An asymptotic numerical algorithm for frictionless contact problems

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ABSTRACT. *Perturbation techniques have been successfully developed to solve problems in non-linear structural mechanics. Based on asymptotic expansions, these techniques lead to analytic representation of the solution branches. In elasticity, when solving contact problems, two non-linearities can occur due to contact constraints and to geometry. The aim of this paper is to propose an asymptotic numerical method for frictionless contact problems. Three examples of 2-D contact problems will be studied to establish the efficiency of our algorithm.*

RÉSUMÉ. *Des techniques de perturbations ont été développées pour le calcul non linéaire des structures élastiques. Ces techniques basées sur des développements asymptotiques permettent d'obtenir une représentation analytique des branches de solutions et un gain de temps de calcul important. Dans les problèmes d'élasticité avec des conditions de contact, il y a deux non-linéarités, la non-linéarité due au contact et la non-linéarité géométrique. L'objectif de ce papier est de présenter une Méthode Asymptotique Numérique pour les problèmes de contact sans frottement. Trois exemples 2D sont étudiés pour montrer l'efficacité de notre algorithme.*

KEY WORDS: *asymptotic numerical method, contact, elastic structures, perturbation technique, Padé approximants.*
MOTS-CLÉS : *méthode asymptotique numérique, contact, structures élastiques, technique de perturbation, approximants de Padé.*

1. Introduction

Elastic problems with contact conditions have been widely treated by classic numerical methods coupled with the penalty method or the Lagrange multiplier one. In the first case, the unilateral conditions are not imposed, but replaced by a large stiffness term, in the second case, one introduces a Lagrange multiplier which is a supplementary variable of the problem [KIK 88][FEN 91][KAL 93]. Our aim is to present an Asymptotic Numerical Method (ANM) to solve the contact problems without friction. The principle of this perturbation method is to associate the asymptotic expansions with the finite element method as presented in [COC 94-1]. Indeed, the unknowns of the non-linear problem are expanded into series expansions with respect to a control parameter. By substituting these asymptotic expansions into the non-linear equations, we obtain a recursive sequence of linear problems which allow ones to calculate the terms of the series. These linear problems admit the same stiffness matrix, so we can compute a large number of terms with a low computational cost. Consequently, we obtain an analytic representation of a part of a solution branch with only one matrix decomposition. The ANM is well adapted to problems with quadratic non-linearity, where the asymptotic expansions are very simple to be set. Compared with the classical iterative methods, the ANM is faster, more reliable and easier to be used.

In the contact problems, the unilateral conditions are not analytic and this property is necessary to apply an asymptotic procedure. To solve this difficulty, we have regularized the contact law and have replaced it by a hyperbolic relation. To establish the efficiency of this technique, some examples will be presented.

2. Contact Problems Without Friction

2.1. *Contact Conditions*

In this study, we consider a contact problem, with possible large deformations between an elastic body and a 2-D rigid surface. At each contact point, we associate a scalar variable h which represents its distance toward the rigid surface. Generally, the contact force $\mathbf{R}$ is a function of h, and in the case of frictionless contact it is parallel to the normal $\mathbf{n}$ on the rigid surface at the contact point.

The unilateral conditions are expressed by:

$$(\mathbf{R}.\mathbf{n})h = 0, \quad \mathbf{R}.\mathbf{n} \geq 0, \quad h \geq 0 \qquad [1]$$

The contact law [1] is not analytic, so we propose to regularize it by the hyperbolic relation:

$$\mathbf{R}h = \eta(\delta - h)\mathbf{n} \qquad [2]$$

where η is a positive and sufficiently small parameter and δ is the initial clearance. For small values of η, the regularized force [2] tends to [1] (Figure 1).

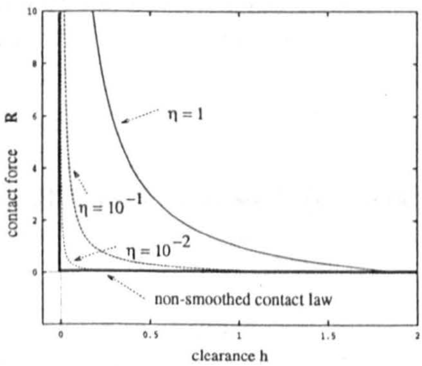


Figure 1. *Regularization with different* η

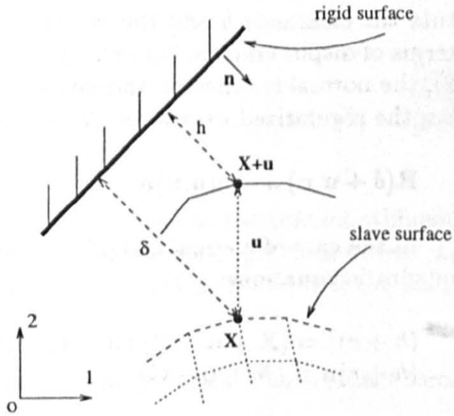


Figure 2. *Plane rigid surface*

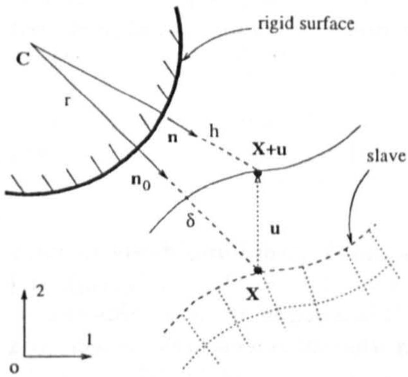


Figure 3. *Circular rigid surface*

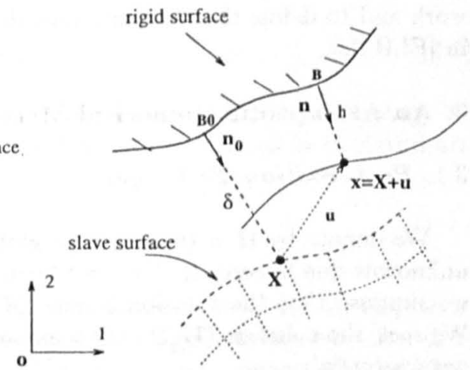


Figure 4. *Arbitrary rigid surface*

2.2. *Mixed Variational Formulation*

We start with the mixed Hellinger-Reissner formulation to which we add the contact energy. The governing equations are obtained by the stationary condition and can be written by introducing the mixed variable $\mathbf{U} = (\mathbf{u}, \mathbf{S})$ [COC 94-1], as follows:

$$L(\mathbf{U}) + Q(\mathbf{U}, \mathbf{U}) = \lambda \mathbf{F} + \mathbf{R}^r \qquad [3]$$

where $\mathbf{u}$ and $\mathbf{S}$ represent respectively the displacement vector and the Piola-

Kirchhoff stress tensor of the second kind, L is a linear operator, Q a symmetric bilinear operator, $\mathbf{F}$ is the external force (see [COC 94-1] for their expressions) and $\mathbf{R}^r$ is the contact force vector at the contact points. In this study, we shall consider the contact with a plane, a circular and an arbitrary rigid surface. The vector $\mathbf{R}$ must be defined in terms of displacements. In this respect, we substitute the clearance h and the normal $\mathbf{n}$ in equation [2] by their expressions in terms of displacements. For example, in the case of a plane rigid surface (Figure 2), the normal is constant and given and the clearance is given by $h = \delta + \mathbf{u.n}$. So, the regularized expression [2] becomes:

$$\mathbf{R}(\delta + \mathbf{u.n}) = -\eta(\mathbf{u.n})\mathbf{n}$$

In the case of a circular rigid surface (Figure 3), h and $\mathbf{n}$ are defined by the quadratic equations:

$$(h+r)^2 = (\mathbf{X}+\mathbf{u}-\mathbf{C}).(\mathbf{X}+\mathbf{u}-\mathbf{C})$$
$$(h+r)\mathbf{n} = (\mathbf{X}+\mathbf{u}-\mathbf{C})$$

For an arbitrary surface, we propose to smooth it by a Bézier polynomial [FAR 90]. We have limited ourselves to a six-degree polynomial (Figure 4). In this case, we introduce some new variables in order to keep a quadratic framework and to define the clearance and the normal, the details beeing presented in [ELH 98].

3. An Asymptotic Numerical Method

3.1. *Perturbation Technique*

We denote by $\Omega = (\mathrm{h}, \mathbf{n}, ...)$ the global variable containing supplementary unknowns due to contact. We start from a given solution $(\mathbf{U_0}, \mathbf{R_0}, \lambda_0, \Omega_0)$, and we suppose that the solution branch of [2][3] is analytic in its neighborhood. We seek the solution $(\mathbf{U}, \mathbf{R}, \lambda)$ by expanding the unknowns with respect to a parameter $'\mathbf{a}'$:

$$\mathbf{U}(a) = \sum_{i=0}^{p} \mathbf{U}_i a^i, \quad \mathbf{R}^r(a) = \sum_{i=0}^{p} \mathbf{R}^r_i a^i, \quad \lambda(a) = \sum_{i=0}^{p} \lambda_i a^i, \quad \Omega(a) = \sum_{i=0}^{p} \Omega_i a^i \quad [4]$$

The parameter $'\mathbf{a}'$ is a supplementary unknown of the problem. We can define it by the projection of the vector $(\mathbf{u}-\mathbf{u_0}, \lambda-\lambda_0)$ on the tangent $(\mathbf{u_1}, \lambda_1)$:

$$a = \langle \mathbf{u}-\mathbf{u}_0, \mathbf{u}_1 \rangle + (\lambda - \lambda_0)\lambda_1$$

By substituting the series [4] into equations [2], [3] and into the definition of $'\mathbf{a}'$ and by identifying like powers of $'\mathbf{a}'$, we obtain the following sequence of the linear problems:

order 1

$$L_t(\mathbf{U}_1) = \lambda_1 \mathbf{F} + \mathbf{R}_1^r$$
$$\langle \mathbf{u}_1, \mathbf{u}_1 \rangle + \lambda_1 \lambda_1 = 1$$
$$\mathbf{R}_1 = L_c \mathbf{u}_1$$

order p

$$L_t(\mathbf{U}_p) = \lambda_p \mathbf{F} + \mathbf{R}_p^r - \sum_{r=1}^{p-1} Q(\mathbf{U}_r, \mathbf{U}_{p-r}) \quad [5.1]$$
$$\langle \mathbf{u}_p, \mathbf{u}_1 \rangle + \lambda_p \lambda_1 = 0 \quad [5.2]$$
$$\mathbf{R}_p = L_c \mathbf{u}_p + \mathbf{f}_p^{nlc} \quad [5.3]$$

where L_c is a linear symmetric operator which represents the contact stiffness, and $\mathbf{f}_p^{nlc}$ is a vector which depends non-linearly on terms at previous orders. L_t is the tangent operator defined by $L_t(.) = L(.) + 2Q(\mathbf{U}_0, .)$, and it is the same as in the case without contact.

In the case of a plane rigid surface, the expansion terms of the contact force at each contact point are given by:

$$\mathbf{R}_1 = \frac{-<\mathbf{u}_1, \mathbf{n}>.(\eta \mathbf{n} + \mathbf{R}_0)}{(\delta + <\mathbf{u}_0, \mathbf{n}>)} \quad [6.1]$$

$$\mathbf{R}_p = \frac{-<\mathbf{u}_p, \mathbf{n}>(\eta \mathbf{n} + \mathbf{R}_0)}{(\delta + <\mathbf{u}_0, \mathbf{n}>)} - \frac{\sum_{r=1}^{p-1} \mathbf{R}_r <\mathbf{u}_{p-r}, \mathbf{n}>}{(\delta + <\mathbf{u}_0, \mathbf{n}>)} \quad [6.2]$$

For the other types of rigid surfaces, we can apply the same strategy to calculate the terms $\mathbf{R}_1$ and $\mathbf{R}_p$ which keep the same form as in [5.3] and are defined by formulas similar as in [6].

3.2. *Finite Element Discretization*

In order to apply a classical finite element method, we eliminate the stress term $\mathbf{S}_p$ by substituting the constitutive law in the equilibrium equation [5.1] at order p [COC 94-1]. Denoting by $[\mathbf{v}]$ the nodal displacements, the problem at order p can be written as:

$$[\mathbf{K}_{Tnlg}][\mathbf{v}_p] = \lambda_p[\mathbf{F}] + [\mathbf{F}_p^{nlg}] + [\mathbf{R}_p^r]$$

where $[\mathbf{K}_{Tnlg}]$ is the contactless tangent stiffness matrix and $[\mathbf{F}_p^{nlg}]$ is a vector which depends non-linearly on terms at previous orders [COC 94-1]. These quantities depend only on the geometrical non-linearity. The vector $[\mathbf{v}_p]$ can be decomposed into two components $[\mathbf{v}_p] = [\mathbf{v}_p^c, \mathbf{v}_p^{nc}]^T$, where $[\mathbf{v}^c]$ is the displacement of the contact nodes and $[\mathbf{v}^{nc}]$ the displacement of the other nodes. We suppose that the contact forces are concentrated at the contact nodes. In this case, the vector $[\mathbf{R}_p]$ can be written via [5.3] as: $[\mathbf{R}_p^r] = [\mathbf{R}_p^c, \mathbf{0}]$, with

$$[\mathbf{R}_p^c] = [\mathbf{K}_{Tc}][\mathbf{v}_p^c] + [\mathbf{F}_p^{nlc}]$$

where $[\mathbf{K}_{Tc}]$ is the symmetric contact stiffness matrix and the vector $[\mathbf{F}_p^{nlc}]$ depends on terms at previous orders. Finally, the displacement problem at order p becomes: find $[\mathbf{v}_p]$ and λ_p solution of

$$[\mathbf{K}_T][\mathbf{v}_p] = \lambda_p[\mathbf{F}] + [\mathbf{F}_p^{nl}] \qquad [\mathbf{v}_p]^t[\mathbf{v}_1] + \lambda_p\lambda_1 = 0.$$

The algorithm to compute a solution branch can be described as follows:

Solution at order 1:

(1) Solve

$$[\hat{\mathbf{v}}] = [\mathbf{K}_T]^{-1}[\mathbf{F}]$$

(2) deduce unknowns at order 1

$$\lambda_1 = \frac{1}{\sqrt{(1+[\hat{\mathbf{v}}]^t[\hat{\mathbf{v}}])}}$$

$$[\mathbf{v}_1] = \lambda_1[\hat{\mathbf{v}}]$$

$$[\mathbf{R}_1^c] = [\mathbf{K}_{Tc}][\mathbf{v}_1^c]$$

Solution at order p:

(1) Calculate

$$[\widetilde{\mathbf{F}}_p^{nlc}] = [\mathbf{F}_p^{nlc}, \mathbf{0}]^T$$

$$[\mathbf{F}_p^{nl}] = [\mathbf{F}_p^{nlg}] + [\widetilde{\mathbf{F}}_p^{nlc}]$$

(2) Solve

$$[\mathbf{v}_p^{nl}] = [\mathbf{K}_T]^{-1}[\mathbf{F}_p^{nl}]$$

(3) deduce unknowns at order p

$$\lambda_p = -\lambda_1[\mathbf{v}_p^{nl}][\mathbf{v}_1]$$

$$[\mathbf{v}_p] = \frac{\lambda_p}{\lambda_1}[\mathbf{v}_1] + [\mathbf{v}_p^{nl}]$$

$$[\mathbf{R}_p^c] = [\mathbf{K}_{Tc}][\mathbf{v}_p^c] + [\mathbf{F}_p^{nlc}]$$

This algorithm needs only one matrix decomposition to compute a large part of the contact solution. However, the computation of the terms of the series becomes expensive if we exceed the order $p = 20$. But in the cases with many degrees of freedom, the series computation requires a rather small CPU time when compared to a matrix decomposition.

3.3. *A Continuation Algorithm*

The domain of validity of the solution branch is limited by the convergence radius of the series. A continuation procedure has been proposed in [COC 94-1], it consists on re-applying the previous algorithm step by step to obtain the entire branch of the solution. We define an end-step criterion which corresponds to a maximum value $a_m = \left(\epsilon \frac{\|\mathbf{v}_1\|}{\|\mathbf{v}_p\|}\right)^{\left(\frac{1}{p-1}\right)}$ of the control parameter '**a**', where ϵ is an accuracy parameter and p is the order of truncature. Generally, we take $\epsilon = 10^{-5}$ and $p = 20$.

3.4. *Improvement of the Series by Padé Approximants*

In order to improve the series representation of the solution previously described, we can use the Padé approximants [BAK 96][COC 94-2]. A technique is fully discussed in [NAJ 98], that transforms the power series in a sum of Padé approximants admitting all the same denominator. So we consider the series, truncated at an order $(n+1)$, of the discretized displacements and the loading parameter:

$$\mathbf{v}(a) = \sum_{i=0}^{n+1} \mathbf{v}_i a^i \qquad \lambda(a) = \sum_{i=0}^{n+1} \lambda_i a^i \qquad [7]$$

From the vectors $\mathbf{v}_i$, we construct an orthogonal basis by the classical Gram-Schmidt procedure. In this new basis, the expansions [7] can be replaced by:

$$P_n(\mathbf{v}(a)) = \sum_{i=0}^{n} \mathbf{v}_i \frac{D_{n-i}}{D_n} a^i \qquad P_n(\lambda(a)) = \sum_{i=0}^{n} \lambda_i \frac{D_{n-i}}{D_n} a^i \qquad [8]$$

where D_n is a polynomial of degree n.

The range of validity of the representation [8] can be restricted by the presence of real roots of the denominator D_n. Thus, we have used a classical algorithm of Bairstow to compute the roots of a polynomial.

3.5. *A Continuation Algorithm Using the Padé Approximants*

In order to have an efficient algorithm, we have to determine automatically the value a_{mp} of the control parameter over which the Padé solution is not acceptable. A very simple way to achieve this, is to define a criterion which requires that the relative difference between two consecutive order solutions remains small. This can be expressed by:

$$\frac{\|(P_n(\mathbf{v}(a)) - P_{n-1}(\mathbf{v}(a)))\|}{\|(P_n(\mathbf{v}(a)) - \mathbf{v}_0\|} \simeq \epsilon \qquad [9]$$

where ϵ is an accuracy parameter. However, the Padé approximants of a power series may have a cluster of poles in $\Re$, but the first real pole is generally greater than the radius of convergence of the approximated series. Consequently, the end of the step a_{mp} is the first value of 'a' which satisfies the criterion [9] in the interval $[a_m, first\ pole]$. It has been defined by some dichotomous iterations, that represents a supplementary but negligeable computing time as compared to that of the series [7].

Sometimes, but rarely, the a_{mp} found is smaller than the estimated radius of convergence a_m of the series. In this case, we leave the Padé approximation and we keep the series solution.

4. Examples and Discussion

We consider an elastic beam subjected to a an external loading $\lambda\mathbf{F}$ and undergoing contact with a rigid surface. The body is discretized in 4 nodes quadrilateral elements. The results of the ANM will be compared with the solution given by ABAQUS which uses a Newton method coupled with the Lagrange multipliers method [ABA 94].

Example 1

In this example, we consider the contact between an elastic beam and a plane rigid surface. The contact nodes have the same initial clearance $\delta = 2mm$ (Figure 5).

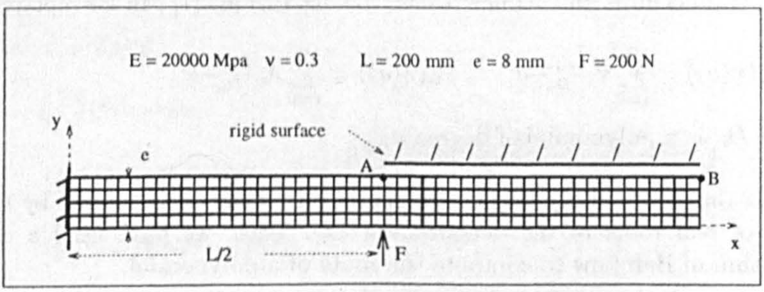


Figure 5. *Contact between an elastic beam and a plane rigid surface*

We denote by AB the contact line containing the nodes which are likely to undergo contact with the rigid surface. The contact starts firstly at the node B and the contact zone moves to attain the node A.

A regularization with high values of η may lead to a bad estimation of the displacements and the force contact forces, but the results are acceptable for the small values. For example, in neighborhood of the unsticking area, where the contact forces are reduced to zero we have an overestimation of the clearance h (Figure 7), but the curves are better for the contacting zone (Figure 6). Concerning the contact forces, the total contact force on the line AB is correct even with high values of η. With a small η, we have also a good distribution of the contact force at the zone where it is large (Figure 8).

However, when η is small the non-linearity becomes very strong and the convergence radius of the asymptotic expansions decreases. Table (1) shows that the number of steps passes from 90 with a small η to 42 with a higher one.

A mesh refinement at the contact surface is necessary to avoid gross interpenetrations. The convergence of the iterative method becomes more difficult when the number of contact nodes increases (Table 1). With the ANM, the mesh refinement permits to decrease the number of steps. Thus, with 658 degrees of freedom and 100 contact nodes, we obtain a good estimation of the solution with 85 matrix decompositions while ABAQUS needs 227 one.

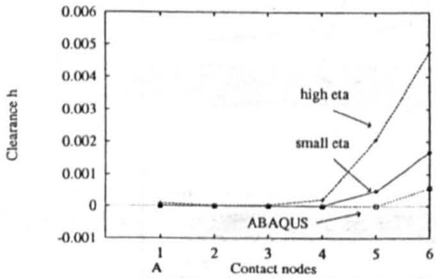


Figure 6. *h, contacting zone*

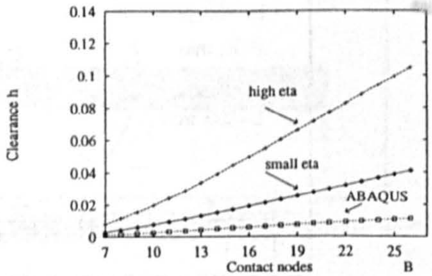


Figure 7. *h, unsticking zone*

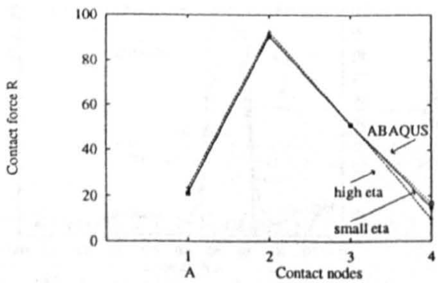


Figure 8. *R, contacting zone*

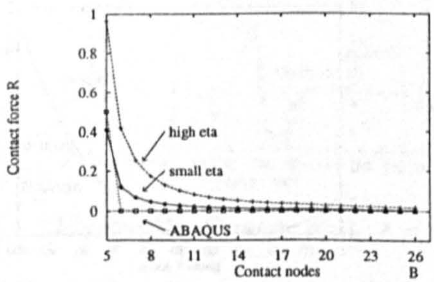


Figure 9. *R, unsticking zone*

ANM (p=20)		Padé (p=20)		ABAQUS
high η	small η	high η	small η	$\eta = 0$
42	90	31	51	128

Table 1. *Comparison between Abaqus, ANM and Padé algorithm - 510 d.o.f and 26 contact nodes*

The Padé technique has permitted to extend the domain of validity of the series with good end-step residuals. Consequently, with the continuation algorithm the number of matrix decompositions is divided by two (Table 1) with respect to the classical ANM formula [7].

Example 2

In this case, the contact line AB interacts with a circular rigid surface involving large displacements(Figure 10). The contact occurs firstly at node A and the contact region moves towards the node B. ABAQUS uses a finite sliding formulation to solve this problem.

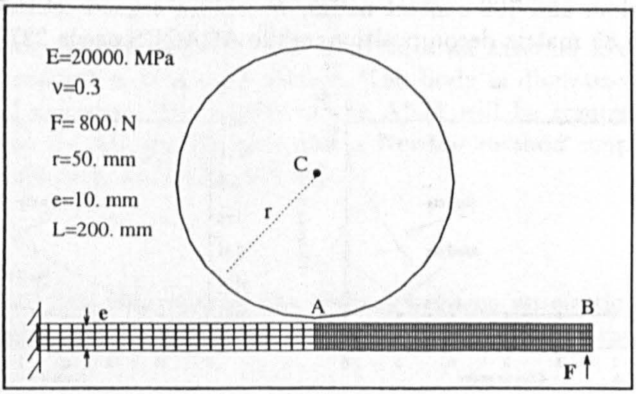
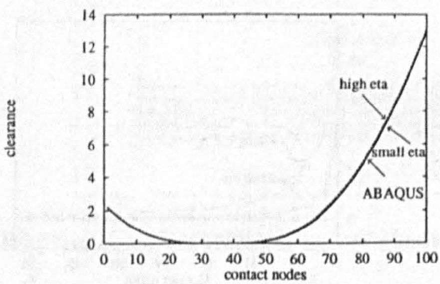


Figure 10. *Contact between an elastic beam and a circular rigid surface*



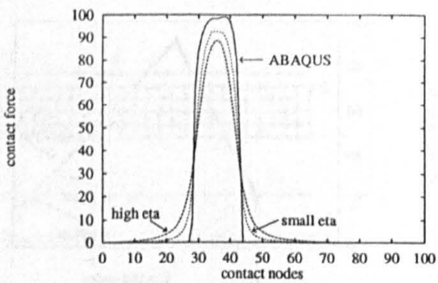


Figure 11. *Contour of h, $\lambda = 1$* **Figure 12.** *Contour of* $\mathbf{R}$, $\lambda = 1$

ANM (p=20)		Padé (p=20)		ABAQUS
high η	small η	high η	small η	$\eta = 0$
25	56	18	37	96

Table 2: *Comparison between Abaqus, ANM and Padé algorithm - 1240 d.o.f and 100 contact nodes*

We have obtained a good estimation of the solution (Figure 11) and (Figure 12) with less computing time than the Lagrange multiplier method (Table 2). The Padé approximants have also permitted to reduce the number of matrix decompositions with respect to the technique by series.

Example 3

In this example, we consider an arbitrary rigid surface composed of a set of six rigid segments. We approximate it by a six-degree Bézier polynomial. The elastic beam is subjected to a distributed loading (Figure 13).

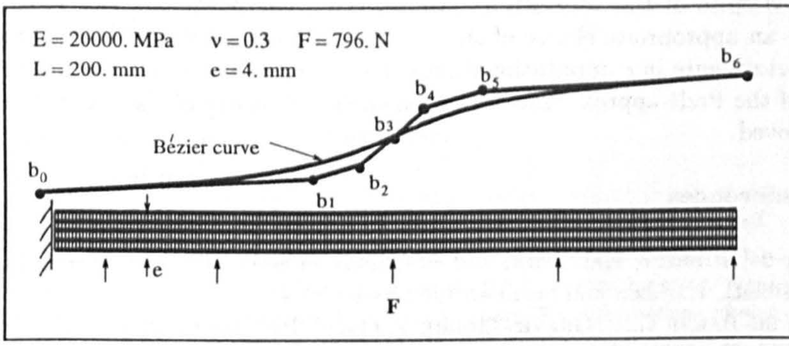


Figure 13. *Contact between an elastic beam and an arbitrary rigid surface*

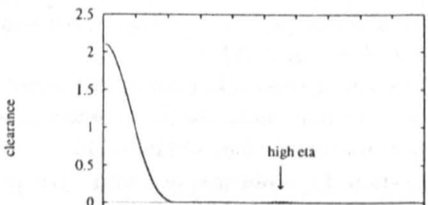


Figure 14. *Contour of h, $\lambda = 1$*

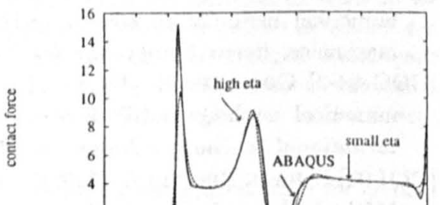


Figure 15. *Contour of* $\mathbf{R}$, $\lambda = 1$

We have obtained a good accuracy of the solution (Figure 14) and (Figure 15) with only 62 matrix decompositions in this case where there are approximately 160 contacting nodes at the end of the loading.

Indeed, the results are similar to those obtained with ABAQUS by representing the rigid surface by a broken line. The computation time with this Lagrange multiplier algorithm was large. One can also remark that the polynomial approximation technique does not require a large computational time as compared to the case of a plane rigid surface (90 matrix decompositions and 26 contacting nodes).

5. Conclusion

In this paper, we have presented an ANM algorithm for contact problems between a 2-D rigid surface and a deformable body undergoing large displacements. Due to the strong non-linear contact law, a regularized force-displacement relationship has been proposed. Then a perturbation technique has been applied to solve the contact problem. This technique based on asymptotic expansions transforms the non-linear problem into a sequence of linear problems admitting the same stiffness matrix. Therefore, a large number of terms can be calculated with small computing time. The tests have established that, despite of the very strong non-linearity, the ANM strategy is efficient. With an appropriate choice of the regularization parameter we can obtain an important gain in computational time, and satisfy the contact conditions. The use of the Padé approximants has allowed the efficiency of the algorithm to be improved.

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