An asymptotic model of laminated plates

Renaud Kail* — Françoise Léné* — Yann-Hervé de Roeck**

** Laboratoire de Modélisation et Mécanique des Structures*
Université Pierre et Marie Curie, tour 66, 5ᵉ étage
4 place Jussieu, 75252 Paris cedex 5
kail@lmmc.jussieu.fr

*** Service Mécanique des Structures et Géotechnique*
IFREMER Centre de Brest, BP 70, 29280 Plouzané
yhdr@ifremer.fr

ABSTRACT. *This model of laminated plate has been developed for approximating the full three-dimensional stress tensor within the structure. In order to link the three-dimensional displacements and stresses to the displacements of the middle surface, the herein described technique consists in performing and asymptotic development of the solution fields with respect to the half-thickness of the plate. The construction of the complete asymptotic development, giving the possibility to choose the degree of the approximation of the solution, together with the very few restrictive assumptions required by this method enable the study of a very large panel of plates. The model is validated by comparison with an experiment, and also against other finite element models and an analytical solution.*

RÉSUMÉ. *Ce modèle de plaque composite a été développé dans le but d'obtenir une approximation de tout le tenseur des contraintes. Pour relier les déplacements et contraintes tridimensionnels aux déplacements de la surface moyenne, la technique décrite ici consiste en un développement asymptotique des champs solutions par rapport à la demi-épaisseur de la plaque. La construction du développement asymptotique complet, laissant le choix du degré de précision de la solution, et les très faibles hypothèses nécessaires permettent l'étude d'un très large type de plaques. Le modèle est validé par comparaison avec une expérience, d'autres codes de calcul et une solution analytique.*

KEY WORDS : *laminate, asymptotic development, finite element.*
MOTS-CLÉS : *stratifiés, développement asymptotique, éléments finis.*

1. Introduction

The study we have carried out on laminates was motivated by the calculation of large naval structures. Concerning shipbuilding, a special interest is drawn in the use of composites, especially of the sandwich type. For such applications, there is a need to obtain a good approximation of the state of local stresses, in order to better apply damage and yield criteria; the final goal aims at being less conservative in design regulations.

The extensive usage of laminates for the design of structures has led to the development of numerous Finite Element (F.E.) formulations. Thereof, a precise review can be read in [RED 89]. To be short, we can isolate two categories of Finite Elements. The first set is a direct extension of homogeneous isotropic F.E. based on classical theories (Kirchhoff-Love or Reissner-Mindlin), where the constitutive law is given in a mixed formulation [BAT 82] [BAT 90] [RED 84]. In this case, the integration of the equilibrium equations is needed to approximate the transverse stresses. The second set includes F.E. especially developed for laminates with kinematical hypotheses through each layer and conditions of continuity of stress and deformation fields at the interfaces [BEA 93] [DI 93] [LAR 89] [RED 84] [REN 86]. In this paper, we have decided to modelise laminates without any *a priori* hypothesis on displacements or stresses.

The model we present is the result of a technique of asymptotic development -using the half-thickness of the laminate as a small parameter- which is applied directly to the local equations of linear elasticity. Also, no hypothesis is made on the structure of the stacking sequence except that each layer has a monoclinic behaviour in the thickness direction. This technique was used before by [CIA 79], for the isotropic case, and [CAI 87], for the anisotropic one, with the aim of conforming the Kirchhoff-Love model as the limit model. In our case, we have built the complete development. Such a technique provides us with:

- at the first order of the development, a Kirchhoff-Love type model that gives a good approximation of the behaviour of the considered laminate,
- the expressions that determine displacements, strains and stresses on each point of the structure and for any order of the development.

In this paper, we first present the methodology we have used and our main theoretical results. We then describe the numerical implementation of the model. Finally, we discuss some results obtained by comparison with an experiment, with calculations obtained from other codes and with an analytical solution.

2. Notations

In this paper, the Latin indices take their values in the set $\{1, 2, 3\}$ and the Greek ones in the set $\{1, 2\}$. We also use Einstein's convention of summation of repeated indices.

2.1. The geometry

Considering that a plate, Figure 1, is a structure with a small thickness, we characterise it by its middle surface, denoted ω, and its half-thickness ε. The plate is then represented by a domain ${}^{\varepsilon}\Omega$ generated in the following manner:

$$\begin{array}{ll} {}^{\varepsilon}\Omega = \omega \times [-\varepsilon, +\varepsilon] & \text{with: } \partial^{\varepsilon}\Omega = \Gamma_{+} \cup {}^{\varepsilon}\Gamma \cup \Gamma_{-}, \\ \Gamma_{\pm} = \omega \times \{\pm\varepsilon\} & \text{upper and bottom faces,} \\ {}^{\varepsilon}\Gamma = \partial\omega \times [-\varepsilon, +\varepsilon] & \text{lateral edge.} \end{array}$$

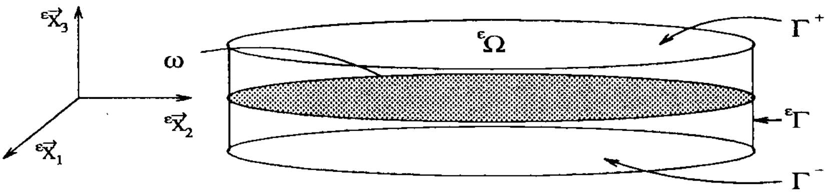

Figure 1: *The geometry*

2.2. The stack

Since we use homogenisation techniques [LEN 84], we consider the case of a plate composed of any stack of anisotropic homogeneous layers, Figure 2. Thus the only hypothesis required about the materials -without loosing generality- is their monoclinicity in the direction of the thickness. In other words, each layer holds its own middle surface as a plane of material symmetry. This hypothesis implies that the stiffness matrix ${}^{\varepsilon}\mathfrak{a}$ verifies [GAY 87]:

$$ {}^{\varepsilon}a^{3\alpha\beta\gamma} = {}^{\varepsilon}a^{333\alpha} = 0 \qquad \forall \alpha, \beta, \gamma . \tag{1}$$

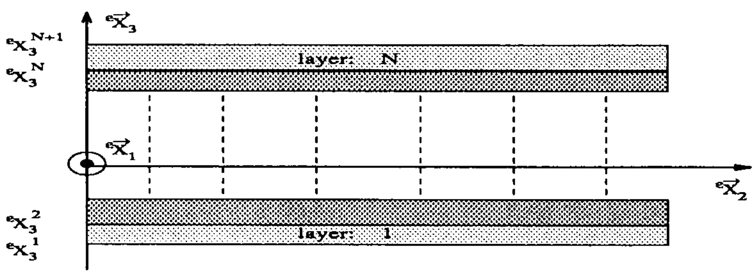

Figure 2: *The laminate.*

2.3. The three-dimensional problem

The framework of our study is the linear elasticity under the hypothesis of small perturbations [DUV 90]. We suppose that a body force density $\overrightarrow{{}^{\varepsilon}f}$ is applied into the domain ${}^\varepsilon\Omega$, surface force densities $\overrightarrow{{}^\varepsilon G_\pm}$ on $\Gamma_\pm$ and $\overrightarrow{{}^\varepsilon F}$ on a part ${}^\varepsilon\Gamma_\sigma$ of the lateral edge ${}^\varepsilon\Gamma$. Finally, we suppose that the domain is bounded on a part ${}^\varepsilon\Gamma_u$ complementary of ${}^\varepsilon\Gamma_\sigma$. Since the plate is laminated, we suppose that all layers adhere perfectly to each other, which means:

$$[\![\overrightarrow{{}^\varepsilon u}]\!] = [\![{}^\varepsilon\sigma(\overrightarrow{{}^\varepsilon x_3})]\!] = 0, \qquad \text{through each interface.} \qquad [2]$$

$\overrightarrow{{}^\varepsilon u}$ represents here the displacement field, ${}^\varepsilon\sigma$ the stress tensor and $[\![.]\!]$ is the symbol of discontinuity. In a distribution sense, the problem is then written:

$$\text{Find } \overrightarrow{{}^\varepsilon u} \text{ solution of } \begin{cases} \overrightarrow{div}\,{}^\varepsilon\sigma + \overrightarrow{{}^\varepsilon f} = \overrightarrow{0} & \text{in } {}^\varepsilon\Omega\,, \\ {}^\varepsilon\sigma = {}^\varepsilon a\,\mathfrak{E}(\overrightarrow{{}^\varepsilon u}) & \text{in } {}^\varepsilon\Omega\,, \\ \overrightarrow{{}^\varepsilon u} = \overrightarrow{0} & \text{on } {}^\varepsilon\Gamma_u\,, \\ {}^\varepsilon\sigma(\overrightarrow{n}) = \overrightarrow{{}^\varepsilon F} & \text{on } {}^\varepsilon\Gamma_\sigma\,, \\ {}^\varepsilon\sigma(\pm\overrightarrow{x_3}) = \overrightarrow{{}^\varepsilon G_\pm} & \text{on } \Gamma_\pm\,. \end{cases} \qquad [3]$$

where $2\mathfrak{E}(\overrightarrow{{}^\varepsilon u}) = \nabla\overrightarrow{{}^\varepsilon u} + {}^t\nabla\overrightarrow{{}^\varepsilon u}$ is the linearised strain tensor. This problem admits a unique solution, the demonstration being established in [DUV 90].

3. The asymptotic model of plates

3.1. The process

In order to find the solution to problem [3] -considering the small thickness of the laminate- we have chosen to use an asymptotic development technique with a small parameter, ε, which has no dimension but which is related to the half-thickness of the plate. In comparison with previous works [CAI 87] or [DES 80], we construct the complete development in order to gain some flexibility in the precision of the results. We then use the following process, for which a more complete and general demonstration is given in [KAI 94]:

- On the first step, a mapping is performed into a reference domain Ω, which is independent from the small parameter, by the linear transformation:

$$\begin{array}{c} {}^\varepsilon\Theta : \mathbb{R}^3 \to \mathbb{R}^3 \\ \overrightarrow{{}^\varepsilon x} = ({}^\varepsilon x_1, {}^\varepsilon x_2, {}^\varepsilon x_3) \longmapsto \overrightarrow{y} = (y_1 = {}^\varepsilon x_1, y_2 = {}^\varepsilon x_2, y_3 = {}^\varepsilon x_3/\varepsilon). \end{array} \qquad [4]$$

  In this domain, each quantity defining the structure is given an order with respect to ε. In this paper, for the external loads, we take the following particular case:

$$\begin{aligned} &{}^{\varepsilon}f^{\alpha} \circ {}^{\varepsilon}\Theta = \varepsilon f^{\alpha(1)} , \quad {}^{\varepsilon}F^{\alpha} \circ {}^{\varepsilon}\Theta = \varepsilon F^{\alpha(1)} , \quad {}^{\varepsilon}G^{\alpha}_{\pm} \circ {}^{\varepsilon}\Theta = \varepsilon^2 G^{\alpha(2)}_{\pm} , \\ &{}^{\varepsilon}f^{3} \circ {}^{\varepsilon}\Theta = \varepsilon^2 f^{3(2)} , \quad {}^{\varepsilon}F^{3} \circ {}^{\varepsilon}\Theta = \varepsilon^2 F^{3(2)} , \quad {}^{\varepsilon}G^{3}_{\pm} \circ {}^{\varepsilon}\Theta = \varepsilon^3 G^{3(3)}_{\pm} . \end{aligned} \qquad [5]$$

This assertion will be discussed on sections 3.2 and 3.3. Then, to the end of this paper, we have:

$$\begin{aligned} &f^{\alpha(n)} = 0 \text{ if } n \neq 1, \quad F^{\alpha(n)} = 0 \text{ if } n \neq 1, \quad G^{\alpha(n)}_{\pm} = 0 \text{ if } n \neq 2, \\ &f^{3(n)} = 0 \text{ if } n \neq 2, \quad F^{3(n)} = 0 \text{ if } n \neq 2, \quad G^{3(n)}_{\pm} = 0 \text{ if } n \neq 3. \end{aligned}$$

We suppose that the behaviour of materials is invariant by transformation ${}^{\varepsilon}\Theta$, therefore:

$${}^{\varepsilon}\mathbf{a} \circ {}^{\varepsilon}\Theta = \mathbf{a} . \qquad [6]$$

Finally, we postulate the existence of a development with respect to the small parameter ε of the three-dimensional displacements and stresses:

$$\overrightarrow{{}^{\varepsilon}u} \circ {}^{\varepsilon}\Theta = \sum_{n=-\infty}^{+\infty} [\varepsilon^n \overrightarrow{u^{(n)}}], \qquad {}^{\varepsilon}\boldsymbol{\sigma} \circ {}^{\varepsilon}\Theta = \sum_{n=-\infty}^{+\infty} [\varepsilon^n \boldsymbol{\sigma}^{(n)}]. \qquad [7]$$

Thus, we can transform problem [3], applied on domain ${}^{\varepsilon}\Omega$, into a problem applied on Ω using formulae [4] to [7]. For example, the third component of the equilibrium equation:

$$\sigma^{i3}_{,i} + f^3 = 0 \quad \text{in } \Omega ,$$

is transformed into:

$$\sum_{n=-\infty}^{+\infty} [\varepsilon^{n+1} \sigma^{33(n)}_{,3} + \varepsilon^n \sigma^{3\alpha(n)}_{,\alpha}] + \varepsilon^2 f^{3(2)} = 0 \quad \text{in } \Omega.$$

By identification of the terms of the same exponent in ε, we obtain:

$$\forall n , \qquad \sigma^{33(n)}_{,3} + \sigma^{3\alpha(n-1)}_{,\alpha} + f^{3(n-1)} = 0 \quad \text{in } \Omega.$$

The same process is used to every equations of problem [3]. We then obtain the following set of relations into Ω:

$$\left\{\begin{array}{rcll} \sigma^{3i\,(n)}_{,3} + \sigma^{i\beta\,(n-1)}_{,\beta} + f^{i\,(n-1)} &=& 0 & \text{in } \Omega\,, \\ \sigma^{\alpha\beta\,(n)} &=& a^{\alpha\beta\gamma\delta} u^{(n)}_{\gamma,\delta} + a^{\alpha\beta 33} u^{(n+1)}_{3,3} & \text{in } \Omega\,, \\ \sigma^{\alpha 3\,(n)} &=& a^{\alpha 3\beta 3}(u^{(n)}_{3,\beta} + u^{(n+1)}_{\beta,3}) & \text{in } \Omega\,, \\ \sigma^{33\,(n)} &=& a^{33\gamma\delta} u^{(n)}_{\gamma,\delta} + a^{3333} u^{(n+1)}_{3,3} & \text{in } \Omega\,, \\ \overrightarrow{u^{(n+1)}} &=& \overrightarrow{0} & \text{on } \Gamma_u\,, \\ \sigma^{i\alpha\,(n)} n_\alpha &=& F^{i\,(n)} & \text{on } \Gamma_\sigma\,, \\ \sigma^{i3\,(n)} &=& \pm G^{i\,(n)}_{\pm} & \text{on } \Gamma_\pm\,. \end{array}\right. \qquad [8]$$

Because problem [3] admits a unique solution, these relations must be verified for each order. Relations [8] are the keystones of our model.

- Some part of these relations can be interpreted as the following problem through the thickness:

$$\left\{\begin{array}{ll} \sigma^{i3\,(n)}_{,3} = -(f^{(n-1)}_i + \sigma^{i\beta\,(n-1)}_{,\beta}) & \text{in } \Omega\,, \\ \sigma^{i3\,(n)} = D^{ij} u^{(n+1)}_{j,3} + s^{i\,(n)} & \text{in } \Omega\,, \\ \sigma^{i3\,(n)} = \pm G^{i\,(n)}_{\pm} & \text{for } y_3 = \pm 1\,. \end{array}\right. \qquad [9]$$

with

$$\mathbb{D} = \begin{pmatrix} a^{1313} & a^{1323} & 0 \\ a^{1323} & a^{2323} & 0 \\ 0 & 0 & a^{3333} \end{pmatrix} \qquad s^{\alpha\,(n)} = a^{\alpha 3\beta 3} u^{(n)}_{3,\beta} \qquad s^{3\,(n)} = a^{33\gamma\delta} u^{(n)}_{\gamma,\delta}\,.$$

A variational formulation easily shows that if the integrability conditions:

$$G^{i\,(n)}_{+} + G^{i\,(n)}_{+} + \int_{-1}^{1} \left(\sigma^{i\beta\,(n-1)}_{,\beta} + f^{i\,(n-1)}\right) dy_3 = 0 \qquad \text{on } \omega, \qquad [10]$$

are satisfied, then the problem admits a solution with the in-plane displacement field $\overrightarrow{\zeta^{(n)}}$ as unknown. This solution is:

$$\left\{\begin{array}{l} u^{(n)}_\gamma(y_1, y_2, y_3) = \zeta^{(n)}_\gamma(y_1, y_2) - y_3 \zeta^{(n-1)}_{3,\gamma}(y_1, y_2) + H^{(n)}_\gamma(y_1, y_2, y_3) \\ u^{(n)}_3(y_1, y_2, y_3) = \zeta^{(n)}_3(y_1, y_2) + H^{(n)}_3(y_1, y_2, y_3) \end{array}\right. \qquad [11]$$

with:

$$
\begin{aligned}
H_1^{(n)} &= \int_0^{y_3} \left(\frac{a^{2323}\sigma^{13(n-1)} - a^{1323}\sigma^{23(n-1)}}{d} - H_{3,1}^{(n-1)} \right) dz \,, \\
H_2^{(n)} &= \int_0^{y_3} \left(\frac{a^{1313}\sigma^{23(n-1)} - a^{1323}\sigma^{13(n-1)}}{d} - H_{3,2}^{(n-1)} \right) dz \,, \\
H_3^{(n)} &= \int_0^{y_3} \left(\frac{\sigma^{33(n-1)}}{a^{3333}} - A^{\gamma\delta} u_{\gamma,\delta}^{(n-1)} \right) dz \,,
\end{aligned}
\qquad [12]
$$

$$
d = a^{1313}a^{2323} - a^{1323}a^{1323} \neq 0 \,, \quad A^{\alpha\beta} = \frac{a^{\alpha\beta 33}}{a^{3333}} \,.
$$

In expression [11], the development of the displacements of the middle surface ω appears with every term $\overrightarrow{\zeta^{(n)}}$. Then, by integrating the equilibrium equations to determine the transverse stresses, we deduce:

$$
\begin{aligned}
\sigma^{\alpha\beta(n)} &= Q^{\alpha\beta\gamma\delta}(\zeta_{\gamma,\delta}^{(n)} - y_3 \zeta_{3,\gamma\delta}^{(n-1)}) + \Psi^{\alpha\beta(n)} \,, \\
\sigma^{3\alpha(n)} &= -G_{-}^{\alpha(n)} - \int_{-1}^{y_3} \left(f^{\alpha(n-1)} + \sigma_{,\beta}^{\alpha\beta(n-1)} \right) dz \,, \\
\sigma^{33(n)} &= -G_{-}^{3(n)} + (y_3 + 1) G_{-,\alpha}^{\alpha(n-1)} - \int_{-1}^{y_3} f^{3(n-1)} dz \\
&\quad + y_3 \int_{-1}^{y_3} \left(f_{,\alpha}^{\alpha(n-2)} + \sigma_{,\alpha\beta}^{\alpha\beta(n-2)} \right) dz - \int_{-1}^{y_3} z \left(f_{,\alpha}^{\alpha(n-2)} + \sigma_{,\alpha\beta}^{\alpha\beta(n-2)} \right) dz \,,
\end{aligned}
\qquad [13]
$$

where $Q^{\alpha\beta\gamma\delta} = a^{\alpha\beta\gamma\delta} - \dfrac{a^{\alpha\beta 33}a^{33\gamma\delta}}{a^{3333}}$ and $\Psi^{\alpha\beta(n)} = \dfrac{a^{\alpha\beta 33}}{a^{3333}}\sigma^{33(n)} + Q^{\alpha\beta\gamma\delta}H_{\gamma,\delta}^{(n)}$. This proper way to calculate transverse stresses is imposed by the resolution of problem [9]. If we use the constitutive law written in [8], we find the unuseful result $\sigma^{i3(n)} = \sigma^{i3(n)}$. The only way to retrieve these stresses is the integration of the equilibrium equations. Relations [11] to [13] imply to change the unknown, which becomes the bidimensional displacement $\overrightarrow{\zeta^{(n)}}$.

- Since relations [8] have to be satisfied, as said before, so have conditions [10]. Using [11] and [13] into [10] and the following boundary conditions:

$$
\overrightarrow{u^{(n+1)}} = \vec{0} \quad \text{on } \Gamma_u \,,
$$

$$
\int_{-1}^{1} \sigma^{\alpha\beta(n+1)} n_\beta dy_3 = \int_{-1}^{1} F^{\alpha(n+1)} dy_3, \quad \int_{-1}^{1} y_3 \sigma^{\alpha\beta(n+1)} n_\beta dy_3 = \int_{-1}^{1} y_3 F^{\alpha(n+1)} dy_3 \,,
$$

$$
\text{and} \int_{-1}^{1} \sigma^{3\alpha(n+2)} n_\alpha dy_3 = \int_{-1}^{1} F^{3(n+2)} dy_3 \quad \text{on } \partial_\sigma\omega \,,
$$

we obtain a two-dimensional problem in $\overrightarrow{\zeta^{(n)}}$ on ω. $\partial_\sigma\omega$ is the projection of ${}^\varepsilon\Gamma_\sigma$ on ω. The kinematic conditions are reduced to:

$$\zeta_\gamma^{(n)} = \zeta_3^{(n-1)} = \zeta_{3,\gamma}^{(n-1)} = 0 \text{ on } \partial_u\omega ,$$

where $\partial_u\omega$ is the projection of ${}^\varepsilon\Gamma_u$ on ω. We now give the expression of this problem.

3.2. The two-dimensional problem of plate

We have seen in [11] and [13] that each term of the development of the three-dimensional quantities depends on the displacements of the middle surface. For each order n, $(\zeta_\gamma^{(n)}, \zeta_3^{(n-1)})$ is the solution of the following problem:

$$\left\{\begin{array}{lr} X^{\alpha\beta(n)}_{,\beta} + \mathcal{F}^{\alpha(n)} = 0 & \text{on } \omega , \\ Y^{\alpha\beta(n)}_{,\alpha\beta} + \mathcal{F}^{3(n)} = 0 & \text{on } \omega , \\ X^{\alpha\beta(n)} = A^{\alpha\beta\gamma\delta}\zeta^{(n)}_{\gamma,\delta} - B^{\alpha\beta\gamma\delta}\zeta^{(n-1)}_{3,\gamma\delta} & \text{on } \omega , \\ Y^{\alpha\beta(n)} = B^{\alpha\beta\gamma\delta}\zeta^{(n)}_{\gamma,\delta} - C^{\alpha\beta\gamma\delta}\zeta^{(n-1)}_{3,\gamma\delta} & \text{on } \omega , \\ \zeta^{(n)}_\gamma = \zeta^{(n-1)}_3 = \zeta^{(n-1)}_{3,\gamma} = 0 & \text{on } \partial_u\omega , \\ X^{\alpha\beta(n)} n_\beta = g^{\alpha(n)} \quad Y^{\alpha\beta(n)} n_\beta = h^{\alpha(n)} \quad Y^{\alpha\beta(n)}_{,\beta} n_\alpha = g^{3(n)} & \text{on } \partial_\sigma\omega . \end{array}\right. \qquad [14]$$

with:

$$\mathcal{F}^{\alpha(n)} = G_+^{\alpha(n+1)} + G_-^{\alpha(n+1)} + \int_{-1}^{1}\left(f^{\alpha(n)} + \Psi^{\alpha\beta(n)}_{,\beta}\right)dy_3 ,$$

$$\mathcal{F}^{3(n)} = G_+^{3(n+2)} + G_-^{3(n+2)} + G_{+,\alpha}^{\alpha(n+1)} - G_{-,\alpha}^{\alpha(n+1)} + \int_{-1}^{1} f^{3(n+1)}dy_3 - \int_{-1}^{1} y_3\left(f^{\alpha(n)}_{,\alpha} + \Psi^{\alpha\beta(n)}_{,\alpha\beta}\right)dy_3 ,$$

$$g^{\alpha(n)} = \int_{-1}^{1}\left(F^{\alpha(n)} - \Psi^{\alpha\beta(n)} n_\beta\right)dy_3 , \qquad [15]$$

$$g^{3(n)} = (G_-^{\alpha(n+1)} - G_+^{\alpha(n+1)})n_\alpha + \int_{-1}^{1}\left(F^{3(n+1)} - y_3(f^{\alpha(n)} + \Psi^{\alpha\beta(n)}_{,\beta})n_\alpha\right)dy_3 ,$$

$$h^{\alpha(n)} = \int_{-1}^{1} y_3\left(F^{\alpha(n)} - \Psi^{\alpha\beta(n)} n_\beta\right)dy_3 .$$

In the case of a stack of N layers, the behaviour of the plate, obtained by a mixture law with conditions [10], can be written:

$$\mathbf{A} = \sum_{k=1}^{N}(z_{k+1} - z_k)\mathbb{Q}^{\,k}, \qquad \mathbb{B} = \frac{1}{2}\sum_{k=1}^{N}(z_{k+1}^2 - z_k^2)\mathbb{Q}^{\,k}, \qquad \mathbb{C} = \frac{1}{3}\sum_{k=1}^{N}(z_{k+1}^3 - z_k^3)\mathbb{Q}^{\,k}, \tag{16}$$

where $\mathbb{Q}^{\,k}$, z_k and z_{k+1} are respectively the matrix of the reduced stiffness, the bottom and top level of layer k. This way, we find the classical expressions of the stiffness matrices of laminates [OCH 92].

Some remarks have to be drawn about problem [14]:

- A coupling of membrane and bending effects appears in problem [14]. This connection depends on the constitution of the laminate through matrix $\mathbb{B}$ [16] and disappears when the stacking sequence is symmetrical with respect to the middle surface.
- We can note that only the right hand side of the equations depends on the order of the calculated term *via* the recurrent terms $\Psi^{(n)}$. Therefore, from a computational point of view, only one stiffness matrix has to be calculated, factorised and stored for the computation of several terms of the development.
- With the expression of the right hand side, we can explain part of assertion [5]. In this assertion, the ratios of all components of any load has been chosen to insure that all external loading effects appear in only one two-dimensional partial differential system [14].
- For an isotropic homogeneous plate, the classical uncoupled Kirchhoff-Love formulation of membrane and bending is deduced. Only the right hand side is changed by recurrent terms $\Psi^{(n)}$.
- Problem [14] admits a unique solution, as it is demonstrated in [KAI 94].

3.3. The asymptotic development

The unicity of the solution of problem [14] implies the unicity of the asymptotic development and also the following property:

Theorem: *it is strictly equivalent to search the developments of $\vec{u}$ and of $\vec{\zeta}$.*

The complete demonstration of this theorem is given in [KAI 94] but it relies essentially in two points:

- we look for the order of the first non trivial problem [14]. Indeed, $(\zeta_\gamma^{(n)}, \zeta_3^{(n-1)})$ is equal to zero until external loads appear, as can be shown by induction. Here, relation [5] leads to $n = 1$.

- Then, we can build all the development using formulae [11], [12], [13] and problem [14].

We obtain:

$$\left\{\begin{array}{ll} u_\gamma &= \varepsilon u_\gamma^{(1)} + \varepsilon^3 u_\gamma^{(3)} + \ldots + \varepsilon^{2n+1} u_\gamma^{(2n+1)} + \ldots\,, \\ u_3 &= u_3^{(0)} + \varepsilon^2 u_3^{(2)} + \ldots + \varepsilon^{2n} u_3^{(2n)} + \ldots\,, \\ \sigma^{\alpha\beta} &= \varepsilon \sigma^{\alpha\beta(1)} + \varepsilon^3 \sigma^{\alpha\beta(3)} + \ldots + \varepsilon^{2n+1} \sigma^{\alpha\beta(2n+1)} + \ldots\,, \\ \sigma^{3\alpha} &= \varepsilon^2 \sigma^{3\alpha(2)} + \varepsilon^4 \sigma^{3\alpha(4)} + \ldots + \varepsilon^{2n} \sigma^{3\alpha(2n)} + \ldots\,, \\ \sigma^{33} &= \varepsilon^3 \sigma^{33(3)} + \varepsilon^5 \sigma^{33(5)} + \ldots + \varepsilon^{2n+1} \sigma^{33(2n+1)} + \ldots\,, \end{array}\right. \qquad [17]$$

where $\overrightarrow{u^{(n)}}$ and $\sigma\!\!\!\sigma^{(n)}$ are defined in [11] and [13] with:

$$H_\gamma^{(1)} = H_3^{(0)} = \Psi^{\alpha\beta(1)} \equiv 0\,. \qquad [18]$$

This last equality [18] is very easy to understand because recurrent terms are equal to zero for the first term of the development. We can draw the following remarks about this development:

- If we restrict ourselves to the first term of the development, the displacements $(u_\gamma^{(1)}, u_3^{(0)})$ are of the Kirchhoff-Love type. If we look for the repartition through the thickness of the laminate, the in-plane stresses $\sigma^{\alpha\beta(1)}$ are piecewise linear, the transverse shear stresses $\sigma^{3\alpha(2)}$ are quadratic and the normal stress $\sigma^{33(3)}$ is cubic.
- When the thickness comes to zero, Destuynder [DES 80], for the isotropic case, and Caillerie [CAI 87], for the anisotropic one, have demonstrated the convergence of the solution of the three-dimensional problem to the main term $(\varepsilon u_\gamma^{(1)}, u_3^{(0)}, \varepsilon\sigma^{\alpha\beta(1)}, \varepsilon^2\sigma^{3\alpha(2)}, \varepsilon^3\sigma^{33(3)})$ of the development.
- From the second term, the displacements consist in bidimensional displacements and recurrent terms. For every order, bidimensional displacements are of the Kirchhoff-Love type. The recurrent terms add at least some linear and quadratic terms in y_3 to each component u_i and $\sigma^{\alpha\beta}$. To the transverse shear stresses, the recurrent terms add some quadratic and cubic pieces, and some cubic and fourth order to the normal stress. But we have to keep in mind that: the higher the order of the term is, the less influential is this term on displacements and stresses.
- The particular choice [5] for the ratios of external loads discards every two term of each field. This way, we have reduced the size of the asymptotic development. Since our framework is the linear elasticity, assertion [5] was made also to find this development which is, in our sense, the most common for a numerical implementation.
- The condition of continuity [2] is assumed if $\vec{u}$ and $\sigma\!\!\!\sigma(\overrightarrow{y_3}) = (\sigma^{13}, \sigma^{23}, \sigma^{33})$ are continuous through each interface. It is easy to verify the continuity

through the thickness of each term $u_i^{(n)}$ and $\sigma^{i3(n)}$ based on definitions [11], [12] and [13]. This last point insures the continuity of the displacement and stress vector at the interfaces.

The boundary conditions [3] on the lateral edge ${}^{\varepsilon}\Gamma$ are very tricky. We are not able to insure these conditions for all the development and at each point of the boundary. The kinematic conditions on ${}^{\varepsilon}\Gamma_u$ are fully satisfied for the first term of the development $(u_\gamma^{(1)}, u_3^{(0)})$. This is assumed by the conditions imposed into problem [14] and relations [11]. But for the following terms, recurrent terms $H_i^{(n)}$ appear, and therefore, the boundary conditions cannot be controlled any more. The construction of the complete asymptotic development induces the loss of the kinematic conditions, which is very unusual. For stresses, the boundary conditions on ${}^{\varepsilon}\Gamma_\sigma$ are verified only in a resultant or moment sense. These restrictions on three-dimensional boundary conditions are useful for plate models. By the transformation into a two-dimensional problem, a model of plate lose the three-dimensional static conditions. Thus we may have edge effects for which it would be advisable to use boundary layer techniques [DUM 90].

4. Numerical tests

4.1. Implementation

At present, only the first term of the development has been implemented, but in the general case of any kind of laminate composed of anisotropic homogeneous layers. Thus, displacements and stresses are:

$$\vec{u} = (\varepsilon u_\gamma^{(1)}, u_3^{(0)}) \qquad \sigma = (\varepsilon \sigma^{\alpha\beta(1)}, \varepsilon^2 \sigma^{3\alpha(2)}, \varepsilon^3 \sigma^{33(3)})$$

where $u_i^{(n)}$ and $\sigma^{ij(n)}$ are defined in [11] and [13] with $H_\gamma^{(1)} = H_3^{(0)} = \Psi^{\alpha\beta(1)} \equiv 0$.

Nevertheless, the implementation of the complete asymptotic development [17] conducts all the algorithmic choices we have made:

- As written above, the construction of the asymptotic development requires the knowledge of bidimensional displacements which are determined by the resolution of problem [14] posed on the middle surface of the laminate. For this reason, we have chosen conforming finite elements. Then, our variational unknown is the bidimensional displacement.
- Since problem [14] contains a differential equation of the fourth order, we have to use an element with a C^1 continuity.
- The need of fourth order derivatives on the deflection appears in the calculation of the three-dimensional stresses [13], so we have to use at least a fourth order polynomial. As we shall have to add some terms to the development, we increase the degree of polynomials of all components of displacements.

For all these reasons and for ease of programming, we have decided to use the Argyris interpolation [ARG 68], Figure 3, for the three directions, to construct the TRIA-P5LP (TRIAngular P5 Laminated Plate) element.

This interpolation may seem too expensive in certain cases, so we have developed a second element, called TRIA-P3LP (TRIAngular P3 Laminated Plate) element. For this one, the transverse stresses are calculated in an approximated way, see Annex, with the two-dimensional stresses **X** and **Y** [14] -in other words the plate is considered as an homogeneous one for these stresses-. This second element uses the reduced Hsieh-Clough-Tocher interpolation [BER 94], Figure 4, in each direction, for the same reasons as TRIA-PL5P.

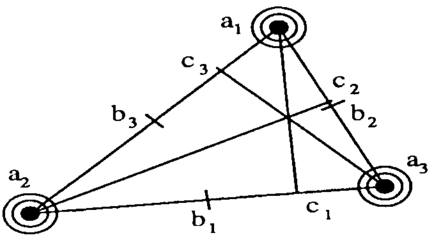

Figure 3: *Argyris interpolation.*

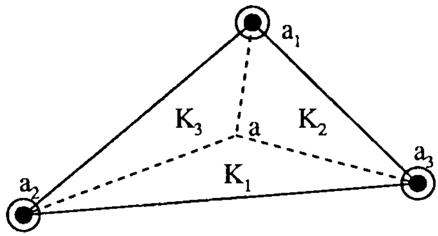

Figure 4: *Reduced HCT interpolation.*

With these two elements, we have constructed, within MODULEF [BER 85] -which is a Fortran Finite Element toolbox for scientific calculation-, the COALA software for the calculation of the laminated structures composed of anisotropic homogeneous layers. COALA uses a bidimensional mesh to solve the plate problem [14]. To provide displacements and stresses on the real three-dimensional structure, COALA asks for some ${}^{\epsilon}x_3$ coordinates and generates by extrusion the corresponding three-dimensional mesh to visualise results through the thickness. As it was shown in [DES 80] and [CAI 87], the first term is adapted to thin structures and allows us to draw some conclusions on the model as shown in the following tests.

4.2. Four point bending

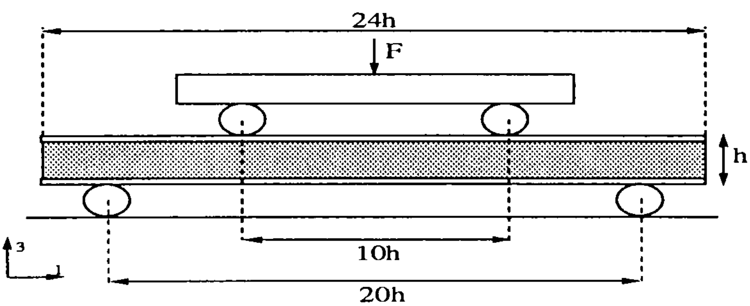

Figure 5: *Four points bending test*

This bending test was realised by Ifremer in the context of a European

Brite-Euram program [BIG 93] [CHO 94]. The material used is a symmetrical sandwich with the following characteristics:

- The core is made of PVC foam, has a thickness of 40mm and a linear isotropic behaviour of coefficients:

$$E = 60MPa \qquad \nu = 0.428 \qquad G = 21MPa,$$

- skins, 3.24mm of thickness, are made of twelve layers of unidirectional material with a sequence of [-45,90,45,0,-45,90,45,0,0,45,90,-45]. Each skin has the following mechanical properties:

$$E_1 = E_2 = 13.8GPa \quad E_3 = 6.7GPa \quad \nu_{12} = 0.287 \quad \nu_{31} = \nu_{32} = 0.159$$

$$G_{12} = 4.7GPa \quad G_{31} = G_{32} = 3.3GPa.$$

We have made a four point bending test under the DIN53293 German norm, Figure 5, where h=46.7mm, the diameter of supports is 35mm and the applied load is F=3950N. The normal displacement and the strain on the bottom face at the centre of the plate had been measured. Since the plate has a ratio $l/h = 24$, it is commonly admitted that the Kirchhoff-Love theory can be used. Experimental results are compared to the following calculation software:

- ADINA: Lagrange type quadrangular shell element with eight nodes under the Mindlin-Reissner [MIN 66] kinematic. This element bears three displacements and rotations per node (we are testing the bending of a plate, therefore the third rotation is equal to zero). The model uses a shear correction factor which depends on the stack. Here we find k=0.0442. We have used the first mesh defined in Figure 6 with 168 d.o.f. for half of the plate.
- SAMCEF: Lagrange type quadrangular shell element (T56) with eight nodes under a discrete Kirchhoff-Love hypothesis. We have three displacements per node and two rotations on each middle of edge. We have also used the first mesh defined on Figure 6 with 116 d.o.f. for half of the plate.
- COALA: our P5LP element which is of Kirchhoff-Love type. For this element we have used the second mesh defined on Figure 6 with 183 d.o.f. for half of the plate.

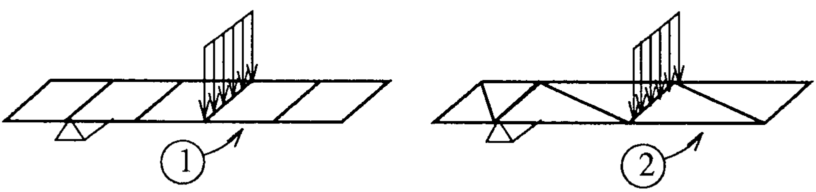

Figure 6: *Meshes of Adina and Samcef (1) and Coala (2)*

The results are:

	experiment	ADINA	SAMCEF	COALA
deflection(mm)	-10	-9.754	-9.439	-9.425
relative error	0	2.46%	5.61%	5.75%
ε_{11} (10^{-3})	1.62	2.12	2.11	2.12
relative error	0	31.05%	30.25%	30.91%

For the deflection, the three models give similar results, close to the experimental measurement. About the strain, the three models again agree but all of them are far from the measurements, with a 30% error. This fact could be explained by the compression of the foam which is not taken into account in all these models. For this case, it will be useful to implement the second term of the development because it induces some variation of the thickness. These comparisons gave a first validation of our work but we have to keep in mind that our goal is to obtain a good approximation of local stresses. Internal stresses are extremely delicate to record. Therefore, instead of experimental measurement, we have chosen to take an analytical solution as a reference.

4.4. A solution of Pagano

We have used an analytical solution developed by Pagano [PAG 69], Figure 7. We suppose that the structure has an infinite dimension along the $\overrightarrow{x_2}$ axis, a length l = 1m along the $\overrightarrow{x_1}$ axis and a thickness of 2ε = 2cm. An external load, $\overrightarrow{Q} = Q_0 sin(\frac{\pi}{l}x_1)\overrightarrow{x_3}$, is applied on the upper face ($Q_0 = 10^4$Pa).

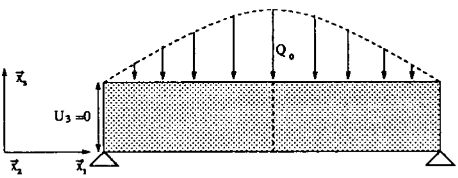

Figure 7: *The problem of Pagano*

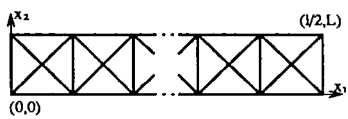

Figure 8: *The 2D-mesh*

Pagano has shown that the solution takes the following form:

$$ {}^{\varepsilon}\overrightarrow{u}(x_1,x_3) = \begin{pmatrix} U(x_3)\cos(\frac{\pi}{l}x_1) \\ 0 \\ W(x_3)\sin(\frac{\pi}{l}x_1) \end{pmatrix}, $$

and

$$ {}^{\varepsilon}\sigma(x_1,x_3) = \begin{pmatrix} S_{11}(x_3)\sin(\frac{\pi}{l}x_1) & 0 & S_{13}(x_3)\cos(\frac{\pi}{l}x_1) \\ & S_{22}(x_3)\sin(\frac{\pi}{l}x_1) & 0 \\ // & & S_{33}(x_3)\sin(\frac{\pi}{l}x_1) \end{pmatrix}, $$

where U,W and S depend on the structure of the stack constituting the laminate.

To look at the distribution of stresses through the thickness, we have chosen to study a non symmetrical sandwich where bottom-middle-top thicknesses are 3mm-15mm-2mm respectively. The skins are made of a unidirectional material having the following properties:

$$E_t = 12GPa\,, \qquad E_l = 45GPa\,, \qquad \nu = 0.31\,, \qquad G_{tt} = 4.58GPa\,, \qquad G_{lt} = 4.5GPa\,, \tag{19}$$

the core is made of a foam:

$$E = 3.04GPa\,, \qquad \nu = 0.31\,. \tag{20}$$

We impose here some symmetry conditions on $x_2 = 0$ and $x_2 = L$ sides. For the numerical solution of the two-dimensional problem, we have used a mesh, Figure 8, with 80 elements -558 d.o.f.- for the P3LP element and 40 elements -789 d.o.f.- for the P5LP element. The number of d.o.f. used here is close to the limit of precision for each kind of element.

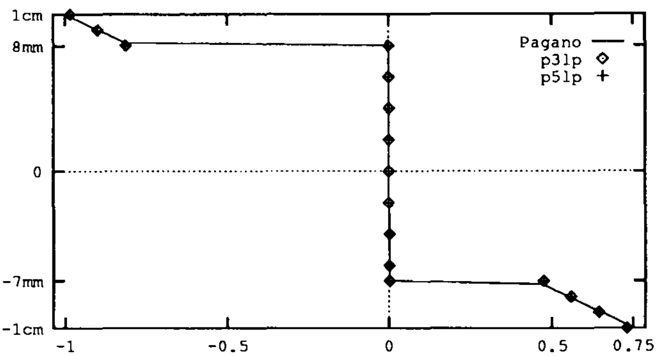

Figure 9: *The stress* $\sigma_{11}(\frac{l}{2}, x_3)$

Figures 9 to 11 show the distribution through the thickness of the calculated stresses with our two elements and the analytical solution. These Figures show the very good results obtained by element P5LP and confirm here that in-plane stresses are piecewise linear functions, that transverse shear stresses are parabolic and that the normal stress is cubic. Element P3LP gives some very good results for in-plane stresses, Figure 9, but the simplified method employed for the calculation of the transverse stresses shows its limitations. Actually the normal stress, Figure 11, is approached with a little error of 5%, while the transverse shear is badly approximated, Figure 10. These errors come from the use of the homogenised behaviour of laminate in the calculation of the shear stresses: we lost the stack in this calculation. Nevertheless, this element can be used for simple cases or for the quick estimation of displacements and in-plane stresses.

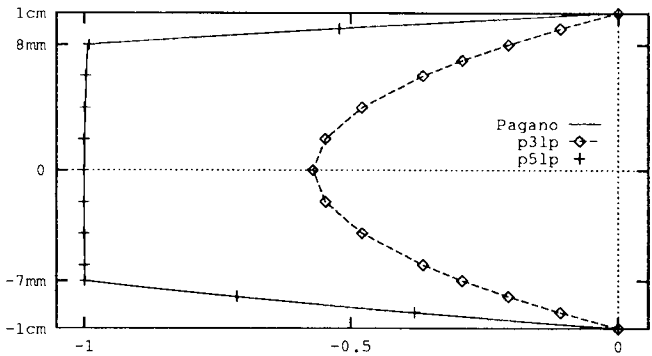

Figure 10: *The stress* $\sigma_{13}(0, x_3)$

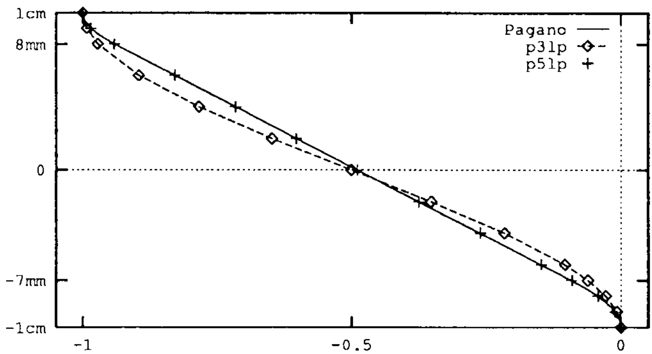

Figure 11: *The stress* $\sigma_{33}(\frac{l}{2}, x_3)$

In order to determine the domain of validity of the first term of the asymptotic development, we have then made a parametrical study. We have used element P5LP with the previous mesh. In a first step, we study, Figure 12, the influence of the thickness of the laminate on the calculation of displacements and stresses. Using materials [19] and [20], we make the thickness vary conserving the following ratios:

- 10% of the total thickness for the top skin,
- 70% of the total thickness for the core,
- 20% of the total thickness for the bottom skin.

Figure 12 shows the $\mathcal{L}^2$ norm of the relative error obtained on the three-dimensional displacements and stresses with respect to the ratio $l/2\varepsilon$. We see that the variation of thickness has very little influence on stresses. But we have to notice here that a Kirchhoff-Love model, like our first term, can not be used for a ratio of $l/2\varepsilon = 5$ where we have 20% of error on the u_3 displacement.

With only the first term, we have to restrict ourselves to plates for which the ratio $l/2\varepsilon$ goes beyond 20. We shall discuss in our conclusion the possibilities implied by the implementation of the other terms of the development.

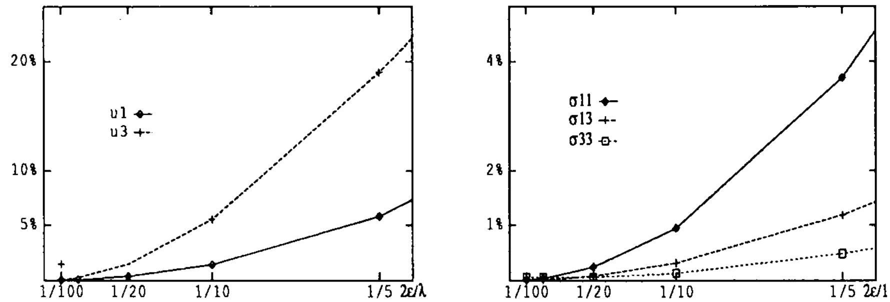

Figure 12: *Influence of the thickness*

In a second step, we study the influence of the core-skin proportion in a symmetrical laminate on the calculation of displacements and stresses. We fix to 2cm ($l/2\varepsilon = 50$) the thickness of the laminate and we progress from a plate only composed of unidirectional [19] to a plate of foam [20].

We show on Figure 13 the $\mathcal{L}^2$ norm of the relative error obtained on displacements and stresses according to the percentage of core in the laminate. The results are excellent whatever the core-skin ratio is -notably when skins are very small or when the core is reduced to a film-.

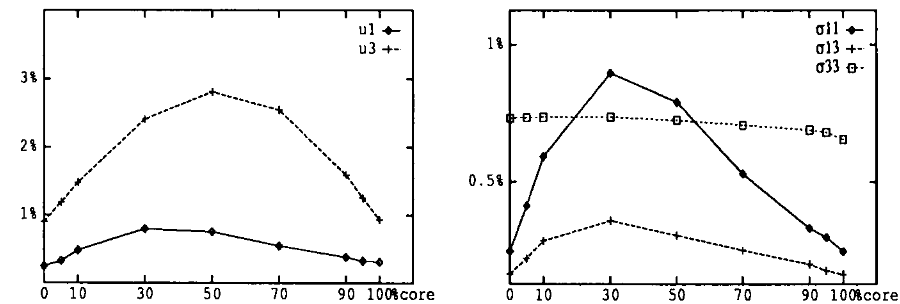

Figure 13: *Influence of the core-skin ratio*

Finally, we study the influence of the ratio of core and skins stiffnesses. We choose the case of a non-symmetrical sandwich where bottom-middle-top thicknesses are 3mm-15mm-2mm respectively. The skins keep the following mechanical properties:

$$E = 68GPa \qquad \nu = 0.33,$$

and we vary the Young's module E_c of the core.

We show on Figure 14 the $\mathcal{L}^2$ norm of the relative error obtained on displacements and stresses according to the ratio E/E_c of stiffnesses. We see that its influence is light until a ratio of 500. Then we deal with a stack of metal and elastomer which is out of the score of our study.

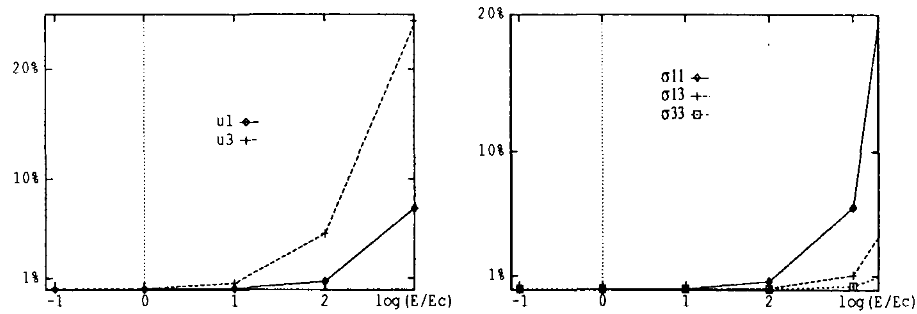

Figure 14: *Influence of the material stiffness ratio*

5. Conclusion

We have constructed a multilayered plate model using an asymptotic development technique -with one small parameter: the half-thickness of the laminate- and the process we used is fully explained. We have seen that the complete development of displacements and stresses can be obtained through the development of displacements of the middle surface only. These displacements are the solutions of two-dimensional problems which only differ by their right hand side members. By this way, we have reduced the dimension. We have highlighted the expression of each term of the development of three-dimensional quantities.

The model offers a great flexibility on the approximation of the solution by the calculation of one or more terms of the development. But we have to point out that every term of two or three dimensional developments depends on previous terms. Then, the implementation of the complete asymptotic development will use an iterative procedure.

Afterwards, we have presented the implementation of the first term of the development and the two elements we have built. Finally, we have shown some results obtained by comparison with an experiment and an analytical solution. These results are very satisfactory in all circumstances for the P5LP element. On the other hand, the less expensive P3LP element uses a simplified method of calculation for the transverse stresses and must be selected only for simple cases or for the estimation of in-plane stresses.

The influence of the thickness of the stack has shown that we have to implement some other terms of the development. As a matter of fact, we hope that the following terms will increase the accuracy of the approximation for all kind of plates. At present, we are able to suppose that the influence of these terms will be stronger for the deflection than for other quantities.

6. Bibliography

[ARG 68]J.H. Argyris, I. Fried, D.W. Scharpf (1968) *The Tuba family of plate elements for the matrix displacement method*, The Aeronautical Journal of the Royal

Aeronautical Society, Vol 72, p701-709.

[BAT 82]K.J. Bathe (1982) *Finite Element Procedures in Engineering Analysis*, Prentice-Hall Inc.

[BAT 90]J.L. Batoz, G. Dhatt (1990) *Modélisation des structures par éléments finis : poutres et plaques*, Tome 2, Hermes

[BEA 93]A. Beakou, M. Touratier (1993) *A rectangular finite element for analysing composite multilayered shallow shells in statics, vibration and buckling*, Int. J. Num. Meth. Eng, Vol 36, p627-653

[BER 85]M. Bernadou et al (1985) *MODULEF: une bibliothèque modulaire d'éléments finis*, INRIA.

[BER 94]M. Bernadou (1994) *Méthodes d'éléments finis pour les problèmes de coques minces*, Masson, Paris.

[BIG 93]B. Bigourdan (1993) *Numerical modelling of composite structures for marine applications*, Rapport IFREMER/DITI/GO/SOM 93-164 COMAST 60/09/-0/28/0094/4/R.

[CAI 87]D. Caillerie (1987) *Models of thin or thick plates and membranes derived from linear elasticity*, "Applications of Multiple Scaling in Mechanics", edited by P.G. Ciarlet et E. Sanchez-Palencia, Masson, Paris.

[CIA 79]P.G. Ciarlet, P. Destuynder (1979) *A justification of the two dimensional plate model*, J. Mécanique 18, p315-344

[CHO 94]D. Choqueuse et al (1994) *Synthesis of performed experimental activities for material and substructure qualification*, IFREMER, BV, CETENA, IPIN, COMAST 60/10/0/28/0095/1/R.

[DI 93]S. Di, E. Ramm (1993) *Hybrid stress formulation for higher-order theory of laminated shell analysis*, Comp. Met. App. Mech. Eng., Vol 109, p359-376

[DUM 90]H. Dumontet (1990) *Homogénéisation et effets de bord dans les matériaux composites*, Thèse d'Etat, Université Pierre et Marie Curie, Paris.

[DUV 90]G. Duvaut (1990) *Mécanique des milieux continus*, Masson, Paris.

[GAY 87]D. Gay (1987) *Matériaux composites*, Hermès, Paris.

[KAI 94]R. Kail (1994) *Modélisation asymptotique et numérique de plaques et coques stratifiées*, Thèse de Doctorat de l'Université Pierre et Marie Curie, Paris.

[LAR 89]P. Lardeur, J.L. Batoz (1989) *Composite plate analysis using a new discrete shear triangular finite element*, Int. j. num. meth. eng., Vol 27, p343-359

[LEN 84]F. Léné (1984) *Contribution à l'étude des matériaux composites et de leur endommagement*, Thèse d'Etat, Université Pierre et Marie Curie, Paris.

[MIN 66]R.D. Mindlin (1966) *Influence of rotary inertia and shear on flexural motions of heterogeneous plates*, International Journal of Solids Structures, Vol 2.

[OCH 92]O.O. Ochoa, J.N. Reddy (1992) *Finite Element Analysis of Composite Laminates*, Kluwet Academic Publishers.

[PAG 69]N.J. Pagano (1969) *Exact solutions for composite laminates in cylindrical bending*, Journal of Composite Materials, Vol 3, p398-411.

[RED 84]J.N. Reddy (1984) *A simple higher-order theory for laminated composite plates*, Journal of Applied Mechanics, Vol 37, p745-752.

[RED 89]J.N. Reddy (1989) *On refined computational models of composite laminates*, Int. j. num. meth. eng., Vol 27, p361-382

[REI 72]E. Reissner (1972) *A consistent treatment of transverse shear deformation in laminated anisotropic plates*, AIAA Vol 10, n°5, p716-718.

[REN 86]J.G. Ren (1986) *A new theory of laminated plate*, Comp. Sci. Tech, Vol 26, p225-239
[TOU 88]M. Touratier (1988) *A refined theory for thick composite plates*, Mechanics Research Communications, Vol 15, p229-236.

Annex:

We use the next linear development of in-plane stresses:

$$\sigma^{\alpha\beta(n)}(y_1, y_2, y_3) = \sigma^{\alpha\beta(n)}(y_1, y_2, 0) + y_3 \frac{\partial}{\partial y_3}\Big(\sigma^{\alpha\beta(n)}(y_1, y_2, 0)\Big),$$

into the definition of:

$$X^{\alpha\beta(n)} = \int_{-1}^{1} \sigma^{\alpha\beta(n)} dy_3 \, , Y^{\alpha\beta(n)} = \int_{-1}^{1} y_3 \sigma^{\alpha\beta(n)} dy_3 \, .$$

By this way, we find:

$$\sigma^{\alpha\beta(n)}(y_1, y_2, y_3) = \frac{1}{2} X^{\alpha\beta(n)}(y_1, y_2) + \frac{3y_3}{2} Y^{\alpha\beta(n)}(y_1, y_2) \, ,$$

expression we use in the equilibrium equations to find the transverse stresses.

Article soumis le 20 décembre 1995.
Version révisée le 20 octobre 1996.