of error indicators on nonconforming discretizations),

– from these indicators, choose the elements which must be cut up and cut them up (but not any other one) into 2^{2k} small triangles by joining the middles of the edges;

– define a decomposition of the initial domain, such that the triangulation in each subdomain is conforming but that meshes on the two sides of an interface do not coincide, and apply the mortar element discretization on this decomposition.

We refer to [BE] for a detailed analysis of the mortar element method in general three-dimensional domains, which should allow for extending our results to this case, and to [OV] for the implementation of the mortar adaptivity in the framework of the $h-p$ version of the finite elements.

In this paper, we perform the numerical analysis of the discretization with a nonconforming triangulation for the model problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad [1]$$

when Ω is a bounded polygon in $\mathbb{R}^2$. The discretization relies on the approximation by continuous finite element approximations on each subdomain which are polynomials of degree at most m on each element of a triangulation, $m \geq 1$. In Section 2, we present the discrete problem and in Section 3, we study the error between the continuous and discrete solutions. Some concluding remarks are given in Section 4.

2. The nonconforming discrete problem

Let $(\mathcal{T}_H)_H$ be a regular family of coarse triangulations of the domain Ω, more precisely a family of sets of triangles covering Ω and satisfying criterions (i) and (ii);

as standard, H stands for the largest diameter of the triangles of $\mathcal{T}_H$. The usual finite element discretization on the triangulation $\mathcal{T}_H$ relies on the space:

$$X_H = \{v \in H_0^1(\Omega);\ \forall K \in \mathcal{T}_H,\ v_{|K} \in \mathcal{P}_m(K)\}, \qquad [2]$$

where $\mathcal{P}_m(K)$ stands for the space of polynomials with total degree $\leq m$ on K, and m is *a priori* ≥ 1. So, the data f belonging for instance to $L^2(\Omega)$, the "coarse" discrete problem reads:

find a function u_H in X_H such that

$$\forall v_H \in X_H, \quad \int_\Omega \mathbf{grad}\, u_H \,.\, \mathbf{grad}\, v_H \, d\mathbf{x} = \int_\Omega f(\mathbf{x}) v_H(\mathbf{x})\, d\mathbf{x}. \qquad [3]$$

It has a unique solution u_H in X_H and, if the solution u of problem [1] belongs to $H^s(\Omega)$, $1 \leq s \leq m+1$, the following error estimate holds:

$$\|u - u_H\|_{H^1(\Omega)} \leq c\, H^{s-1}\, \|u\|_{H^s(\Omega)}.$$

Now, we assume that an arbitrary number of triangles K of $\mathcal{T}_H$ have been cut up into 2^{2k} triangles according to the algorithm described in the introduction, for a fixed integer $k \geq 1$, and we set: $h = 2^{-k}\, H$. We introduce a decomposition of Ω into Ω_0 and $\Omega_1 = \Omega \setminus \overline{\Omega}_0$ such that $\overline{\Omega}_0$ is the union of the non-refined triangles of $\mathcal{T}_H$. We define $\mathcal{T}_h$ as the set of the new "small" triangles obtained by cutting up those of $\mathcal{T}_H$ which are contained in $\overline{\Omega}_1$. We denote by Γ_{Hj}, $1 \leq j \leq J(H)$, the edges of triangles K in $\mathcal{T}_H$ which are contained in $\partial\Omega_0 \setminus \partial\Omega$, and by $\mathcal{S}_H$ their union. The discretization parameter is now the pair $\delta = (H, h)$.

REMARK. — The parameter k is fixed independently of the triangles only to simplify the notation, without restriction: indeed, there is no difficulty in considering more general refinements, a subdomain Ω_k must then be associated with each value of k.

The new discrete problem reads:

find a function u_δ in X_δ such that

$$\forall v_\delta \in X_\delta, \quad \sum_{i=0}^{1} \int_{\Omega_i} \mathbf{grad}\, u_\delta \,.\, \mathbf{grad}\, v_\delta \, d\mathbf{x} = \sum_{i=0}^{1} \int_{\Omega_i} f(\mathbf{x}) v_\delta(\mathbf{x})\, d\mathbf{x}. \qquad [4]$$

So, we are left with defining the "mortar" discrete space X_δ. We first define the spaces of finite element functions on each Ω_i:

$$Y_{\delta,0} = \{v \in H^1(\Omega_0);\ \forall K \in \mathcal{T}_H,\ v_{|K} \in \mathcal{P}_m(K)\},$$

$$Y_{\delta,1} = \{v \in H^1(\Omega_1);\ \forall K \in \mathcal{T}_h,\ v_{|K} \in \mathcal{P}_m(K)\}.$$

Next, for $1 \leq j \leq J(H)$, we define $W_{\delta,j}^i$ as the space of traces on Γ_{Hj} of functions in $Y_{\delta,i}$ and, on each Γ_{Hj}, we choose $W_{\delta,j}$ equal to one of the $W_{\delta,j}^i$ (of course, the final discrete space depends on this choice). The space of mortar functions W_δ is the space of functions on $\mathcal{S}_H$ such that their restrictions to Γ_{Hj} belong to $W_{\delta,j}$, $1 \le j \le J(H)$.

Finally, we define for $i = 0$ and 1, a subspace $\tilde{W}^i_{\delta,j}$ of codimension 1 or 2 in $W^i_{\delta,j}$:

(i) $\tilde{W}^0_{\delta,j}$ coincides with the space $\mathcal{P}_{m-1}(\Gamma_{Hj})$ of polynomials with degree $\le m-1$ on Γ_{Hj},

(ii) $\tilde{W}^1_{\delta,j}$ coincides with the space of continuous functions on Γ_{Hj} such that their restrictions to each edge F (contained in $\overline{\Gamma}_{Hj}$) of a triangle of $\mathcal{T}_h$ belong to $\mathcal{P}_m(F)$ if F does not contain any endpoint of Γ_{Hj} and to $\mathcal{P}_{m-1}(F)$ if it contains an endpoint.
The two choices $W^0_{\delta,j}$ and $W^1_{\delta,j}$, and the associated subspaces $\tilde{W}^0_{\delta,j}$ and $\tilde{W}^1_{\delta,j}$, are represented in the following figure.

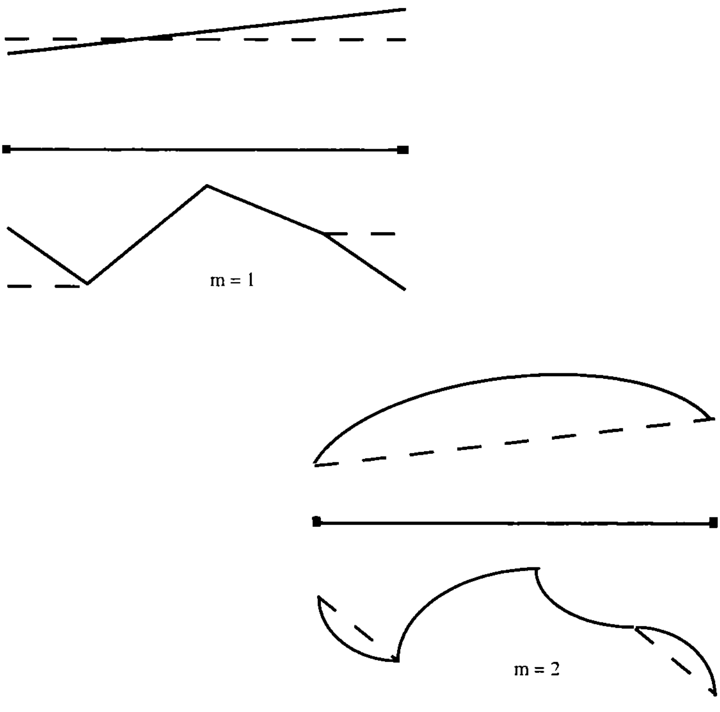

Figure 2. The discrete spaces of traces and their subspaces

DEFINITION 2.1. — The space X_δ is the space of functions v_δ:
– such that their restrictions to Ω_i, $i = 0$ and 1, belong to $Y_{\delta,i}$,
– which vanish on $\partial\Omega$,
– such that, the mortar function φ being defined in W_δ as the trace on each Γ_{Hj} of $v_{\delta\,|\Omega_i}$ for the i such that $W_{\delta,j}$ coincides with $W^i_{\delta,j}$, the following integral condition holds for $1 \le j \le J(H)$ and $i = 0$ and 1:

$$\forall \psi \in \tilde{W}^i_{\delta,j}, \quad \int_{\Gamma_{Hj}} (v_{\delta\,|\Omega_i} - \varphi)(\tau)\psi(\tau)\,d\tau = 0 \qquad [5]$$

(of course, this condition is obvious for one value of i, corresponding to the choice of $W_{\delta,j} = W^i_{\delta,j}$).

REMARK. — In the first version of the mortar element method [BMP1], the space $\tilde{W}^0_{\delta,j}$ was chosen as $\mathcal{P}_{m-2}(\Gamma_{Hj})$ and a further condition was enforced in the definition of the space X_δ: at each corner $\mathbf{a}$ of Ω_i which does not belong to $\partial\Omega$,

$$v_{\delta\,|\Omega_i}(\mathbf{a}) = \varphi(\mathbf{a}). \qquad [6]$$

Enforcing condition [6] clearly justifies the choice of a subspace $\tilde{W}^i_{\delta,j}$ with codimension 2 in $W^i_{\delta,j}$, in order not to have more conditions than degrees of freedom on each edge Γ_{Hj}, and this choice is usually kept even without condition [6]. The numerical analysis leads to similar results for the two versions, however giving up the condition [6] induces a discretization which is much easier to implement and more efficient, especially on parallel computers and in three dimensions. So, the choice $\tilde{W}^0_{\delta,j} = \mathcal{P}_{m-1}(\Gamma_{Hj})$ is not standard, however the conformity of the initial discretization allows for it and it is necessary for deriving optimal estimates, especially in the case $m = 1$, as will appear later on.

REMARK. — The discretization that we propose is nonconforming, i.e., the space X_δ is not included in $H^1_0(\Omega)$ in the general case since no condition is enforced at the endpoints of Γ_{Hj}. However, it is "more conforming" with the choice $W_{\delta,j} = W^0_{\delta,j}$ than with the choice $W_{\delta,j} = W^1_{\delta,j}$, in the sense that $2^k\, m - 1$ matching conditions are enforced on each Γ_{Hj} instead of m.

Problem [4] is now completely described thanks to Definition 2.1. Clearly, its well-posedness relies of the ellipticity of the left-hand member with respect to the broken norm $\|.\|_{1*}$, where the broken seminorm and norm are defined by:

$$|v|_{1*} = \Big(\sum_{i=0}^{1} |v|^2_{H^1(\Omega_i)}\Big)^{\frac{1}{2}} \quad \text{and} \quad \|v\|_{1*} = \Big(\sum_{i=0}^{1} \|v\|^2_{H^1(\Omega_i)}\Big)^{\frac{1}{2}}. \qquad [7]$$

This ellipticity is a consequence of the following proposition.

PROPOSITION 2.2. — *For each value of δ, there exists a positive constant c_δ such that, for all v_δ in X_δ,*

$$\|v_\delta\|_{L^2(\Omega)} \le c_\delta \Big(\sum_{i=0}^{1} |v|^2_{H^1(\Omega_i)}\Big)^{\frac{1}{2}}. \qquad [8]$$

PROOF: If [8] does not hold, there is a function v_δ in X_δ such that:

$$\|v_\delta\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \sum_{i=0}^{1} |v|^2_{H^1(\Omega_i)} = 0. \qquad [9]$$

The second equation implies that v_δ has a null gradient, so it is equal to a constant on each connected component of each Ω_i. Due to the boundary conditions, this constant is zero if the connected component has a part of its boundary (of positive length) contained in $\partial\Omega$. So, it remains to prove that all the constants are equal, which results from the following argument: if c and c' are the constants on each side of Γ_{Hj}, one of them coincides with the mortar function on Γ_{Hj} and, since the constant functions belong to both $\tilde{W}^i_{\delta,j}$, condition [5] implies:

$$\int_{\Gamma_{Hj}} (c - c')\, d\tau = 0.$$

So, the function v_δ is equal to 0, which is in contradiction with the first part of [9].

So, the following corollary is now obvious.

COROLLARY 2.3. — *For any data f in $L^2(\Omega)$, problem* [4] *has a unique solution u_δ in X_δ. This solution satisfies:*

$$\|u_\delta\|_{1*} \leq c_\delta \sqrt{1 + c_\delta^2}\, \|f\|_{L^2(\Omega)}. \qquad [10]$$

3. Analysis of the error

Due to the nonconformity of the discretization, the abstract error estimate reads:

$$\|u - u_\delta\|_{1*} \leq 3\sqrt{1 + c_\delta^2}$$

$$\Big(\inf_{v_\delta \in X_\delta} |u - v_\delta|_{1*} + \sup_{w_\delta \in X_\delta} \frac{\sum_{j=1}^{J(H)} \int_{\Gamma_{Hj}} (\partial_n u)(\tau)[w_\delta](\tau)\, d\tau}{|w_\delta|_{1*}}\Big), \qquad [11]$$

where ∂_n stands for the normal derivative to Γ_{Hj} and $[\cdot]$ denotes the jump through Γ_{Hj}. The first term in the right-hand side is the approximation error, the second one is the consistency error. So the analysis of the error is parted in three steps: uniform ellipticity (which means that all the c_δ are smaller than a positive constant independent of δ), study of the consistency error, study of the approximation error.

3.1. *Uniform ellipticity*

We prove a modified version of Proposition 2.2, where the constant is independent of δ. However the arguments are more technical. We use the standard finite element notation by a hat for: the reference triangle $\hat{K}$, the functions $\hat{v}$ on this triangle associated to functions v on a fixed triangle K through the affine mapping and the operators $\hat{\pi}$ acting on functions $\hat{v}$.

We denote by H_j the length of Γ_{Hj}, $1 \leq j \leq J(H)$. With each Γ_{Hj}, we associate the two triangles K_j and K'_j of the initial triangulation $\mathcal{T}_H$ which are on the two sides of Γ_{Hj}.

PROPOSITION 3.1. — *There exists a positive constant c such that, for each value of* δ *and for all* v_δ *in* X_δ,

$$\|v_\delta\|_{L^2(\Omega)} \le c\big(\sum_{i=0}^{1} |v|^2_{H^1(\Omega_i)}\big)^{\frac{1}{2}}. \qquad [12]$$

To prove this statement, without restriction, we assume that the domain Ω is a rectangle $]a, a'[\times]b, b'[$: indeed, both triangulations $\mathcal{T}_H$ and $\mathcal{T}_h$ can be extended in a conforming way to any rectangle containing Ω; it allows for constructing a discrete space which contains the extensions by zero of functions in X_δ, so that proving [12] for this new space yields [12] on X_δ. Next, for $b < y < b'$, let $a^y_{j_k}$, $1 \le k \le K(y)$, denote the x–coordinates, in increasing order, of the points where the segment $]a, a'[\times\{y\}$ crosses any edge Γ_{Hj} (here, "crossing" means going from one Ω_i to the other one), and let H_{j_k} stand for the length of the corresponding edge Γ_{Hj}. We set: $a^y_{j_0} = a$ and $a^y_{j_{K(y)+1}} = a'$. We begin with a "geometric" lemma.

LEMME 3.2. — *For any* y, $b < y < b'$, *the quantity* $\sum_{k=1}^{K(y)} H_{j_k}$ *is bounded by a constant c independent of* δ.

PROOF: The first point $(a^y_{j_1}, y)$ belongs to Γ_{Hj_1}, we denote by $\mathbf{b}_{j_1}$ the endpoint of Γ_{Hj_1} which is the closest to $(a^y_{j_1}, y)$. Next, it follows from criterion (ii) that:

• all the angles of the triangles are larger than a minimal angle α_0 independent of H, so that the number of triangles of $\mathcal{T}_H$ which share the same corner is bounded by an integer $M < \frac{2\pi}{\alpha_0}$;

• the ratio of the diameters of two triangles which share the same corner is smaller than a constant c_0 independent of H.

Hence, if $(a^y_{j_1}, y), \dots, (a^y_{j_\ell}, y)$, with $1 \le \ell \le M$, belong to edges Γ_j which contain $\mathbf{b}_{j_1}$, and not $(a^y_{j_{\ell+1}}, y)$, it is readily checked that the distance of $(a^y_{j_1}, y)$ either to (a, y) or to $(a^y_{j_{\ell+1}}, y)$ is larger than $c\, H_{j_1}$ for a constant c depending only on α_0 and c_0. Then, we have:

$$\sum_{k=1}^{K(y)} H_{j_k} \le \sum_{k=1}^{\ell} c_0\, H_{j_1} + \sum_{k=\ell+1}^{K(y)} H_{j_k} \le M\frac{c_0}{c}(a^y_{j_{\ell+1}} - a) + \sum_{k=\ell+1}^{K(y)} H_{j_k}.$$

Iterating this argument, we derive that:

$$\sum_{k=1}^{K(y)} H_{j_k} \le 2M\,\frac{c_0}{c}(a' - a).$$

PROOF OF PROPOSITION 3.1: Let v_δ be in X_δ. For any (x, y) in Ω, we write:

$$v_\delta(x, y) = \int_a^{a^y_{j_1}} (\frac{\partial v_\delta}{\partial x})(t, y)\, dt + [v_\delta](a^y_{j_1}, y) + \int_{a^y_{j_1}}^{a^y_{j_2}} (\frac{\partial v_\delta}{\partial x})(t, y)\, dt + \cdots$$
$$+ [v_\delta](a^y_{j_\ell}, y) + \int_{a^y_{j_\ell}}^{x} (\frac{\partial v_\delta}{\partial x})(t, y)\, dt,$$

where $a^y_{j_\ell}$ now stands for the largest $a^y_{j_k} < x$. This gives:

$$|v_\delta(x,y)| \leq \sum_{k=1}^{K(y)+1} \Big| \int_{a^y_{j_{k-1}}}^{a^y_{j_k}} (\frac{\partial v_\delta}{\partial x})(t,y)\,dt \Big| + \sum_{k=1}^{K(y)} |[v_\delta](a^y_{j_\ell},y)|.$$

Integrating the square of this inequality with respect to x and y and using a Cauchy–Schwarz inequality, we derive

$$\|v_\delta\|^2_{L^2(\Omega)} \leq (a'-a) \int_\Omega \sum_{k=1}^{K(y)+1} \int_{a^y_{j_{k-1}}}^{a^y_{j_k}} (\frac{\partial v_\delta}{\partial x})^2(t,y)\,dt\,dx\,dy$$

$$+ \int_\Omega \Big(\sum_{k=1}^{K(y)} H_{j_k}\Big)\Big(\sum_{k=1}^{K(y)} H_{j_k}^{-1}\,[v_\delta]^2(a^y_{j_\ell},y)\Big)\,dx\,dy.$$

In order to bound the last term, we use Lemma 3.2 and observe that each Γ_{Hj_k} is contained in a non horizontal line $x = \lambda_j y + \mu_j$, so that $d\tau$ is equal to $\sqrt{1+\lambda_j^2}\,dy \geq dy$. Thus, we obtain:

$$\|v_\delta\|^2_{L^2(\Omega)} \leq (a'-a)^2 \sum_{i=0}^{1} \int_{\Omega_i} (\frac{\partial v_\delta}{\partial x})^2(t,y)\,dt\,dy + c \sum_{j=1}^{J(H)} H_j^{-1} \int_{\Gamma_{Hj}} [v_\delta]^2(\tau)\,d\tau.$$

Next, on each Γ_{Hj}, the mortar function associated with v_δ coincides with one of the traces of v_δ restricted to each side of Γ_j and, since each $\tilde{W}^i_{Hj}$ contains the constants, condition [5] implies that the integral of $[v_\delta]$ on Γ_{Hj} is equal to 0. As a consequence, denoting by c_j the mean value of $v_{\delta\,|K_j}$ and $v_{\delta\,|K'_j}$ on Γ_{Hj}, we have:

$$\int_{\Gamma_{Hj}} [v_\delta]^2(\tau)\,d\tau \leq 2\Big(\int_{\Gamma_{Hj}} (v_{\delta\,|K_j} - c_j)^2(\tau)\,d\tau + \int_{\Gamma_{Hj}} (v_{\delta\,|K'_j} - c_j)^2(\tau)\,d\tau\Big).$$

On the reference triangle $\hat{K}$, the semi-norm $|.|_{H^1(\hat{K})}$ is a norm on the subspace of $H^1(\hat{K})$ made of functions with a null integral on one edge $\hat{\Gamma}$ of $\hat{K}$. So, using the equivalence of norms on finite-dimensional subspaces, we derive:

$$\int_{\Gamma_{Hj}} (v_{\delta\,|K_j} - c_j)^2(\tau)\,d\tau = H_j \int_{\hat{\Gamma}} (\hat{v} - c_j)^2(\hat{\tau})\,d\hat{\tau}$$

$$\leq \hat{c}\,H_j\,|\hat{v}|^2_{H^1(\hat{K})} \leq c'\,H_j\,|v_\delta|^2_{H^1(K_j)},$$

and the similar inequality with K_j replaced by K'_j. As a consequence,

$$\|v_\delta\|^2_{L^2(\Omega)} \leq (a'-a)^2 |v|^2_{1*} + c \sum_{j=1}^{J(H)} \big(|v_\delta|^2_{H^1(K_j)} + |v_\delta|^2_{H^1(K'_j)}\big).$$

Since each triangle of $\mathcal{T}_H$ appears at most thrice in the last sum, the proof of the proposition is complete.

3.2. *Consistency error*

The estimate is a consequence of condition [5], it is proven in the following proposition. We begin with a lemma concerning the approximation properties of the orthogonal projection operator $\pi^i_{\delta,j}$ from $L^2(\Gamma_{Hj})$ onto $\tilde{W}^i_{\delta,j}$.

LEMME 3.3. — *For $1 \le j \le J(H)$ and for $i = 0, 1$, there exists a positive constant c such that, for any function ψ in $H^s(K_j)$, $0 \le s \le m$,*

$$\|\psi - \pi^i_{\delta,j}\psi\|_{L^2(\Gamma_{Hj})} \le c\, h_i^{s-\frac{1}{2}}\, |\psi|_{H^s(K_j)}, \qquad [13]$$

with $h_0 = H$ and $h_1 = h$.

PROOF: We treat separately the cases $i = 0$ and $i = 1$.

1) In the case $i = 0$, we simply write

$$\|\psi - \pi^0_{\delta,j}\psi\|_{L^2(\Gamma_{Hj})} \le \inf_{\chi\in\mathcal{P}_{m-1}(K_j)} \|\psi - \chi\|_{L^2(\Gamma_{Hj})}.$$

As previously, using the reference triangle $\hat{K}$, we observe that the norm $\|.\|_{L^2(\hat{\Gamma})}$ is smaller than the norm $|.|_{H^s(\hat{K})}$ on $H^s(\hat{K})/\mathcal{P}_{m-1}(\hat{K})$ for $s \le m$, so that going back to K_j yields the desired estimate.

2) We only sketch the proof in the case $i = 1$ which is more standard. When m is ≥ 2, assuming firstly that s is > 1, we have by the definition of $\pi^1_{\delta,j}$:

$$\|\psi - \pi^1_{\delta,j}\psi\|_{L^2(\Gamma_{Hj})} \le \|\psi - i^1_{\delta,j}\psi\|_{L^2(\Gamma_{Hj})},$$

where $i^1_{\delta,j}$ is now the Lagrange interpolation operator in

$$\{\varphi \in H^1(\Gamma_{Hj});\ \forall K \in \mathcal{T}_h,\ v_{|K\cap\Gamma_{Hj}} \in \mathcal{P}_{m-1}(K \cap \Gamma_{Hj})\}.$$

On each triangle K contained in K_j such that $K \cap \Gamma_{Hj}$ is not empty and with obvious notation,

$$\|\psi - i^1_{\delta,j}\psi\|_{L^2(K\cap\Gamma_{Hj})} \le c\, h^{\frac{1}{2}} \inf_{\hat{\chi}\in\mathcal{P}_{m-1}(\hat{K})} \|\hat{\psi} - \hat{\chi} - \hat{i}(\hat{\psi} - \hat{\chi})\|_{L^2(\hat{\Gamma})}$$

$$\le c'\, h^{\frac{1}{2}}\, |\hat{\psi}|_{H^s(\hat{K})} \le c''\, h^{\frac{1}{2}}\, h^{s-1}\, |\psi|_{H^s(K)},$$

so that the desired estimate is obtained by summing up the square of this inequality on the K. When s is ≤ 1, the proof is similar, however the interpolation operator $i^1_{\delta,j}$ has to be replaced by the regularization operator which is described for instance in [BG]. In the case $m = 1$, we refer to [BMP1] for the study of the projection operator $\pi^1_{\delta,j}$ since the space $\tilde{W}^1_{\delta,j}$ is not usual.

PROPOSITION 3.4. — *For any function u in $H^s(\Omega)$, $s > \frac{3}{2}$, such that each $u_{|\Omega_i}$ belongs to $H^{s_i}(\Omega_i)$, $\frac{3}{2} < s_i \le m+1$, the following estimate holds for the consistency error:*

$$\sup_{w_\delta \in X_\delta} \frac{\sum_{j=1}^{J(H)} \int_{\Gamma_{Hj}} (\partial_n u)(\tau)[w_\delta](\tau)\,d\tau}{|w_\delta|_{1*}}$$

$$\leq c\Big(H^{s_0-1}\,\|u\|_{H^{s_0}(\Omega_0)} + h^{s_1-1}\,\|u\|_{H^{s_1}(\Omega_1)}\Big). \qquad [14]$$

PROOF: Let w_δ be any function in X_δ. On each Γ_{Hj}, the mortar function φ associated with w_δ is equal to one of the $w_{\delta\,|\Omega_i}$, so that $[w_\delta]$ is equal to $w_{\delta\,|\Omega_{|i-1|}} - \varphi$, up to the sign. Thus, applying [5] implies that:

$$\int_{\Gamma_{Hj}} (\partial_n u)(\tau)[w_\delta](\tau)\,d\tau = \int_{\Gamma_{Hj}} \big(\partial_n u - \pi_{\delta,j}^{|i-1|}(\partial_n u)\big)(\tau)[w_\delta](\tau)\,d\tau$$

$$= \int_{\Gamma_{Hj}} \big(\partial_n u - \pi_{\delta,j}^{|i-1|}(\partial_n u)\big)(\tau)\big([w_\delta] - \pi_{\delta,j}^{|i-1|}[w_\delta]\big)(\tau)\,d\tau.$$

Assuming for instance that K_j is contained in $\overline{\Omega}_i$, we derive from Lemma 3.3 that:

$$\int_{\Gamma_{Hj}} (\partial_n u)(\tau)[w_\delta](\tau)\,d\tau$$

$$\leq \|\partial_n u - \pi_{\delta,j}^{|i-1|}(\partial_n u)\|_{L^2(\Gamma_j)} \sum_{\ell=0}^{1} \|w_{\delta\,|\Omega_\ell} - \pi_{\delta,j}^{|i-1|} w_{\delta\,|\Omega_\ell}\|_{L^2(\Gamma_j)}$$

$$\leq h_{|i-1|}^{s_{|i-1|}-\frac{3}{2}}\,\|u\|_{H^{s_{|i-1|}}(K_j')}\,h_{|i-1|}^{\frac{1}{2}}\,\big(|w_\delta|_{H^1(K_j)} + |w_\delta|_{H^1(K_j')}\big).$$

Summing up on j and noting that each K_j or K_j' appears at most thrice in the sum, we obtain the desired result.

REMARK. — If $\mathcal{P}_{m-1}(\Gamma_{Hj})$ were replaced by $\mathcal{P}_{m-2}(\Gamma_{Hj})$ in the choice of $\tilde{W}_{\delta,j}$, estimate [14] would only hold with some $s_i \leq m$, so that it cannot lead to a completely optimal error estimate which must exploit the full accuracy of the discretization.

3.3. *Approximation error*

In order to estimate the distance of a function u to X_δ, we introduce the orthogonal projection operator Π_δ from $H_0^1(\Omega)$ onto X_δ, associated with the semi-norm $|\cdot|_{1*}$.

We firstly recall a result concerning the lifting of finite element traces which is proven in [BG, Thm 5.1] according to an argument in [WI].

PROPOSITION 3.5. — *Let $\mathcal{O}$ be a bounded polygon in $\mathbb{R}^2$, and let $\mathcal{S}_h$ be a regular family of triangulations of $\mathcal{O}$. We define the space*

$$Y_h(\mathcal{O}) = \big\{v_h \in H^1(\mathcal{O});\ \forall K \in \mathcal{S}_h,\ v_{h\,|K} \in \mathcal{P}_m(\mathcal{O})\big\}.$$

There exists an operator R_h from the space $W_h(\partial\mathcal{O})$ of traces of $Y_h(\mathcal{O})$ on $\partial\mathcal{O}$ with values in $Y_h(\mathcal{O})$, such that:
(i) for any φ_h in $W_h(\partial\mathcal{O})$, the trace of $R_h\varphi_h$ on $\partial\mathcal{O}$ coincides with φ_h,
(ii) the following stability property holds for a constant c independent of h:

$$\forall\varphi_h \in W_h(\partial\mathcal{O}), \quad \|R_h\varphi_h\|_{H^1(\mathcal{O})} \leq c\,\|\varphi_h\|_{H^{\frac{1}{2}}(\partial\mathcal{O})}. \qquad [15]$$

However, in the previous statement, the dependency of the stability constant with respect to the geometry of $\mathcal{O}$ is not given. So we need a corollary. Here, we assume that the triangle K_j is included in $\overline{\Omega}_1$.

COROLLARY 3.6. — *There exists an operator $R_{\delta,j}$ from the subspace $W_{\delta,j,0}$ of $W^1_{\delta,j}$ made of functions vanishing at both endpoints of Γ_{Hj}, with values in $Y_{\delta,1}$, such that:*
(i) for any φ in $W_{\delta,j,0}$, the trace of $R_{\delta,j}\varphi$ on ∂K_j coincides with φ on Γ_{Hj} and $R_{\delta,j}\varphi$ vanishes on $\Omega_1 \setminus K_j$,
(ii) the following stability property holds for a constant c independent of δ:

$$\forall\varphi \in W_{\delta,j,0}, \quad |R_{\delta,j}\varphi|_{H^1(K_j)} \leq c\,|\varphi|_{H^{\frac{1}{2}}_{00}(\Gamma_{Hj})}. \qquad [16]$$

PROOF: Let F_j be the affine transformation which maps $\hat{K}$ onto K_j and $\hat{\Gamma}$ onto Γ_{Hj}. Using the operator $R_{\frac{h}{H}}$ of Proposition 3.5 with $\mathcal{O} = \hat{K}$, and denoting by $\overline{\varphi}$ the extension by zero of any function in $W_{\delta,j,0}$ to ∂K_j, we define the operator $R_{\delta,j}$ by the formula:

$$R_{\delta,j}\varphi = \begin{cases} \left(R_{\frac{h}{H}}(\overline{\varphi}\circ F_j)\right)\circ F_j^{-1} & \text{on } K_j, \\ 0 & \text{on } \Omega_1 \setminus K_j. \end{cases} \qquad [17]$$

Then, it is readily checked that property (i) is satisfied. Moreover, due to the nullity properties of φ, estimate [15] yields:

$$|R_{\frac{h}{H}}(\overline{\varphi}\circ F_j)|_{H^1(\hat{K})} \leq c\,|\varphi\circ F_j|_{H^{\frac{1}{2}}_{00}(\hat{\Gamma})},$$

whence [16] thanks to the properties of the mapping F_j.

PROPOSITION 3.7. — *For any function u in $H^1_0(\Omega)$ such that each $u_{|\Omega_i}$ belongs to $H^{s_i}(\Omega_i)$, $1 \leq s_i \leq m+1$, the following approximation estimate holds:*

$$|u - \Pi_\delta u|_{1*} \leq c\left(H^{s_0-1}\,\|u\|_{H^{s_0}(\Omega_0)} + h^{s_1-1}\,\|u\|_{H^{s_1}(\Omega_1)}\right). \qquad [18]$$

PROOF: We can assume that both s_i are ≥ 2 since the general result follows from the definition of Π_δ thanks to an interpolation argument. Then, we have, for any w_δ in X_δ,

$$|u - \Pi_\delta u_\delta|_{1*} \leq |u - w_\delta|_{1*}.$$

Then, if $\mathcal{I}_\delta^i$ stands for the standard Lagrange interpolation operator with values in Y_δ^i, the idea is to choose:

$$w_\delta = \begin{cases} \mathcal{I}_\delta^0 u & \text{on } \Omega_0, \\ \mathcal{I}_\delta^1 u + \sum_{j=1}^{J(H)} R_{\delta,j}\big((\mathcal{I}_\delta^0 u)_{|\Gamma_{Hj}} - (\mathcal{I}_\delta^1 u)_{|\Gamma_{Hj}}\big) & \text{on } \Omega_1. \end{cases} \quad [19]$$

This function is well-defined and continuous since the trace $\mathcal{I}_\delta^0 u - \mathcal{I}_\delta^1 u$ on Γ_{Hj} belongs to $W_{\delta,j,0}$. From Corollary 3.6, we derive that:

$$|u - \Pi_\delta u|_{1*} \leq \sum_{j=0}^{1} |u - \mathcal{I}_\delta^i u|_{H^1(\Omega_i)} + c \sum_{j=1}^{J(H)} |(\mathcal{I}_\delta^0 u)_{|\Gamma_{Hj}} - (\mathcal{I}_\delta^1 u)_{|\Gamma_{Hj}}|_{H_{00}^{\frac{1}{2}}(\Gamma_{Hj})}.$$

So the desired estimate follows from the estimate, for all function φ in $H^1(K_j)$ such that its trace on Γ_{Hj} belongs to $H_{00}^{\frac{1}{2}}(\Gamma_{Hj})$,

$$|\varphi|_{H_{00}^{\frac{1}{2}}(\Gamma_{Hj})} \leq c\, |\varphi|_{H^1(K_j)}$$

(which is proven by using the transformation F_j), by its analogue with K_j replaced by K_j', together with the usual properties of the operators $\mathcal{I}_\delta^i$.

3.4. *The final error estimate*

As a conclusion, inserting the results of Propositions 3.1, 3.4 and 3.7 into formula [11], we derive the error estimate.

THEOREM 3.8. — *If the solution u of problem [1] is such that each $u_{|\Omega_i}$ belongs to $H^{s_i}(\Omega_i)$, $\frac{3}{2} < s_i \leq m+1$, the following error estimate holds for problem [4]:*

$$\|u - u_\delta\|_{1*} \leq c\left(H^{s_0 - 1}\, \|u\|_{H^{s_0}(\Omega_0)} + h^{s_1 - 1}\, \|u\|_{H^{s_1}(\Omega_1)}\right). \quad [20]$$

Note that this estimate involves the regularity of the exact solution separately on each subdomain, which is interesting since it is always necessary to cut up the elements where the solution is less regular.

REMARK. — In the case $m = 1$ of piecewise affine functions, assuming for instance that the non convex corners of Ω are all in Ω_1 and that the largest angle is ω, we obtain:

$$\|u - u_\delta\|_{1*} \leq c(H + h^{\frac{\pi}{\omega}})\, \|f\|_{L^2(\Omega)}, \quad [21]$$

so that an appropriate choice of the refinement parameter k ensures that the discretization converges at least as $c\,H$.

REMARK. — Of course, this technique can be extended to a number of more general elliptic equations with more complicated boundary conditions. The numerical analysis is even easier for instance in the case of the Helmoltz equation $-\Delta u + u = f$ with Neumann boundary conditions, where the ellipticity property is obvious.

4. Concluding remarks

The error estimate [20] is optimal with the two choices $W_{\delta,j} = W^0_{\delta,j}$ and $W_{\delta,j} = W^1_{\delta,j}$, so that we need a further criterion to decide on the choice. It can be observed that, when $\tilde{W}_{\delta,j}$ coincides with $\tilde{W}^0_{\delta,j}$, the matching conditions on each edge Γ_{Hj} writes:

$$\forall \psi \in \tilde{W}^1_{\delta,j}, \quad \int_{\Gamma_{Hj}} (v_{\delta\,|\Omega_1} - v_{\delta\,|\Omega_0})(\tau)\psi(\tau)\,d\tau = 0.$$

Their number is $2^k\,m - 1$, and the number of degrees of freedom on $\Gamma_{H,j}$ is $2^k + 1$ such that the method is "nearly conforming" in the following sense: except for two degrees of freedom (or even less if for instance one of the two end points of Γ_{Hj} is on the boundary $\partial\Omega$), the piecewise polynomial trace $v_{\delta\,|\Omega_1}$ is obliged to degenerate into a polynomial. In the case $k = 2$, the refinement is lost!

As a consequence, the choice $W_{\delta,j} = W^1_{\delta,j}$ seems to be more efficient for the refinement, and it must be chosen whenever possible. However, if the choice $W_{\delta,j} = W^0_{\delta,j}$ has to be preferred for exterior reasons, we propose the following heuristic modification of the algorithm: the subdomain Ω_1 being defined as previously, the triangles of Ω_0 which has an edge in $\partial\Omega_1$ are also cut up and the new subdomain $\overline{\Omega}_1^*$ is the union of $\overline{\Omega}_1$ and of these further triangles. This of course is expensive!

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