An Asymptotic Numerical Method for non-linear transient dynamics

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ABSTRACT. *The main objective of this presentation is to show that a perburbation method can be very effective for solving a large class of transient non-linear dynamic problems. We describe an algorithm which has three part : apply a perturbation technique to transform a non-linear problem into a series of linear ones, use an FEM and a time stepping scheme to solve the linear problems, perform the summation of the series to get the solution. Usually, the perturbation series has a finite radius of convergence, and the algorithm has to be restarted several times to get the solution on the whole time interval. However, as compared to a classical conbination of time stepping and Newton-Raphson method, the present algorithm requires much less stiffness matrix evaluations and triangulations. The performances of the proposed algorithm will be demonstrated with an example.*

RÉSUMÉ. *L'objectif premier de ce papier est de montrer que les méthodes de perturbation peuvent être très efficaces pour résoudre des problèmes de dynamique transitoire en non-linéaire. Nous décrivons un algorithme dont le principe est : appliquer une méthode de perturbation pour transformer un problème non linéaire en une succession de problèmes linéaires, utiliser une méthode d'éléments finis et un shéma d'intégration temporelle pour résoudre ces problèmes de dynamique linéaire, procéder à la sommation des séries pour former les solutions. En général, les séries ont un rayon de convergence fini et la procédure doit être réappliquée successivement pour obtenir la solution sur tout l'intervalle de temps. Cependant, comparée à une méthode d'intégration temporelle implicite en non linéaire, cet algorithme réclame nettement moins d'évaluations et de décompositions de matrices de rigidité tangente. Les performances seront illustrées sur un exemple.*

KEYWORDS: *Perturbation methods , Non-linear Algorithm, Structural Dynamics, F.E.M., Series.*

MOTS-CLÉS : *Méthodes de perturbation, Algorithme Non-linéaire, Dynamique des structures, M.E.F., Séries.*

1. Introduction

The so-called perturbation methods are well-known techniques for decomposing a difficult problem (for instance non-linear) into an infinite series of simpler ones (linear). The key point of these methods is to assume a power series expansion of the unknown field $u = u_0+\varepsilon u_1+\varepsilon^2 u_2+\varepsilon^3 u_3+...$ with respect to a well chosen parameter ε that either appears naturally in the problem or that is artificially introduced. If the series are truncated at order 2 or 3, they provide only a qualitative solution, but if a large number of terms of the series are determinated, a quantitative solution can readily be obtained. The main objective of this paper is to show that a perturbation technique can be the basis of efficient numerical algorithms for transient non-linear elastodynamic problems.

Perturbation methods [NAY 73, NAY 89] have almost never been used as a mean to solve numerically a non linear problem in the field of Computational Mechanics. The first attempt to combine a series expansion and a finite element method for non-linear static structural problems is presented in [THO 68, WAL 69]. However, their method was not suited to getting a large number of terms of the series and could hardly compete with classical incremental-iterative schemes of the Newton type. This was also the conclusion of a review paper on perturbation methods for non-linear mechanics by Gallagher [GAL 75]. It seems that the first reliable algorithm has been proposed by Noor and co-workers between 1979 and 1985[NOO 80a, NOO 80b, NOO 85], the so-called "reduced basis technique". The idea is to use the first terms of a perturbation series as Ritz vectors in a classical Ritz reduction technique. Applications to static and dynamic structural mechanics have been proposed. However, the method does not take advantage of having an analytical solution, and it suffers from a costly evaluation of the reduced problem, as has been reported by Riks [RIK 84] and confirmed in [NAJ 98].

During the nineties, the use of a perturbation technique as a non-linear solver has been revisited in [DAM 90, AZR 93, COC 94c, COC 94b, COC 94a, ZAH 99, ELH 98]. These authors have shown that an expansion technique combined with a finite element method can be very effective to solve non-linear structural problems, even for large complex models with several non-linearities, geometrical, material, contact conditions. The key is to be able to compute a large number of terms of the series expansions in order to get an accurate analytical solution. Doing so, the performances of the so-called Asymptotic Numerical Methods (ANM) appear to be greater than those of classical incremental-iterative methods such as the well-known Newton Raphson algorithm. See also the works by Dhatt and Ammar [AMM 96] and Fafard et al [FAF 97] for linear dynamics.

In this paper, we describe an ANM algorithm for computing transient solutions in non-linear elasto-dynamics. The perturbation method is very similar to those used in the field of non-linear vibrations for finding periodic solutions. We refer to the books by Nayfeh and Mook [NAY 89] and Szemplinska-Stupnicka [SWE 90] for the details on the so-called straighforward expansion, Lindsteht-Poincare methods, aver-

aging techniques, and for an extensive bibliography on perturbation methods in dynamics.

2. A transient algorithm based on the asymptotic numerical method

2.1. *Elastodynamic governing equations*

We consider an elastic body Ω which is submitted to a prescribed displacement $\mathbf{u}_d$ on $\partial\Omega_u$, surface forces $\mathbf{F}$ on $\partial\Omega_F$ and body forces $\mathbf{f}$. We assume that this body undergoes large displacements but small deformations during the motion. Hence, we use the framework of geometrical non-linear elasticity with a linear elastic behaviour for the material. The motion equations and the constitutive law can be written as:

$$\left\{ \begin{array}{l} \int_{\Omega_0} \rho \ddot{\mathbf{u}} \delta\mathbf{u} dv + \int_{\Omega_0} \mathbf{S} : \delta\mathbf{E}(\mathbf{u}) dv - P_e(\delta\mathbf{u}) = 0 \\ \mathbf{S} = \mathbf{D} : (\mathbf{E}_l(\mathbf{u}) + \mathbf{E}_{nl}(\mathbf{u}, \mathbf{u})) \end{array} \right. \quad [1]$$

where $\mathbf{u}$ is the displacement, $\mathbf{E}$ the Green-Lagrange strain, $\mathbf{S}$ the second Piola-Kirchhoff stress, $\mathbf{D}$ the elasticity tensor, and $P_e(\delta\mathbf{u}) = \int \mathbf{F}\delta\mathbf{u} dS + \int \mathbf{f}\delta\mathbf{u} dv$ the virtual work performed by the given loads. The Green-Lagrange strain is decomposed into its linear part $\mathbf{E}_l(\mathbf{u})$ and its quadratic one $\mathbf{E}_{nl}(\mathbf{u}, \mathbf{u})$. Its variation is denoted by $\delta\mathbf{E}(\mathbf{u}) = \mathbf{E}_l(\delta\mathbf{u}) + 2\mathbf{E}_{nl}(\mathbf{u}, \delta\mathbf{u})$.

It is very important to notice that these two equations are only quadratic with respect to the variables $\mathbf{u}$ and $\mathbf{S}$. In order to apply a perturbation procedure, it is worth introducing the new unknown vector

$$U = \{\mathbf{u}, \mathbf{S}\}^t$$

and rewriting the equations as

$$M(\ddot{U}) + L(U) + Q(U, U) = F \quad [2]$$

where L and M are linear operators, and Q a quadratic one. We use the following formal notations:

$$M(\ddot{U}) = \left\{ \begin{array}{c} \int_{\Omega_0} \rho \ddot{\mathbf{u}} \delta\mathbf{u}\, dv \\ 0 \end{array} \right. \quad [3]$$

$$L(U) = \left\{ \begin{array}{c} \int_{\Omega_0} \mathbf{S} : \mathbf{E}_l(\delta\mathbf{u})\, dv \\ \mathbf{S} - \mathbf{D} : \mathbf{E}_l(\delta\mathbf{u}) \end{array} \right. \quad [4]$$

$$Q(U, U) = \left\{ \begin{array}{c} \int_{\Omega_0} \mathbf{S} : 2\mathbf{E}_{nl}(\mathbf{u}, \delta\mathbf{u})\, dv \\ -\mathbf{D} : \mathbf{E}_{nl}(\mathbf{u}, \mathbf{u}) \end{array} \right. \quad [5]$$

$$F = \left\{ \begin{array}{c} \int_{\Omega_0} \rho \mathbf{f}\delta\mathbf{u}\, dv + \int_{\partial\Omega_F} \mathbf{F}.\delta\mathbf{u}\, ds \\ 0 \end{array} \right. \quad [6]$$

With the initial conditions denoted by the subscript '0', the non-linear dynamic problem becomes:

Find $U = \{\mathbf{u}, \mathbf{S}\}^t$ such that:

$$\begin{cases} \mathbf{u} = \mathbf{u}_d \, on \, \partial\Omega_u \\ M(\ddot{U}) + L(U) + Q(U,U) = F \\ \mathbf{u}(t=0) = \mathbf{u}_0 \, , \dot{\mathbf{u}}(t=0) = \mathbf{v}_0 \end{cases} \quad [7]$$

2.2. *Perturbation of the equations*

In the previous system of equations, there is no small parameter that would be a self evident parameter for expanding the unknown vector U. However, a perturbation technique can still be applied provided that an artificial expansion parameter is introduced. For instance, in a non-linear problem, it is usual to put an ε parameter in front of the non-linear term in order to get a new problem which has the following properties: for $\varepsilon = 0$, it is linear and for $\varepsilon = 1$ the original problem is retrieved. By applying a perturbation technique to this new problem, ie, by expanding the unknown in a power series with respect to ε, we are led to solve a series of linear problems that are generally simpler than the original one. If the radius of convergence of the series is greater than one, the required solution is obtained by summing the series for $\varepsilon = 1$. Otherwise, a special procedure for summing divergent series should be used, or a continuation with respect to ε has to be performed.

In the above problem, the direct introduction of an ε parameter in front of the non-linear term would not be efficient. First, it would lead to a series of linear dynamic problems where the usual small displacement stiffness matrix appears, whereas it would be preferable to have the tangent stiffness at $\mathbf{u}_0 = \mathbf{u}(t=0)$. Also, we would have no guarantees about the convergence of the series. Hence, it is more advisable to first make the following change of variable :

$$U(t) = U_0 + \hat{U}(t) \quad [8]$$

in order to separate the linear and non-linear terms with respect to $\hat{U}$, which represents the increment of displacement and stress from the initial state at $t = 0$. In equation [8], $U_0 = \{\mathbf{u}_0, \mathbf{D} : (\mathbf{E}_l(\mathbf{u}_0) + \mathbf{E}_{nl}(\mathbf{u}_0, \mathbf{u}_0))\}^t$. In the following, we forget the hat on U to simplify the notations. The problem now becomes:

$$\begin{cases} \mathbf{u} = \mathbf{u}_d \, on \, \partial\Omega_u \\ M(\ddot{U}) + L_t(U) + 2Q(U,U) = F' \\ \mathbf{u}(t=0) = 0 \, , \dot{\mathbf{u}}(t=0) = \mathbf{v}_0 \end{cases} \quad [9]$$

with

$$L_t(.) = L(.) + 2Q(., U_0) \quad [10]$$

$$F' = F - L(U_0) - 2Q(U_0, U_0) \quad [11]$$

The new tangent operator $L_t(.)$ is linear with respect to U, and F' is a new load vector. We now introduce the artificial perturbation in front of the non-linear term $Q(U,U)$ to get:

$$\left\{\begin{array}{l} \mathbf{u} = \mathbf{u}_d \, on \, \partial\Omega_u \\ M(\ddot{U}) + L_t(U) + 2\varepsilon Q(U,U) = F' \\ \mathbf{u}(t=0) = 0\,, \dot{\mathbf{u}}(t=0) = \mathbf{v}_0 \end{array}\right. \quad [12]$$

We assume that the solution can be expanded into a power series with respect to the perturbation parameter ε.

$$U(t) = \sum_{i=1}^{\infty} U_i(t).\varepsilon^{i-1} = U_1 + \varepsilon U_2 + \varepsilon^2 U_3 + ... \quad [13]$$

By reporting the series into the problem and equating like powers of ε, we get a set of *linear* dynamic problems of the form:

$$order\ 1\ (\varepsilon^0) \left\{\begin{array}{l} \mathbf{u}_1 = \mathbf{u}_d \, on \, \partial\Omega_u \\ M(\ddot{U}_1) + L_t(U_1) = F' \\ \mathbf{u}_1(t=0) = 0\,, \dot{\mathbf{u}}_1(t=0) = \mathbf{v}_0 \end{array}\right. \quad [14]$$

$$order\ 2\ (\varepsilon^1) \left\{\begin{array}{l} \mathbf{u}_2 = 0 \, on \, \partial\Omega_u \\ M(\ddot{U}_2) + L_t(U_2) = -Q(U_1,U_1) \\ \mathbf{u}_2(t=0) = 0\,, \dot{\mathbf{u}}_2(t=0) = 0 \end{array}\right. \quad [15]$$

$$order\ p\ (\varepsilon^{p-1}) \left\{\begin{array}{l} \mathbf{u}_p = 0 \, on \, \partial\Omega_u \\ M(\ddot{U}_p) + L_t(U_p) = -\sum_{r=1}^{p-1} Q(U_r, U_{p-r}) \\ \mathbf{u}_p(t=0) = 0\,, \dot{\mathbf{u}}_p(t=0) = 0 \end{array}\right. \quad [16]$$

2.3. *Finite element solutions of the linear problems*

In order to solve the problem [16], it is convenient to leave the formal notation $M(.), L(.), Q(.,.)$ and to come back to an explicit form of each equation. We have:

$$\int_{\Omega_0} \rho\ddot{\mathbf{u}}\delta\mathbf{u}\, dv + \int_{\Omega_0} \mathbf{E}_l(\delta\mathbf{u}) + 2\mathbf{E}_{nl}(\mathbf{u}_0, \delta\mathbf{u})) : \mathbf{D} : (\mathbf{E}_l(\mathbf{u}_p) + 2\mathbf{E}_{nl}(\mathbf{u}_p, \delta\mathbf{u}))\, dv$$

$$+2\int_{\Omega_0} (\mathbf{S}_0 : \mathbf{E}_{nl}(\mathbf{u}_p, \delta\mathbf{u}))\, dv =$$

$$-\int_{\Omega_0} \{\sum_{r=1}^{p-1} (\mathbf{S}_r : 2\mathbf{E}_{nl}(\mathbf{u}_{p-r}, \delta\mathbf{u}) + \mathbf{S}_p^{nl} : (\mathbf{E}_l(\delta\mathbf{u}) + 2\mathbf{E}_{nl}(\mathbf{u}_0, \delta\mathbf{u}))\}\, dv \quad [17]$$

for the equation of motion and

$$\mathbf{S}_p = \mathbf{D} : (\mathbf{E}_l(\mathbf{u}_p) + 2\mathbf{E}_{nl}(\mathbf{u}_0, \mathbf{u}_p)) + \mathbf{S}_p^{nl} \quad [18]$$

for the constitutive law, with

$$\mathbf{S}_p^{nl} = \mathbf{D} : \sum_{i=1}^{p-1} \mathbf{E}_{nl}(\mathbf{u}_r, \mathbf{u}_{p-r}) \qquad [19]$$

Using a classical FEM, the discretized problems are:

$$Order\ 1 \begin{cases} [M]\{\ddot{u}_1\} + [K_t]\{u_1\} = \{F\} \\ \{S_1\} = [D][B]\{u_1\} \\ \{u_1\} = 0, \, , \, \{\dot{u}_1\} = \{\dot{u}_0\} \end{cases} \qquad [20]$$

$$Order\ 2 \begin{cases} [M]\{\ddot{u}_2\} + [K_t]\{u_2\} = -\{F_2^{nl}\} \\ \{S_2\} = [D][B]\{u_2\} + \{S_2^{nl}\} \\ \{u_2\} = 0\,, \{\dot{u}_2\} = 0 \end{cases} \qquad [21]$$

$$Order\ p \begin{cases} [M]\{\ddot{u}_p\} + [K_t]\{u_p\} = -\{F_p^{nl}\} \\ \{S_p\} = [D][B]\{u_p\} + \{S_p^{nl}\} \\ \{u_p\} = 0\,, \{\dot{u}_p\} = 0 \end{cases} \qquad [22]$$

The first equation in [22] contains the classical mass matrix, the tangent stiffness matrix evaluated at the starting point U_0 , and a r-h-s force vector which only depends on the previous order solutions. The constitutive law (second equation in [22]) includes a pre-stress term which also depends only on the previous order solutions.

2.4. *An introductory example*

A duffing oscillator is simulated by using two quadrilateral plane stress elements (figure 1). At time $t = 0$, the two nodes 3 and 4 are subjected to a constant force in the x direction.

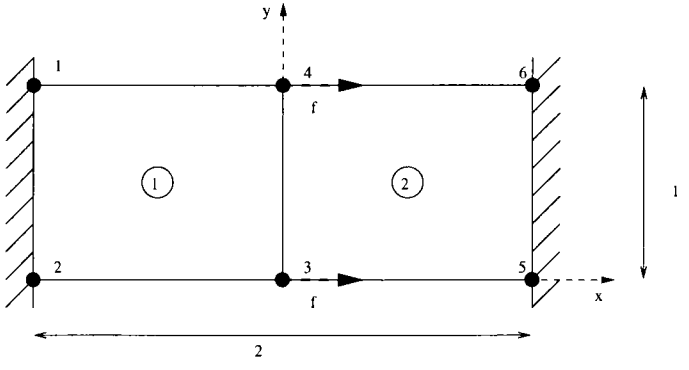

Figure 1. *Finite element duffing oscillator*

The equation of motion of this simple one d.o.f. example is:

$$\begin{cases} \ddot{u} + 3u + \varepsilon \frac{3}{2} u^3 = f \\ t = 0\,, u = 0 \, and \, \dot{u} = 0 \end{cases} \qquad [23]$$

For $f = 0.1$, the problem is poorly non-linear. The solutions of the linear problems at order 1, 2 and 3 are reported in figures 2 and 3, showing the appearance of well-known secular terms [NAY 89]. By summing ten terms of the series [13], we get an accurate solution up to t=15 s.

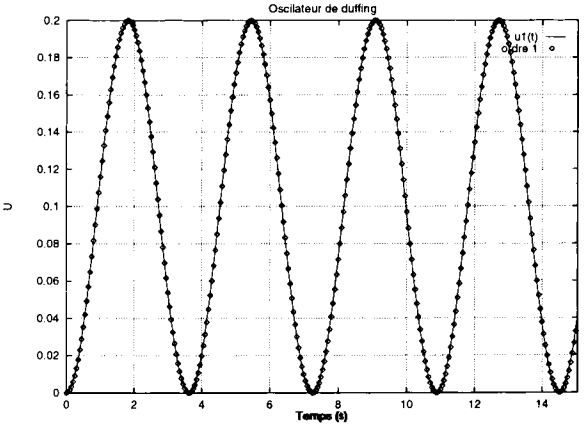

Figure 2. *Order 1. Solution of [20]*

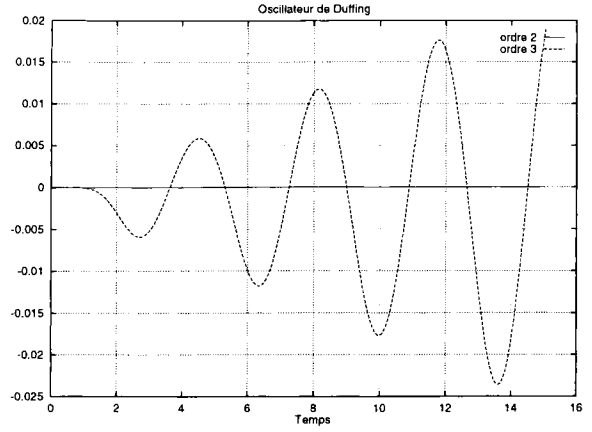

Figure 3. *Orders 2 and 3*

For $f = 0.4$, the problem becomes more non-linear. The asymptotic solutions at orders 10 and 15 are compared to the exact solution in figure 4. We can see that the

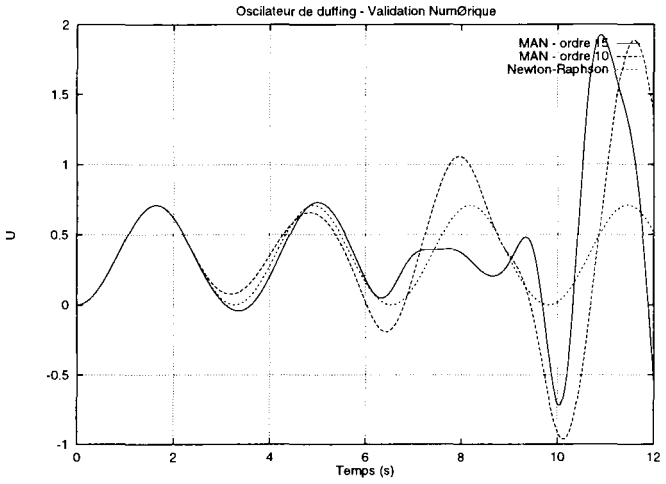

Figure 4. *Asymptotic solution at orders 10 and 15 compared to the exact solution*

asymptotic solution ($\varepsilon = 1$) converges to the exact one only if time t is lower than about three seconds, which is, here, the practical range of validity of the perturbations series.

It is a well-known feature that the convergence is not uniform with respect to time [NAY 89] and that the range of validity of the asymptotic solution hardly depends on the value of ε: it is infinite for $\varepsilon = 0$ and it decreases monotonically when ε increases.

2.5. *Time resolution*

All the linear dynamic problems [20],[21],[22] are solved using a time stepping algorithm. For simplicity, we choose here the classical average acceleration Newmark algorithm. It is well known that, with a constant step, only one effective stiffness matrix has to be evaluated and triangulated to solve a linear dynamic problem. Here, all the linear dynamic systems have the same mass and stiffness matrix, and lead to the same effective stiffness matrix. Hence, the computation of the series (13) requires only one stiffness matrix triangulation and several forward-backward substitutions.

There are two differents ways of solving the linear systems [20],[21],[22]. One can first solve [20] on a given time interval, then solve [21] for the same time interval, and so on until order p. On the other hand, one can perform a single time step successively for [20],[21] and [22] up to a given order p, and go to the next time increment. This second way is more convenient since it does not require to give a time interval and it permits to efficiently control the convergence as explained below.

2.6. *Evaluation of the range of validity in time and restart of the algorithm*

In practice, we need to know the range of times for which the perturbation series converge. This can be easily achieved by controlling the radius of convergence in ε after each time increment. This radius of convergence can be evaluated using classical estimates [COC 94a] or by controlling the growth of the norm of the residual vector of the asymptotic solution. When the radius of convergence is almost reached, the time stepping is stopped and the algorithm is restarted at the very beginning, taking the last point of solution as a new initial condition. This restart involves a new evaluation and triangulation of the effective stiffness matrix.

3. Example and discussion

An elastic slender structure (beam like) is meshed with twelve exactely integrated Q8 elements for 2D plane stress elasticity (figure 5). It is submitted to a compressive axial force which is 1.5 times higher than its buckling load. Hence, the motion of the beam consists in large oscillations between its initial straight position and a stable bent position. A small disymmetric force is added to force the motion always to the same side.

The first five seconds of motion are computed using a time step of 0.02 second. The results obtained with the proposed ANM algorithm are reported in figure 8 and 9, the ones with a classical combination of Newmark and Newton-Raphson method in figures 10, 11 and 12. In both algorithms, the tolerance for the residual vector is set to 10^{-3} N (the applied force is about 750 N).

The different parameters are:

– mechanical parameters: $E = 100000Mpa$, $\rho = 10^{-4}Kg/m^3$, $\nu = 0.3$

– buckling load: $F_b = \frac{\pi^2 E(\frac{h^3}{12})}{4L^2} = 514N$

– periods of free vibration: $T_1 = 0.783s$, $T_2 = 0.125s$, $T_3 = 0.045s$, $T_4 = 0.023s$, $T_5 = 0.014s$

– time step: $\Delta T = 0.02s$

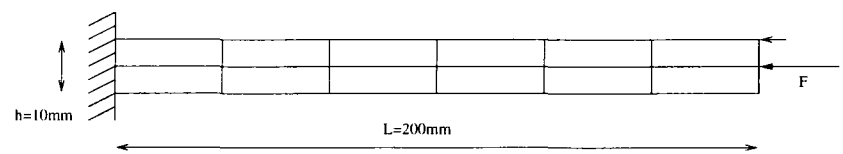

Figure 5. *Undeformed beam*

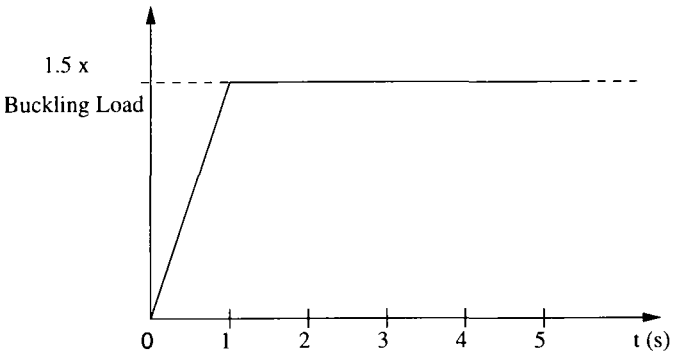

Figure 6. *Compressive axial force*

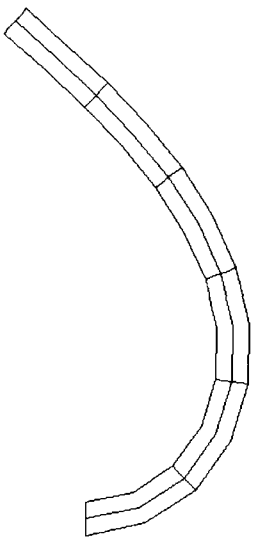

Figure 7. *Deformed beam - Magnification=1*

We can see in figure 9 that the norm of the residual vector of the asymptotic solution is between 10^{-11} and 10^{-8} at the beginning of the computation. This means that the convergence is very good and a very accurate solution is obtained. Around $t = 0.8s$, a sudden growth of the residual vector occurs, indicating that the radius of convergence of the series is to be reached. The program stops the time stepping just before the residual tolerance is reached, and restarts the algorithm at $t = 0.94s$.

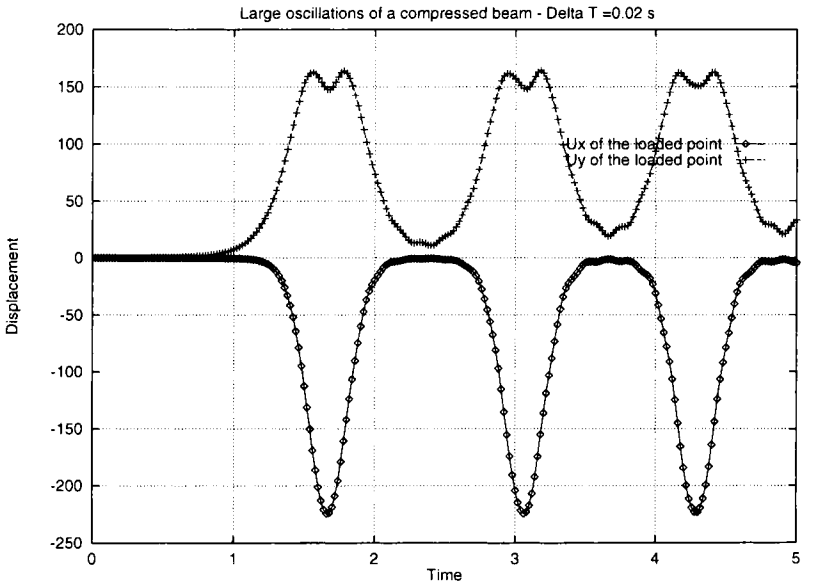

Figure 8. *Displacement of the loaded node - Asymptotic Numerical Method - Order 30*

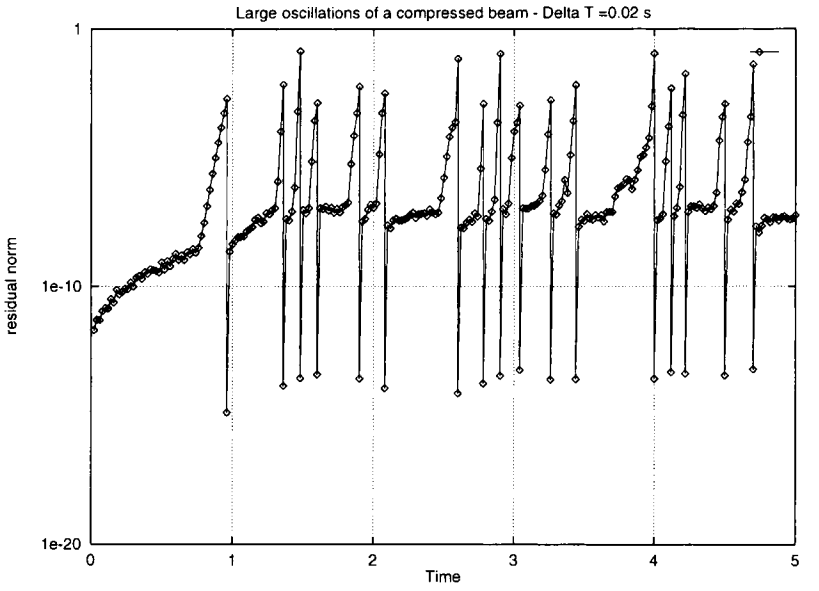

Figure 9. *Residual norm - Asymptotic Numerical Method - Order 30*

Finally, the computation of the solution up to five seconds requires 17 restarts, involving 18 effective stiffness matrix evaluations. On average, each effective stiffness matrix permits to advance the solution by 14 time steps. The following comments should be made :

– Using a smaller time step would result in exactly the same number of restarts. Morover, using another way of solving the linear problems, the modal method for instance, would also lead to the same number. The reason is that the radius of convergence of the series does not depend on the way the linear systems are solved.

– Truncating the series at another order would give exactly the same solution. The order of truncature only influences the number of restarts and the CPU time. Indeed, with a lower order, the sudden growth of the residual occurs sooner, and the number of restarts increases. However, the increase of stiffness matrix to be evaluated and triangulated is compensated by the fact that less r-h-s vectors have to be built. The optimal order, with respect to CPU time, does depend on the problem size. Practically, all the orders between 10 and 30 are satisfactory.

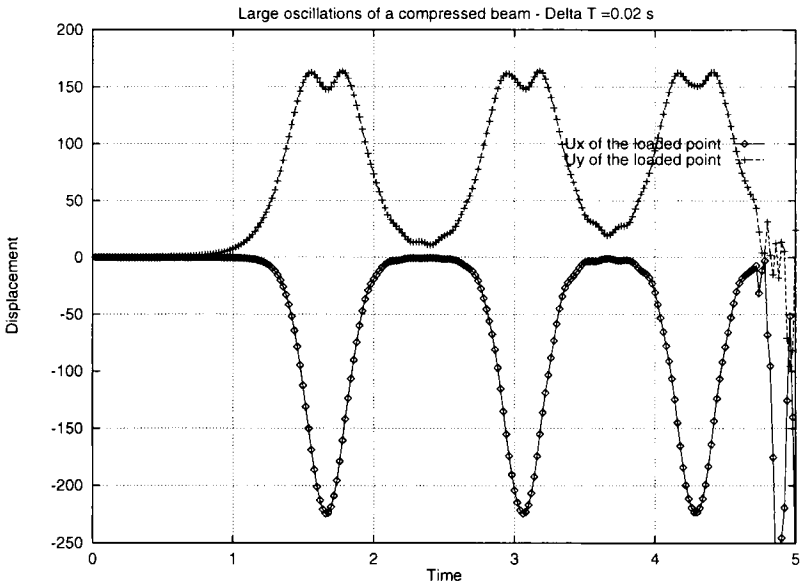

Figure 10. *Displacement of the loaded node - Newton Raphson*

Comparing figures 8 and 10, we can see that the two algorithms give, at least for the beginning, the same solution. Theoretically, the two methods should give rigorously the same solution. This holds because the same Newmark formulae are used and the same discrete residual vector is set to zero in both methods. However, the residual vector is set to zero only to a finite accuracy, which produces small differences between the solution at the beginning, and larger ones after a large number of steps.

A typical breakdown of the Newton-Rapshon algorithm occurs around $t = 4.8s$. This well-known unstable phenomenom has been reported by Simo and Criesfield, among others. It can be eliminated by using a suitable energy conserving algorithm [SIM 96, KHU 96, CRI 97]. The same breakdown also occurs with the ANM solution but later for $t = 8s$. This is probably due to the fact that, for the same tolerance residual requirement, the ANM solution has finally lower residual vectors.

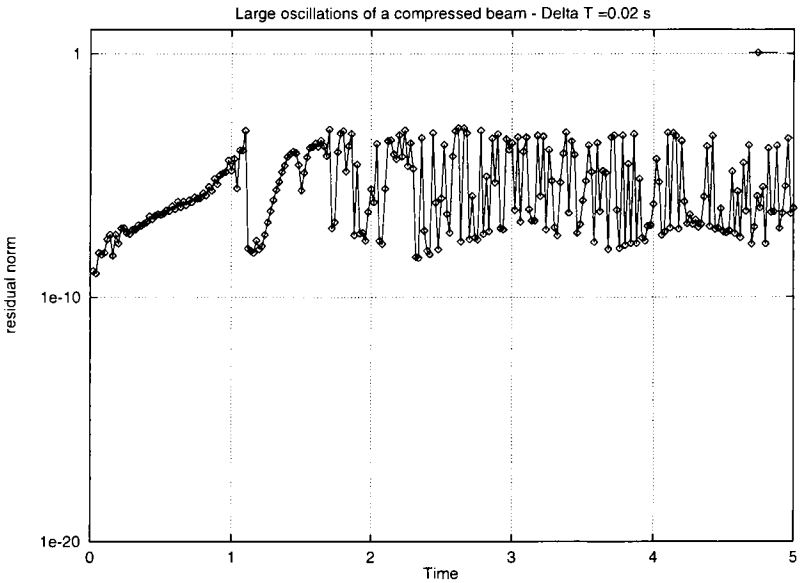

Figure 11. *Residual norm - Newton Raphson*

The number of iterations of the Newton process is presented in figure 12. We can see that about 900 iterations are used on the whole time interval, which involves the use of 900 stiffness matrix. This is in sharp contrast to the proposed algorithm that uses only 18. However, we must say immediately that the Newton process requires only 900 r-h-s vectors, whereas the ANM algorithm uses (250 steps * 31 r-h-s vectors per step) 7750 r-h-s vectors.

In summary, the ANM algorithm requires much less stiffness matrix evaluations and much more r-h-s vectors. Everything occurs as if the non-linear terms were eliminated from the left hand side of the equations and put on the right hand side. Significant CPU reductions can be expected for large finite element models, where the CPU time for building a r-h-s vector is very small as compared to the time to assemble and triangulate a stiffness matrix.

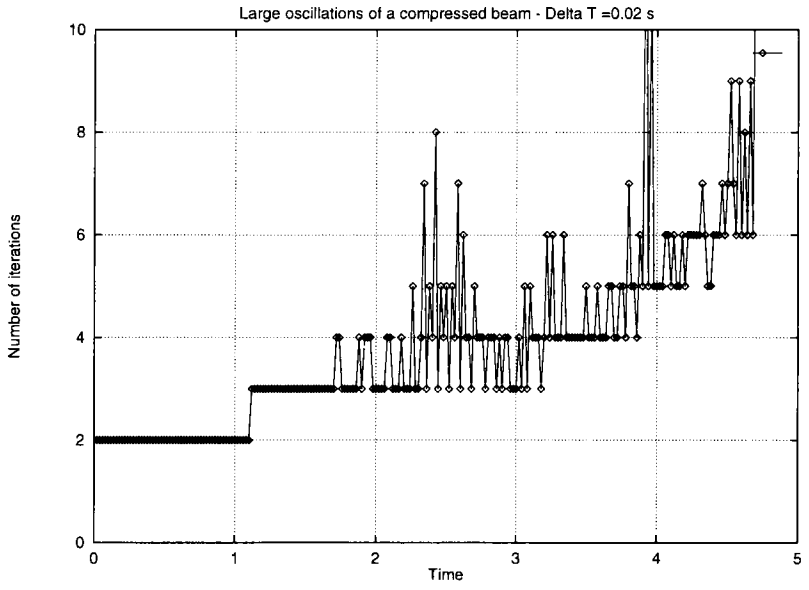

Figure 12. *Number of iterations - Newton Raphson*

4. Conclusions

In this paper we describe an asymptotic numerical algorithm to compute transient responses in non-linear elastodynamics. The key point of the algorithm is to apply a perturbation method in order to replace the non-linear problem by a collection of linear ones, that are solved using classical numerical tools. The performance is tested on an academic example, showing that, as compared to the classical combination of Newton-Raphson and Newmark method, the number of stiffness matrix to be used is dramatically decreased.

Various improvements and extensions are now scheduled:

– The Newmark algorithm should be replaced by energy conserving algorithms [SIM 96, KHU 96, CRI 97] in order to avoid instability of the time stepping;

– Since the stiffness matrix can be kept for several steps, the opportunity of using a modal method (discoupled system) for solving the linear problems should be investigated;

– The improvement of the series by using Padé approximants has to be tested in depth. It will permit to reduce again the number of stiffness matrix;

– The perturbation method used is a kind of straightforward expansion. Other perturbation techniques, which account for the frequency-amplitude dependance, should be tested in order to enlarge the radius of convergence of the series [NAY 89, SWE 90];

– In this paper, we only have taken into account of geometrical non-linearities. The inclusion of physical non-linearities and contact conditions, as done in [BRA 97, POT 97, ZAH 99, ELH 98], is now scheduled.

With all these improvements in hand, we will be able to test the performance of ANM algorithms for solving complex industrial problems.

5. References

[AMM 96] AMMAR S., *Methode asymptotique perturbee appliquee a la resolution de problemes non lineaires en grandes rotations et deplacements*, PhD Thesis, Laval University, QUEBEC, 1996.

[AZR 93] AZRAR L., COCHELIN B., DAMIL N., POTIER-FERRY M., "An asymptotic numerical method to compute the post-buckling behavior of elastic plates and shells", *Int. J. Numer. Meth. Eng.*, vol. 36, 1993, p. 1251–1277.

[BRA 97] BRAIKAT B., DAMIL N., POTIER-FERRY M., "Methodes asymptotiques numeriques pour la plasticite", *Revue Europeenne des Elements Finis*, vol. 6, 1997, p. 337-357.

[COC 94a] COCHELIN B., "A path following technique via an asymptotic numerical method", *Computers & Structures*, vol. 29, 1994, p. 1181-1192.

[COC 94b] COCHELIN B., DAMIL N., POTIER-FERRY M., "The Asymptotic-Numerical-Method: an efficient perturbation technique for nonlinear structural mechanics", *Revue Europeenne des Elements Finis*, vol. 3, 1994, p. 281-297.

[COC 94c] COCHELIN B., DAMIL N., POTIER-FERRY M., "Asymptotic-Numerical Methods and Pade approximants for nonlinear elastic structures", *Int. J. Numer. Meth. Eng.*, vol. 37, 1994, p. 1181-1192.

[CRI 97] CRISFIELD M. A., *Nonlinear Finite Element Analysis of Solids ans Structures, Vol.2*, John Wiley and Sons, 1st edition, 1997.

[DAM 90] DAMIL N., POTIER-FERRY M., "A new method to compute perturbed bifurcations: application to the buckling of imperfect elastic structures", *International Journal of Engineering Sciences - N9*, vol. 28, 1990, p. 943–957.

[ELH 98] ELHAGE-HUSSEIN A., DAMIL N., POTIER-FERRY M., "An asymptotic numerical algorithm for frictionless contact problems", *Revue Europeenne des Elements Finis*, vol. 7, 1998, p. 119-130.

[FAF 97] FAFARD M., AMMAR S., HENCHI K., GENDRON G., "Application of an asymptotic method to transient dynamic problems", *Journal of Sound and Vibration*, vol. 208, 1997, p. 73-99.

[GAL 75] GALLAGHER R., "Perturbation procedures in nonlinear finite element structural analysis", *Computational Mechanics - Lecture Notes in Mathematics*, vol. 461, 1975, p. 75–89.

[KHU 96] KHUL D., RAMM E., "Constraint Energy Momentum Algorithm and its application to nonlinear dynamics of shells", *Comput. Methods Appl. Mech. Engrg.*, vol. 136, 1996, p. 293-315.

[NAJ 98] NAJAH A., COCHELIN B.AND DAMIL N., POTIER-FERRY M., "A critical review of Asymptotic Numerical Methods", *Archives of Computational Methods in Engineering*, vol. 5, 1998, p. 31-50.

[NAY 73] NAYFEH A., *Perturbation methods*, John Wiley and Sons - New York, 1973.

[NAY 89] NAYFEH A., MOOK T., *Nonlinear Oscillations*, Whiley Classic Library, 2nd edition, 1989.

[NOO 80a] NOOR A., PETERS J., "Reduced basis technique for nonlinear analysis of structures", *AIAA Journal - N4*, vol. 18, 1980, p. 455–462.

[NOO 80b] NOOR A., PETERS J., "Tracing Post-Limit-Point Paths with reduced Basis Technique", *Comput. Methods Appl. Mech. Engrg.*, vol. 28, 1980, p. 217–240.

[NOO 85] NOOR A., "Hybrid analytical technique for nonlinear analysis of structures", *AIAA Journal - N6*, vol. 23, 1985, p. 938–946.

[POT 97] POTIER-FERRY M., DAMIL N., BRAIKAT B., DESCAMPS J., CADOU J., CAO H., ELHAGEHUSSEIN A., "Traitement des fortes non-linearite par la methode asymptotique numerique", *C. R. Acad. Sci. Paris*, vol. t 314, 1997, p. 171-177, Serie II b.

[RIK 84] RIKS E., "Some Computational aspects of stability of nonlinear structures", *Comput. Methods Appl. Mech. Engrg.*, vol. 47, 1984, p. 219–259.

[SIM 96] SIMO J., TARNOW N., "The discrete Energy-Momentum method. Conserving algorithms for nonlinear elastodynamics", *Int. J. Numer. Meth. Eng.*, vol. 136, 1996, p. 293-315.

[SWE 90] SWEMPLINSKA-STUPNICKA W., *The Behavior of Nonlinear Vibrating Systems - Vol. 1 and 2*, Kluwer Academic Publisher, 1st edition, 1990.

[THO 68] THOMPSON J., WALKER A., "The nonlinear perturbation analysis of discrete structural systems", *Int. J. Solids Structures*, vol. 4, 1968, p. 757–758.

[WAL 69] WALKER A., "A nonlinear finite element analysis of shallow circular arches", *Int. J. Solids Structures*, vol. 5, 1969, p. 97–107.

[ZAH 99] ZAHROUNI H., COCHELIN B., POTIER-FERRY M., "Asymptotic-numerical methods for shells with finite rotations", *Comput. Methods Appl. Mech. Engrg.*, vol. 175, 1999, p. 71-85.