Adaptative finite element analysis for strongly heterogeneous elasticity problems

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ABSTRACT. *We present a new a posteriori error estimate for strongly heterogeneous elasticity problems. This new approach is based on a simple modification of the well known residual estimate, but with the nice property that it is correctly dimensionalised with respect to the physical data.*

RÉSUMÉ. *Dans ce travail on présente un nouvel estimateur d'erreur a posteriori pour des problèmes d'élasticité avec coefficients élastiques fortement hétérogènes. La nouvelle approche, qui est une variation de l'estimateur par résidu, a la propriété d'être correctement dimensionné par rapport aux données physiques.*

KEY WORDS : *a posteriori error estimate, Poisson's equation, linear elasticity, residuals, heterogeneity.*
MOTS-CLÉS : *estimateur d'erreur a posteriori, équation de Poisson, problèmes elliptiques fortement hétérogènes, élasticité, résidus.*

1. Introduction

Recent accidents have clearly demonstrated that reliable *a posteriori* error estimates and mesh adaption techniques were imperatively needed when computing large scale structures. From the theoretical point of view, this problem can be solved either by using consistent residual estimates (see [BAB 78]) or by solving local auxiliary equilibrium problems (see [BAB 93], [ZIE 87], [LAD 91]) at the element level.

The literature on *a posteriori* error estimation for finite element is very vast, for example see the excellent and recent survey work [AIN 97] and the extensive bibliography cited therein.

The purpose of this paper is to describe and study a new version of a local *a posteriori* error estimates. This estimate uses a weighted measure of element and interface residuals, and can be proved to be correctly dimensionalized with respect to the physical data, and to be uniformly valid with respect to material heterogeneities. For simplicity, the technique is introduced and analyzed for a simple Poisson type equation discretized by triangular or tetrahedric finite element grids and then extended and tested numerically to elasticity problems.

This work is organized as follows. In section 2 we present the model problem, its finite element discretization and some technical results. In section 3 we present the error estimate and prove its robustness. In section 4 and 5 we generalize our approach to the elasticity problem and present some numerical examples. Finally in section 6 we outline some conclusions and open problems.

2. Model problem and notation

2.1. *The continuous problem*

Let Ω be a bounded domain of $\mathbb{R}^n$ (n= 2 or 3), with Lipschitz continuous boundary $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$. Let $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\Gamma_N)$ be given data. We then consider the following model problem

$$[P] \qquad \left\{ \begin{aligned} -\text{div}(\kappa \nabla u) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ \kappa \frac{\partial u}{\partial n} &= g && \text{on } \Gamma_N. \end{aligned} \right.$$

Here the scalar coefficient κ is supposed to be piecewise constant. In the simplest case this means that the domain Ω is split into two subdomains Ω_1 and Ω_2 with interface Γ_{12} (see Figure 1) and that we have

$$\kappa = \left\{ \begin{array}{ll} \kappa_1 & \text{in } \Omega_1, \\ \kappa_2 & \text{in } \Omega_2, \end{array} \right.$$

with $\kappa_1, \kappa_2 > 0$.

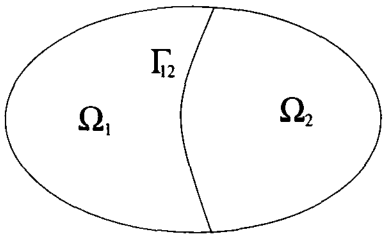

Figure 1. *The domain* Ω

The standard weak formulation of the problem [P] (see [CIA 78]) is then: *Find* $u \in \mathrm{H}$ *such that*

$$a(u,v) = < F, v > \quad , \forall v \in \mathrm{H}, \tag{1}$$

where

$$\begin{aligned} \mathrm{H} &= \{v \in \mathrm{H}^1(\Omega) \,/\, v = 0 \text{ on } \Gamma_D\}, \\ a(u,v) &= \int_{\Omega_1} \kappa_1 \nabla u \cdot \nabla v + \int_{\Omega_2} \kappa_2 \nabla u \cdot \nabla v, \\ < F, v > &= \int_\Omega f v + \int_{\Gamma_N} g v \,. \end{aligned}$$

This space is endowed with the natural *energy norm*

$$||v||_\Omega = \sqrt{a(v,v)}\,. \tag{2}$$

2.2. *Finite element discretization*

Let h be a positive discretization parameter, and consider a triangulation $\mathcal{T}_h$ of $\bar{\Omega}$, that is a partition of $\bar{\Omega}$ into non degenerate triangles T (resp. tetrahedra in dimension 3), with diameter bounded by h, and such that each pair of elements T_1 and T_2 of $\mathcal{T}_h$ are either disjoint or share a vertex, an edge or a complete face. We denote by h_T the diameter of T, by ρ_T the diameter of the circle (resp. sphere) inscribed in T and we set

$$\sigma_T = \frac{h_T}{\rho_T}\,.$$

We assume that the family of triangulations $(\mathcal{T}_h)_h$ is *shape regular*, i.e., there exists a constant σ, independent of h, such that

$$\sigma_T \le \sigma \qquad , \forall T \in \mathcal{T}_h\,. \tag{3}$$

On each element T we then introduce a local finite element space $\mathbb{P}_k(T)$ of polynomial functions defined on the element T and with degree less than or equal to k. With this notation, we define the finite element space H_h by

$$\mathrm{H}_h = \{v_h \in \mathcal{C}(\Omega) \,/\, v_h = 0 \text{ on } \Gamma_D \,,\, v_h|_T \in \mathbb{P}_k(T) \,, \forall T \in \mathcal{T}_h \,\} \quad (k \geq 1) \,.$$

Then the approximate problem of [1] is: *Find* $u_h \in \mathrm{H}_h$ *such that*

$$a(u_h, v_h) =< F, v_h > \quad , \forall v_h \in \mathrm{H}_h \,. \qquad [4]$$

In what follows we use the following notation

$$\begin{aligned} a &\preceq b \Longleftrightarrow a \leq Cb \\ a &\simeq b \Longleftrightarrow a \preceq b \text{ and } b \preceq a \,, \end{aligned}$$

where the constant C is independent of h and κ.

2.3. *Edges and vertices*

For any $T \in \mathcal{T}_h$ we denote by $\mathcal{E}(T)$ and $\mathcal{N}(T)$ the set of its edges (faces) and vertices, respectively, and set [VER 96]

$$\mathcal{E}_{h,\Omega} = \bigcup_{T \in \mathcal{T}_h} \mathcal{E}(T) \,.$$

We split $\mathcal{E}_{h,\Omega}$ into

$$\mathcal{E}_{h,\Omega} = (\mathcal{E}_h \backslash \mathcal{E}_{12}) \cup \mathcal{E}_{12} \cup \mathcal{E}_N \cup \mathcal{E}_D \,,$$

with

$$\begin{aligned} \mathcal{E}_N &= \{E \in \mathcal{E}_{h,\Omega} \,/\, E \subset \Gamma_N\} \quad , \quad \mathcal{E}_D = \{E \in \mathcal{E}_{h,\Omega} \,/\, E \subset \Gamma_D\} \,, \\ \mathcal{E}_{12} &= \{E \in \mathcal{E}_{h,\Omega} \,/\, E \subset \Gamma_{12}\} \,. \end{aligned}$$

Given an $E \in \mathcal{E}_{h,\Omega}$ we denote by $\mathcal{N}(E)$ the set of its vertices. For $T \in \mathcal{T}_h$ and $E \in \mathcal{E}_{h,\Omega}$ we define their neighborhoods (see Figure 2)

$$\begin{aligned} w_T &= \bigcup_{\mathcal{E}(T) \cap \mathcal{E}(T') \neq \emptyset} T' \,,\; w_E = \bigcup_{E \in \mathcal{E}(T')} T' \,, \\ \tilde{w}_T &= \bigcup_{\mathcal{N}(T) \cap \mathcal{N}(T') \neq \emptyset} T' \,,\; \tilde{w}_E = \bigcup_{\mathcal{N}(E) \cap \mathcal{N}(T') \neq \emptyset} T' \,. \end{aligned}$$

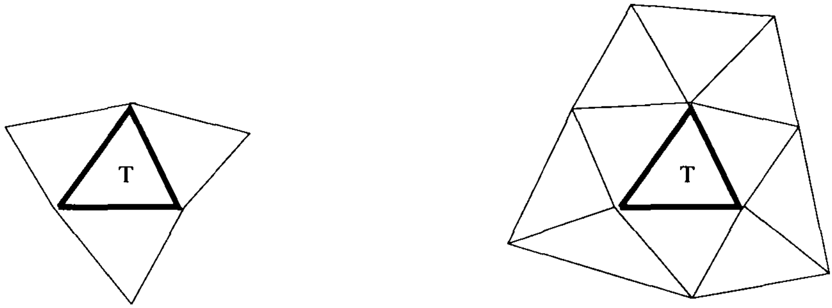

Figure 2. w_T *and* $\tilde{w}_T$*, respectively*

Remark. Condition [3] implies that $h_T/h_E, T \in \mathcal{T}_h, E \in \mathcal{E}(T)$, and $h_T/h_{T'}$, $T, T' \in \mathcal{T}_h, \mathcal{N}(T) \cap \mathcal{N}(T') \neq \emptyset$, are bounded from below and from above by constants which only depend on σ.

2.4. *Bubble functions*

For each element $T \in \mathcal{T}_h$ we can define the *element bubble* function b_T by

$$b_T = (n+1)^{n+1} \prod_{i=1}^{n+1} \lambda_{T,i} \, .$$

Above $\lambda_{T,j}(M)$ denote the j *barycentric coordinates* of the point M in T. Similarly, to each edge (face) $E \in \mathcal{E}_{h,\Omega}$, we can define the *edge (face) bubble* function

$$b_E = n^n \prod_{i=1}^{n} \lambda_{T,i} \, ,$$

with $T \in \omega_E$.

The above definition of b_E assumes that, for example if $n = 2$, in each triangle of ω_E, the edge E is associated to the vertices with local numbers 1 and 2.

By construction, we have the following properties of the bubble functions b_T and b_E.

Lemma 1 *Let* $T \in \mathcal{T}_h$ *and* $E \in \mathcal{E}_{h,\Omega}$ *be arbitrary, then*

$$\operatorname{supp} b_T \subset T, \quad 0 \leq b_T \leq 1, \quad \max_{x \in T} b_T(x) = 1, \qquad [5]$$

$$\operatorname{supp} b_E \subset w_E, \quad 0 \leq b_E \leq 1, \quad \max_{x \in E} b_E(x) = 1 \, . \qquad [6]$$

Moreover, using standard discrete norm equivalence arguments, we can prove (see [ARA 97])

Lemma 2 *The following estimate holds for any local function $f \in \mathbb{P}_{k-2}(T)$ and $g \in \mathbb{P}_{k-1}(E)$*

$$\begin{aligned}
\int_T b_T f^2 &\simeq \|f\|_{0,2,T}^2 , \\
\int_E b_E g^2 &\simeq \|g\|_{0,2,E}^2 , \\
\int_T |\nabla(b_T f)|^2 &\preceq h_T^{-2} \int_T f^2 , \\
\int_{T\in w_E} |\nabla(b_E g)|^2 &\preceq h_E^{-1} \int_E g^2 , \\
\int_{T\in w_E} |b_E g|^2 &\preceq h_E \int_E g^2 .
\end{aligned}$$

Like a non trivial extension of a local projection (see [ARA 97], [BER 95], [CLÉ 75], [NEP 97], [SCO 90]) we obtain

Lemma 3 *There exists a projection operator $R_h : \mathrm{H} \to \mathrm{H}_h$ such that for all $v \in \mathrm{H}$*

$$\begin{aligned}
\|v - R_h v\|_{0,2,T} &\preceq h_T |v|_{1,2,\tilde{\omega}_T \cap \Omega_i} \\
\|v - R_h v\|_{0,2,E} &\preceq h_E^{1/2} |v|_{1,2,\tilde{\omega}_E \cap \Omega_i} \quad , \forall i = 1,2
\end{aligned} \qquad [7]$$

where $T \in \mathcal{T}_h$, $E \in \mathcal{E}_N \cup \mathcal{E}_h$.

3. A posteriori error estimates

3.1. *Construction of the estimate*

The purpose of this section is to propose a local explicit evaluation of the error between the exact solution u of our original problem [P] and the approximate solution u_h of the finite element problem [4]. This error estimate should be easy to compute, should only involve the data and the approximate solution u_h, and its efficiency should be independent of the choice of the physical parameters κ_i. As classically observed in the literature (cf. [AIN 93], [BAB 93]) the dual energy norm of the residual gives a good indication of the error. The challenge is then to obtain an explicit local approximation of this norm, uniformly valid with respect to the coefficients κ_i.

For this purpose, on each edge (face) $E \in \mathcal{E}_h$ separating the elements T_1 and T_2, we first introduce weighting factors $\alpha(T_i, E), i = 1, 2$, such that

$$\alpha(T_1, E) + \alpha(T_2, E) = 1, \qquad [8]$$

$$\frac{\alpha(T_1, E)^2}{\kappa_{T_1}} + \frac{\alpha(T_2, E)^2}{\kappa_{T_2}} \leq \frac{1}{\kappa_{T_i}} \quad , \forall i = 1, 2, \qquad [9]$$

where κ_{T_i} denotes the restriction of the physical coefficient κ to the element T_i. A good choice of coefficients is

$$\alpha(T_i, E) = \frac{\kappa_{T_i}}{\kappa_{T_1} + \kappa_{T_2}},$$

which obviously satisfies

$$\frac{\alpha(T_1, E)^2}{\kappa_{T_1}} + \frac{\alpha(T_2, E)^2}{\kappa_{T_2}} = \frac{1}{\kappa_{T_1} + \kappa_{T_2}}.$$

Next, we introduce the piecewise projections f_T and g_E of right hand sides f and g on each element or edge (face) subspace $\mathbb{P}_{k-2}(T), T \in \mathcal{T}_h$ or $\mathbb{P}_{k-1}(E), E \in \mathcal{E}_{h,\Omega}$ defined by

$$\int_T f_T q = \int_T f q \quad , \forall q \in \mathbb{P}_{k-2}(T), f_T \in \mathbb{P}_{k-2}(T), \qquad [10]$$

$$\int_E g_E q = \int_E g q \quad , \forall q \in \mathbb{P}_{k-1}(E), g_E \in \mathbb{P}_{k-1}(E). \qquad [11]$$

Thus we define the *weighted element residuals* $\eta_{R,T}$ by

$$\eta_{R,T} = \left\{ \frac{h_T^2}{\kappa_T} ||f_T + \kappa_T \Delta u_h||_{0,2,T}^2 + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_N} \frac{h_E}{\kappa_T} ||g_E - \kappa_T \partial_n u_h||_{0,2,E}^2 \right.$$
$$\left. + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_h} \frac{\alpha(T, E)^2}{\kappa_T} h_E ||[\kappa_T \partial_n u_h]||_{0,2,E}^2 \right\}^{1/2}. \qquad [12]$$

Observe that the value of $\eta_{R,T}$ scales exactly like the solution energy norm when changing the physical scales or units. Above, we have used the standard notation for jumps in normal derivatives

$$[\kappa_T \partial_n u_h] = \kappa_{T_1} \partial_n u_h - \kappa_{T_2} \partial_n u_h.$$

With this notation we can prove

Theorem 4 *The following error estimate holds*

$$||u - u_h||_\Omega \preceq \left\{ \sum_{T \in \mathcal{T}_h} \eta_{R,T}^2 + \sum_{T \in \mathcal{T}_h} \frac{h_T^2}{\kappa_T} ||f_T - f||_{0,2,T}^2 \right.$$
$$\left. + \sum_{E \in \mathcal{E}_N} \frac{h_E}{\kappa_T} ||g_E - g||_{0,2,E}^2 \right\}^{1/2} \qquad [13]$$

and

$$\eta_{R,T} \preceq \left\{ ||u-u_h||^2_{\omega_T} + \sum_{T'\subset\omega_T} \frac{h^2_{T'}}{\kappa_{T'}} ||f_{T'} - f||^2_{0,2,T'} + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_N} \frac{h_E}{\kappa_T} ||g_E - g||^2_{0,2,E} \right\}^{1/2}. \qquad [14]$$

Proof

As usual, the proof is split into three parts : an algebraic manipulation of the residual, the derivation of the upper bound [13] , and of the lower bound [14].

Step 1: residual transform

By construction of the continuous and discrete problems [1] and [4], and after integration by parts on each element T, we can write for any v in H

$$\begin{aligned} a(u-u_h, v) &= \int_\Omega fv + \int_{\Gamma_N} gv - a(u_h, v) \\ &= \int_\Omega fv + \int_{\Gamma_N} gv - (\int_{\Omega_1} \kappa_1 \nabla u_h \cdot \nabla v + \int_{\Omega_2} \kappa_2 \nabla u_h \cdot \nabla v) \\ &= \sum_{T\in\mathcal{T}_h} \int_T (f + \kappa_T \Delta u_h) v + \sum_{E\in\mathcal{E}_N} \int_E (g - \kappa\partial_n u_h) v \\ &\quad - \sum_{E\in\mathcal{E}_h} \int_E [\kappa\partial_n u_h] v\,. \end{aligned} \qquad [15]$$

Then using the fact that $a(u-u_h, v_h) = 0\,, \forall v_h \in \mathrm{H}_h$ and the partition of unity [8], we obtain

$$\begin{aligned} a(u-u_h, v) &= \sum_{T\in\mathcal{T}_h} \int_T (f + \kappa\Delta u_h)(v - v_h) + \sum_{E\in\mathcal{E}_N} \int_E (g - \kappa\partial_n u_h)(v - v_h) \\ &\quad - \sum_{E\in\mathcal{E}_h} \int_E [\kappa\partial_n u_h](v - v_h) \\ &= \sum_{T\in\mathcal{T}_h} \left\{ \int_T \frac{1}{\sqrt{\kappa_T}} (f + \kappa_T \Delta u_h) \sqrt{\kappa_T} (v - v_h) \right. \\ &\quad + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_N} \int_E \frac{1}{\sqrt{\kappa_T}} (g - \kappa_T \partial_n u_h) \sqrt{\kappa_T} (v - v_h) \\ &\quad \left. - \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_h} \int_E \frac{\alpha(T,E)}{\sqrt{\kappa_T}} [\kappa\partial_n u_h] \sqrt{\kappa_T} (v - v_h) \right\}. \end{aligned} \qquad [16]$$

Step 2: upper bound

Let us now take $v_h = R_h v$. Using [2], [7] and the Cauchy-Schwarz inequality, the residual [16] can be bounded by

$$a(u-u_h,v) \preceq \sum_i \sum_{T\in\mathcal{T}_h\cap\Omega_i} \Big(\frac{h_T}{\sqrt{\kappa_i}} ||f+\kappa_i\Delta u_h||_{0,2,T}\sqrt{\kappa_i}|v|_{1,2,\tilde{\omega}_T\cap\Omega_i}$$

$$+ \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_N} \frac{h_E^{1/2}}{\sqrt{\kappa_i}} ||g-\kappa_i\partial_n u_h||_{0,2,E}\sqrt{\kappa_i}|v|_{1,2,\tilde{\omega}_E\cap\Omega_i}$$

$$+ \sum_{E\in\mathcal{E}(T)\cap(\mathcal{E}_h\setminus\mathcal{E}_{12})} \frac{\alpha(T,E)h_E^{1/2}}{\sqrt{\kappa_i}} ||[\kappa_i\partial_n u_h]||_{0,2,E}\sqrt{\kappa_i}|v|_{1,2,\tilde{\omega}_E\cap\Omega_i}\Big)$$

$$+\sum_i \sum_{E\in\mathcal{E}_{12}} \sum_{T_i\in\omega_E\cap\Omega_i} \frac{\alpha(T_i,E)h_E^{1/2}}{\sqrt{\kappa_i}} ||\kappa_1\partial_n u_h - \kappa_2\partial_n u_h||_{0,2,E}\sqrt{\kappa_i}|v|_{1,2,\tilde{\omega}_E\cap\Omega_i}$$

$$\preceq \Big\{ \sum_{T\in\mathcal{T}_h} \frac{h_T^2}{\kappa_T}||f+\kappa_T\Delta u_h||_{0,2,T}^2 + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_N} \frac{h_E}{\kappa_T}||g-\kappa_T\partial_n u_h||_{0,2,E}^2$$

$$+ \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_h} \frac{\alpha(T,E)^2 h_E}{\kappa_T}||[\kappa\partial_n u_h]||_{0,2,E}^2 \Big\}^{1/2}$$

$$\cdot\Big\{ \sum_i \Big(\sum_{T\in\mathcal{T}_h\cap\Omega_i} \kappa_i|v|_{1,2,\tilde{\omega}_T\cap\Omega_i}^2 + \sum_i \sum_{E\in((\mathcal{E}_h\setminus\mathcal{E}_{12})\cup\mathcal{E}_N)} \kappa_i|v|_{1,2,\tilde{\omega}_E\cap\Omega_i}^2 \Big)$$

$$+ \sum_{E\in\mathcal{E}_{12}} \big(\kappa_1|v|_{1,2,\tilde{\omega}_E\cap\Omega_1}^2 + \kappa_2|v|_{1,2,\tilde{\omega}_E\cap\Omega_2}\big)\Big\}^{1/2}$$

$$\preceq \Big\{ \sum_{T\in\mathcal{T}_h} \eta_{R,T}^2 + \sum_{T\in\mathcal{T}_h} \frac{h_T^2}{\kappa_T}||f_T - f||_{0,2,T}^2 + \sum_{E\in\mathcal{E}_N} \frac{h_E}{\kappa_T}||g_E - g||_{0,2,E}^2 \Big\}^{1/2} ||v||_\Omega .$$

To obtain [13], we just have to take $v = u - u_h$ and divide each term of the above inequality by $||u-u_h||_\Omega$, completing then the proof of step 2.

Step 3: inverse bound

Let us consider the local bubble test function $v_T = (f_T + \kappa_T\Delta u_h)b_T$. Then using Lemma 2, we obtain

$$||f_T+\kappa_T\Delta u_h||_{0,2,T}^2 \preceq \int_T (f_T+\kappa_T\Delta u_h)v_T \qquad [17]$$

and

$$\begin{aligned}||\nabla v_T||_{0,2,T} &\preceq h_T^{-1}||f_T + \kappa_T \Delta u_h||_{0,2,T}, \\ ||v_T||_{0,2,T} &\preceq ||f_T + \kappa_T \Delta u_h||_{0,2,T}.\end{aligned}$$

On the other hand, since the support of the function v_T is included in T and u is a solution of the continuous problem, we have

$$\begin{aligned}&\int_T (f_T + \kappa_T \Delta u_h) v_T = \int_T f v_T + \int_T \kappa_T \Delta u_h v_T + \int_T (f_T - f) v_T \\ = &\int_\Omega f v_T + \int_{\Gamma_N} g v_T - \int_\Omega \kappa \nabla u_h \cdot \nabla v_T + \int_T (f_T - f) v_T \\ = &\int_T \kappa_T \nabla (u - u_h) \cdot \nabla v_T + \int_T (f_T - f) v_T \\ \leq &\sqrt{\kappa_T} |u - u_h|_{1,2,T} \sqrt{\kappa_T} ||\nabla v_T||_{0,2,T} + ||f_T - f||_{0,2,T} ||v_T||_{0,2,T}.\end{aligned}$$

Thus

$$\begin{aligned}\int_T (f_T + \kappa_T \Delta u_h) v_T \preceq &||f_T + \kappa_T \Delta u_h||_{0,2,T} (h_T^{-1} \kappa_T^{1/2} ||u - u_h||_T \\ &+ ||f - f_T||_{0,2,T}).\end{aligned}$$

Hence, combining the two inequalities, we finally obtain

$$||f_T + \kappa_T \Delta u_h||_{0,2,T} \preceq h_T^{-1} \kappa_T^{1/2} ||u - u_h||_T + ||f_T - f||_{0,2,T}. \quad [18]$$

Next, we consider an arbitrary boundary edge (face) $E \in \mathcal{E}_N$ and define

$$v_E = (g_E - \kappa_T \partial_n u_h) b_E.$$

From our Lemma 2, we have

$$||g_E - \kappa_T \partial_n u_h||_{0,2,E}^2 \preceq \int_E (g_E - \kappa_T \partial_n u_h) v_E. \quad [19]$$

On the other hand, using the construction of u, v_E and our basic inverse inequalities, we obtain

$$\begin{aligned}&\int_E (g_E - \kappa_T \partial_n u_h) v_E = \int_E (g - \kappa_T \partial_n u_h) v_E + \int_E (g_E - g) v_E \\ = &\int_\Omega f v_E + \int_{\Gamma_N} g v_E - \int_\Omega \kappa_T \nabla u_h \cdot \nabla v_E - \sum_{T' \in w_E} \int_{T'} (f + \kappa \Delta u_h) v_E \\ &+ \int_E (g_E - g) v_E\end{aligned}$$

$$
\begin{aligned}
&\leq \sqrt{\kappa_T}|u - u_h|_{1,2,w_E}\sqrt{\kappa_T}||\nabla v_E||_{0,2,w_E} + \sum_{T' \in w_E} ||f + \kappa\Delta u_h||_{0,2,T'}||v_E||_{0,2,T'} \\
&\quad + ||g_E - g||_{0,2,E}||v_E||_{0,2,E} \\
&\preceq h_E^{-1/2}\kappa_T^{1/2}||u - u_h||_{w_E}||g_E - \kappa_T\partial_n u_h||_{0,2,E} \\
&\quad + h_E^{1/2}\sum_{T' \in w_E} ||f + \kappa\Delta u_h||_{0,2,T'}||g_E - \kappa_T\partial_n u_h||_{0,2,E} \\
&\quad + ||g_E - g||_{0,2,E}||g_E - \kappa_T\partial_n u_h||_{0,2,E}\,.
\end{aligned}
$$

Thus, by combining the above two inequalities, we get

$$
\begin{aligned}
||g_E - \kappa_T\partial_n u_h||_{0,2,E} &\preceq h_E^{-1/2}\kappa_T^{1/2}||u - u_h||_{w_E} \\
&\quad + h_E^{1/2}\sum_{T' \subset w_E} (||f - f_{T'}||_{0,2,T'} + ||f_{T'} + \kappa\Delta u_h||_{0,2,T'}) \\
&\quad + ||g_E - g||_{0,2,E}\,. \qquad [20]
\end{aligned}
$$

Finally, let us consider an internal edge (face) $E \in \mathcal{E}_h$ separating the elements T_1 and T_2 and define

$$
v_E = \left(\frac{\alpha^2(T_1, E)}{\kappa_{T_1}} + \frac{\alpha^2(T_2, E)}{\kappa_{T_2}}\right)^{1/2}(\kappa_{T_1}\partial_n u_h - \kappa_{T_2}\partial_n u_h)b_E\,.
$$

From Lemma 2, we first have

$$
\left(\frac{\alpha^2(T_1, E)}{\kappa_{T_1}} + \frac{\alpha^2(T_2, E)}{\kappa_{T_2}}\right)^{1/2}||[\kappa\partial_n u_h]||_{0,2,E}^2 \preceq \int_E (\kappa_{T_1}\partial_n u_h - \kappa_{T_2}\partial_n u_h)v_E\,. \qquad [21]
$$

Now, using the fact that the support of v_E is included in w_E, the construction of u and the equivalence of norms, we obtain

$$
\begin{aligned}
&\int_E (\kappa_{T_1}\partial_n u_h - \kappa_{T_2}\partial_n u_h)v_E \\
&= \sum_{T \subset w_E} \int_T (f + \kappa_T\Delta u_h)v_E - \int_\Omega f v_E - \int_{\Gamma_N} g v_E + \int_\Omega \kappa\nabla u_h \cdot \nabla v_E \\
&= \sum_{T \subset w_E} \int_T (f + \kappa_T\Delta u_h)v_E - \sum_i \int_{w_E \cap \Omega_i} \kappa_i\nabla(u - u_h) \cdot \nabla v_E
\end{aligned}
$$

$$\leq \sum_i \sqrt{\kappa_i}|u - u_h|_{1,2,w_E\cap\Omega_i}\sqrt{\kappa_i}||\nabla v_E||_{0,2,w_E\cap\Omega_i}$$

$$+||(f + \kappa\Delta u_h)||_{0,2,w_E\cap\Omega_i}||v_E||_{0,2,w_E\cap\Omega_i}$$

$$\preceq \sum_i h_E^{-1/2}\sqrt{\kappa_i}||u - u_h||_{w_E\cap\Omega_i}||v_E||_{0,2,E}$$

$$+||f + \kappa\Delta u_h||_{0,2,w_E\cap\Omega_i}||v_E||_{0,2,w_E\cap\Omega_i}$$

$$\preceq \sqrt{h_E}||\kappa_1\partial_n u_h - \kappa_2\partial_n u_h||_{0,2,E}\Big(\sum_i h_E^{-1}||u - u_h||_{w_E\cap\Omega_i}\sqrt{\kappa_i}(\frac{\alpha_1^2}{\kappa_1} + \frac{\alpha_2^2}{\kappa_2})^{1/2}$$

$$+(\frac{\alpha_1^2}{\kappa_1} + \frac{\alpha_2^2}{\kappa_2})^{1/2}||f + \kappa\Delta u_h||_{0,2,w_E\cap\Omega_i}\Big).$$

By construction of coefficients α_i, we have

$$\sqrt{\kappa_i}(\frac{\alpha_1^2}{\kappa_1} + \frac{\alpha_2^2}{\kappa_2})^{1/2} \leq 1$$

and hence by using the above inequalities and [18] we obtain

$$(\frac{\alpha^2(T_1,E)}{\kappa_{T_1}} + \frac{\alpha^2(T_2,E)}{\kappa_{T_2}})^{1/2}||[\kappa\partial_n u_h]||_{0,2,E}$$

$$\preceq h_E^{-1/2}||u - u_h||_{w_E} + \sum_{T'\subset w_E} h_E^{1/2}\frac{1}{\sqrt{\kappa_{T'}}}||f - f_{T'}||_{0,2,T'}. \qquad [22]$$

Then from [18], [20] and [22], we get [14] □

4. The elasticity problem

Now, we will try to extend the previous approach to linear elasticity problems. Let be Ω a Lipschitz, bounded domain of $\mathbb{R}^n$ with an interior boundary denoted Γ_{12}. This boundary represents the interface between two elastic, isotropic and homogeneous materials, noted Ω_1 and Ω_2, respectively. Let $\Gamma = \partial\Omega$ such that $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, with $\partial\Omega_i \cap \Gamma_D \neq \emptyset$, $i = 1, 2$. This means that we assume for the time being that each subdomain is fixed on part of its boundary. This assumption will be useful to relate the H^1 semi-norm used in the local interpolation [7] and the local energy norm. It can be relaxed if we can prove this interpolation result directly with the $|\varepsilon(\cdot)|_{0,2}$ norm. In this framework, we consider the following elasticity problem

$$[\text{P}] \quad \left\{ \begin{array}{rcl} -\text{div}\,\boldsymbol{\sigma} = & \mathbf{f} & \text{in}\,\Omega\,, \\ \mathbf{u} = & \mathbf{0} & \text{on}\,\Gamma_D\,, \\ \boldsymbol{\sigma}\cdot\mathbf{n} = & \mathbf{g} & \text{on}\,\Gamma_N\,, \end{array} \right.$$

where $\mathbf{f} \in L^2(\Omega)^n$ and $\mathbf{g} \in L^2(\Gamma_D)^n$ are the external forces and $\boldsymbol{\sigma}$ is the stress tensor. Assuming isotropy, this tensor satisfies the constitutive law

$$\boldsymbol{\sigma} = (\sigma_{ij}) = (\lambda_k \varepsilon_{pp}(\mathbf{u})\delta_{ij} + 2\mu_k \varepsilon_{ij}(\mathbf{u})),$$

with $\lambda_k, \mu_k > 0$ the Lame's coefficients of the material Ω_k and $\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ the components of the linearized strain tensor $\boldsymbol{\varepsilon}(\mathbf{u})$ associated to $\mathbf{u}$.

Remark. There is extensive work relating linear elasticity and *a posteriori* estimates, see by example [AIN 94], [JOH 92], [LAD 91], [LAD 83], [MÜC 95] and [SZA 90].

The standard weak formulation of the problem [P] is then: *Find* $\mathbf{u} \in \mathbf{H}$ *such that*

$$a(\mathbf{u}, \mathbf{v}) = < F, \mathbf{v} > \quad, \forall \mathbf{v} \in \mathbf{H} \qquad [23]$$

where

$$\begin{aligned} \mathbf{H} &= \{\mathbf{v} \in H^1(\Omega)^n \,/\, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\} \\ a(\mathbf{u}, \mathbf{v}) &= \int_{\Omega_1} \boldsymbol{\sigma}_1(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) + \int_{\Omega_2} \boldsymbol{\sigma}_2(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \\ < F, \mathbf{v} > &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v}\,. \end{aligned}$$

Let $\mathbf{H}_h$ the finite element space defined by

$$\mathbf{H}_h = \{\mathbf{v}_h \in \mathcal{C}(\Omega)^n \,/\, \mathbf{v}_h = \mathbf{0} \text{ on } \Gamma_D\,, \mathbf{v}_h|_T \in \mathbb{P}_k(T)^n\,, \forall T \in \mathcal{T}_h\} \quad (k \geq 1)\,.$$

Then the approximate problem of [23] is: *Find* $\mathbf{u}_h \in \mathbf{H}_h$ *such that*

$$a(\mathbf{u}_h, \mathbf{v}_h) = < F, \mathbf{v}_h > \quad, \forall \mathbf{v}_h \in \mathbf{H}_h\,. \qquad [24]$$

We define the *energy norm* by:

$$||\mathbf{v}||_\Omega = \{\sum_i ||\mathbf{v}||^2_{\Omega_i}\}^{1/2} = \{\sum_i \int_{\Omega_i} \sigma_i(\mathbf{v}) : \varepsilon(\mathbf{v})\}^{1/2}\,, \forall \mathbf{v} \in \mathbf{H}\,.$$

By the Korn's inequality, since $\partial\Omega_i \cap \Gamma_D$ has a non empty measure, there exist two positive constants C_{Ω_1} and C_{Ω_2}, depending only on the geometry of Ω_1 and Ω_2, respectively, such that

$$||\mathbf{v}||_{1,2,\Omega_i} \leq C_{\Omega_i} ||\varepsilon(\mathbf{v})||_{0,2,\Omega_i} \quad \forall \mathbf{v} \in \mathbf{H},\ i = 1, 2\,.$$

Finally, there exits an interpolation operator (see Lemma 2) $R_h : \mathbf{H} \to \mathbf{H}_h$ such that for all $\mathbf{v} \in \mathbf{H}$, $T \in \mathcal{T}_h$, $E \in \mathcal{E}_N \cup \mathcal{E}_h$

$$\begin{aligned} ||\mathbf{v} - R_h \mathbf{v}||_{0,2,T} &\preceq h_T |\mathbf{v}|_{1,2,\tilde{\omega}_T \cap \Omega_i} \\ ||\mathbf{v} - R_h \mathbf{v}||_{0,2,E} &\preceq h_E^{1/2} |\mathbf{v}|_{1,2,\tilde{\omega}_E \cap \Omega_i}\,. \end{aligned}$$

With the above definitions we can prove exactly the same results as in section 3. We only have to pay attention to:

— replace κ_T by E_T where E_T is the Young modulus of the material that composes the element T,

— note that in the proof of step 3 we find a factor $\frac{1}{1-2\nu_i}$ that we include in a constant. It's clear that if a material is quasi-incompressible then this constant explodes. This means that our development is valid only for compressible materials. In fact, for practical purposes we assume that $0 < \nu_i \leq 0.45$.

Altogether, defining the local weighted residual by

$$\begin{aligned}\eta_{R,T} &= \left\{ \frac{h_T^2}{E_T} ||\mathbf{f}_T + \nabla \cdot \sigma(\mathbf{u}_h)||_{0,2,T}^2 + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_N} \frac{h_E}{E_T} ||\mathbf{g}_E - \sigma(\mathbf{u}_h) \cdot \mathbf{n}||_{0,2,E}^2 \right. \\ &\left. + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_h} \frac{\alpha(T,E)^2}{E_T} h_E ||[\sigma(\mathbf{u}_h) \cdot \mathbf{n}]||_{0,2,E}^2 \right\}^{1/2},\end{aligned}$$

we can prove

Theorem 5 *Let* $\mathbf{u}$ *be the solution to the continuous problem* [23] *and* $\mathbf{u}_h$ *the solution to the approximate problem* [24]. *Then the following estimate holds*

$$\begin{aligned}||\mathbf{u} - \mathbf{u}_h||_\Omega &\preceq \left\{ \sum_{T \in \mathcal{T}_h} \eta_{R,T}^2 + \sum_{T \in \mathcal{T}_h} \frac{h_T^2}{E_T} ||\mathbf{f}_T - \mathbf{f}||_{0,2,T}^2 \right. \\ &\left. + \sum_{E \in \mathcal{E}_N} \frac{h_E}{E_T} ||\mathbf{g}_E - \mathbf{g}||_{0,2,E}^2 \right\}^{1/2},\end{aligned}$$

$$\begin{aligned}\eta_{R,T} &\preceq \left\{ ||\mathbf{u} - \mathbf{u}_h||_{\omega_T}^2 + \sum_{T' \subset \omega_T} \frac{h_{T'}^2}{E_{T'}} ||\mathbf{f}_{T'} - \mathbf{f}||_{0,2,T'}^2 \right. \\ &\left. + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_N} \frac{h_E}{E_T} ||\mathbf{g}_E - \mathbf{g}||_{0,2,E}^2 \right\}^{1/2}.\end{aligned}$$

All the constants are independent of the mesh size h and the Young's moduli E_i, *but they can depend on the factor* $\frac{1}{1-2\nu_i}$.

5. Numerical results

In this section we will apply the results of section 4 to two elasticity problems. We will try here to monitor the evolution of the residual as the mesh is refined.

Remark. In order to obtain a optimal mesh refinement procedure (cf. [LAD 91]), let ε_0 be the accuracy required by the user, we say that the mesh $\mathcal{T}^*$ is *optimal* if its elements number N^* is minimum and it provides a global error ε^* equal to ε_0. In this framework, for each element $T \in \mathcal{T}$, we compute a *refinement factor*:

$$r_T = \frac{h_T^*}{h_T}$$

where h_T is the size of the element T of $\mathcal{T}$, and h_T^* the size of the elements of $\mathcal{T}^*$ within the T area (in 2D case).

If no strong gradients appear in the solution (see [COO 93]) then a priori error estimates indicate that the local contribution to the error should scale like

$$\frac{\eta_T^*}{\eta_T} = \left(\frac{h_T^*}{h_T}\right)^p = r_T^p$$

where p depends on the element type ($p = 1$ for linear and $p = 2$ for quadratics elements). Thus we have the following minimization problem:

$$\min N^* = \sum_T \frac{1}{r_T^n} \quad \text{with} \quad \sum_T r_T^{2p} \eta_T^2 = \varepsilon_0^2 .$$

This problem admits the explicit solution:

$$r_T = \frac{\varepsilon_0^{1/p}}{\eta_T^{2/(2p+n)} \left[\sum_T \eta_T^{2n/(2p+n)}\right]^{1/2p}} .$$

The new mesh is then obtained by a metric controlled Delaunay mesh generator (cf. [BOR 96]) constrained to generate local equilateral triangles of size $r_T h_T$.

In the first example (Figure 3), we consider a soft material neighboring a more rigid isotropic material.

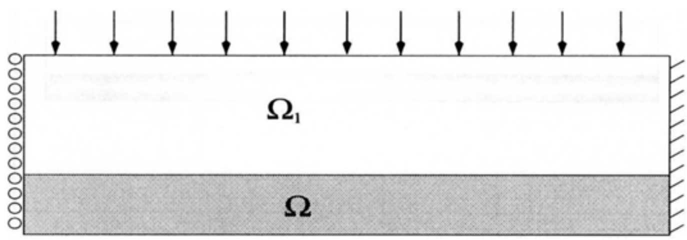

Figure 3. *Example 1*

This problem is discretized using 3-node triangles, but the same type of result is valid for quadrilaterals (we have tested the same examples using Q_2 quadrilaterals).

In Figures 4-5 we can see the initial and adapted meshes and the distribution of the estimator η.

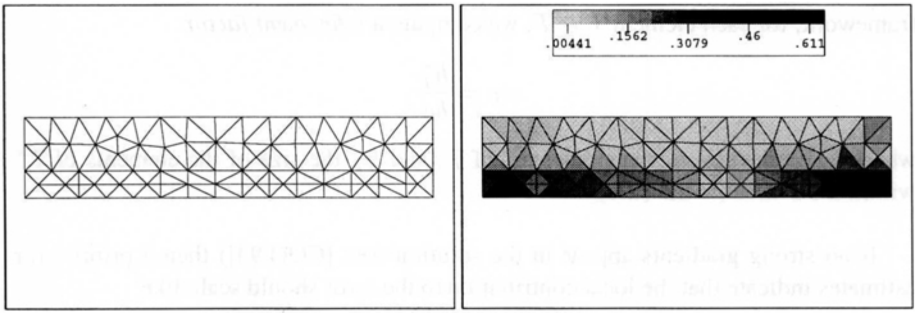

Figure 4. *Initial mesh (146 elements) and distribution of the error estimator* η

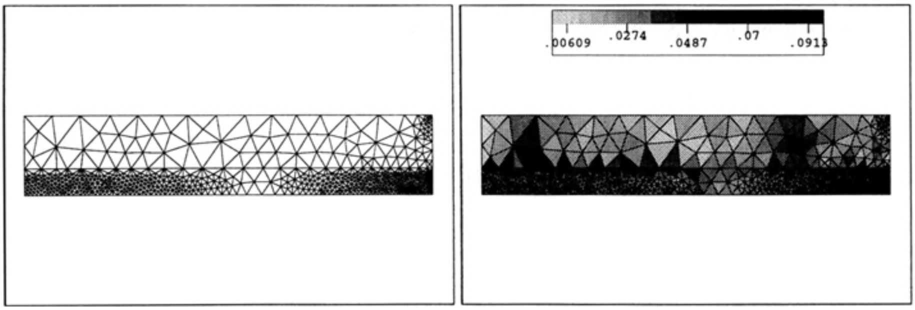

Figure 5. *Adapted mesh (1430 elements) and distribution of the error estimator* η

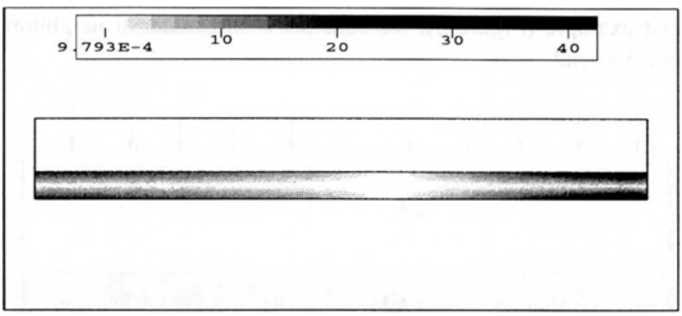

Figure 6. *Global view of Von Mises stress field for Example 1*

Finally we show a comparison of the approximate solution in the initial mesh, the adapted mesh and our reference solution (calculated in a uniformly refined mesh) and the evolution of our estimator and the standard one.

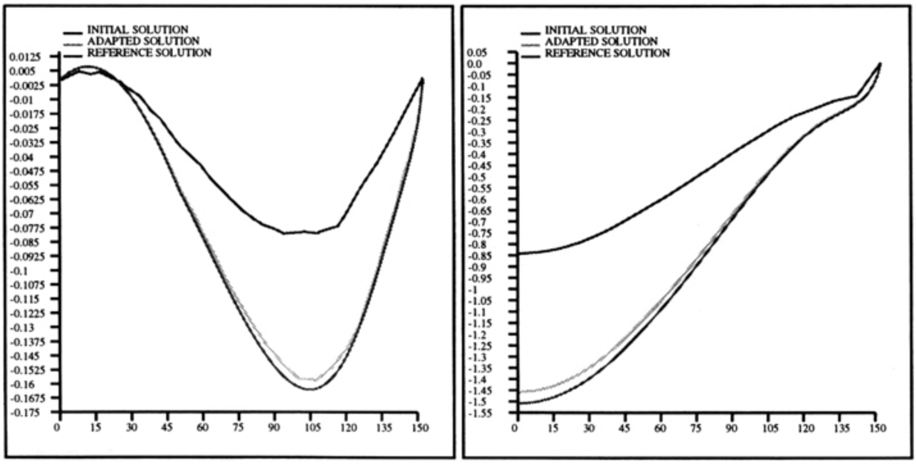

Figure 7. *Comparison of the different solutions of Example 1 in a diagonal cut*

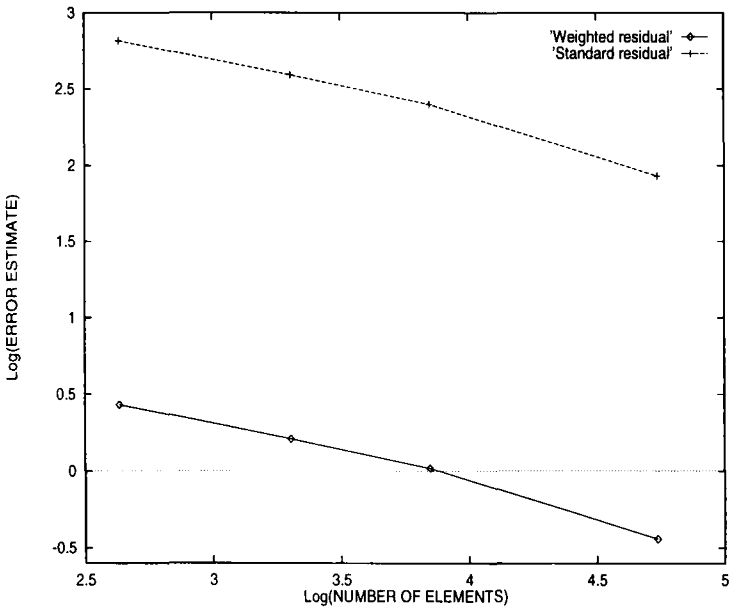

Figure 8. *Comparison of standard and weighted residuals for Example 1*

Example 2 (Figure 9), considers a bi-material dam discretized with the same finite element as in example 1.

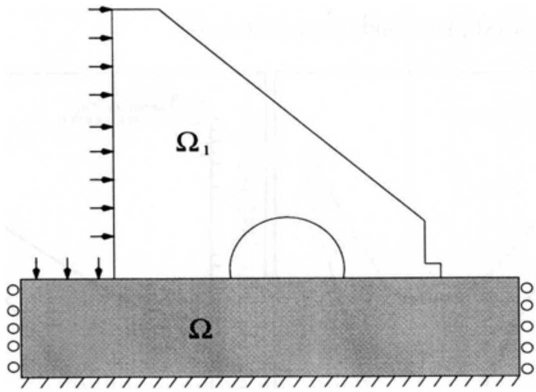

Figure 9. *Example 2*

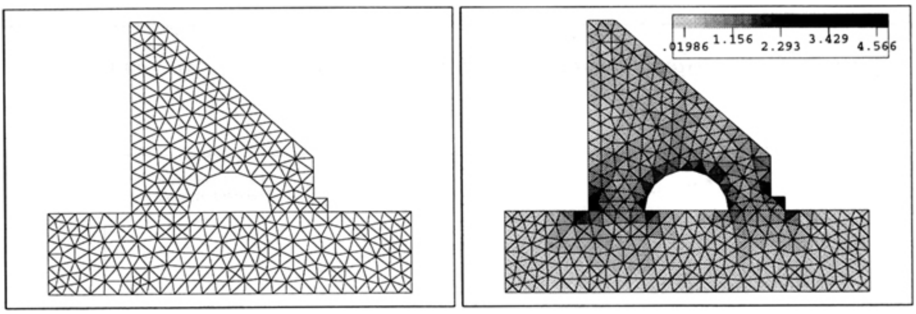

Figure 10. *Initial mesh (639 elements) and distribution of the error estimator η*

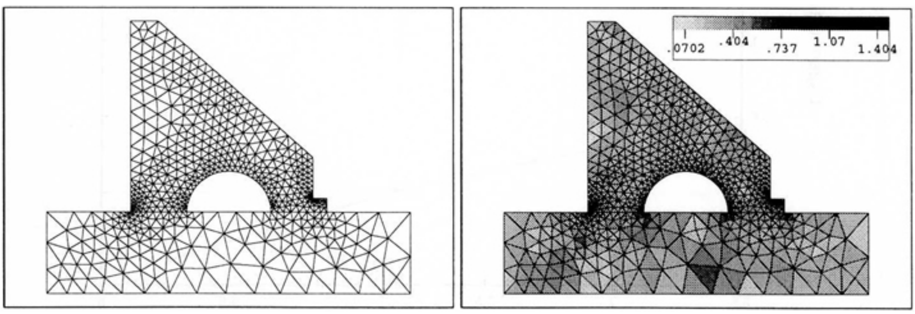

Figure 11. *Adapted mesh (1105 elements) and distribution of the error estimator η*

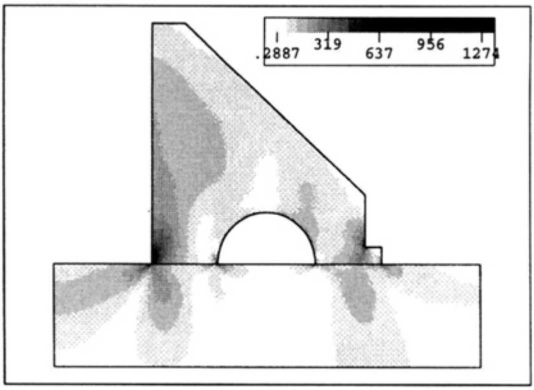

Figure 12. *Global view of Von Mises stress field for Example 2*

Finally, like in example 1, we show a cut of the approximate solution in the initial mesh, the adapted mesh and our reference solution (calculated in a uniformly refined mesh).

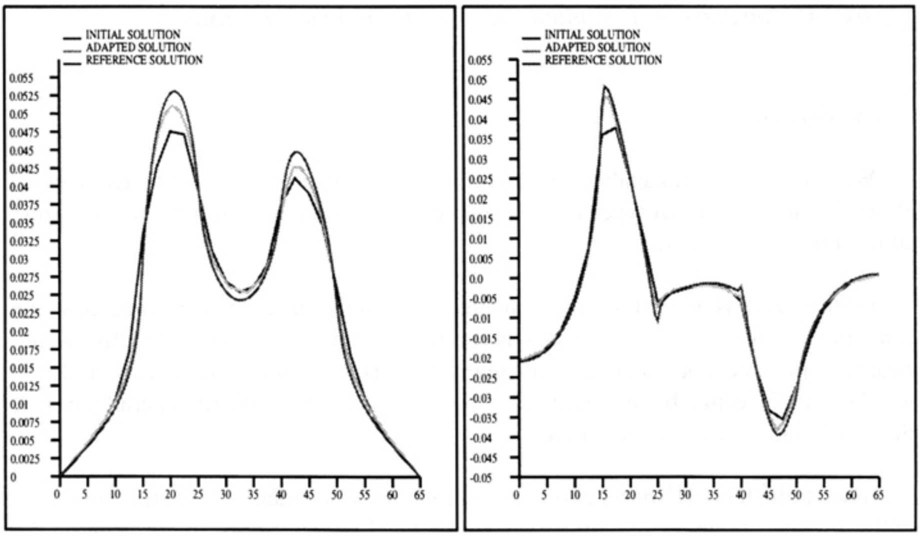

Figure 13. *Comparison of the different solutions of Example 2 in a cut*

The relative error that we obtain in this example is globally near to 5 %. Finally, the last figure shows a comparison between our error estimator and the standard residual when the number of elements increase (i.e., $h \to 0$).

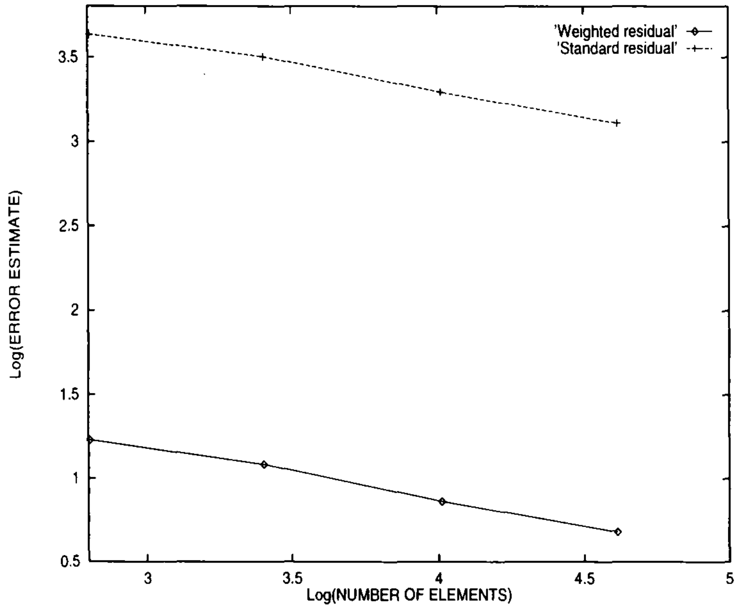

Figure 14. *Comparison of standard and weighted residuals for Example 2*

6. Conclusions

We have derived and analyzed a local *a posteriori* error estimate for heterogeneous elastic bodies of residual type. The first numerical tests are encouraging for compressible isotropic materials.

Further work is needed to handle anisotropic heterogeneous materials because we cannot prove the same type of results as for the isotropic case; nevertheless the numerical tests indicate that our error estimate might work even in this framework. If this is not the case, it seems that it would be necessary to use some kind of generalization of the equilibration residual technique.

Indeed, the local H^1 norm appearing in the inverse inequality for estimating ∇v_T will no longer be uniformly equivalent to the local energy norm. In our opinion the local energy norm of the residual can only be properly obtained by solving a local (cf. [AIN 97]) Neumann problem.

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Bibliographie

[AIN 93] AINSWORTH M., ODEN J., « A unified approach to *a posteriori* error estimation using element residual methods ». *Numer. Math.*, vol. 65, p. 23–50, 1993.

[AIN 94] AINSWORTH M., ODEN J., WU W., « *A posteriori* error estimation for *h-p* approximations in elastostatics ». *Appl. Numer. Math.*, vol. 14, p. 23–54, 1994.

[AIN 97] AINSWORTH M., ODEN J., « *A posteriori* error estimation in finite element analysis ». *Comput. Methods Appl. Mech. Engrg.*, vol. 142, p. 1–88, 1997.

[ARA 97] ARAYA R., LE TALLEC P., « A robust a posteriori error estimate for elliptic nonhomogeneous equations ». Rapport de recherche, n° 3279, INRIA, 1997.

[BAB 78] BABUŠKA I., RHEINBOLDT W., « A posteriori error estimates for the finite element method ». *Int. J. Numer. Methods Engrg.*, vol. 12, p. 1597–1615, 1978.

[BAB 93] BABUŠKA I., RODRÍGUEZ R., « The problem of the selection of an *a posteriori* error indicator based on smoothening techniques ». *Int. J. Numer. Methods Engrg.*, vol. 36, p. 539–567, 1993.

[BER 95] BERNARDI C., GIRAULT V., « A local regularization operator for triangular and quadrilateral finite elements ». Rapport de recherche, n° 95036, UPMC, Paris, 1995.

[BOR 96] BOROUCHAKI H., LAUG P., « The BL2D mesh generator: beginner's guide, user's guide and programmer's manual ». Rapport de recherche, n° 0194, INRIA, 1996.

[CIA 78] CIARLET P. G., *The finite element method for elliptic problems*. Editions North-Holland, Amsterdam, 1978.

[CLÉ 75] CLÉMENT P., « Approximation by finite element functions using local regularization ». *RAIRO Anal. Numér.*, vol. 2, p. 77–84, 1975.

[COO 93] COOREVITS P., « *Maillage adaptatif anisotrope: application aux problemes de dynamique* ». Thèse de doctorat, Ecole Normale Superieure de Cachan, 1993.

[JOH 92] JOHNSON C., HANSBO P., « Adaptative finite elements methods in computational mechanics ». *Comput. Methods Appl. Mech. Engrg.*, vol. 101, p. 143–181, 1992.

[LAD 83] LADEZEVE P., LEGUILLON D., « Error estimate procedure in the finite element method and applications ». *SIAM J. Numer. Anal.*, vol. 20, p. 485–509, 1983.

[LAD 91] LADEVEZE P., PELLE J., ROUGEOT P., « Error estimation and mesh optimization for classical finite elements ». *Engrg. Comput.*, vol. 8, p. 69–80, 1991.

[MÜC 95] MÜCKE R., WHITEMAN J., « *A posteriori* error estimates and adaptivity for finite element solutions in finite elasticity ». *Int. J. Numer. Methods Engrg.*, vol. 38, p. 775–795, 1995.

[NEP 97] NEPOMNYASCHIKH S., « A local projection operator ». Personal communication, 1997.

[SCO 90] SCOTT L., ZHANG S., « Finite element interpolation of nonsmooth functions satisfying boundary conditions ». *Math. Comp.*, vol. 54, n° 190, p. 483–493, 1990.

[SZA 90] SZABÓ B., « The *p* and *h-p* versions of the finite element method in solid mechanics ». *Comput. Methods Appl. Mech. Engrg.*, vol. 80, p. 185–195, 1990.

[VER 96] VERFÜRTH R., *A review of a posteriori error estimation and adaptative mesh-refinement techniques*. Editions Wiley and Teubner, Chichester-Stuttgart, 1996.

[ZIE 87] ZIENKIEWICZ O., ZHU J., « A simple error estimator and adaptative procedure for practical engineering analysis ». *Int. J. Numer. Methods Engrg.*, vol. 24, p. 333–357, 1987.

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