Modal effective parameters in structural dynamics

Alain Girard — Nicolas A. Roy

INTESPACE
Research and Development Division
18, Avenue Edouard-Belin, BP 4356, 31029 Toulouse cedex 4

ABSTRACT. *Modal superposition techniques are generally used to analyse low frequency dynamic responses of complex structures. Within this context, modal effective parameters allow us to characterize in a comprehensive and intrinsic manner the eigenmodes, and to compute frequency response functions while controlling truncation effects. The general formulation is first given, introducing the effective flexibilities, masses and transmissibilities. These parameters provide a physical understanding of the dynamic phenomena; they also respect summation rules related to particular static properties so that truncation effects can be estimated in terms of residuals. Finally, they can be directly used for many purposes : selection of important modes, comparison of modal bases, elaboration of equivalent models, and computation of responses of any type.*

RÉSUMÉ. *Les techniques de superposition modale sont généralement utilisées pour analyser les réponses dynamiques des structures complexes aux basses fréquences. Dans ce contexte, les paramètres modaux effectifs permettent de caractériser les modes propres de manière compréhensive et intrinsèque, et de calculer les réponses fréquentielles tout en contrôlant les effets de troncature. La formulation générale, introduisant les flexibilités, masses et transmissibilités effectives est d'abord présentée. Ces paramètres donnent en premier lieu une interprétation physique des phénomènes dynamiques ; ils suivent aussi des règles de sommation faisant intervenir des propriétés statiques, permettant d'estimer les effets de troncature en termes de paramètres résiduels. En définitive, ils peuvent être utilisés directement à diverses fins : sélection de modes importants, comparaison de bases modales, élaboration de modèles équivalents et calcul de réponses de tous types.*

KEY WORDS : *dynamic structural analysis, vibration mode, modal response, parameter identification.*
MOTS-CLÉS : *analyse dynamique des structures, mode de vibration, réponse modale, identification de paramètres.*

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Nomenclature

Convention

X_{ij}	:	matrix X with rows and columns related to the i-set and the j-set of degrees of freedom respectively - by reciprocity : $X_{ji} = X_{ij}^T$ and X_{ii} symmetric

Scalars and matrices

A	:	rigid body mode filtering transformation matrix
C, c	:	viscous damping matrix
d	:	distance
F	:	vector of applied forces
G	:	flexibility, center of gravity
H	:	dynamic amplification factor
I	:	unit matrix, scalar inertia
i	:	$\sqrt{-1}$
K, k	:	stiffness matrix
L	:	modal participation factor
M, m	:	mass, mass matrix
O	:	reference point
Q	:	amplification factor at resonance = $1/2\zeta$
q	:	vector of modal coordinates
r	:	rotation
T	:	transmissibility
t	:	translation
u	:	vector of displacements
W	:	power spectral density function
X	:	general matrix or parameter
ζ	:	viscous damping ratio
Φ	:	matrix of constrained junction normal modes
Ψ	:	matrix of constraint modes
ψ	:	RMS value
ω	:	circular frequency

Subscripts

c	:	constrained degree of freedom (c + l = i)
i	:	internal degree of freedom
j	:	junction (interface) degree of freedom
k	:	normal mode
l	:	unconstrained degree of freedom (l + c = i)
r	:	rigid (statically determinate) junction degree of freedom
res	:	residual
s	:	selected
x	:	excitation
y	:	response

Superscripts

$\dot{X}$: time derivation
X^T : transpose of X ($X_{ij}^T = X_{ji}$)
$\tilde{X}$: effective parameter
$\bar{X}$: condensed matrix
$\hat{X}$: variant of X or $\bar{X}$
$\vec{X}$: vector in 3 dimensions

Abbreviations

DOF : Degree Of Freedom
FRF : Frequency Response Function
PSD : Power Spectral Density

1. Introduction

In the lower frequency range, the current practice for dynamic analysis of linear structures is based on the computation of dynamic response of its eigenmodes, considered as uncoupled one-DOF (Degree Of Freedom) systems. Dynamic responses are then obtained by summing the contribution of the first few modes only. This modal truncation decreases the computational effort, but may involve significant errors if not treated adequately. In addition, even if the number of selected modes is small with respect to the size of the system, they involve a large amount of data difficult to analyse for physical understanding. To overcome these difficulties, it is important prior to the dynamic response phase of the analysis, to identify the significant parameters which govern the dynamic phenomena. This leads to the general concept of modal effective parameters.

Various attempts have already been made in this area. The mode-acceleration method [WIL 45] [CRA 81] was used early to compensate the truncation errors by improving the convergence. A semi-graphical method involving mobility calculations was introduced in the mid-fifties [PLU 54]. The effective mass concept was introduced and developed in the sixties and seventies [NEU 64] [BAM 71] [WAD 72] and used intensely, in particular in the aerospace industry [IMB 78a] [IMB 78b] [MOR 79], but it covers only one aspect of the problem. An extension to other modal parameters was then made in the mid-eighties [GIR 86] [GIR 87] to present a unified approach for these parameters. Since that time, extensive use of this concept has been made in various fields of structural dynamics. In fact, as it will be seen, the modal effective parameters have a simple physical meaning and can be used for many purposes : detection and selection of important modes for a given response, evaluation of truncation effects, comparison of modal bases from tests and/or analyses, elaboration of simple equivalent models, and comprehensive computation of responses of any type.

In the present paper the fundamentals of the concept are reviewed, considering the general case of a structure excited by internal forces and/or interface motion and producing internal motion or interface force response (chapter 2). Modal decomposition of the various types of frequency response functions yields the definition of the effective parameters (chapter 3). Numerical examples on beams are given for direct illustration, completed by a more complex case in an industrial context (chapter 4).

2. Dynamic force and displacement responses

2.1. *Frequency response functions*

Let us consider a n-DOF linear structure with given boundary conditions (interface with the outer world, named "junction" in the following, for mnemonic reasons). The set of DOFs is therefore partitioned in two subsets, as illustrated in Figure 1 :

- the junction or interface DOFs (j-subset, null for a free-free structure),
- the internal DOFs (i-subset).

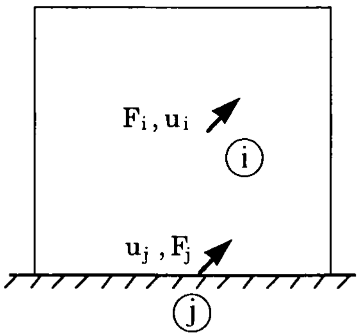


Figure 1. *Structural displacements and forces*

This structure is assumed to be subjected to prescribed junction motion (displacements u_j for example) and/or internal forces F_i. The dynamic analysis objective is to provide estimates of the responses of unknown internal displacements u_i and/or junction forces F_j. In the frequency domain ω, the relationship between the possible excitations F_i, u_j, and the possible responses u_i, F_j, can be written using FRF (Frequency Response Functions) :

$$\begin{bmatrix} u_i(\omega) \\ F_j(\omega) \end{bmatrix} = \begin{bmatrix} G_{ii}(\omega) & T_{ij}(\omega) \\ -T_{ji}(\omega) & K_{jj}(\omega) \end{bmatrix} \begin{bmatrix} F_i(\omega) \\ u_j(\omega) \end{bmatrix} \tag{1}$$

where G, T and K are the dynamic flexibility, transmissibility and stiffness matrices respectively ($T_{ji} = T_{ij}^T$ from reciprocal principle).

Let's recall, as mentioned in the nomenclature, that the subscripts correspond to the rows and columns of the matrices. For example, G_{ii} designates a square matrix where the rows, as the columns, are related to the i-set, while T_{ij} is a rectangular matrix where the rows are related to the i-set and the columns to the j-set.

2.2. *Equations of motion*

The previous FRF can be expressed using adequate modes of the structure and integrating the equations of motion. For that purpose, let's take the discrete equations of motion written as :

$$M \ddot{u} + C \dot{u} + K u = F \tag{2}$$

where M, C, and K are the mass, damping and stiffness matrices, u the vector of physical displacements, and F the vector of applied forces. The internal motion of the structure can then be written as the sum of the motion due to the junction and the internal motion expressed in the basis of the constrained junction normal modes :

$$u_i = \Psi_{ij}\, u_j + \Phi_{ik}\, q_k \tag{3}$$

where Ψ is the matrix of constraint modes and Φ is the matrix of normal modes, given by :

$$\Psi_{ij} = - K_{ii}^{-1} K_{ij} \tag{4}$$

$$(-\omega_k^2 M_{ii} + K_{ii})\, \Phi_{ik} = 0 \tag{5}$$

assuming that the j-set is sufficient to suppress the rigid body modes, redering the K_{ii} matrix invertible.

The following transformation :

$$\begin{bmatrix} u_j \\ u_i \end{bmatrix} = \begin{bmatrix} I & 0 \\ \Psi_{ij} & \Phi_{ik} \end{bmatrix} \begin{bmatrix} u_j \\ q_k \end{bmatrix} \tag{6}$$

applied to Eq. (2) results in

$$\begin{bmatrix} \overline{M}_{jj} & L_{jk} \\ L_{kj} & m_k \end{bmatrix} \begin{bmatrix} \ddot{u}_j \\ \ddot{q}_k \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_k \end{bmatrix} \begin{bmatrix} \dot{u}_j \\ \dot{q}_k \end{bmatrix} + \begin{bmatrix} \overline{K}_{jj} & 0 \\ 0 & k_k \end{bmatrix} \begin{bmatrix} u_j \\ q_k \end{bmatrix} = \begin{bmatrix} F_j + \Psi_{ji} F_i \\ \Phi_{ki} F_i \end{bmatrix} \tag{7}$$

where $\overline{M}_{jj}$ and $\overline{K}_{jj}$ are the mass and stiffness matrices condensed on the junction, m_k, c_k, k_k are the generalized mass, damping and stiffness diagonal matrices (uncoupled modal viscous damping [CRA 81]), and L_{kj} is the matrix of modal participation factors, given by :

$$L_{kj} = \Phi_{ki}\, (M_{ii}\, \Psi_{ij} + M_{ij}) \tag{8}$$

2.3. *Solutions*

Solving Eq. (7) leads to the following results for the FRF of Eq. (1) (details in [GIR 85]) :

$$G_{ii}(\omega) = \sum_{k=1}^{n} H_k(\omega) \frac{\Phi_{ik}\,\Phi_{ki}}{\omega_k^2\, m_k} \tag{9}$$

$$T_{ij}(\omega) = \sum_{k=1}^{n} T_k(\omega) \frac{\Phi_{ik}\, L_{kj}}{m_k} - M_{ii}^{-1} M_{ij} \tag{10}$$

$$K_{jj}(\omega) = -\,\omega^2 \left(\sum_{k=1}^{n} T_k(\omega) \frac{L_{jk}\, L_{kj}}{m_k} + M_{jj} - M_{ji} M_{ii}^{-1} M_{ij} \right) + \overline{K}_{jj} \tag{11}$$

where H_k and T_k are the amplification and transmissibility factors of mode k, related to its circular frequency ω_k and its viscous damping ratio ζ_k :

$$H_k(\omega) = \frac{1}{1 - \left(\frac{\omega}{\omega_k}\right)^2 + i\, 2\zeta_k \left(\frac{\omega}{\omega_k}\right)} \tag{12}$$

$$T_k(\omega) = \frac{1 + i\, 2\zeta_k \left(\frac{\omega}{\omega_k}\right)}{1 - \left(\frac{\omega}{\omega_k}\right)^2 + i\, 2\zeta_k \left(\frac{\omega}{\omega_k}\right)} \tag{13}$$

In the case of structural damping, $2\zeta_k\ (\omega/\omega_k)$ is replaced by the modal loss factor η_k in Eqs. (12) and (13). In Eqs. (9), (10) and (11), the summation clearly expresses the modal superposition, each mode behaving like a 1-DOF system. Every n-DOF system is in fact equivalent to n 1-DOF systems in parallel [MAC 71]. For free structures, the DOFs reduce to the i-set , and only Eq. (9) is relevant, with the contribution of the rigid body modes reducing to $\Phi_{ik}\, \Phi_{ki}/(-\omega^2\, m_k)$.

Equation (11) may be rewritten :

$$K_{jj}(\omega) = -\,\omega^2\, M_{jj}(\omega) + \overline{K}_{jj} \tag{14}$$

where :

$$M_{jj}(\omega) = \sum_{k=1}^{n} T_k(\omega) \frac{L_{jk}\, L_{kj}}{m_k} + M_{jj} - M_{ji} M_{ii}^{-1} M_{ij} \tag{15}$$

$M_{jj}(\omega)$ is the dynamic mass matrix which excludes $\overline{K}_{jj}$ representing the static contribution from statically indeterminate junctions (otherwise $\overline{K}_{jj}$ null). The static terms outside of the summation over the modes in Eqs. (10) and (15) are directly related to the junction DOFs and result from the discretization itself. They tend towards zero as the mesh is refined and may often be neglected in practice. In the case of a continuous approach, they are null.

3. Modal effective parameters

The three FRF matrices G, T and M (Eqs. (9), (10) and (15)) have the same basic form expressing each normal mode contribution as the product of an amplification factor, $H_k(\omega)$ or $T_k(\omega)$, and a matrix of terms independent of ω known as "effective" parameters :

$$G_{ii}(\omega) = \sum_{k=1}^{n} H_k(\omega)\, \widetilde{G}_{ii,k} \tag{16}$$

$$T_{ij}(\omega) = \sum_{k=1}^{n} T_k(\omega)\, \widetilde{T}_{ij,k} - M_{ii}^{-1} M_{ij} \tag{17}$$

$$M_{jj}(\omega) = \sum_{k=1}^{n} T_k(\omega)\, \widetilde{M}_{jj,k} + M_{jj} - M_{ji} M_{ii}^{-1} M_{ij} \tag{18}$$

where

$$\widetilde{G}_{ii,k} = \frac{\Phi_{ik}\, \Phi_{ki}}{\omega_k^2\, m_k} \tag{19}$$

$$\widetilde{T}_{ij,k} = \frac{\Phi_{ik}\, L_{kj}}{m_k} \tag{20}$$

$$\widetilde{M}_{jj,k} = \frac{L_{jk}\, L_{kj}}{m_k} \tag{21}$$

are the effective flexibility, transmissibility and mass matrices of mode k. These effective parameters are independent of the normalization of the eigenvectors. They have the same dimension as the corresponding FRF and are directly related to physical quantities such as static flexibilities, constraint modes and masses. Moreover, they characterize the importance of the mode in the overall behaviour, independently of the nature of the excitation.

3.1. *Summation rules*

From Eqs. (16) to (21), it can be shown that the effective parameters follow specific summation rules. In particular, the direct summation may be derived from Eq. (9), (10) and (15) with $\omega = 0$, leading to :

$$\sum_{k=1}^{n} \widetilde{G}_{ii,k} = G_{ii} \tag{22}$$

$$\sum_{k=1}^{n} \widetilde{T}_{ij,k} = \widehat{\Psi}_{ij} = \Psi_{ij} + M_{ii}^{-1} M_{ij} \tag{23}$$

$$\sum_{k=1}^{n} \widetilde{M}_{jj,k} = \widehat{M}_{jj} = \overline{M}_{jj} - \left(M_{jj} - M_{ji} M_{ii}^{-1} M_{ij}\right) \tag{24}$$

where $G_{ii} = K_{ii}^{-1}$ is the static flexibility matrix corresponding to the i-set, discussed later, representing, along with Ψ_{ij} and $\overline{M}_{jj}$, the physical static properties of the structure with respect to the i- and j-sets (the additional terms resulting from discretization, as previously mentioned). Eqs. (22) to (24) can be interpreted as a projection of the static terms in the normal modes basis. In particular, Eq. (23) expresses the constraint modes as a linear combination of the normal modes :

$$\Psi_{ij} = \sum_k \Phi_{ik} \frac{L_{kj}}{m_k} - M_{ii}^{-1} M_{ij} \qquad (25)$$

The summation rules can be generalized to include effective parameters multiplied by powers of $\omega_k{}^2$, as shown in Table 1. This is useful for taking into account higher-order terms such as inertia effects.

$\widetilde{X}_k$	$\widetilde{G}_{ii,k}$	$\widetilde{T}_{ij,k}$	$\widetilde{M}_{jj,k}$
$\sum_{k=1}^{n} \omega_k^2 \widetilde{X}_k$	M_{ii}^{-1}	$M_{ii}^{-1} K_{ii} \widehat{\Psi}_{ij}$	$\widehat{\Psi}_{ji} K_{ii} \widehat{\Psi}_{ij}$
$\sum_{k=1}^{n} \widetilde{X}_k$	K_{ii}^{-1}	$\widehat{\Psi}_{ij} = \Psi_{ij} + M_{ii}^{-1} M_{ij}$	$\widehat{M}_{jj} = \widehat{\Psi}_{ji} M_{ii} \widehat{\Psi}_{ij}$
$\sum_{k=1}^{n} \frac{1}{\omega_k^2} \widetilde{X}_k$	$K_{ii}^{-1} M_{ii} K_{ii}^{-1}$	$K_{ii}^{-1} M_{ii} \widehat{\Psi}_{ij}$	$\widehat{\Psi}_{ji} M_{ii} K_{ii}^{-1} M_{ii} \widehat{\Psi}_{ij}$

Table 1. *Summation rules for effective parameters*

3.2. *Truncation effects*

When computing responses with a truncated set of modes, the contribution of the excluded modes may be approximated using the previous considerations. Since their eigenfrequencies are significantly higher than the excitation frequency, their amplification factors (Eqs. (12) and (13)) approach 1 and their contributions to the FRF in Eqs. (16) to (18) reduce to their effective parameters (first order approximation), leading to :

$$G_{ii}(\omega) \approx \sum_{k=1}^{m} H_k(\omega)\, \widetilde{G}_{ii,k} + G_{ii,res} \qquad (26)$$

$$T_{ij}(\omega) \approx \sum_{k=1}^{m} T_k(\omega)\, \widetilde{T}_{ij,k} + T_{ij,res} \qquad (27)$$

$$M_{jj}(\omega) \approx \sum_{k=1}^{m} T_k(\omega)\, \widetilde{M}_{jj,k} + \overline{M}_{jj,res} \qquad (28)$$

where k denotes the m-set of truncated modes and $G_{ii,res}$, $T_{ij,res}$, $\overline{M}_{jj,res}$ are residual terms representing the contribution of the higher modes and the junction due to discretization :

$$G_{ii,res} = \sum_{k=m+1}^{n} \tilde{G}_{ii,k} \tag{29}$$

$$T_{ij,res} = \sum_{k=m+1}^{n} \tilde{T}_{ij,k} - M_{ii}^{-1} M_{ij} \tag{30}$$

$$\overline{M}_{jj,res} = \sum_{k=m+1}^{n} \tilde{M}_{jj,k} + M_{jj} - M_{ji} M_{ii}^{-1} M_{ij} \tag{31}$$

These residual terms can be directly derived from the static terms and the effective parameters of the retained modes, from Eqs. 22 to 24 :

$$G_{ii,res} = G_{ii} - \sum_{k=1}^{m} \tilde{G}_{ii,k} \tag{32}$$

$$T_{ij,res} = \Psi_{ij} - \sum_{k=1}^{m} \tilde{T}_{ij,k} \tag{33}$$

$$\overline{M}_{jj,res} = \overline{M}_{jj} - \sum_{k=1}^{m} \tilde{M}_{jj,k} \tag{34}$$

Neglecting these terms in the computation of the responses can lead to significant errors. Using them minimizes truncation effects by taking into account the static contribution of the higher modes, while neglecting only the additional contribution due to their dynamic amplifications, which is small if the truncation is coherent with the frequency content of the excitation.

3.3. *Statically determinate junction*

The particular case of a statically determinate, or rigid, junction with (up to) 6 DOFs does not affect the flexibilities, but simplifies the basic terms for masses and transmissibilities as shown in table 2 (j-subset renamed r-subset).

General case	Rigid junction	Interpretation
j	r	(up to) 6 rigid junction DOFs
Ψ_{ij}	Ψ_{ir}	6 rigid body modes
$\overline{K}_{jj}$	0	condensed stiffness matrix null at junction
$\overline{M}_{jj}$	$\overline{M}_{rr}$	6x6 rigid body mass matrix
$\tilde{M}_{jj,k}$	$\tilde{M}_{rr,k}$	6x6 "physical" effective mass matrix

Table 2. *Particular case of statically determinate (rigid) junction*

In general, the r-subset may be composed of DOFs of several distinct nodes. However, it can always be reduced to the 6 DOFs of a single reference node by means of a rigid body transformation, as will be supposed in the following.

Each modal effective mass matrix $\widetilde{M}_{rr,k}$ (6x6) has only 1 non null eigenvalue which is in fact the "mass" of mode k acting in the eigendirection L_{rk} (see Eq. (21)). To illustrate this point, let's suppose that the r-set is related to the 6 DOFs of a node O, (translations 1, 2, 3 and rotations 4, 5, 6), as shown in Figure 2. In this case, $\widetilde{M}_{rr,k}$ represents the masses, moments and inertia of the mode with respect to O, including :

$$\widetilde{M}_k = \widetilde{M}_{11,k} + \widetilde{M}_{22,k} + \widetilde{M}_{33,k} \tag{35}$$

$$\widetilde{I}_{Ok} = \widetilde{M}_{44,k} + \widetilde{M}_{55,k} + \widetilde{M}_{66,k} \tag{36}$$

where $\widetilde{M}_k$ is the scalar mass acting in direction $\vec{t}_k$ (L_{k1},L_{k2},L_{k3}), and $\widetilde{I}_{Ok}$ the scalar inertia acting around the direction $\vec{r}_k$ (L_{k4},L_{k5},L_{k6}) [BAM 71] [WAD 72].

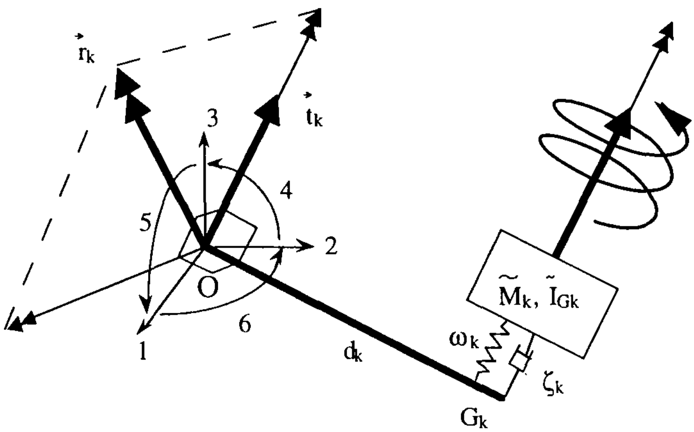


Figure 2. *1-DOF system from effective masses*

Additional considerations can be derived concerning the physical interpretation [GIR 91]. The center of gravity G_k of $\widetilde{M}_k$ may be found by considering its moment with respect to O, giving :

$$\overrightarrow{OG_k} = \frac{\vec{t}_k \wedge \vec{r}_k}{\left|\vec{t}_k\right|^2} \tag{37}$$

and the inertia with respect to G_k is given by :

$$\tilde{I}_{Gk} = \frac{\left| \vec{t}_k \cdot \vec{r}_k \right|^2}{\left| \vec{t}_k \right|^2 m_k} \tag{38}$$

A physical representation of each mode may then be derived. Each mode k is equivalent to a 1-DOF system having the mass/inertia ($\tilde{M}_k$, $\tilde{I}_{Gk}$), moving at G_k like a corkscrew in the direction $\vec{t}_k$ with a coupled translation and rotation in the ratio $(\vec{t}_k \cdot \vec{r}_k)/\left|\vec{t}_k\right|^2 = \pm\sqrt{\tilde{I}_{Gk}/\tilde{M}_k}$, on a spring giving the frequency ω_k and a dashpot giving the viscous damping ratio ζ_k, as depicted in Figure 2. This leads to an equivalent model for the structure with respect to the point O made of a collection of 1-DOF systems completed by a residual term (from Eq. 34). The general case is difficult to depict graphically, but practical cases can be easily represented such as axial models (DOF 1) shown in Figure 3-a, and lateral models (DOFs 2 and 6) shown in Figure 3-b where the pure masses $\tilde{M}_{22,k}$ ($\tilde{I}_{Gk}$ = 0 in Eq. (38)) act in direction 2 at distances $d_k = L_{k6}/L_{k2}$ from O (from Eq. (37)). Similar results may be found for any r-set.

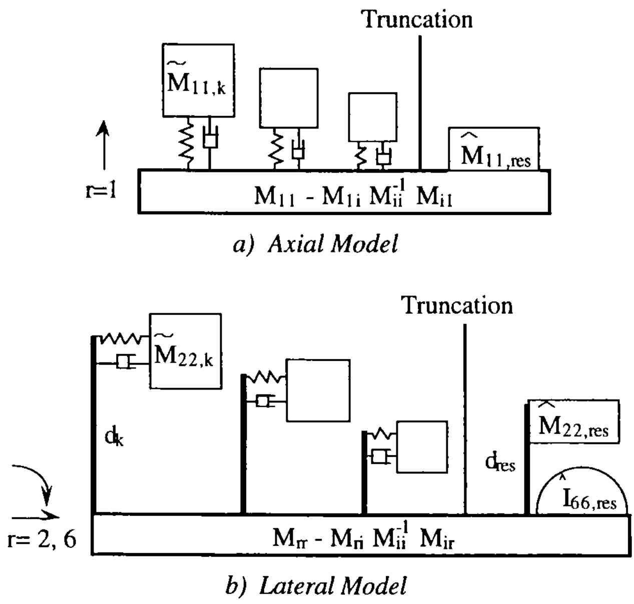


Figure 3. *Equivalent effective mass models*

These "effective mass models" are very useful in defining equivalent models with respect to the junction. They can easily be connected to the adjacent structure for coupled analysis. Although they are related to the junction and therefore excited by u_r , they can also be used in the general case of excitations F_i and u_r. The force F_i

in this case is proportionally distributed on each effective mass $\tilde{M}_{rr,k}$ according to the corresponding effective transmissibility $\tilde{T}_{ri,k}$:

$$\tilde{F}_{r,k} = \tilde{T}_{ri,k}\, F_i \qquad (39)$$

as depicted in Figure 4.

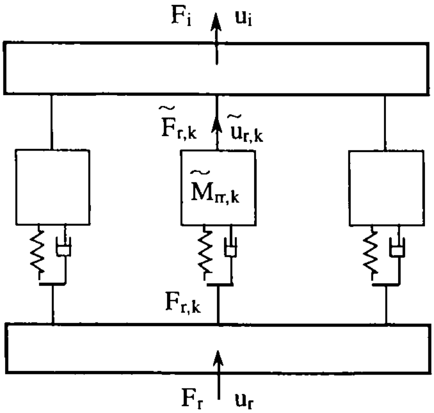


Figure 4. *Effective mass model - General case*

This is equivalent to considering an effective force/moment ($\vec{\tilde{\tilde{F}}}_k$, $\vec{\tilde{\tilde{M}}}_k$) applied directly to the mass/inertia ($\tilde{M}_k$, $\tilde{I}_{Gk}$) , with :

$$\vec{\tilde{\tilde{F}}}_k = \frac{\vec{t}_k}{m_k}\, \Phi_{ki}\, F_i \qquad (40)$$

$$\vec{\tilde{\tilde{M}}}_k = \frac{\vec{t}_k \cdot \vec{r}_k}{\left|\vec{t}_k\right|^2}\, \vec{\tilde{\tilde{F}}}_k \qquad (41)$$

which represents the generalized force properly normalized. The effective mass excited by F_i and u_r results in a translation/rotation giving the displacement $\tilde{u}_{r,k}$ and the reaction force $F_{r,k}$. The physical displacements u_i are then recovered by summing the $\tilde{u}_{r,k}$ weighted by the effective transmissibilities, whereas the reaction forces are recovered by summing directly the $F_{r,k}$:

$$u_i = \sum_k \tilde{T}_{ir,k}\, \tilde{u}_{r,k} \qquad (42)$$

$$F_r = \sum_k F_{r,k} \tag{43}$$

In case of modal truncation, residual terms have to be added :

$$\begin{bmatrix} u_{i,res} \\ F_{r,res} \end{bmatrix} = \begin{bmatrix} G_{ii,res} & T_{ir,res} \\ -T_{ri,res} & -\omega^2 \overline{M}_{rr,res} \end{bmatrix} \begin{bmatrix} F_i \\ u_r \end{bmatrix} \tag{44}$$

This can be directly extended to any statically indeterminate j-set, even if the physical interpretation is more difficult.

3.4. *Static and pseudo-static flexibilities*

In dealing with constrained structures, the static flexibility matrix G_{ii} introduced in Eq. (22) is the inverse of the stiffness matrix K_{ii}. In practice, only a subset s (selected) of DOFs i is concerned by excitation forces. In this case, the problem reduces to :

$$K_{ii} \; G_{is} = I_{is} \tag{45}$$

where I_{is} is an identity matrix partitioned on the s-set. Eq. (45) can be solved efficiently using a forward/backward substitution applied to the s-set.

In the case of free structures, the summation of Eq. (22) is limited to the elastic modes. The stiffness matrix K_{ii} is singular and the rigid body modes must be filtered to obtain the pseudo-flexibility matrix representing the flexibilities of the structure around its center of gravity. A classical approach is to constrain the structure at arbitrary DOFs c which prohibit all rigid body motion in order to obtain a "constrained flexibility matrix" $\widehat{G}_{ii}$. The pseudo-flexibility matrix G_{ii} is derived using the rigid body mode filtering transformation A (non symmetric) :

$$G_{ii} = A_{ii}^T \, \widehat{G}_{ii} \, A_{ii} \tag{46}$$

where

$$A_{ii} = I_{ii} - M_{ii} \, \Psi_{ic} \, \overline{M}_{cc}^{-1} \, \Psi_{ci} \tag{47}$$

with Ψ_{ic} the rigid body constraint modes. A procedure similar to Eq. (45) may be applied to find more efficiently the submatrix G_{is} in two steps using the partitioned stiffness matrix K_{ll} with $l = i - c$. First, $\widehat{G}_{is}$ is obtained by the following two relations [RUB 75] [CRA 81] :

$$\widehat{G}_{cs} = 0_{cs} \quad \text{and} \quad K_{ll} \, \widehat{G}_{ls} = A_{ls} \tag{48}$$

Secondly, G_{is} is derived by filtering the rigid body contribution from $\widehat{G}_{is}$:

$$G_{is} = A_{ii}^T \widehat{G}_{is} \tag{49}$$

The columns of G_{is} contain the inertia relief modes of the free structure with respect to the s-set.

3.5. *Dynamic Responses*

In the frequency domain ω, a dynamic response is obtained by multiplying the dynamic excitation by the corresponding FRF, as in Eq. (1). This covers the case of sinusoidal forces of pulsation ω, and also the case of transient forces by using Fourier transforms. The basic FRF given in Eqs. (16) to (18) have the same shape due to their similar expressions, as shown in Figure 5. Each mode behaves as a 1-DOF system, and contributes by its amplification factor multiplied by its effective parameter. At ω = 0, the FRF is equal to the accumulated effective parameters, i.e. the static value. Near each natural frequency, the corresponding mode is generally predominant (resonance) and the FRF is close to its contribution. Between two resonances, the bordering modes create a minimum : an antiresonance if the effective parameters of the bordering modes have the same sign, a trough if the effective parameters have opposite signs. Many other features concerning the FRF shape may be found by interpreting Eqs. (16) to (18). In addition, the shapes may be qualitatively found by considering only the natural frequencies, the damping ratios and the effective parameters.

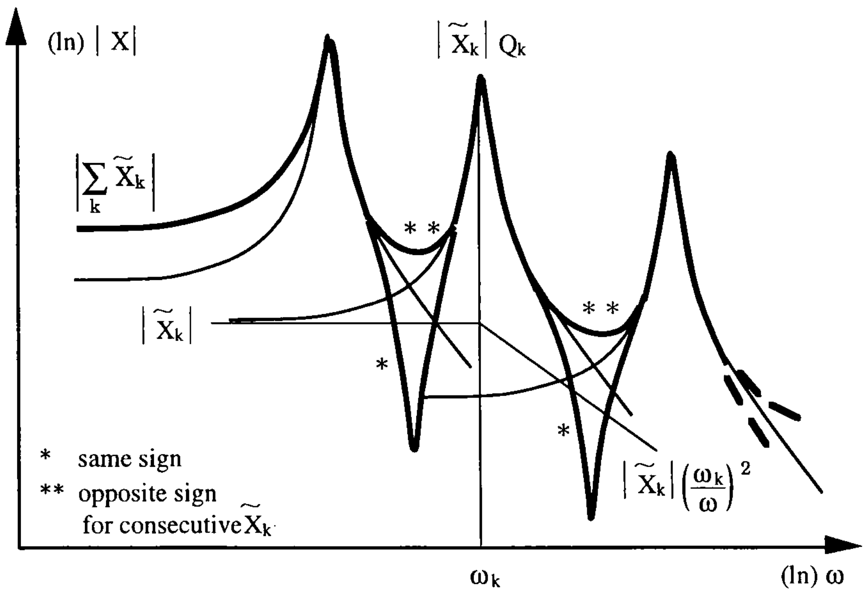


Figure 5. *FRF and effective parameters*

In the case of a transient response, an equivalent effective mass model as defined previously and manipulated through Eqs. (39) to (44) may be used for frequency response spectrum calculations to provide directly the maximum contribution of each mode to the internal displacements and reaction forces.

In the case of a random excitation x described by its PSD (Power Spectral Density) $W_x(\omega)$ in a given frequency range, the PSD $W_y(\omega)$ of a random response y is given by :

$$W_y(\omega) = \left| X_{yx}(\omega) \right|^2 W_x(\omega) \tag{50}$$

where X_{yx} is the FRF between x and y. The mean square value of the response ψ_y^2 is obtained by integrating W_y in the entire frequency band. If X_{yx} is one of the three basic FRF of Eqs. (16) to (18), ψ_y^2 is generally well approximated by the following formula, assuming that W_x is slowly varying in the vicinity of the natural frequencies and that the modes are relatively well separated :

$$\psi_y^2 = \sum_k \frac{\pi}{2} f_k Q_k \tilde{X}_k^2 W_x(f_k) + X_{res}^2 \psi_x^2 \tag{51}$$

where k is the set of modes included in the frequency band, f_k the natural frequencies, $Q_k = 1/2\zeta_k$ the amplification factors at f_k, $\tilde{X}_k$ the effective parameters related to X_{yx}, X_{res} the corresponding residual term (see Eqs. (32) to (34)), $W_x(f_k)$ the excitation PSD at f_k, and ψ_x^2 the mean square value of the excitation. As in the case of FRFs, each mode contributes as a 1-DOF system, and the last term of Eq. (51) represents the static contribution of the higher modes.

3.6. *Conclusions*

Modal effective flexibilities, transmissibilities and masses are inherent to solving the equations of motion in a modal basis. These parameters follow specific summation rules to recover relevant static properties so that truncation effects can be estimated by means of residuals. Consequently, they play a fundamental role in the context of modal superposition, providing physical understanding of dynamic responses. They can be used for many purposes as described in the introduction.

These basic considerations are derived from the equations of motion for the reaction forces and displacements related to the structural DOFs. It is possible to extend these considerations to the next step of analysis, i.e. the recovery step at the element level, especially for the dynamic stresses which are often of primary importance [GIR 86] [GIR 87].

4. Examples

4.1. *The Cantilever beam example*

Let's consider the cantilever beam of Figure 6 in pure bending, with length L, mass M, and bending stiffness EI (shear effect neglected). The r-set (statically determinate junction) is composed of DOFs 2 and 6 at the clamped end, and the i-set considered here is just composed of DOFs 2 and 6 at the free end. The static properties (Eqs. (22) to (24)) related to the i- and r-sets are as follows :

$$G_{ii} = \begin{bmatrix} \frac{L^3}{3\,EI} & \frac{L^2}{2\,EI} \\ \frac{L^2}{2\,EI} & \frac{L}{EI} \end{bmatrix} \qquad \Psi_{ir} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \qquad \overline{M}_{rr} = \begin{bmatrix} M & \frac{M\,L}{2} \\ \frac{M\,L}{2} & \frac{M\,L^2}{3} \end{bmatrix} \tag{52}$$

Numerical results related to dimensionless modal parameters are given in Table 3 (continuous system formulation - see details in [GIR 85] [GIR 86] [GIR 87]). Accumulated effective parameters are plotted in Figure 6 versus frequency, showing the convergence with respect to the summation rules (Eqs. (22) to (24)).

For example, concerning the effective masses $\tilde{M}_{22,k}$ the first mode concentrates 61 % of the total mass M, the second mode 19 %, etc. These values can be directly used to derive the equivalent model of Figure 3b, the parameters d_k being given by $\tilde{M}_{26,k} / \tilde{M}_{22,k}$, i.e. 0.7265 L for the first mode, 0.2092 L for the second mode, etc.

4.2. *Industrial example*

For complex structures, modal effective parameters are extremely useful in detecting modal contribution and assessing truncation errors. An industrial example is presented using a symmetrically modeled car seat and floorboard shown in Figure 7. The finite element model obtained using MSC/NASTRAN™ [MSC 95] contains over 50 000 degrees of freedom. It is clamped along the edges of the floorboard and "guided" in the plane of symmetry, thus generating 77 symmetrical modes up to 300 Hz.

Modal effective flexibilities computed using PROTO-Dynamique [ITS 97] are illustrated in Figure 8a for a point dynamic flexibility at the base of the car seat. Individual modal parameters are shown by circles and spikes, whereas the cumulative sum is shown as a staircase plot. The horizontal line represents the corresponding static term. Note that for the given transfer, over 70 % of the total contribution (given by the static term) is coming from only 6 modes. Moreover, the contribution of the 77 modes results in a truncation error of approximately 10 % with respect to the static flexibility. The corresponding dynamic flexibility calculated with 2% modal damping and including the static residual flexibility is plotted in Figure 8b.

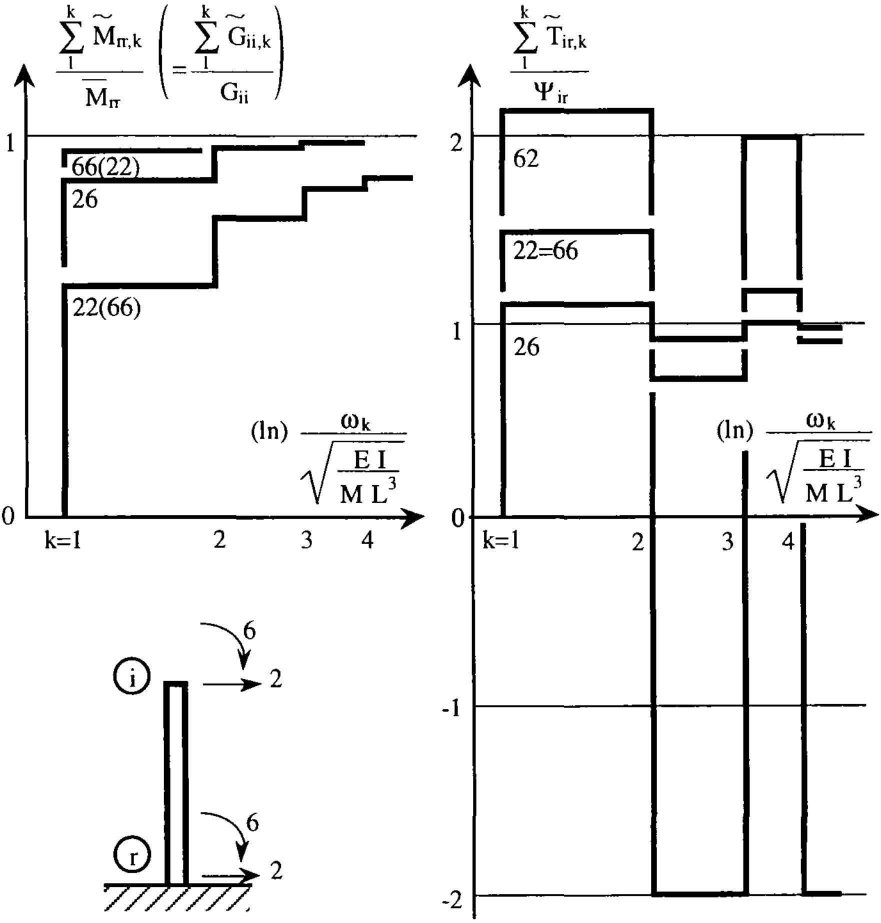


Figure 6. *Cantilever beam - Accumulated effective parameters*

k	$\frac{\omega_k}{\sqrt{\frac{EI}{ML^3}}}$	$\widetilde{M}_{22,k}/\overline{M}_{22}$ $= \widetilde{G}_{66,k}/G_{66}$	$\widetilde{M}_{26,k}/\overline{M}_{26}$ $= \widetilde{G}_{26,k}/G_{26}$	$\widetilde{M}_{66,k}/\overline{M}_{66}$ $= \widetilde{G}_{22,k}/G_{22}$	$\widetilde{T}_{22,k}$ $= \widetilde{T}_{66,k}$	$\frac{\widetilde{T}_{26,k}}{L}$	$\widetilde{T}_{62,k}$
1	3.516	0.6131	0.8908	0.9707	+ 1.5660	+1.1377	+ 2.1556
2	22.03	0.1883	0.0788	0.0247	- 0.8679	- 0.1815	- 4.1494
3	61.70	0.0647	0.0165	0.0032	+ 0.5088	+0.0648	+ 3.9936
4	120.9	0.0331	0.0060	0.0008	- 0.3638	-0.0331	-4.0002
>4	$\left(\frac{(2k-1)\pi}{2}\right)^2$	$\left(\frac{4}{(2k-1)\pi}\right)^2$	$\left(\frac{4}{(2k-1)\pi}\right)^3$	$\frac{3}{4}\left(\frac{4}{(2k-1)\pi}\right)^4$	$\frac{-8(-1)^k}{(2k-1)\pi}$	$\frac{-16(-1)^k}{((2k-1)\pi)^2}$	$-4(-1)^k$
Σ	-	1	1	1	1	1	± 2

Table 3. *Cantilever beam - Dimensionless modal parameters*

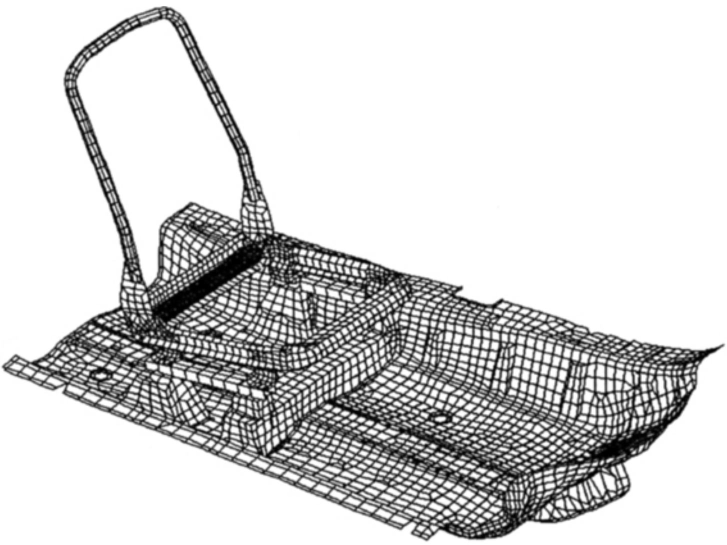


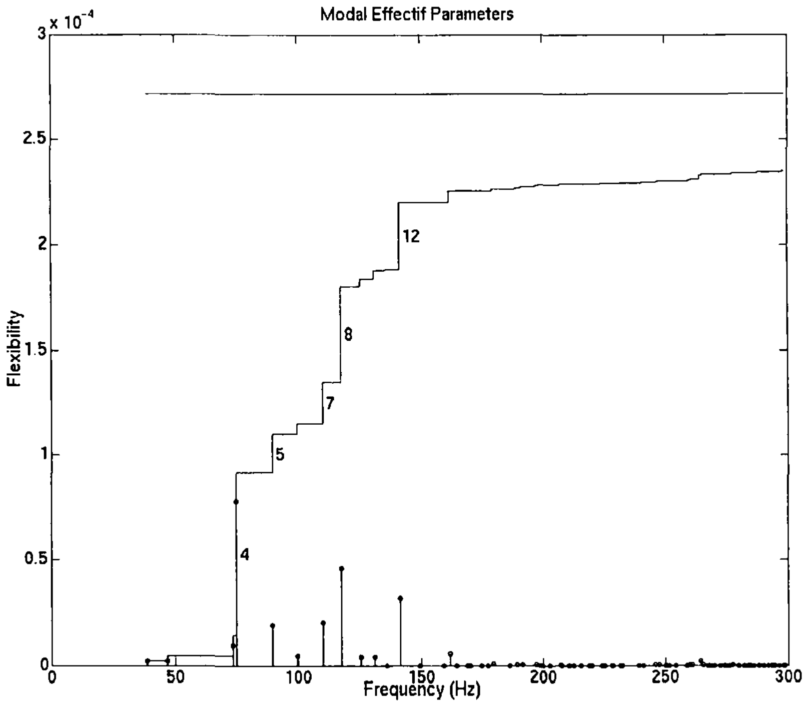
Figure 8. *Industrial example - Car seat and floorboard model*

A second transfer function involving the transmissibility between the base of the car seat and the rigid junction of the floorboard is illustrated in Figure 9. The modal effective transmissibilities shown in Figure 9a have both positive and negative values leading to a non-monotonic cumulative sum which "overshoots" the static term of 1 (unit translation). As before, a small number of modes account for the majority of the total static contribution. The dynamic transmissibility is plotted in Figure 9b and displays a characteristically intricate behavior including antiresonances as a result of the sign changes in the modal effective parameters. Note that at low excitation frequencies, the response converges to the exact static value thanks to the addition of the residual term.

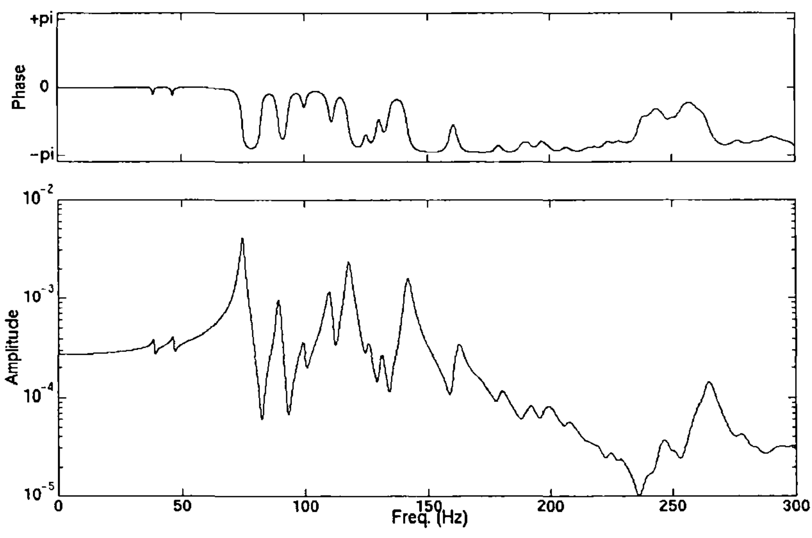
5. Conclusions

Modal effective parameters within the context of modal superposition techniques play an important and integral role in terms of better understanding the behavior and limitations of the modal representation of a physical structure. Independent of mode shape normalization, the modal effective parameters bridge the gap between the modal and physical (static) worlds and allow the engineer to quantitatively identify important modes and compensate for mode truncation effects. In addition to their diagnostic features, the modal effective parameters can be directly exploited in response calculations including stresses, and used to elaborate equivalent models for analysis or substructuring.

The authors hope that this paper will allow engineers to benefit from the simplifying and practical aspects of modal effective parameters and use them as a general and efficient tool in dynamic analysis and design problems.

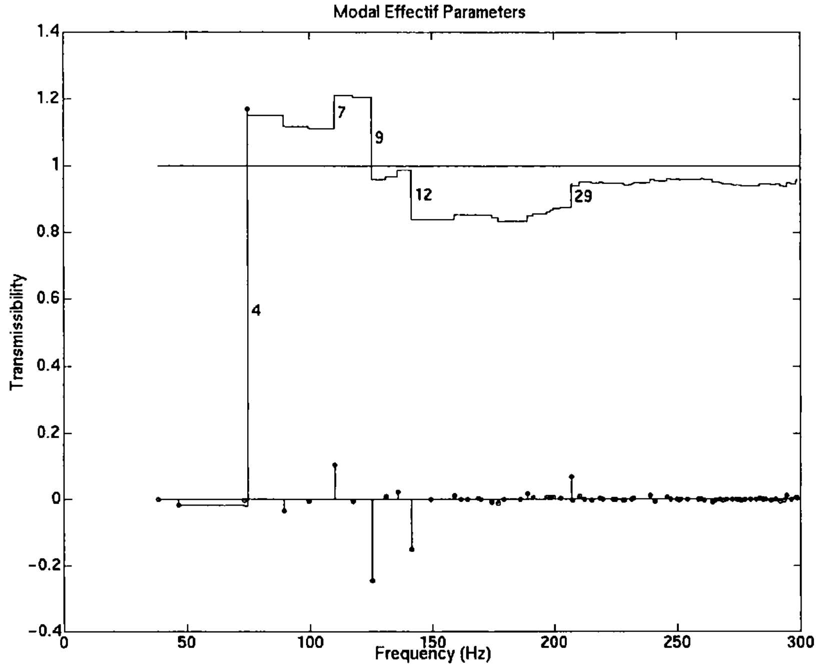


a) Modal effective flexibilities and static term

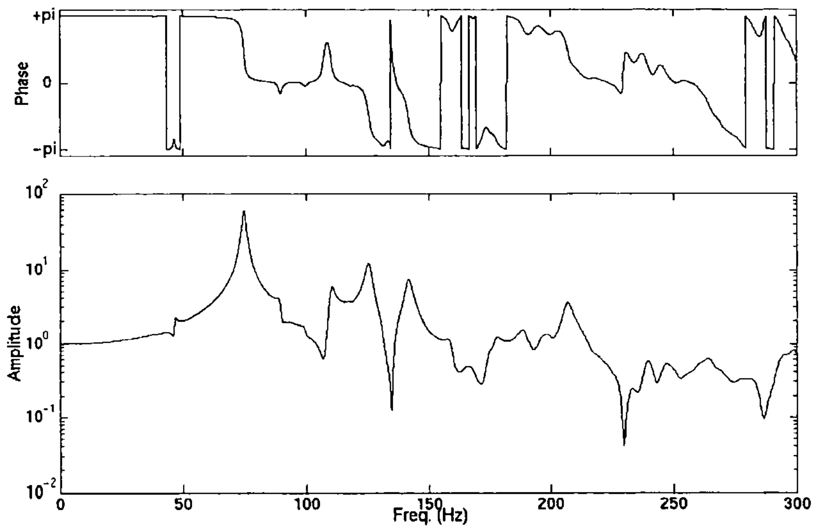


b) Corresponding dynamic flexibility

Figure 8. *Industrial example - Point flexibility*



a) Modal effective transmissibilities and static term



b) Corresponding dynamic transmissibility

Figure 9. *Industrial example - Transmissibility*

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