Parametrization of finite rotations in computational dynamics : a review

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ABSTRACT. Finite rotations are traditionnally regarded as geometric operations on vectors. By adopting an algebraic point of view, they many also be regarded as linear transformations with invariance properties. They can thus be described in terms of a minimal set of parameters, the choice of which is very wide. The objective of the paper is to make a general presentation of finite and differential motion kinematics in algebraic form and to discuss different methods of parametrization. The proposed concepts are then applied to develop an energy conserving time integration strategy to compute the long term response of a spinning top in a gravity field.

RÉSUMÉ. Les rotations finies sont généralement assimilées à des opérations géométriques sur des vecteurs. D'un point de vue algébrique, on peut aussi les voir comme des transformations linéaires dotées de propriétés d'invariance. Elles peuvent donc être décrites en termes d'un ensemble minimal de paramètres que l'on peut choisir de multiples façons. L'objectif de l'article est de faire une présentation générale de la cinématique des mouvements fini et différentiel sous forme algébrique et de discuter les différentes formes de paramétrisation. Les concepts proposés sont ensuite appliqués au développement d'une stratégie d'intégration temporelle préservant l'énergie du système pour le calcul de la réponse à long terme d'une toupie dans un champ de gravité.

KEY WORDS : finite rotation, kinematics, time integration.
MOTS-CLÉS : rotation finie, cinématique, intégration temporelle.

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1. Introduction

The kinematics of finite motion and, in particular, finite rotations, has been a topic of continuous interest since the pioneering work of Euler [EUL76] in the development of classical dynamics. L. Euler's objective was the study of motion undergone by particles, rigid bodies and systems thereof.

In particular, he has been the first to recognize the importance of *spherical motion* defined as the pure rotation motion occuring in a body fixed at one point. He also observed that a spherical motion can always be described as a unique rotation about an axis of given orientation in space.

Many textbooks of classical dynamics cover the subject of kinematics of rigid motion, but most of the time with the only objective of treating the classical problem of the dynamics of a single rigid body in space. Let us mention in particular the classical works of Whittaker [WHI65], Goldstein [GOL64], Lur'é [LUR68] and Meirovitch [MEI70].

Two remarkable texts which are devoted exclusively to the study of kinematics ought to be mentioned : the now classical book of Bottema and Roth [BOT79] and a more recent monography by Angeles [ANG88] which have both been used extensively throughout the elaboration of the present review.

Several points of view may be adopted to represent large rotations in a three-dimensional space [GER88, CAR89].

The geometrical point of view, which is found in most textbooks of classical dynamics [GOL64, MEI70, LUR68, WHI65], consists to express an arbitrary rotation in terms of elementary rotations about well defined axes. Euler himself introduced a set of angles, well known as Euler angles, which is particularly well suited to the study of spinning bodies such as tops and gyroscopes.

The community of flight mechanics [ETK72, HUG86] has adopted another set of purely geometrical parameters, known as Bryant or nautical angles, which describe the rotation in terms of angular quantities known as roll, pitch and yaw. The latter have an easy geometric interpretation in the very context of flight mechanics. The same choice has been made in the early development of robotics to identify the motion of the end-effector of the robot relatively to its base [PAU81].

Despite their straightforward physical interpretation, geometric parameters have also serious drawbacks. They may lead to singularities in specific situations, and due to their trigonometric nature they are not computationally efficient to describe the arbitrarily large rotations which can be encountered in very complex systems such as articulated systems made of interconnected rigid and flexible bodies. Therefore, the main interest of geometric representations lies today in the post-processing of results following a numerical simulation.

The development of the algebraic approach is based on the very fundamental observation that spherical motion preserves the length of the position vector of any point undergoing the rotation. The well known orthonormality property of the rotation operator follows immediatly and allows in turn to express its algebraic structure in terms of invariants [ANG88]. A large variety of parameters sets can be proposed to describe these invariants, the most well known being the Euler parameters and the Rodrigues parameters. A significant part of our review work will be devoted to the development of this algebraic approach and to analyzing the algebraic properties and computational merits of the various parameter sets which can be adopted.

The use of the algebraic approach in a numerical context has become common practice with the development of the multibody dynamics discipline. Efficient use of Euler parameters for numerical simulation in rigid multibody dynamics can be found in textbooks such as [NIK88, HAU89]. The methodology was extended later to flexible multibody systems by several authors [GER86, VAN84].

The interest of the continuum mechanics community for the kinematic description of rotational motion is more recent, due to the slow development of geometrically nonlinear analysis of engineering structures until the early eighties. In this context, appropriate formalism is however needed to describe locally the motion undergone by the continuum. Among the contributions of continuum mechanics experts, one can mention in particular remarkable synthesis works by Argyris [ARG82] and Atluri [ATL95].

The most fundamental contribution in the context of the application to structural mechanics is certainly that contained in the impressive set of articles by J. Simo and his co-workers [SIM86, SIM85, SIM95, SIM91, SIM89]. J. Simo has raised the level of abstraction of the algebraic approach by making the choice to represent finite rotations as objects belonging to a non-linear manifold, the special orthogonal Lie group [SIM86]. The special orthogonal group concept has been used to develop a geometrically exact modelling of beams [SIM86] and shells [SIM89]. Special attention was brought to time integration aspects, showing in particular how to increment finite rotations according to the finite rotation composition rule. He also demonstrated that energy conserving algorithms can be developed even in presence of large rotations, therefore requiring a specific incrementation procedure for finite rotations [SIM91, SIM95, SIM92].

The work accomplished by Cardona and Géradin [CAR88, CAR89, GER93a] is much inspired by the work of Simo, but extends and systematizes the concept of nonlinear finite element to flexible multibody dynamics. Their formalism has provided a general basis for the development of a large class of element models, including rigid bodies, elastic beams [CAR88] and specialized types of joints [GER93b, GER93a]. They have also proposed a very general methodology for substructuring in flexible multibody dynamics which avoids cumbersome computations for the evaluation of subsystem kinetic energy [GER91].

The present paper deals exclusively with rigid body dynamics aspects and is organized as follows.

Section 2 is devoted to the kinematic description of rigid body motion. It is much inspired from ref. [ANG88] for the development of the algebraic approach to spherical motion starting from the concept of rotation invariants. Like in [ANG88], systematic use is made of the matrix notation to describe vector operations in the Euclidian space. The explicit expression of the rotation operator is given under various forms, and the interpretation of general rigid body motion as a screw motion is also recalled.

Section 3 deals with velocity analysis. It mainly establishes the material and spatial expressions of angular velocities in terms of rotations invariants and their time derivatives. It also introduces the concept of instantaneous screw axis for velocity analysis of general rigid body motion.

Section 4 goes through the same steps for acceleration analysis.

Section 5 discusses an important aspect of computational kinematics : incremental rotational motion. It is shown that rotation increments behave as angular velocities. Special situations such as updating the rotation of the reference frame and motion analysis in a non inertial frame are also discussed.

Sections 7 to 10 deal with the problem of selecting parameters to represent the spherical part of rigid body motion. Section 7 presents the most usual parameter sets of algebraic nature: the Cartesian rotation vector, the Rodrigues parameters and Euler parameters. Section 8 establishes the link between Euler parameters and quaternion algebra and derives additional sets of parameters: the conformal rotation vector (CRV) and the linear parameters. Section 10 comes back to the problem of geometric interpretation of finite rotations by describing the concepts of Euler angles and Bryant angles.

Section 11 presents an application of academic nature which has the merit to make use of many concepts presented in the previous sections. It demonstrates how an energy conserving integration strategy can be devised to numerically integrate the motion equations of a spinning top in a gravity field. The time integration scheme adopted is the mid-point rule, and its application requires the splitting of the total rotation increment in two equal parts to express dynamic equilibrium at mid-point [BAU95, GER94]. It is shown that this is best achieved by making use of Euler parameters.

The numerical tests achieved demonstrate the effectiveness of this computational approach. Physical interpretation of the results is obtained by post-processing them in terms of Euler angles.

2. Kinematic description of rigid body motion

In the following presentation of kinematics of rigid motion, matrix notation has been systematically adopted. Vectors of the euclidian space $\mathcal{E}^3$ are represented by column matrices collecting their components in a given frame, and linear transformations are described by vector-matrix products. The notations adopted and the fundamental operations of linear algebra which will be invoked are given in appendix A.

Unless specified otherwise, the column matrix collecting the spatial (inertial) components of a vector is denoted by a lower case letter while the column matrix collecting the material (body fixed) components of the same vector is denoted by the same letter in upper case.

2.1. Spherical motion

Spherical motion corresponds to the rotation of a rigid body about a fixed point in space. In other words, we can regard it as the motion of the end point of a vector with its origin remaining fixed (see figure 1). It is thus characterized by

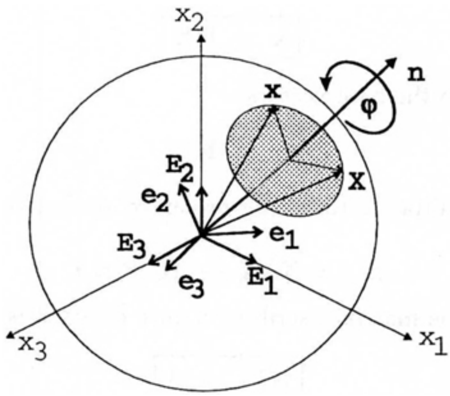

Figure 1: *Spherical motion*

two invariance properties:

(i) the length of the position vector of a given point P attached to the rigid body remains unaffected by the pure rotation;

(ii) the relative angle between any two directions attached to the body remains constant under the transformation.

Let us define

$\vec{\mathbf{X}}$ the position vector of point P in the reference configuration, of Cartesian components

$$\mathbf{X} = [X_1\ X_2\ X_3]^T \tag{1}$$

$\vec{\mathbf{x}}$ the position vector of point P after transformation, of Cartesian components

$$\mathbf{x} = [x_1\ x_2\ x_3]^T \tag{2}$$

$[X_1\ X_2\ X_3]$ and $[x_1\ x_2\ x_3]$ will be respectively referred to as the material and spatial coordinates of point P.
Similarly, we define ,,,

$[\vec{\mathbf{E}}_1\ \vec{\mathbf{E}}_2\ \vec{\mathbf{E}}_3]$ a set of orthonormal base vectors attached to the body in the reference configuration. They form the so-called *absolute or spatial frame.*

$[\vec{e}_1\ \vec{e}_2\ \vec{e}_3]$ the same set of orthonormal base vectors after transformation. They remain attached to the body in its current configuration and form thus the *material or body frame.*

The pure rotation can be expressed as a linear transformation

$$\boxed{\mathbf{x} = \mathbf{R}\mathbf{X}} \tag{3}$$

which also applies to the base vectors

$$\mathbf{e}_i = \mathbf{R}\mathbf{E}_i \tag{4}$$

The length of the initial vector being preserved during the spherical motion, this implies

$$\mathbf{x}^T\mathbf{x} = \mathbf{X}^T\mathbf{X} \longrightarrow \mathbf{R}^T\mathbf{R} = \mathbf{I} \tag{5}$$

which means that the matrix describing a pure rotation is orthogonal

$$\boxed{\mathbf{R}^T = \mathbf{R}^{-1}} \tag{6}$$

The base vectors before transformation forming an orthonormal set, we have

$$\mathbf{E}_i^T\mathbf{E}_j = \delta_{ij} \qquad \text{and} \qquad \mathbf{E}_3 = \tilde{\mathbf{E}}_1\mathbf{E}_2 \tag{7}$$

and therefore

$$(\tilde{\mathbf{E}}_1\mathbf{E}_2)^T\mathbf{E}_3 = 1 \tag{8}$$

We have likewise by hypothesis

$$\mathbf{e}_i^T\mathbf{e}_j = \delta_{ij} \qquad \text{and} \qquad \mathbf{e}_3 = \tilde{\mathbf{e}}_1\mathbf{e}_2 \tag{9}$$

and therefore

$$(\tilde{\mathbf{e}}_1\mathbf{e}_2)^T\mathbf{e}_3 = 1 \tag{10}$$

Let us now define the base vector matrices

$$\mathbf{A} = [\,\mathbf{E}_1 \quad \mathbf{E}_2 \quad \mathbf{E}_3\,] \qquad \text{and} \qquad \mathbf{B} = [\,\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3\,] \tag{11}$$

which, owing to (8) and (10), have a unit determinant

$$\det(\mathbf{A}) = 1 \qquad \text{and} \qquad \det(\mathbf{B}) = 1 \tag{12}$$

They are such as

$$\mathbf{B} = \mathbf{R}\mathbf{A} \tag{13}$$

and therefore we have

$$\det(\mathbf{R}) = 1 \tag{14}$$

which shows that the rotation matrix **R** is a proper orthogonal matrix*. [1]

[1] It can be shown that an unproper orthogonal matrix, characterized by $\det(\mathbf{R}) = -1$, would generate a reflection with respect to the plane orthogonal to **n**.

Let us now examine the properties of the eigensolutions of matrix $\mathbf{R}$. Owing to (14) we have

$$\det(\mathbf{R}) = \lambda_1\lambda_2\lambda_3 = 1. \tag{15}$$

The rotation matrix $\mathbf{R}$ also admits the spectral expansion

$$\mathbf{R} = \mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1} \qquad \text{with} \qquad \mathbf{X}^{\star}\mathbf{X} = \mathbf{I} \tag{16}$$

and likewise, since $\mathbf{R}$ is real

$$\mathbf{R}^T = \mathbf{R}^{\star} = \mathbf{X}^{-\star}\mathbf{\Lambda}^{\star}\mathbf{X}^{\star} \tag{17}$$

and therefore

$$\mathbf{R}^T\mathbf{R} = \mathbf{X}^{-\star}\mathbf{\Lambda}^{\star}\mathbf{\Lambda}\mathbf{X}^{\star} = \mathbf{I} \tag{18}$$

which implies

$$\mathrm{tr}(\mathbf{R}^T\mathbf{R}) = \mathrm{tr}(\mathbf{\Lambda}^{\star}\mathbf{\Lambda}) = |\lambda_1^2| + |\lambda_2^2| + |\lambda_3^2| = \mathrm{tr}(\mathbf{I}) = 3 \tag{19}$$

It is easily verified that the set of equations

$$\begin{cases} |\lambda_1^2| + |\lambda_2^2| + |\lambda_3^2| & = 3 \\ \lambda_1\lambda_2\lambda_3 & = 1 \end{cases} \tag{20}$$

has for solution

$$\lambda_1 = 1 \qquad \lambda_{2,3} = \exp(\pm i\phi) \qquad (\phi \text{ arbitrary}) \tag{21}$$

and therefore the proper orthogonal matrix $\mathbf{R}$ admits at least one eigenvector $\mathbf{n}$ such that

$$\mathbf{R}\mathbf{n} = \mathbf{n} \tag{22}$$

which remains unaffected by the transformation: the unit vector $\mathbf{n}$ gives thus the rotation axis.
The eigenvectors associated to λ_2 and λ_3 being necessarily complex conjugate, let us express the modal matrix $\mathbf{X}$ in the form

$$\mathbf{X} = [\mathbf{n} \quad \mathbf{u} + i\mathbf{v} \quad \mathbf{u} - i\mathbf{v}] \tag{23}$$

and thus

$$\mathbf{X}^{*} = \begin{bmatrix} \mathbf{n}^T \\ (\mathbf{u} - i\mathbf{v})^T \\ (\mathbf{u} + i\mathbf{v})^T \end{bmatrix} \tag{24}$$

Owing to property (16.b) we have

$$\mathbf{X}^{*}\mathbf{X} = \begin{bmatrix} \mathbf{n}^T\mathbf{n} & \mathbf{n}^T(\mathbf{u} + i\mathbf{v}) & \mathbf{n}^T(\mathbf{u} - i\mathbf{v}) \\ (\mathbf{u} - i\mathbf{v})^T\mathbf{n} & (\mathbf{u} - i\mathbf{v})^T(\mathbf{u} + i\mathbf{v}) & (\mathbf{u} - i\mathbf{v})^T(\mathbf{u} - i\mathbf{v}) \\ (\mathbf{u} + i\mathbf{v})^T\mathbf{n} & (\mathbf{u} + i\mathbf{v})^T(\mathbf{u} + i\mathbf{v}) & (\mathbf{u} + i\mathbf{v})^T(\mathbf{u} - i\mathbf{v}) \end{bmatrix} = \mathbf{I} \tag{25}$$

which implies

$$\begin{aligned} \mathbf{n}^T\mathbf{u} = \mathbf{n}^T\mathbf{v} &= 0 \\ \mathbf{u}^T\mathbf{v} &= 0 \\ \mathbf{u}^T\mathbf{u} + \mathbf{v}^T\mathbf{v} &= 1 \end{aligned} \tag{26}$$

From the properties (26) we deduce that vectors $\mathbf{u}$ and $\mathbf{v}$ form an orthogonal set of base vectors in the plane perpendicular to rotation axis $\mathbf{n}$.
We next observe that the vectors

$$\begin{aligned} \mathbf{R}(\mathbf{u} + i\mathbf{v}) &= \exp(i\phi)(\mathbf{u} + i\mathbf{v}) \\ \mathbf{R}(\mathbf{u} - i\mathbf{v}) &= \exp(-i\phi)(\mathbf{u} - i\mathbf{v}) \end{aligned} \tag{27}$$

can be developed in the form

$$\begin{aligned} \mathbf{R}\mathbf{u} + i\mathbf{R}\mathbf{v} &= (\mathbf{u}\cos\phi - \mathbf{v}\sin\phi) + i(\mathbf{u}\sin\phi + \mathbf{v}\cos\phi) \\ \mathbf{R}\mathbf{u} - i\mathbf{R}\mathbf{v} &= (\mathbf{u}\cos\phi - \mathbf{v}\sin\phi) - i(\mathbf{u}\sin\phi + \mathbf{v}\cos\phi) \end{aligned} \tag{28}$$

and therefore

$$\begin{aligned} \mathbf{R}\mathbf{u} &= \mathbf{u}\cos\phi - \mathbf{v}\sin\phi \\ \mathbf{R}\mathbf{v} &= \mathbf{u}\sin\phi + \mathbf{v}\cos\phi \end{aligned} \tag{29}$$

It shows that both vectors $\mathbf{u}$ and $\mathbf{v}$ undergo a plane rotation of angle ϕ in the plane perpendicular to the rotation axis.
This result is known as the *Theorem of Euler* [EUL76] on finite rotations:

> *If a rigid body undergoes a motion leaving fixed one of its points, then a set of points of the body, lying on a line that passes through that point, remains fixed as well.*

2.2. Explicit expressions of the rotation operator

The orthonormality property (14) implies the six constraints

$$\mathbf{r}_i^T\mathbf{r}_j = \delta_{ij} \qquad (i = 1, \ldots j,\ j = 1, 2, 3) \tag{30}$$

and therefore the rotation matrix can be expressed in the form

$$\mathbf{R} = \mathbf{R}(\alpha_1, \alpha_2, \alpha_3) \tag{31}$$

where $\alpha_1, \alpha_2, \alpha_3$ are 3 independent rotation parameters which can be chosen in many ways.
Different explicit expressions of the rotation operator with direct geometric meaning can be deduced from the results of the previous section.
A first series of expressions may be obtained in terms of the initial and transformed base vectors. They do not involve a minimal set of parameters as indicated by eqn (31), but their physical interpretation is immediate.
A second series of expressions is obtained in terms of the linear invariants of the rotation $\mathbf{n}$ and ϕ. The latter form an almost minimal set in the sense that they are simply linked by the normality constraint $||\mathbf{n}|| = 1$.

The outer product expression The outer product expression of the rotation operator is a direct consequence of eqn (13). By making use of eqn (11) we obtain

$$\boxed{\mathbf{R} = \sum_i \mathbf{e}_i \mathbf{E}_i^T} \tag{32}$$

Expression in terms of direction cosines It is obtained by resolving the linear transformation describing the rotation into Cartesian components

$$\mathbf{x} = \mathbf{R}\mathbf{X} = \sum_j x_j \mathbf{E}_j = \sum_j X_j \mathbf{e}_j = \sum_j X_j \mathbf{R}\mathbf{E}_j \tag{33}$$

and therefore

$$x_i = \mathbf{E}_i^T \mathbf{x} = \sum_j \mathbf{E}_i^T \mathbf{e}_j X_j = \sum_j r_{ij} X_j \tag{34}$$

Hence the rotation operator explicit expression

$$\boxed{\mathbf{R} = \begin{bmatrix} \mathbf{E}_1^T\mathbf{e}_1 & \mathbf{E}_1^T\mathbf{e}_2 & \mathbf{E}_1^T\mathbf{e}_3 \\ \mathbf{E}_2^T\mathbf{e}_1 & \mathbf{E}_2^T\mathbf{e}_2 & \mathbf{E}_2^T\mathbf{e}_3 \\ \mathbf{E}_3^T\mathbf{e}_1 & \mathbf{E}_3^T\mathbf{e}_2 & \mathbf{E}_3^T\mathbf{e}_3 \end{bmatrix}} \tag{35}$$

in terms of inner products or, equivalently, direction cosines between the base vectors.

Expression in terms of the linear invariants $\mathbf{n}$ *and* ϕ By making use of the projection operator (366), the vectors $\mathbf{X}$ and $\mathbf{x} = \mathbf{R}\mathbf{X}$ may be decomposed into their parts along and orthogonal to the rotation axis

$$\begin{aligned} \mathbf{X} &= (\mathbf{I} - \mathbf{P}_n)\mathbf{X} + \mathbf{P}_n\mathbf{X} \\ \mathbf{x} &= (\mathbf{I} - \mathbf{P}_n)\mathbf{x} + \mathbf{P}_n\mathbf{x} \end{aligned} \tag{36}$$

By observing that

$$(\mathbf{I} - \mathbf{P}_n)\mathbf{R} = \mathbf{n}\mathbf{n}^T\mathbf{R} = \mathbf{n}\mathbf{n}^T = \mathbf{I} - \mathbf{P}_n \tag{37}$$

and with the definitions

$$\mathbf{Y} = \mathbf{P}_n\mathbf{X} \qquad \text{and} \qquad \mathbf{y} = \mathbf{P}_n\mathbf{x} = \mathbf{P}_n\mathbf{R}\mathbf{X} \tag{38}$$

Eqns (36) may be rewritten in the form

$$\begin{aligned} \mathbf{X} &= (\mathbf{I} - \mathbf{P}_n)\mathbf{X} + \mathbf{Y} \\ \mathbf{x} &= (\mathbf{I} - \mathbf{P}_n)\mathbf{X} + \mathbf{y} \end{aligned} \tag{39}$$

The orthogonal part $\mathbf{y}$ of the transformed vector may be computed by observing that it undergoes a pure rotation in the plane orthogonal to $\mathbf{n}$, and thus

$$\begin{aligned} \tilde{\mathbf{Y}}\mathbf{y} &= \|\mathbf{Y}\|^2 \mathbf{n} \sin\phi \\ \mathbf{Y}^T\mathbf{y} &= \|\mathbf{Y}\|^2 \cos\phi \end{aligned} \tag{40}$$

and therefore is solution of the (4 × 3) system

$$\begin{bmatrix} \tilde{\mathbf{Y}} \\ \mathbf{Y}^T \end{bmatrix} \mathbf{y} = ||\mathbf{Y}||^2 \begin{bmatrix} \mathbf{n}\sin\phi \\ \cos\phi \end{bmatrix} \tag{41}$$

Matrix $\tilde{\mathbf{Y}}$ being of rank 2, it has a unique solution which may be computed by taking the More-Penrose inverse of the system matrix. Multiplying by the transposed system matrix yields

$$(\tilde{\mathbf{Y}}^T \tilde{\mathbf{Y}} + \mathbf{Y}\mathbf{Y}^T)\mathbf{y} = ||\mathbf{Y}||^2 \begin{bmatrix} \tilde{\mathbf{Y}}^T & \mathbf{Y} \end{bmatrix} \begin{bmatrix} \mathbf{n}\sin\phi \\ \cos\phi \end{bmatrix} \tag{42}$$

Owing to (364) we have

$$\tilde{\mathbf{Y}}^T \tilde{\mathbf{Y}} + \mathbf{Y}\mathbf{Y}^T = ||\mathbf{Y}||^2 \mathbf{I} \tag{43}$$

and thus

$$\mathbf{y} = \begin{bmatrix} \tilde{\mathbf{Y}}^T & \mathbf{Y} \end{bmatrix} \begin{bmatrix} \mathbf{n}\sin\phi \\ \cos\phi \end{bmatrix} = (\tilde{\mathbf{n}}\sin\phi + \mathbf{I}\cos\phi)\mathbf{Y} = [\tilde{\mathbf{n}}\sin\phi + (\mathbf{I} - \mathbf{n}\mathbf{n}^T)\cos\phi]\mathbf{X} \tag{44}$$

We finally get

$$\mathbf{x} = [\mathbf{I}\cos\phi + (1 - \cos\phi)\mathbf{n}\mathbf{n}^T + \tilde{\mathbf{n}}\sin\phi]\mathbf{X} \tag{45}$$

from which we extract the geometric expression of the rotation operator in terms of $\mathbf{n}$ and ϕ

$$\boxed{\mathbf{R} = \mathbf{I}\cos\phi + (1 - \cos\phi)\mathbf{n}\mathbf{n}^T + \tilde{\mathbf{n}}\sin\phi} \tag{46}$$

It has the linear invariants

$$\begin{aligned} \operatorname{tr}(\mathbf{R}) = \lambda_1 + \lambda_2 + \lambda_3 \quad &= 1 + 2\cos\phi \\ \operatorname{vect}(\mathbf{R}) \quad &= \mathbf{n}\sin\phi \end{aligned} \tag{47}$$

The exponential map Let us start again from the linear transformation describing spherical motion

$$\mathbf{x} = \mathbf{R}\mathbf{X} \tag{48}$$

and let us derive it with respect to the rotation angle

$$\frac{d\mathbf{x}}{d\phi} = \frac{d\mathbf{R}}{d\phi}\mathbf{X} = \frac{d\mathbf{R}}{d\phi}\mathbf{R}^T\mathbf{x} \tag{49}$$

with

$$\frac{d\mathbf{R}}{d\phi} = \tilde{\mathbf{n}}\cos\phi - (\mathbf{I} - \mathbf{n}\mathbf{n}^T)\sin\phi \tag{50}$$

It is easily verified that

$$\frac{d\mathbf{R}}{d\phi}\mathbf{R}^T = \tilde{\mathbf{n}} \tag{51}$$

and therefore the spherical motion is governed by the differential equation

$$\frac{d\mathbf{x}}{d\phi} - \tilde{\mathbf{n}}\mathbf{x} = 0 \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{X} \tag{52}$$

Its solution is

$$\mathbf{x} = \exp(\tilde{\mathbf{n}}\phi)\ \mathbf{X} \tag{53}$$

and yields the exponential representation of the rotation operator

$$\boxed{\mathbf{R} = \exp(\tilde{\mathbf{n}}\phi)} \tag{54}$$

2.3. General motion of a rigid body

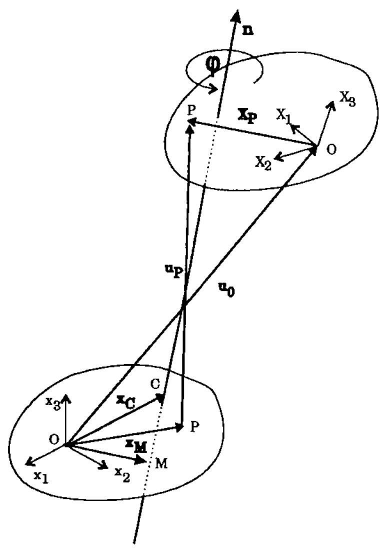

Figure 2: *Screw description of rigid body general motion*

Let us now consider the case of a rigid body undergoing translation and rotation motion simultaneously. The spatial position of any point P on the body can be described by a frame transformation (figure 2)

$$\boxed{\mathbf{x}_P = \mathbf{x}_0 + \mathbf{R}\mathbf{X}_P} \tag{55}$$

where

O is a reference point on the body, adopted as origin of the material frame $[\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3]$.

$\mathbf{x}_0$ describes the spatial position of this origin.

$\mathbf{X}_P$ contains the material coordinates of point P on the body.

$\mathbf{R}$ is the rotation operator from space frame $[\mathbf{E}_1\ \mathbf{E}_2\ \mathbf{E}_3]$ to material frame $[\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3]$.

We can also express the displacement of point P from its reference position in the form

$$\mathbf{u}_P = \mathbf{x}_P - \mathbf{X}_P = \mathbf{u}_0 + \mathbf{D}\mathbf{X}_P \tag{56}$$

where $\mathbf{u}_0 = \mathbf{x}_0$ is the displacement of the origin, and where

$$\mathbf{D} = \mathbf{R} - \mathbf{I} \tag{57}$$

is the *rotation displacement matrix*. It has not maximum rank since $\mathbf{n}$ is in its null space

$$\mathbf{D}\mathbf{n} = \mathbf{D}^T\mathbf{n} = 0 \tag{58}$$

Therefore the system

$$\mathbf{D}\mathbf{X}_P + \mathbf{u}_0 = 0 \tag{59}$$

has no solution in general, which means that there is generally no point on the body which remains fixed under the frame transformation (55).
Let us thus look for a point C which would undergo minimum displacement $\mathbf{u}_C$. Its position can be determined by solving the least squares problem

$$||\mathbf{u}_C|| = \min_{\mathbf{X}} ||\mathbf{D}\mathbf{X} + \mathbf{u}_0||^2 \tag{60}$$

Its solution verifies the following equation

$$\mathbf{D}^T(\mathbf{D}\mathbf{X}_C + \mathbf{u}_0) = 0 \tag{61}$$

which shows that the displacement of point C is in the null space of $\mathbf{D}^T$

$$\mathbf{u}_C = \mathbf{D}\mathbf{X}_C + \mathbf{u}_0 = k\mathbf{n} \tag{62}$$

where k is a constant. It is obtained by multiplying eqn (62) by $\mathbf{n}^T$

$$k = \mathbf{n}^T\mathbf{u}_0 \tag{63}$$

and the position of point C is thus solution of

$$\mathbf{D}\mathbf{X}_C = (\mathbf{n}\mathbf{n}^T - \mathbf{I})\mathbf{u}_0 \tag{64}$$

Combining eqns (56) and (62) allows to rewrite the displacement of point P in the form

$$\boxed{\mathbf{u}_P = k\mathbf{n} + \mathbf{D}(\mathbf{X}_P - \mathbf{X}_C)} \tag{65}$$

It expresses that the displacement of any point P on the body can be decomposed into a finite rotation about the rotation axis $\mathbf{n}$ followed by a translation about the same axis. It is equivalent to the motion undergone by the nut of a

screw, and therefore a general rigid body motion as described by eqn (65) is often referred to as *screw motion* (figure 2).
In order to remove the undeterminacy upon the choice of point C on the rotation axis, let us choose on the screw axis the point M which is at the shortest distance from the origin O. Its vector position M, which is defined as the *pose* of the screw axis, verifies the constraint

$$\mathbf{n}^T \mathbf{X}_M = 0 \tag{66}$$

and is solution of the 4×3 system

$$\begin{bmatrix} \mathbf{D} \\ \mathbf{n}^T \end{bmatrix} \mathbf{X}_M = \begin{bmatrix} \mathbf{n}\mathbf{n}^T - \mathbf{I} \\ 0 \end{bmatrix} \mathbf{u}_0 \tag{67}$$

$\mathbf{D}$ being of rank 2, the solution of (67) is unique and is obtained from

$$[\mathbf{D}^T \quad \mathbf{n}] \begin{bmatrix} \mathbf{D} \\ \mathbf{n}^T \end{bmatrix} \mathbf{X}_M = -\mathbf{D}^T \mathbf{u}_0 \tag{68}$$

By making use of (57) and of the expression (46) of the rotation operator in terms of $[\mathbf{n}\ \phi]$, we obtain the pose vector of the screw

$$\mathbf{X}_M = -\frac{\mathbf{D}^T \mathbf{u}_0}{2(1 - \cos\phi)} \tag{69}$$

It is also immediatly verified that the displacement of point M is a pure translation

$$\mathbf{u}_M = k\mathbf{n} + \mathbf{D}(\mathbf{X}_M - \mathbf{X}_C) = k\mathbf{n} \tag{70}$$

The *pitch* of the screw is defined as the ratio

$$p = \frac{2\pi k}{\phi} \tag{71}$$

and corresponds to the amount of translation produced by a rotation of 1 radian.
The *screw* nature of a general rigid body motion as expressed by eqn (65) had already been observed by Chasles and formulated by the theorem

> *Under the most general motion of a rigid body, a set of points of the body, namely a line parallel to the axis of rotation involved, undergo a displacement of minimum magnitude that is parallel to that axis. Moreover, the axis passes through a point whose position vector is given by eqn. (69).*

3. Velocity analysis of rigid motion

3.1. Velocity analysis of spherical motion

Let us again consider the case of finite motion about a fixed point where the motion af an arbitrary point P on the rigid body is governed by

$$\mathbf{x}_P = \mathbf{R}\mathbf{X}_P \qquad \text{with} \qquad \mathbf{R}^T = \mathbf{R}^{-1} \tag{72}$$

The velocity vector of point P expressed in *spatial coordinates* is obtained by taking the time derivative of (72)

$$\mathbf{v}_P = \dot{\mathbf{x}}_P = \dot{\mathbf{R}}\mathbf{X}_P = \dot{\mathbf{R}}\mathbf{R}^T\mathbf{x}_P \tag{73}$$

since its material coordinates $\mathbf{X}_P$ remain invariant with respect to time. The matrix $\dot{\mathbf{R}}\mathbf{R}^T$ is skew symmetric since

$$\frac{d}{dt}(\mathbf{R}\mathbf{R}^T) = \dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T = \dot{\mathbf{R}}\mathbf{R}^T + (\dot{\mathbf{R}}\mathbf{R}^T)^T = 0 \tag{74}$$

and therefore let us define

$$\tilde{\omega} = \dot{\mathbf{R}}\mathbf{R}^T \tag{75}$$

It can be regarded as the matrix of angular velocities expressed in spatial coordinates, since equation (73) may be rewritten in the form

$$\mathbf{v}_P = \tilde{\omega}\mathbf{x}_P \tag{76}$$

and is the matrix analog of the vector relationship

$$\mathbf{v}_P = \omega \times \mathbf{x}_P \tag{77}$$

The vector part of $\tilde{\omega}$ provides the spatial expression of the angular velocity vector

$$\omega = \text{vect}(\dot{\mathbf{R}}\mathbf{R}^T) \tag{78}$$

The velocity vector of point P may also be transformed to material coordinates

$$\mathbf{V}_P = \mathbf{R}^T\mathbf{v}_P = \mathbf{R}^T\dot{\mathbf{R}}\mathbf{X}_P \tag{79}$$

and likewise we define

$$\tilde{\Omega} = \mathbf{R}^T\dot{\mathbf{R}} \tag{80}$$

which may be interpreted as the matrix of material angular velocities. The vector of material angular velocities is thus obtained in the form

$$\Omega = \text{vect}(\mathbf{R}^T\dot{\mathbf{R}}) \tag{81}$$

3.2. Explicit expression of the angular velocities

We are now looking for an explicit expression of the angular velocity vectors ω and $\boldsymbol{\Omega}$ in terms of the invariants of the rotation $(\mathbf{n},\ \phi)$ and their time derivatives $(\dot{\mathbf{n}},\ \dot{\phi})$.

To this purpose, let us start from the invariance properties

$$\mathbf{R}\mathbf{n} = \mathbf{n} \qquad \mathbf{R}^T\mathbf{n} = \mathbf{n} \tag{82}$$

of the rotation operator. Their time derivative yields

$$(\mathbf{R} - \mathbf{I})\dot{\mathbf{n}} = -\dot{\mathbf{R}}\mathbf{n} \qquad (\mathbf{R}^T - \mathbf{I})\dot{\mathbf{n}} = -\dot{\mathbf{R}}^T\mathbf{n} \tag{83}$$

If we premultiply the first eqn (83) by $\mathbf{R}^T$ and the second one by $\mathbf{R}$, we get

$$(\mathbf{R}^T - \mathbf{I})\dot{\mathbf{n}} = \mathbf{R}^T\dot{\mathbf{R}}\mathbf{n} = \tilde{\boldsymbol{\Omega}}\mathbf{n} \qquad (\mathbf{I} - \mathbf{R})\dot{\mathbf{n}} = -\mathbf{R}\dot{\mathbf{R}}^T\mathbf{n} = \tilde{\omega}\mathbf{n} \tag{84}$$

and thus

$$(\mathbf{I} - \mathbf{R}^T)\dot{\mathbf{n}} = \tilde{\mathbf{n}}\boldsymbol{\Omega} \qquad (\mathbf{R} - \mathbf{I})\dot{\mathbf{n}} = \tilde{\mathbf{n}}\omega \tag{85}$$

In order to solve the equations (85) with respect to $\dot{\mathbf{n}}$, let us take into account the constraint

$$\frac{d}{dt}(\mathbf{n}^T\mathbf{n}) = 2\mathbf{n}^T\dot{\mathbf{n}} = 0 \tag{86}$$

$\dot{\mathbf{n}}$ is thus solution of either one of the (4×3) systems

$$\begin{bmatrix} \mathbf{I} - \mathbf{R}^T \\ \mathbf{n}^T \end{bmatrix} \dot{\mathbf{n}} = \begin{bmatrix} \tilde{\mathbf{n}}\boldsymbol{\Omega} \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{R} - \mathbf{I} \\ \mathbf{n}^T \end{bmatrix} \dot{\mathbf{n}} = \begin{bmatrix} \tilde{\mathbf{n}}\omega \\ 0 \end{bmatrix} \tag{87}$$

They can be solved by taking the More-Penrose inverse of the left-hand side

$$\begin{aligned} [2\mathbf{I} - (\mathbf{R} + \mathbf{R}^T) + \mathbf{n}\mathbf{n}^T]\dot{\mathbf{n}} &= (\mathbf{I} - \mathbf{R})\tilde{\mathbf{n}}\boldsymbol{\Omega} \\ [2\mathbf{I} - (\mathbf{R} + \mathbf{R}^T) + \mathbf{n}\mathbf{n}^T]\dot{\mathbf{n}} &= (\mathbf{R}^T - \mathbf{I})\tilde{\mathbf{n}}\omega \end{aligned} \tag{88}$$

Owing to (46), it is immediatly verified that

$$[2\mathbf{I} - (\mathbf{R} + \mathbf{R}^T) + \mathbf{n}\mathbf{n}^T]\dot{\mathbf{n}} = 2(1 - \cos\phi)\dot{\mathbf{n}} \tag{89}$$

and also

$$\begin{aligned} (\mathbf{I} - \mathbf{R})\tilde{\mathbf{n}} &= (1 - \cos\phi)\tilde{\mathbf{n}} - \sin\phi\tilde{\mathbf{n}}\tilde{\mathbf{n}} \\ (\mathbf{R}^T - \mathbf{I})\tilde{\mathbf{n}} &= -(1 - \cos\phi)\tilde{\mathbf{n}} - \sin\phi\tilde{\mathbf{n}}\tilde{\mathbf{n}} \end{aligned} \tag{90}$$

We thus get the relationships

$$\mathbf{B}\boldsymbol{\Omega} = \dot{\mathbf{n}} \qquad \mathbf{B}^T\omega = \dot{\mathbf{n}} \tag{91}$$

with the matrix

$$\mathbf{B} = \left[\frac{1}{2}\tilde{\mathbf{n}} + \frac{\sin\phi}{2(1 - \cos\phi)}(\mathbf{I} - \mathbf{n}\mathbf{n}^T)\right] \tag{92}$$

which has not maximal rank and thus cannot be inverted to compute $\boldsymbol{\Omega}$ and ω.

Extra relationships which relate $\boldsymbol{\Omega}$ and ω respectively to $\dot{\phi}$ can be obtained by noticing that

$$\begin{aligned} \mathrm{tr}(\dot{\mathbf{R}}) &= \tfrac{d}{dt}(\mathrm{tr}(\mathbf{R})) &&= -2\dot{\phi}\sin\phi \\ &= \mathrm{tr}(\mathbf{R}\tilde{\boldsymbol{\Omega}}) &&= \mathrm{tr}(\tilde{\omega}\mathbf{R}) \end{aligned} \tag{93}$$

Let us decompose $\mathbf{R}$ into its symmetric and skew symmetric parts

$$\mathbf{R} = \mathbf{S} + \tilde{\mathbf{a}} \qquad \text{with} \qquad \mathbf{a} = \mathbf{n}\sin\phi \tag{94}$$

it is easily verified that for any symmetric matrix $\mathbf{S}$

$$\mathrm{tr}(\mathbf{S}\tilde{\mathbf{u}}) = \mathrm{tr}(\tilde{\mathbf{u}}\mathbf{S}) = 0 \tag{95}$$

while eqn (364) yields

$$\mathrm{tr}(\tilde{\mathbf{a}}\tilde{\mathbf{u}}) = \mathrm{tr}(\tilde{\mathbf{u}}\tilde{\mathbf{a}}) = -2\mathbf{a}^T\mathbf{u} \tag{96}$$

We thus get the results

$$\begin{aligned} \mathrm{tr}(\mathbf{R}\tilde{\boldsymbol{\Omega}}) &= -2\dot{\phi}\sin\phi &&= -2\mathbf{n}^T\boldsymbol{\Omega}\sin\phi \\ \mathrm{tr}(\tilde{\omega}\mathbf{R}) &= -2\dot{\phi}\sin\phi &&= -2\mathbf{n}^T\omega\sin\phi \end{aligned} \tag{97}$$

$\boldsymbol{\Omega}$ and ω are then obtained by solving the (4×3) systems

$$\begin{bmatrix} \mathbf{B} \\ \mathbf{n}^T \end{bmatrix} \boldsymbol{\Omega} = \begin{bmatrix} \dot{\mathbf{n}} \\ \dot{\phi} \end{bmatrix} \tag{98}$$

and

$$\begin{bmatrix} \mathbf{B}^T \\ \mathbf{n}^T \end{bmatrix} \omega = \begin{bmatrix} \dot{\mathbf{n}} \\ \dot{\phi} \end{bmatrix} \tag{99}$$

They can be inverted in the form

$$\boldsymbol{\Omega} = \mathbf{A}^{-1}[\mathbf{B}^T \quad \mathbf{n}] \begin{bmatrix} \dot{\mathbf{n}} \\ \dot{\phi} \end{bmatrix} \tag{100}$$

and

$$\omega = \mathbf{A}^{-1}[\mathbf{B} \quad \mathbf{n}] \begin{bmatrix} \dot{\mathbf{n}} \\ \dot{\phi} \end{bmatrix} \tag{101}$$

with the matrix $\mathbf{A}$

$$\mathbf{A} = \mathbf{B}^T\mathbf{B} + \mathbf{n}\mathbf{n}^T = \left[\frac{1}{2(1-\cos\phi)}(\mathbf{I} - \mathbf{n}\mathbf{n}^T) + \mathbf{n}\mathbf{n}^T \right] \tag{102}$$

and its inverse

$$\mathbf{A}^{-1} = [\,2(1-\cos\phi)(\mathbf{I} - \mathbf{n}\mathbf{n}^T) + \mathbf{n}\mathbf{n}^T\,] \tag{103}$$

We finally get the expressions of the angular velocity vectors

$$\boxed{\begin{aligned} \boldsymbol{\Omega} &= \mathbf{M}\dot{\mathbf{n}} + \mathbf{n}\dot{\phi} \\ \omega &= \mathbf{M}^T\dot{\mathbf{n}} + \mathbf{n}\dot{\phi} \end{aligned}} \tag{104}$$

with the matrix

$$\boxed{\mathbf{M} = \sin\phi\mathbf{I} - (1-\cos\phi)\tilde{\mathbf{n}}} \tag{105}$$

3.3. *Velocity analysis of arbitrary motion. Instantaneous screw axis*

The arbitrary instantaneous motion of a rigid body is now studied in the same way as the displacement analysis was performed in section 2.3..
Let us start from the description of general rigid body motion in spatial coordinates

$$\mathbf{x}_P = \mathbf{x}_0 + \mathbf{R}\mathbf{X}_P \tag{106}$$

P being an arbitrary point on the rigid body.
The velocity $\mathbf{v}_P$ of point P is obtained by time differentiating eqn (106)

$$\mathbf{v}_P = \dot{\mathbf{x}}_P = \dot{\mathbf{x}}_0 + \dot{\mathbf{R}}\mathbf{X}_P \tag{107}$$

By inverting eqn (107) in the form

$$\mathbf{X}_P = \mathbf{R}^T(\mathbf{x}_P - \mathbf{x}_0) \tag{108}$$

it may also be expressed in terms of the spatial coordinates

$$\mathbf{v}_P = \mathbf{v}_0 + \tilde{\omega}(\mathbf{x}_P - \mathbf{x}_0) \tag{109}$$

where the first term $\mathbf{v}_0 = \dot{\mathbf{x}}_0$ represents the velocity of the reference point O, and the second one expresses the relative velocity of point P with respect to O.
It makes clear that the instantaneous motion of a rigid body is known if the position and velocity of one of is points, as well as its angular velocity, are known.
The reference point O being not unique, the above description is not unique. We thus may aim at obtaining an invariant description of instantaneous motion instead. It is then described via its *instantaneous screw parameters* as established below.
To this purpose, let us look for a point C on the body which undergoes minimum velocity

$$\min_{\mathbf{x}_P}\left\{\frac{1}{2}\mathbf{v}_P^2\right\} = \min_{\mathbf{x}_P}\left\{\frac{1}{2}||\dot{\mathbf{x}}_0 + \tilde{\omega}(\mathbf{x}_P - \mathbf{x}_0)||^2\right\} = \frac{1}{2}\mathbf{v}_C^2 \tag{110}$$

Its coordinates $\mathbf{x}_C$ verify the condition

$$\tilde{\omega}^T\left[\tilde{\omega}(\mathbf{x}_C - \mathbf{x}_0) + \mathbf{v}_0\right] = 0 \tag{111}$$

and its velocity vector being in the null space of matrix $\tilde{\omega}$, is thus parallel to the direction of angular velocity

$$\mathbf{v}_C = \tilde{\omega}(\mathbf{x}_C - \mathbf{x}_0) + \mathbf{v}_0 = u\frac{\omega}{||\omega||} \tag{112}$$

The translation velocity u is obtained by projecting the velocity vector (112) on the direction ω

$$u = \frac{1}{||\omega||}\omega^T\mathbf{v}_0 \tag{113}$$

The position $\mathbf{x}_C$ of point C is obtained by attempting to solve

$$\tilde{\omega}\mathbf{x}_C = \tilde{\omega}\mathbf{x}_0 - (\mathbf{I} - \frac{\omega\omega^T}{||\omega||^2})\mathbf{v}_0 \tag{114}$$

It is however not unique since matrix $\tilde{\omega}$ has not maximum rank.
Let us assume that a particular solution $\mathbf{x}_M$ to eqn (114) has been obtained: the points

$$\mathbf{x}_C = \mathbf{x}_M + \alpha\omega \tag{115}$$

where α is an arbitrary parameter, all have the same minimum velocity (112). Equation (115) represents a line within the body issued from $\mathbf{x}_M$ and directed along the angular velocity direction ω. It is the *instantaneous screw axis* of the rigid body.
The origin is found by imposing to it to be at minimum distance from the reference point

$$\omega^T\mathbf{x}_M = 0 \tag{116}$$

It is then solution of the (4×3) system

$$\begin{bmatrix} \tilde{\omega} \\ \omega^T \end{bmatrix} \mathbf{x}_M = \begin{bmatrix} \tilde{\omega}\mathbf{x}_0 - (\mathbf{I} - \frac{\omega\omega^T}{||\omega||^2})\mathbf{v}_0 \\ 0 \end{bmatrix} \tag{117}$$

Premultiplying by $\begin{bmatrix} \tilde{\omega}^T & \omega \end{bmatrix}$ yields

$$||\omega||^2\mathbf{x}_M = \tilde{\omega}^T\tilde{\omega}\mathbf{x}_0 + \tilde{\omega}\mathbf{v}_0 \tag{118}$$

and finally

$$\mathbf{x}_M = \frac{\tilde{\omega}}{||\omega||^2}\mathbf{v}_0 + (\mathbf{I} - \frac{\omega\omega^T}{||\omega||^2})\mathbf{x}_0 \tag{119}$$

One thus gets the following theorem

> The locus of points of a rigid body having same minimum velocity is a line parallel to the angular velocity vector that passes through a point $\mathbf{x}_M$ whose position vector is given by eqn (119).

The quantities $\mathbf{x}_M$ and ω, which define the instantaneous screw axis in position and direction, are its *Plücker coordinates*.
The velocity u given by eqn (113) is the *sliding* of the instantaneous screw, and the ratio

$$p = \frac{2\pi u}{||\omega||} \tag{120}$$

is the *pitch* of the screw.
The velocity of an arbitrary point on the body takes the final form

$$\mathbf{v}_P = u\frac{\omega}{||\omega||} + \tilde{\omega}(\mathbf{x}_P - \mathbf{x}_M) \tag{121}$$

showing that the velocity field on the rigid body is composed of a translation component along the instantaneous rotation axis and a rotation component about the same axis.

4. Acceleration analysis of rigid motion

4.1. Acceleration analysis of spherical motion

The acceleration vector of point P expressed in *spatial coordinates* is obtained by differentiating (72) twice with respect to time

$$\mathbf{a}_P = \ddot{\mathbf{x}}_P = \ddot{\mathbf{R}}\mathbf{X}_P = \ddot{\mathbf{R}}\mathbf{R}^T\mathbf{x}_P \tag{122}$$

The matrix $\ddot{\mathbf{R}}\mathbf{R}^T$ of spatial angular accelerations can be computed from the matrix (75) of spatial angular velocities in the form

$$\alpha = \ddot{\mathbf{R}}\mathbf{R}^T = \frac{d}{dt}(\dot{\mathbf{R}}\mathbf{R}^T) - \dot{\mathbf{R}}\dot{\mathbf{R}}^T = \dot{\tilde{\omega}} + \tilde{\omega}\tilde{\omega} \tag{123}$$

The first term, which is skew symmetric, represents the rate of change of angular velocities. The second one is symmetric and collects the centrifugal acceleration terms.

The spatial expression of the angular acceleration vector is now defined as the vector part of matrix α and is clearly the vector part of $\dot{\tilde{\omega}}$ alone

$$\dot{\omega} = \mathrm{vect}(\alpha) = \mathrm{vect}(\dot{\tilde{\omega}}) \tag{124}$$

The acceleration vector of point P may also be expressed in *material coordinates*

$$\mathbf{A}_P = \mathbf{R}^T\mathbf{a}_P = \mathbf{R}^T\ddot{\mathbf{R}}\mathbf{X}_P \tag{125}$$

and likewise we define the matrix $\mathbf{A}$ of material angular accelerations which may be computed from the matrix of material angular velocities (80) in the form

$$\mathbf{A} = \mathbf{R}^T\ddot{\mathbf{R}} = \frac{d}{dt}(\mathbf{R}^T\dot{\mathbf{R}}) - \dot{\mathbf{R}}^T\dot{\mathbf{R}} = \dot{\tilde{\mathbf{\Omega}}} + \tilde{\mathbf{\Omega}}\tilde{\mathbf{\Omega}} \tag{126}$$

The material expression of the angular acceleration vector is similarly defined as

$$\dot{\mathbf{\Omega}} = \mathrm{vect}(\mathbf{A}) = \mathrm{vect}(\dot{\tilde{\mathbf{\Omega}}}) \tag{127}$$

4.2. Explicit expression of angular accelerations

The explicit expressions of the angular acceleration vectors $\dot{\omega}$ and $\dot{\mathbf{\Omega}}$ in terms of the invariants of the rotation $(\mathbf{n},\ \phi)$ and their time derivatives are directly obtained from the angular velocity vectors (104).

The time differentiation of (104) provides

$$\begin{aligned} \dot{\mathbf{\Omega}} &= \mathbf{M}\ddot{\mathbf{n}} + \mathbf{n}\ddot{\phi} + \dot{\mathbf{M}}\dot{\mathbf{n}} + \dot{\mathbf{n}}\dot{\phi} \\ \dot{\omega} &= \mathbf{M}^T\ddot{\mathbf{n}} + \mathbf{n}\ddot{\phi} + \dot{\mathbf{M}}^T\dot{\mathbf{n}} + \dot{\mathbf{n}}\dot{\phi} \end{aligned} \tag{128}$$

The computation may be further simplified by observing that

$$\dot{\tilde{\mathbf{n}}}\dot{\mathbf{n}} = 0 \tag{129}$$

and thus

$$\begin{aligned} \dot{\Omega} &= \mathbf{M}\ddot{\mathbf{n}} + \mathbf{n}\ddot{\phi} + \mathbf{N}\dot{\mathbf{n}} + \dot{\mathbf{n}}\dot{\phi} \\ \dot{\omega} &= \mathbf{M}^T\ddot{\mathbf{n}} + \mathbf{n}\ddot{\phi} + \mathbf{N}^T\dot{\mathbf{n}} + \dot{\mathbf{n}}\dot{\phi} \end{aligned} \tag{130}$$

with the matrix

$$\mathbf{N} = \dot{\phi}(\cos\phi \mathbf{I} - \sin\phi \tilde{\mathbf{n}}) \tag{131}$$

4.3. Time rate of change of instantaneous rotation axis

In a spherical motion, the direction of the instantaneous rotation axis, which is defined by the unit vector

$$\mathbf{f} = \frac{\omega}{||\omega||} \tag{132}$$

does generally not remain constant in time.
It obviously is in the null space of matrix $\tilde{\omega}$

$$\tilde{\omega}\mathbf{f} = 0 \tag{133}$$

and its time derivative is such that

$$\tilde{\omega}\dot{\mathbf{f}} = -\dot{\tilde{\omega}}\mathbf{f}\mathbf{f}^T\dot{\mathbf{f}} = 0 \tag{134}$$

The solution to system (133) can be obtained by taking the More-Penrose inverse of the system matrix in the form

$$\left[-\tilde{\omega}\tilde{\omega} + \mathbf{f}\mathbf{f}^T\right]\dot{\mathbf{f}} = \tilde{\omega}\dot{\tilde{\omega}}\mathbf{f} \tag{135}$$

If we now return to the definition (132) of $\mathbf{f}$ and make use of (365), we get

$$\left[-\tilde{\omega}\tilde{\omega} + \mathbf{f}\mathbf{f}^T\right] = ||\omega||^2(\mathbf{I} - \mathbf{f}\mathbf{f}^T) + \mathbf{f}\mathbf{f}^T \tag{136}$$

and

$$\left[-\tilde{\omega}\tilde{\omega} + \mathbf{f}\mathbf{f}^T\right]^{-1} = \frac{1}{||\omega||^2}(\mathbf{I} - \mathbf{f}\mathbf{f}^T) + \mathbf{f}\mathbf{f}^T \tag{137}$$

The time rate of change in the direction of the instantaneous rotation axis is thus obtained in the form

$$\dot{\mathbf{f}} = \frac{1}{||\omega||^2}\tilde{\omega}(\dot{\tilde{\omega}}\mathbf{f}) = \frac{1}{||\omega||}\tilde{\mathbf{f}}\dot{\tilde{\omega}}\mathbf{f} = (\mathbf{I} - \mathbf{f}\mathbf{f}^T)\frac{\dot{\omega}}{||\omega||} = \mathbf{P}_{\mathbf{f}}\,\frac{\dot{\omega}}{||\omega||} \tag{138}$$

which shows that it remains orthogonal to the rotation axis ω.

5. Infinitesimal spherical motion and rotation increments

The description of infinitesimal rotations and the concept of incremental motion are of primary importance in the formulation of kinematic and dynamic problems. Infinitesimal rotations are used to formulate virtual work expressions, while incremental relationships for rotations, angular velocities and accelerations are implied in the linearization of kinematic and equilibrium equations.

5.1. Spatial and material infinitesimal rotations

Let us start again from the linear transformation describing spherical motion

$$\mathbf{x} = \mathbf{R}\mathbf{X} \tag{139}$$

The associated virtual displacement is obtained through variation of this expression

$$\delta\mathbf{x} = \delta\mathbf{R}\mathbf{X} \tag{140}$$

and can be recast in either one of the forms

$$\delta\mathbf{x} = \delta\mathbf{R}\mathbf{R}^T\mathbf{x} = \delta\tilde{\theta}\mathbf{x} \qquad \delta\mathbf{x} = \mathbf{R}(\mathbf{R}^T\delta\mathbf{R})\mathbf{X} = \mathbf{R}\delta\tilde{\Theta}\mathbf{X} \tag{141}$$

with the skew matrices

$$\boxed{\delta\tilde{\theta} = \delta\mathbf{R}\mathbf{R}^T \qquad \delta\tilde{\Theta} = \mathbf{R}^T\delta\mathbf{R}} \tag{142}$$

which have respectively the meaning of spatial and material matrices of infinitesimal rotations. They are related together by

$$\delta\tilde{\Theta} = \mathbf{R}^T\delta\tilde{\theta}\,\mathbf{R} \tag{143}$$

and the associated axial vectors of infinitesimal angular displacements

$$\delta\theta = \mathrm{vect}(\delta\tilde{\theta}) \qquad \text{and} \qquad \delta\Theta = \mathrm{vect}(\delta\tilde{\Theta}) \tag{144}$$

are likewise related by

$$\delta\Theta = \mathbf{R}^T\delta\theta \tag{145}$$

5.2. Variation of angular velocities

Let us consider the expression of the variation of the angular velocity matrices (75) and (80)

$$\begin{aligned} \delta\tilde{\Omega} &= \delta\mathbf{R}^T\dot{\mathbf{R}} + \mathbf{R}^T\delta\dot{\mathbf{R}} \\ \delta\tilde{\omega} &= \delta\dot{\mathbf{R}}\mathbf{R}^T + \dot{\mathbf{R}}\delta\mathbf{R}^T \end{aligned} \tag{146}$$

They can be related in different ways to the time derivatives of infinitesimal rotation angles

$$\begin{aligned} \delta\dot{\tilde{\Theta}} &= \mathbf{R}^T\delta\dot{\mathbf{R}} + \dot{\mathbf{R}}^T\delta\mathbf{R} \\ \delta\dot{\tilde{\theta}} &= \delta\mathbf{R}\dot{\mathbf{R}}^T + \delta\dot{\mathbf{R}}\mathbf{R}^T \end{aligned} \tag{147}$$

If we combine (146.a) with (147.a) and (146.b) with (147.b), we get

$$\begin{aligned} \delta\tilde{\Omega} &= \delta\dot{\tilde{\Theta}} + \delta\mathbf{R}^T\dot{\mathbf{R}} - \dot{\mathbf{R}}^T\delta\mathbf{R} = & \delta\dot{\tilde{\Theta}} - \delta\tilde{\Theta}\tilde{\Omega} + \tilde{\Omega}\delta\tilde{\Theta} \\ \delta\tilde{\omega} &= \delta\dot{\tilde{\theta}} + \dot{\mathbf{R}}\delta\mathbf{R}^T - \delta\mathbf{R}\dot{\mathbf{R}}^T = & \delta\dot{\tilde{\theta}} + \delta\tilde{\theta}\tilde{\omega} - \tilde{\omega}\delta\tilde{\theta} \end{aligned} \tag{148}$$

If we then notice that according to (364) we have

$$\mathrm{vect}(\tilde{\mathbf{u}}\tilde{\mathbf{v}} - \tilde{\mathbf{v}}\tilde{\mathbf{u}}) = \mathrm{vect}(\mathbf{u}\mathbf{v}^T - \mathbf{v}\mathbf{u}^T) = \widetilde{(\tilde{\mathbf{u}}\mathbf{v})} \tag{149}$$

we get a first expression of the variations of angular velocities

$$\boxed{\begin{aligned} \delta\Omega &= \delta\dot{\Theta} + \tilde{\Omega}\delta\Theta \\ \delta\omega &= \delta\dot{\theta} - \tilde{\omega}\delta\theta \end{aligned}} \tag{150}$$

A second set of results can be obtained by transforming both equations (147) by the rotation operator

$$\begin{aligned} \mathbf{R}\delta\dot{\tilde{\Theta}}\mathbf{R}^T &= \delta\dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T\delta\mathbf{R}\mathbf{R}^T \\ \mathbf{R}^T\delta\dot{\tilde{\theta}}\mathbf{R} &= \mathbf{R}^T\delta\dot{\mathbf{R}} + \mathbf{R}^T\delta\mathbf{R}\dot{\mathbf{R}}^T\mathbf{R} \end{aligned} \tag{151}$$

By making use of (146.a) and (146.b), we get

$$\begin{aligned} \mathbf{R}\delta\dot{\tilde{\Theta}}\mathbf{R}^T &= \delta\tilde{\omega} - \dot{\mathbf{R}}\delta\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T\delta\mathbf{R}\mathbf{R}^T = \delta\tilde{\omega} \\ \mathbf{R}^T\delta\dot{\tilde{\theta}}\mathbf{R} &= \delta\tilde{\Omega} - \delta\mathbf{R}^T\dot{\mathbf{R}} + \mathbf{R}^T\delta\mathbf{R}\dot{\mathbf{R}}^T\mathbf{R} = \delta\tilde{\Omega} \end{aligned} \tag{152}$$

We have thus also the crossed relationships between material and spatial quantities

$$\boxed{\begin{aligned} \delta\omega &= \mathbf{R}\delta\dot{\Theta} \\ \delta\Omega &= \mathbf{R}^T\delta\dot{\theta} \end{aligned}} \tag{153}$$

The variations of angular accelerations result from a direct time differentiation of eqns (150). They have for expressions

$$\begin{aligned} \delta\dot{\Omega} &= \delta\ddot{\Theta} + \tilde{\Omega}\delta\dot{\Theta} + \dot{\tilde{\Omega}}\delta\Theta \\ \delta\dot{\omega} &= \delta\ddot{\theta} - \tilde{\omega}\delta\dot{\theta} - \dot{\tilde{\omega}}\delta\theta \end{aligned} \tag{154}$$

5.3. Angular velocities and accelerations in a moving frame

In various applications, one needs to compute inertial velocities and accelerations from measures taken in a moving frame. It is generally the case when studying the dynamics of a space vehicle with moving components. It is also the case when applying some kind of component mode representation for the elastic deformation of a large body undergoing arbirarily large motion. The actual rotation results then from the composition of the rotation of a reference frame $\mathbf{R}_0$ in which an additional rotation $\mathbf{R}_{rel}$ occurs

$$\mathbf{R} = \mathbf{R}_0\mathbf{R}_{rel} \tag{155}$$

The reference rotation $\mathbf{R}_0$ being time dependent, the computation involves terms arising from the variation of $\mathbf{R}_0$.

Let us for example compute the material variation of angular displacements. By differentiating both sides of eqn (155) and multiplying by $\mathbf{R}^T$ we get

$$\delta\tilde{\Theta} = \mathbf{R}_{rel}^T\delta\tilde{\Theta}_0\mathbf{R}_{rel} + \delta\tilde{\Theta}_{rel} \tag{156}$$

and the corresponding axial vector takes the form

$$\delta\mathbf{\Theta} = \mathbf{R}_{rel}^T \delta\mathbf{\Theta}_0 + \delta\mathbf{\Theta}_{rel} \tag{157}$$

The same procedure can be followed to compute the material angular velocity in terms of the velocity of the reference frame and of the relative velocity.

$$\boldsymbol{\Omega} = \mathbf{R}_{rel}^T \boldsymbol{\Omega}_0 + \boldsymbol{\Omega}_{rel} \tag{158}$$

Similar formulas can be obtained for the corresponding spatial quantities.

5.4. Incremental rotations as unknowns

Nonlinear structural and multibody problems, either in statics or dynamics, are formulated in a step-by-step form : the final solution is obtained by solving a sequence of partial problems. For instance, step-by-step algorithms are used to time integrate the nonlinear differential equations that constitute the nonlinear dynamic problem. The problem is then solved in an incremental way : starting from a known configuration we want to determine the increment necessary to obtain a new equilibrium configuration.

In this context, rotations are treated incrementally. At each stage of the solution process one determines the incremental rotation necessary to carry from the previously converged configuration (taken as reference) to the current one :

$$\mathbf{R} = \mathbf{R}_0 \mathbf{R}_{inc} \tag{159}$$

In that case, the reference configuration $\mathbf{R}_0$ can be regarded as fixed. The differentiation or variation of eqn (159) yields

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_{inc} \qquad \text{and} \qquad \delta\mathbf{\Theta} = \delta\mathbf{\Theta}_{inc} \tag{160}$$

which means that the use of an intermediate configuration has no influence on the computation of material angular velocities and and rotation increments. This result is of fundamental importance for the application of an updated Lagrangian description for the incremental procedure.

6. Parametrization of rigid body spherical motion

The parametrization of rigid body spherical motion is an important issue in practice since the efficiency of nonlinear computations involving large rotations depends largely on the adequacy of the set of parameters adopted.

The choice of a given set of parameters may be governed by various criteria such as independence (3 parameters are sufficient), mathematical form (transcendental versus purely algebraic), possible existence of singularities on the geometric domain of interest, computational efficiency, composition law, geometric interpretation, adequacy to describe a given kinematic situation, etc.

The parametrization of spherical motion results most of the time from the choice of an independent set of three parameters $\mathbf{a}^T = [\alpha_1\ \alpha_2\ \alpha_3]$ such that the rotation operator can be expressed in the form

$$\mathbf{R} = \mathbf{R}(\mathbf{a}) \tag{161}$$

In some cases however, it is found more convenient to use a larger set of parameters which are then dependent but are linked by additional constraints.

According to (104) the associated angular velocities will take the form of linear expressions of the time derivatives of the parameters and may thus be written in the general form

$$\boxed{\omega = \mathbf{P}(\mathbf{a})\dot{\mathbf{a}} \qquad \boldsymbol{\Omega} = \mathbf{Q}(\mathbf{a})\dot{\mathbf{a}}} \tag{162}$$

Provided that the parameters adopted form an independent set, the matrices $\mathbf{P}$ and $\mathbf{Q}$ have maximal rank. Since the angular velocities are linked by the frame transformation $\omega = \mathbf{R}\boldsymbol{\Omega}$, they verify the important relationship

$$\boxed{\mathbf{R} = \mathbf{P}\mathbf{Q}^{-1}} \tag{163}$$

The most classical choices of rotation parameters are described in the next sections. In each case, the fundamental relationships are established and some hints are given to compare the advantages of the different representations.

6.1. The Cartesian rotation vector

Starting from the general expression of the rotation operator in terms of the direction $\mathbf{n}$ of the rotation and its amplitude ϕ, the parametrization of spherical motion in terms of the Cartesian rotation vector is certainly the most natural one. It has also several advantages such as the number of parameters which remains minimal, an easy geometric interpretation and the absence of kinematic singularities.

The Cartesian rotation vector is defined as the vector which has the direction of the rotation axis and a length equal to the amplitude of the rotation

$$\boxed{\boldsymbol{\Psi} = \mathbf{n}\phi} \tag{164}$$

and therefore, the rotation operator can be expressed directly either in trigonometric form starting from the general expression (46)

$$\boxed{\mathbf{R} = \mathbf{I} + \frac{\sin||\boldsymbol{\Psi}||}{||\boldsymbol{\Psi}||}\widetilde{\boldsymbol{\Psi}} + \frac{1-\cos||\boldsymbol{\Psi}||}{||\boldsymbol{\Psi}||^2}\widetilde{\boldsymbol{\Psi}}\widetilde{\boldsymbol{\Psi}}} \tag{165}$$

or in exponential form by making use of the exponential map representation (54)

$$\boxed{\mathbf{R} = \exp(\widetilde{\boldsymbol{\Psi}})} \tag{166}$$

It admits the series expansion

$$\mathbf{R} = \mathbf{I} + \widetilde{\boldsymbol{\Psi}} + \frac{1}{2!}\widetilde{\boldsymbol{\Psi}}^2 + \frac{1}{3!}\widetilde{\boldsymbol{\Psi}}^3 + \ldots + \frac{1}{n!}\widetilde{\boldsymbol{\Psi}}^n \ldots \tag{167}$$

The expressions of material and angular velocities are easily obtained in terms of the time derivatives of the Cartesian rotation vector. To that purpose, let us note that

$$\dot{\boldsymbol{\Psi}} = \mathbf{n}\dot{\phi} + \dot{\mathbf{n}}\phi \qquad \text{and} \qquad \mathbf{n}^T\dot{\mathbf{n}} = 0 \tag{168}$$

If we invert the linear system (168), we get

$$\begin{aligned} \dot{\phi} &= \frac{1}{\phi}\mathbf{n}^T\dot{\boldsymbol{\Psi}} &&= \frac{1}{\|\boldsymbol{\Psi}\|^2}\boldsymbol{\Psi}^T\dot{\boldsymbol{\Psi}} \\ \dot{\mathbf{n}} &= (\mathbf{I} - \mathbf{n}\mathbf{n}^T)\frac{\dot{\boldsymbol{\Psi}}}{\phi} &&= \frac{\widetilde{\boldsymbol{\Psi}}\widetilde{\boldsymbol{\Psi}}}{\|\boldsymbol{\Psi}\|^3}\dot{\boldsymbol{\Psi}} \end{aligned} \tag{169}$$

and the substitution of eqns (169) into eqns (104) yields the parameterized expressions of the material and angular velocities

$$\boxed{\Omega = \mathbf{T}(\boldsymbol{\Psi})\dot{\boldsymbol{\Psi}} \qquad\qquad \omega = \mathbf{T}^T(\boldsymbol{\Psi})\dot{\boldsymbol{\Psi}}} \tag{170}$$

where $\mathbf{T}(\boldsymbol{\Psi})$ is the so-called tangent operator. By making use of (105) it is obtained in the form

$$\boxed{\mathbf{T}(\boldsymbol{\Psi}) = \mathbf{I} + \left(\frac{\cos\|\boldsymbol{\Psi}\| - 1}{\|\boldsymbol{\Psi}\|^2}\right)\widetilde{\boldsymbol{\Psi}} + \left(1 - \frac{\sin\|\boldsymbol{\Psi}\|}{\|\boldsymbol{\Psi}\|}\right)\frac{\widetilde{\boldsymbol{\Psi}}\widetilde{\boldsymbol{\Psi}}}{\|\boldsymbol{\Psi}\|^2}} \tag{171}$$

The apparent singularity in $\|\boldsymbol{\Psi}\| = 0$ which appears in both rotation matrix and tangent operator expressions (165) and (171) is easily removed by noticing that

$$\lim_{\|\boldsymbol{\Psi}\| \Rightarrow 0} \mathbf{T}(\boldsymbol{\Psi}) = \lim_{\|\boldsymbol{\Psi}\| \Rightarrow 0} \mathbf{R}(\boldsymbol{\Psi}) = \mathbf{I} \tag{172}$$

Whenever possible, the tangent operator can be replaced by its series expansion

$$\mathbf{T}(\boldsymbol{\Psi}) = \mathbf{I} - \frac{1}{2}\widetilde{\boldsymbol{\Psi}} + \frac{1}{6}\widetilde{\boldsymbol{\Psi}}\widetilde{\boldsymbol{\Psi}} + O(\boldsymbol{\Psi}^3) \tag{173}$$

which considerably simplifies the computation of linearized expressions.
The angular accelerations are obtained through further differentiation

$$\boxed{\dot{\Omega} = \mathbf{T}(\boldsymbol{\Psi})\ddot{\boldsymbol{\Psi}} + \dot{\mathbf{T}}(\boldsymbol{\Psi})\dot{\boldsymbol{\Psi}} \qquad\qquad \dot{\omega} = \mathbf{T}^T(\boldsymbol{\Psi})\ddot{\boldsymbol{\Psi}} + \dot{\mathbf{T}}^T(\boldsymbol{\Psi})\dot{\boldsymbol{\Psi}}} \tag{174}$$

Let us finally note that the material and spatial rotation increments of rotation are likewise computed in terms of the tangent operator. By replacing the time derivative with the variation operator we get

$$\delta\boldsymbol{\Theta} = \mathbf{T}(\boldsymbol{\Psi})\delta\boldsymbol{\Psi} \qquad \text{and} \qquad \delta\boldsymbol{\theta} = \mathbf{T}^T(\boldsymbol{\Psi})\delta\boldsymbol{\Psi} \tag{175}$$

6.2. Cayley form of rotation matrix - Rodrigues parameters

Starting again from the general transformation describing spherical motion

$$\mathbf{x} = \mathbf{R}\mathbf{X} \qquad \text{with} \qquad \mathbf{R}^T\mathbf{R} = \mathbf{I} \tag{176}$$

Let us express explicitly that the length of the initial vector is preserved

$$\mathbf{x}^T\mathbf{x} - \mathbf{X}^T\mathbf{X} = (\mathbf{x} + \mathbf{X})^T(\mathbf{x} - \mathbf{X}) = 0 \tag{177}$$

let us next define the two vectors

$$\begin{array}{lll} \mathbf{f} = & \mathbf{x} - \mathbf{X} & = (\mathbf{R} - \mathbf{I})\mathbf{X} \\ \mathbf{g} = & \mathbf{x} + \mathbf{X} & = (\mathbf{R} + \mathbf{I})\mathbf{X} \end{array} \tag{178}$$

Expressing their orthogonality as implied by (177) yields the condition

$$\mathbf{f}^T\mathbf{g} = \mathbf{g}^T\mathbf{B}\mathbf{g} = 0 \tag{179}$$

where matrix $\mathbf{B}$ is necessarily skew-symmetric

$$\mathbf{B} = (\mathbf{R} - \mathbf{I})(\mathbf{R} + \mathbf{I})^{-1} = \tilde{\mathbf{b}} \tag{180}$$

since the necessary and sufficient condition under which any quadratic form of a given matrix vanishes is that this matrix is skew-symmmetric.
The relationship (180) can be inverted in the form

$$\mathbf{R} = (\mathbf{I} - \tilde{\mathbf{b}})^{-1}(\mathbf{I} + \tilde{\mathbf{b}}) \tag{181}$$

It is easily verified that

$$(\mathbf{I} - \tilde{\mathbf{b}})^{-1} = \frac{1}{1 + ||\mathbf{b}||^2}(\mathbf{I} + \tilde{\mathbf{b}} + \mathbf{b}\mathbf{b}^T) \tag{182}$$

in which case we get the expression of the rotation operator

$$\boxed{\mathbf{R} = \mathbf{I} + \frac{2}{1 + ||\mathbf{b}||^2}(\tilde{\mathbf{b}} + \tilde{\mathbf{b}}\tilde{\mathbf{b}})} \tag{183}$$

in terms of the three parameters $\mathbf{b}^T = [b_1 \; b_2 \; b_3]$ forming the skew-symmetric matrix $\tilde{\mathbf{b}}$.
From the comparison of (183) with (46) we deduce that the latter are related to the invariants $(\mathbf{n}, \ \phi)$ of the rotation by

$$\frac{2\mathbf{b}}{1 + ||\mathbf{b}||^2} = \mathbf{n} \sin\phi \tag{184}$$

and thus

$$\boxed{\mathbf{b} = \mathbf{n} \tan\frac{\phi}{2}} \tag{185}$$

This set of three independent parameters is known as the *Rodrigues Parameters*. They provide a very simple expression of the rotation operator for which the associated inversion procedure is obtained by computing the trace and the vector part. By making use of (364) we get

$$\mathrm{tr}(\mathbf{R}) = \frac{3 - ||\mathbf{b}||^2}{1 + ||\mathbf{b}||^2} \qquad \Rightarrow \qquad ||\mathbf{b}||^2 = \frac{3 - \mathrm{tr}(\mathbf{R})}{1 + \mathrm{tr}(\mathbf{R})} \tag{186}$$

and

$$\mathrm{vect}(\mathbf{R}) = \frac{\mathbf{b}}{1 + ||\mathbf{b}||^2} \qquad \Rightarrow \qquad \mathbf{b} = \frac{1}{2}(1 + ||\mathbf{b}||^2)\mathrm{vect}(\mathbf{R}) \tag{187}$$

The inversion procedure presents a singularity when $\phi = \pm\pi$.
The computation of angular velocities in terms of Rodrigues parameters is straightforward. Time differentiation of (185) yields

$$\dot{\mathbf{b}} = \dot{\mathbf{n}} \tan\frac{\phi}{2} + \mathbf{n}\frac{\dot{\phi}}{2}\frac{1}{\cos^2\frac{\phi}{2}} \tag{188}$$

with the condition $\mathbf{n}^T\dot{\mathbf{b}} = 0$. We get thus the system of equations

$$\begin{bmatrix} \mathbf{I}\tan\frac{\phi}{2} & \frac{\mathbf{n}}{\cos^2\frac{\phi}{2}} \\ \mathbf{n}^T & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{n}} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{b}} \\ 0 \end{bmatrix} \tag{189}$$

which is solved to yield

$$\dot{\phi} = 2\cos^2\frac{\phi}{2}\mathbf{n}^T\dot{\mathbf{b}} \qquad \text{and} \qquad \dot{\mathbf{n}} = \frac{1}{\tan\frac{\phi}{2}}[\mathbf{I} - \mathbf{n}\mathbf{n}^T]\dot{\mathbf{b}} \tag{190}$$

The computation of the material and angular velocities from (104), (105), (185) and (190) yields then the final results

$$\boxed{\boldsymbol{\Omega} = \mathbf{T}(\mathbf{b})\dot{\mathbf{b}} \qquad \omega = \mathbf{T}^T(\mathbf{b})\dot{\mathbf{b}}} \tag{191}$$

with the expression of the tangent operator in terms of Rodrigues parameters

$$\boxed{\mathbf{T} = \frac{2}{1 + ||\mathbf{b}||^2}(\mathbf{I} - \tilde{\mathbf{b}})} \tag{192}$$

6.3. Finite rotations in terms of Euler parameters

Euler parameters are naturally introduced by starting from Euler's representation of the rotation operator. They simply result from a change of variables in terms of half the rotation angle which gives equal roles to all 4 parameters. In contrast to all set of parameters that we have introduced sofar, they are purely algebraic quantities.

Definition Euler parameters are defined as a 4-dimensional vector

$$\mathbf{p}^T = [\, e_0 \quad \mathbf{e}^T \,] = [\, e_0 \quad e_1 \quad e_2 \quad e_3 \,] \tag{193}$$

with the components

$$\boxed{e_0 \;=\; \cos\frac{\phi}{2} \qquad\qquad \mathbf{e} \;=\; \mathbf{n}\sin\frac{\phi}{2}} \tag{194}$$

which are thus such that

$$-1 \leq e_i \leq 1 \qquad (i = 0 \ldots 3) \tag{195}$$

They are not independent quantities since they are linked by the constraint

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 \;=\; 1 \tag{196}$$

They are obviouly related to the Rodrigues parameters by

$$\mathbf{b} = \frac{1}{e_0}\mathbf{e} \tag{197}$$

Expression of the rotation operator In order to obtain the explicit form of the rotation operator in terms of Euler parameters, let us start from the general expression (46) with the trigonometric transformations

$$\begin{array}{lll} \cos\phi & = 2\cos^2\frac{\phi}{2} - 1 & = 2e_0^2 - 1 \\ 1 - \cos\phi & = 2\sin^2\frac{\phi}{2} & \\ \sin\phi & = 2\sin\frac{\phi}{2}e_0 & \end{array} \tag{198}$$

The final form of the rotation operator is then the quadratic expression

$$\boxed{\mathbf{R} = (2e_0^2 - 1)\mathbf{I} + 2\mathbf{e}\mathbf{e}^T + 2e_0\tilde{\mathbf{e}}} \tag{199}$$

It can also be expressed as a product of two (4×3) matrices which are linear expressions of the Euler parameters

$$\boxed{\mathbf{R} = \mathbf{H}\mathbf{G}^T} \tag{200}$$

with

$$\mathbf{H} = [\, -\mathbf{e} \quad e_0\mathbf{I} + \tilde{\mathbf{e}} \,] \qquad\qquad \mathbf{G} = [\, -\mathbf{e} \quad e_0\mathbf{I} - \tilde{\mathbf{e}} \,] \tag{201}$$

Inversion formulas The inversion procedure from the numerical value of the rotation operator to Euler parameters may be achieved by the formulas

$$\begin{array}{lll} e_0 & = & \frac{1}{2}\sqrt{1 + \mathrm{tr}(\mathbf{R})} \\ e_k & = & \frac{1}{2}\mathrm{sign}([\mathrm{vect}(\mathbf{R})]_k)\sqrt{1 + 2r_{kk} - \mathrm{tr}(\mathbf{R})} \qquad (k = 1 \ldots 3) \end{array} \tag{202}$$

This procedure is however not optimal from a numerical point of view.

The following formulas provide a means of performing this inversion with maximum numerical accuracy and efficiency [SPU78].

step 1 : consists to construct the 4×4 symmetric matrix $\mathbf{S}$ obtained from $\mathbf{R}$

$$\mathbf{S} = \begin{bmatrix} 1 + r_{11} + r_{22} + r_{33} & r_{32} - r_{23} & r_{13} - r_{31} & r_{21} - r_{12} \\ r_{32} - r_{23} & 1 + r_{11} - r_{22} - r_{33} & r_{12} + r_{21} & r_{13} + r_{31} \\ r_{13} - r_{31} & r_{21} + r_{12} & 1 - r_{11} + r_{22} - r_{33} & r_{23} + r_{32} \\ r_{21} - r_{12} & r_{13} + r_{31} & r_{23} + r_{32} & 1 - r_{11} - r_{22} + r_{33} \end{bmatrix} \tag{203}$$

step 2 : consists to observe that $\mathbf{S}$ is a quadratic expression of Euler parameters

$$\mathbf{S} = 4 \begin{bmatrix} e_0^2 & e_0 e_1 & e_0 e_2 & e_0 e_3 \\ e_0 e_1 & e_1^2 & e_1 e_2 & e_1 e_3 \\ e_0 e_2 & e_1 e_2 & e_2^2 & e_2 e_3 \\ e_0 e_3 & e_1 e_3 & e_2 e_3 & e_3^2 \end{bmatrix} \tag{204}$$

step 3 : one uses use the row of (204) with maximum diagonal term to compute the parameters

$$S_{ii} = \max_k \{S_{kk}\} \qquad \Rightarrow \qquad \begin{cases} e_i = \frac{1}{2}\sqrt{S_{ii}} & \\ e_k = \dfrac{S_{ik}}{4e_i} & k = 0 \ldots 3 \\ & k \neq i \end{cases} \tag{205}$$

Angular velocities In order to compute the spatial and material angular velocities in terms of the time derivatives of Euler parameters, let us derive eqns (193) with respect to time. We get the system of equations

$$\begin{bmatrix} \dot{\mathbf{e}} \\ \dot{e}_0 \end{bmatrix} = \begin{bmatrix} \sin\frac{\phi}{2}\mathbf{I} & \frac{1}{2}\cos\frac{\phi}{2}\mathbf{n} \\ 0\,0\,0 & -\frac{1}{2}\sin\frac{\phi}{2} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{n}} \\ \dot{\phi} \end{bmatrix} \tag{206}$$

which can be inverted in the form

$$\begin{bmatrix} \dot{\mathbf{n}} \\ \dot{\phi} \end{bmatrix} = \frac{1}{\sin\frac{\phi}{2}} \begin{bmatrix} \mathbf{I} & \frac{\cos\frac{\phi}{2}}{\sin\frac{\phi}{2}}\mathbf{n} \\ 0\,0\,0 & -2 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{e}} \\ \dot{e}_0 \end{bmatrix} \tag{207}$$

and making use of (104) together with (105) yields the angular velocity relationships

$$\boxed{\begin{aligned} \mathbf{\Omega} &= 2\mathbf{G}\dot{\mathbf{p}} \\ \omega &= 2\mathbf{H}\dot{\mathbf{p}} \end{aligned}} \tag{208}$$

It is of interest noticing that compared to the other representation methods described sofar, Euler parameters have the following attractive properties :

- no singularity occurs in their inversion procedure,
- they are purely algebraic quantities ($\mathbf{R}$ quadratic),
- as will be seen in section 7., they obey to the quaternion multiplication rule to form compound rotations.

Their main drawback is that they form a set of 4 parameters linked by 1 contraint.

7. Quaternion algebra and finite rotations

Quaternion algebra [HAM99, WEH84] provides a very elegant way of describing finite rotations. It leads in a totally different manner to the same concept of Euler parameters which has been introduced in the previous section. At the same time, the fundamental rule of quaternion multiplication provides an efficient means to express the angular velocities and how to combine successive rotations.

7.1. Quaternion algebra : definition and properties

Arbitrary quaternion A quaternion is defined as 4-D complex number

$$\boxed{\widehat{q} = q_0 + iq_1 + jq_2 + kq_3} \tag{209}$$

where i, j and k are imaginary numbers such that

$$\begin{array}{llll} i^2 & = j^2 & = k^2 = -1 \\ jk & = -kj & = i \\ ki & = -ik & = j \\ ij & = -ji & = k \end{array} \tag{210}$$

One may also adopt the vector notation

$$\boxed{\widehat{q} = q_0 + \mathbf{q}} \tag{211}$$

where q_0 is the scalar part of the quaternion and $\mathbf{q}$ its vector part.

Multiplication rule The quaternion multiplication rule derives then from the property (210) of the imaginary numbers

$$\boxed{\widehat{r} = \widehat{p}\widehat{q} = p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + p_0 \mathbf{q} + q_0 \mathbf{p} + \tilde{\mathbf{p}}\mathbf{q}} \tag{212}$$

It is fundamental noticing that the multiplication rule is non commutative due to the presence of the cross product whose sign is changed by interverting the roles of $\widehat{p}$ and $\widehat{q}$.

Conjugate quaternion The conjugate quaternion to (209) is defined as

$$\widehat{q}^* = q_0 - iq_1 - jq_2 - kq_3 = q_0 - \mathbf{q} \tag{213}$$

Norm of a quaternion It is computed by

$$||\widehat{q}||^2 = \widehat{q}\widehat{q}^* = q_0^2 + \mathbf{q} \cdot \mathbf{q} \tag{214}$$

Unit quaternion It is defined as a quaternion $\hat{e} = e_0 + \mathbf{e}$ such that

$$||\hat{e}|| = 1 \tag{215}$$

Vector quaternion A pure vector quaternion is a quaternion $\hat{v}$ such that

$$\hat{v} = 0 + \mathbf{v} \tag{216}$$

It verifies thus the property

$$\hat{v} + \hat{v}^* = 0 \tag{217}$$

7.2. Representation of finite rotations in terms of quaternions

Given

$$\begin{array}{lll} \widehat{e} & = e_0 + \mathbf{e} & \text{unit quaternion} \\ \widehat{X} & = 0 + \mathbf{X} & \text{vector quaternion} \end{array} \tag{218}$$

it is easy to verify that the finite rotation of $\mathbf{X}$ to a new position $\mathbf{x}$ is given by

$$\boxed{\widehat{x} = \widehat{e}\widehat{X}\widehat{e}^*} \tag{219}$$

The proof holds by verifying that

1. the norm of $\widehat{X}$ is preserved by the transformation :

$$\begin{aligned} ||\widehat{x}||^2 &= \widehat{x}\widehat{x}^* & &= (\widehat{e}\widehat{X}\widehat{e}^*)(\widehat{e}\widehat{X}\widehat{e}^*)^* \\ &= \widehat{e}\widehat{X}\widehat{e}^*\widehat{e}\widehat{X}^*\widehat{e}^* & &= \widehat{e}||\widehat{X}||^2\widehat{e}^* \\ &= ||\widehat{X}||^2 \end{aligned} \tag{220}$$

2. the resulting quaternion $\widehat{x}$ is also a vector quaternion :

$$\begin{aligned} \widehat{x} + \widehat{x}^* &= \widehat{e}\widehat{X}\widehat{e}^* + (\widehat{e}\widehat{X}\widehat{e}^*)^* & &= \widehat{e}\widehat{X}\widehat{e}^* + \widehat{e}\widehat{X}^*\widehat{e}^* \\ &= \widehat{e}(\widehat{X} + \widehat{X}^*)\widehat{e}^* & &= 0 \end{aligned} \tag{221}$$

The inverse rotation operation may be put in the similar form

$$\widehat{X} = \widehat{e}^*\widehat{x}\widehat{e} \tag{222}$$

indicating that transposition reverses the sense of the rotation.

Equivalence with Euler parameters It is observed by putting the unit quaternion in the form

$$\widehat{e} = \cos\alpha + \mathbf{n}\sin\alpha \qquad \text{with } ||\mathbf{n}|| = 1 \tag{223}$$

It provides the expression

$$\widehat{x} = 0 + (\cos 2\alpha\mathbf{I} + (1 - \cos 2\alpha)\mathbf{n}\mathbf{n}^T + \sin 2\alpha\tilde{\mathbf{n}})\mathbf{X} \tag{224}$$

which is the same as (46) provided that we take $\alpha = \frac{\phi}{2}$.

Composition rule of quaternions Let us perform two successive rotations $\hat{e}_1$ and $\hat{e}_2$

$$\begin{aligned} \widehat{x}_1 &= \widehat{e}_1 \widehat{X} \widehat{e}_1^{\,*} \\ \widehat{x}_2 &= \widehat{e}_2 \widehat{x}_1 \widehat{e}_2^{\,*} = (\widehat{e}_2 \widehat{e}_1) \widehat{X} (\widehat{e}_2 \widehat{e}_1)^* \end{aligned} \tag{225}$$

They generate the resulting rotation

$$\widehat{x} = \widehat{e} \widehat{X} \widehat{e}^* \qquad \text{with} \qquad \widehat{e} = \widehat{e}_2 \widehat{e}_1 \tag{226}$$

It provides the important result that

> *two successive rotations may be combined by multiplying the corresponding quaternions in the appropriate order.*

Angular velocities Let us derive (219) with respect to time, **X** being assumed constant

$$\dot{\widehat{x}} = \dot{\widehat{e}} \widehat{X} \widehat{e}^* + \widehat{e} \widehat{X} \dot{\widehat{e}}^* \tag{227}$$

Sustituting (222) into (227) and taking account of (215) yields

$$\dot{\widehat{x}} = \dot{\widehat{e}} \widehat{e}^* \widehat{x} + \widehat{x} \widehat{e} \dot{\widehat{e}}^* \tag{228}$$

The time differentiation of (215) yields also

$$\dot{\widehat{e}} \widehat{e}^* + \widehat{e} \dot{\widehat{e}}^* = \dot{\widehat{e}} \widehat{e}^* + (\dot{\widehat{e}} \widehat{e}^*)^* = 0 \tag{229}$$

so that $\dot{\widehat{e}} \widehat{e}^*$ is a vector quaternion which we may express in the form

$$\dot{\widehat{e}} \widehat{e}^* = \hat{\omega} = 0 + \frac{1}{2} \omega \tag{230}$$

The susbstitution of (230) into (228) yields

$$\dot{\widehat{x}} = \frac{1}{2} (\hat{\omega} \widehat{x} - \widehat{x} \hat{\omega}) = 0 + \frac{1}{2} (\tilde{\omega} \mathbf{x} - \tilde{\mathbf{x}} \omega) \tag{231}$$

and thus

$$\boxed{\dot{\mathbf{x}} = \tilde{\omega} \mathbf{x}} \tag{232}$$

showing that the vector part of $\hat{\omega}$ is nothing else than the spatial angular velocity vector which, according to (212), has for expression

$$\boxed{\omega = 2(e_0 \dot{\mathbf{e}} - \dot{e}_0 \mathbf{e} + \tilde{\mathbf{e}} \dot{\mathbf{e}})} \tag{233}$$

We may likewise obtain the material expression of angular velocities by projecting (227) into material coordinates

$$\widehat{V} = \widehat{e}^* \dot{\widehat{x}} \widehat{e} = \widehat{e}^* \dot{\widehat{e}} \widehat{X} + \widehat{X} \dot{\widehat{e}}^* \widehat{e} \tag{234}$$

We express then the quaternion vector $\widehat{e^*}\dot{\widehat{e}}$ in the form

$$\widehat{e^*}\dot{\widehat{e}} = \frac{1}{2}\hat{\Omega} = 0 + \frac{1}{2}\boldsymbol{\Omega} \tag{235}$$

and substituting (235) into (234) yields

$$\widehat{V} = \frac{1}{2}(\hat{\Omega}\widehat{X} - \widehat{X}\hat{\Omega}) = 0 + \frac{1}{2}(\tilde{\boldsymbol{\Omega}}\mathbf{X} - \tilde{\mathbf{X}}\boldsymbol{\Omega}) \tag{236}$$

and thus

$$\boxed{\mathbf{V} = \tilde{\boldsymbol{\Omega}}\mathbf{X}} \tag{237}$$

with the material angular velocity vector

$$\boxed{\boldsymbol{\Omega} = 2(e_0\dot{\mathbf{e}} - \dot{e}_0\mathbf{e} - \tilde{\mathbf{e}}\dot{\mathbf{e}})} \tag{238}$$

7.3. Matrix representation of quaternions

In view of computer implementation of quaternion algebra, let us represent a quaternion in matrix form by the 4-dimensional column matrix

$$\widehat{\mathbf{q}} = [q_0 \quad q_1 \quad q_2 \quad q_3]^T \tag{239}$$

A quaternion product

$$\widehat{a} = \widehat{p}\widehat{q} \tag{240}$$

may then be expressed in the form

$$\widehat{\mathbf{a}} = \mathbf{A}_p\widehat{\mathbf{q}} = \mathbf{B}_q\widehat{\mathbf{p}} \tag{241}$$

with the 4×4 matrices

$$\mathbf{A}_p = \begin{bmatrix} p_0 & -\mathbf{p}^T \\ \mathbf{p} & p_0\mathbf{I} + \tilde{\mathbf{p}} \end{bmatrix} \qquad \text{and } \mathbf{B}_q = \begin{bmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & q_0\mathbf{I} - \tilde{\mathbf{q}} \end{bmatrix} \tag{242}$$

and where $\tilde{\mathbf{q}}$ is the antisymmetric matrix obtained from the vector part of $\widehat{q}$

$$\tilde{\mathbf{q}} = \text{spin}(\mathbf{q}) \tag{243}$$

In matrix form, the rotation operation (219) on a vector quaternion becomes

$$\widehat{\mathbf{x}} = \mathbf{A}_e\mathbf{B}_e^T\widehat{\mathbf{X}} \tag{244}$$

with the matrix product

$$\mathbf{A}_e\mathbf{B}_e^T = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \tag{245}$$

from which one extracts then the 3×3 matrix

$$\mathbf{R} = (2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}}) \tag{246}$$

According to (245) it can also be put in the simplified form

$$\boxed{\mathbf{R} = \mathbf{H}\mathbf{G}^T} \tag{247}$$

where the matrices **H** and **G**, which were already introduced at section 6.3., are 3×4 matrices extracted from **A** and **B**

$$\boxed{\begin{aligned} \mathbf{H} &= [-\mathrm{e} \quad e_0\mathbf{I} + \tilde{\mathrm{e}}] \\ \mathbf{G} &= [-\mathrm{e} \quad e_0\mathbf{I} - \tilde{\mathrm{e}}] \end{aligned}} \tag{248}$$

Likewise, the angular velocity quaternions (230) and (235) can be written in the matrix form

$$\begin{aligned} \hat{\Omega} &= 2\mathbf{A}^T\dot{\hat{\mathrm{e}}} \\ \hat{\omega} &= 2\mathbf{B}^T\dot{\hat{\mathrm{e}}} \end{aligned} \tag{249}$$

Their vector parts provide the matrix forms of the angular velocity vectors (238) and (233)

$$\boxed{\begin{aligned} \Omega &= 2\mathbf{G}\dot{\hat{\mathrm{e}}} \\ \omega &= 2\mathbf{H}\dot{\hat{\mathrm{e}}} \end{aligned}} \tag{250}$$

It is easy to show that **H** and **G** verify the useful relationships

$$\begin{aligned} \mathbf{H}\mathbf{H}^T &= \mathbf{G}\mathbf{G}^T &&= \mathbf{I} \\ \mathbf{H}^T\mathbf{H} &= \mathbf{G}^T\mathbf{G} &&= \mathbf{I} - \hat{\mathrm{e}}\hat{\mathrm{e}}^T \\ \mathbf{H}\hat{\mathrm{e}} &= \mathbf{G}\hat{\mathrm{e}} &&= \mathbf{0} \end{aligned} \tag{251}$$

8. The conformal rotation vector (CRV)

The conformal rotation vector (CRV) is obtained through a conformal rotation applied to Euler parameters [MIL82]

$$c_i = \frac{4e_i}{1 + e_0} \qquad (i = 0, 1, 2, 3) \tag{252}$$

It produces a set of three independent parameters involving the fourth of the rotation angle

$$\mathrm{c} = 4\mathbf{n}\tan\frac{\phi}{4} \tag{253}$$

with the additional quantity

$$c_0 = \frac{1}{8}[16 - \|\mathrm{c}\|^2] \tag{254}$$

By opposition with Rodrigues parameters, they produce no singularity in the rotation interval $\phi \in [-\pi, +\pi]$ since

$$\left\{ \begin{aligned} 0 \leq \quad & c_0 \leq 2 \\ -4 \leq \quad & c_i \leq +4 \qquad (i = 1, 2, 3) \end{aligned} \right. \tag{255}$$

The explicit expression of the rotation operator can directly be computed from the general formula (46). After some algebra and making use of (254) we get the expression

$$\mathbf{R} = \frac{1}{(4-c_0)^2}\left[(c_0^2 + 8c_0 - 16)\mathbf{I} + 2\mathbf{c}\mathbf{c}^T + 2c_0\tilde{\mathbf{c}}\right] \tag{256}$$

However, the most remarkable property of th CRV is the fact that the total rotation (46) can be split into two equal rotations $\mathbf{F}$ of amplitude $\frac{1}{2}\phi$ which can then be computed in terms of Rodrigues parameters of the half rotation :

$$\boxed{\mathbf{R} = \mathbf{F}^2} \tag{257}$$

$\mathbf{F}$ being computed from (192) by making the substitution $\mathbf{b} = \frac{1}{4}\mathbf{c}$:

$$\mathbf{F} = \mathbf{I} + \frac{8}{16 + ||\mathbf{c}||^2}(\tilde{\mathbf{c}} + \tilde{\mathbf{c}}\tilde{\mathbf{c}}) \tag{258}$$

By making use of (254) and (364) we get the final expression

$$\boxed{\mathbf{F} = \frac{1}{4-c_0}\left[c_0\mathbf{I} + \frac{1}{4}\mathbf{c}\mathbf{c}^T + \tilde{\mathbf{c}}\right]} \tag{259}$$

The expression of material and angular velocities is obtained by time differentiation of (253)

$$\dot{\mathbf{c}} = 4\dot{\mathbf{n}}\tan\frac{\phi}{4} + \frac{\mathbf{n}}{\cos^2\frac{\phi}{4}}\dot{\phi} \tag{260}$$

By making also use of $\mathbf{n}^T\dot{\mathbf{n}} = 0$ we get the linear system

$$\begin{bmatrix} 4\tan\frac{\phi}{4} & \frac{\mathbf{n}}{\cos^2\frac{\phi}{4}} \\ \mathbf{n}^T & 0 \end{bmatrix}\begin{bmatrix} \dot{\mathbf{n}} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{c}} \\ 0 \end{bmatrix} \tag{261}$$

which has for solution

$$\dot{\phi} = \cos^2\frac{\phi}{4}\mathbf{n}^T\dot{\mathbf{c}} \qquad \text{and } \dot{\mathbf{n}} = \frac{1}{4\tan\frac{\phi}{4}}\left[\mathbf{I} - \mathbf{n}\mathbf{n}^T\right]\dot{\mathbf{c}} \tag{262}$$

In order to compute the material and spatial angular velocities from (104), we still make use of the relationships

$$\cos^2\frac{\phi}{4} = \frac{2}{4-c_0} \qquad \sin^2\frac{\phi}{4} = \frac{2-c_0}{4-c_0} \tag{263}$$

and

$$\sin\phi = 4\sin\frac{\phi}{4}\cos\frac{\phi}{4}(2\cos^2\frac{\phi}{4} - 1) \qquad 1 - \cos\phi = 8\sin^2\frac{\phi}{4}\cos^2\frac{\phi}{4} \tag{264}$$

We finally get the results

$$\boxed{\Omega = \mathbf{T}(\mathbf{c})\dot{\mathbf{c}} \qquad \omega = \mathbf{T}^T(\mathbf{c})\dot{\mathbf{c}}} \tag{265}$$

with the expression of the tangent operator

$$\boxed{\mathbf{T} = \frac{2}{(4-c_0)^2}\left[c_0\mathbf{I} + \frac{1}{4}\mathbf{c}\mathbf{c}^T - \tilde{\mathbf{c}}\right]} \tag{266}$$

The CRV parameters are undubitably a good choice to computational represent finite rotations : they form a rather simple set of algebraic quantities which presents no singularity over one full rotation and they have a very linear behavior in terms of the rotation angle. Moreover, their property to allow a separation of the full rotation into two half rotations is specially of interest when performing a time integration with the so-called mid-point rule.

9. The linear parameters

The linear parameters obey to a definition very similar to that of Euler parameters

$$s_0 = \cos\phi \qquad \mathbf{s} = \mathbf{n}\sin\phi \tag{267}$$

and they satisfy the normality constraint

$$s_0^2 + ||\mathbf{s}||^2 = 1 \tag{268}$$

By direct substitution of (267) into (46) the resulting form of the rotation operator is

$$\mathbf{R} = s_0\mathbf{I} + \frac{1}{1+s_0}\mathbf{s}\mathbf{s}^T + \tilde{\mathbf{s}} \tag{269}$$

It is obvious that they exhibit a singularity at $\phi = \pm\pi$.

In order to obtain the explicit expression of angular velocities, let us time differentiate (267) and make use of $\mathbf{n}^T\dot{\mathbf{n}} = 0$ to get

$$\begin{bmatrix} \sin\phi\mathbf{I} & \mathbf{n}\cos\phi \\ \mathbf{n}^T & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{n}} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{s}} \\ 0 \end{bmatrix} \tag{270}$$

We then get

$$\dot{\phi} = \frac{1}{\cos\phi}\mathbf{n}^T\dot{\mathbf{s}} \qquad \dot{\mathbf{n}} = \frac{1}{\sin\phi}(1 - \mathbf{n}\mathbf{n}^T)\dot{\mathbf{s}} \tag{271}$$

The substitution of (271) into (104) yields

$$\Omega = \mathbf{T}(\mathbf{s})\dot{\mathbf{s}} \qquad \omega = \mathbf{T}^T(\mathbf{s})\dot{\mathbf{s}} \tag{272}$$

with the expression of the tangent operator

$$\mathbf{T} = \mathbf{I} + \frac{1}{1+s_0}\left(\frac{\mathbf{s}\mathbf{s}^T}{s_0} + \tilde{\mathbf{s}}\right) \tag{273}$$

The main interest of the linear parameters is their ability to provide a decomposition of the rotation operator into two half rotations, this time expressed in terms of Euler parameters. By setting $s_0 = e_0$ and $\mathbf{s} = \mathbf{e}$ into (269) we get

$$\boxed{\mathbf{R} = \mathbf{F}^2} \tag{274}$$

with

$$\mathbf{F} = e_0 \mathbf{I} + \frac{1}{1 + e_0} \mathbf{e}\mathbf{e}^T + \tilde{\mathbf{e}} \tag{275}$$

10. Geometric description of finite rotations

The geometric representation of spherical motion in terms of successive elementary rotations about coordinate axes is probably the most popular one to mechanical engineers. All textbooks of classical mechanics provide a description of finite rotation transformations in terms of Euler angles, because Euler angles provide the most intuitive kinematic description of rotating systems such as a spinning top in a gravity field. The same is true with Bryant angles when studying flying systems such as an airplane in relative motion with respect to the earth or the the end effector of an industrial robot with respect to its working environment.

10.1. Euler angles

Euler angles provide a system of three independent parameters which consists of expressing the transformation from spatial frame $\mathbf{E}_1$ $\mathbf{E}_2$ $\mathbf{E}_3$ to material frame $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ as a sequence of three elementary rotations about *successive material axes* (figure 3).

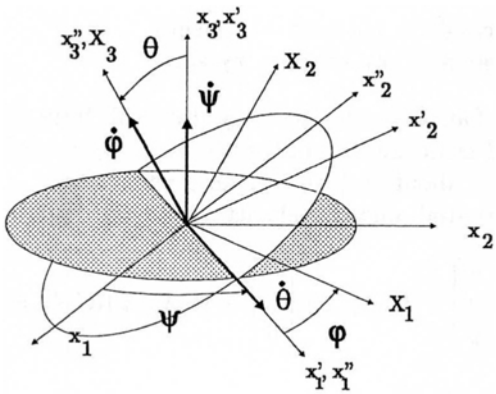

Figure 3: *Finite rotation in terms of Euler angles*

- a ψ rotation about Ox_3: $\mathbf{R}(X_3, \psi)$

- a θ rotation about Ox_1': $\mathbf{R}(X_1, \theta)$
- a ϕ rotation about Ox_3'': $\mathbf{R}(X_3, \phi)$

The resulting frame transformation is

$$\mathbf{x} = \mathbf{R}(X_3, \psi)\ \mathbf{R}(X_1, \theta)\ \mathbf{R}(X_3, \phi)\ \mathbf{X} = \mathbf{R}\ \mathbf{X} \tag{276}$$

The rotation operator is thus obtained as the result of three elementary rotations

$$\boxed{\mathbf{R} = \mathbf{R}(X_3, \psi)\ \mathbf{R}(X_1, \theta)\ \mathbf{R}(X_3, \phi)} \tag{277}$$

It has for explicit expression

$$\mathbf{R} = \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & -\cos\psi\sin\phi - \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ \sin\psi\cos\phi + \cos\psi\cos\theta\sin\phi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & -\cos\psi\sin\theta \\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \end{bmatrix} \tag{278}$$

Singularities may occur in the transformation since colinearity of rotation axes arises when

$$\theta = 0 \quad \text{or} \quad \pi$$

A satisfactory solution to the kinematic inversion is obtained when using the function $\tan^{-1}$

$$\phi = \tan^{-1}\left(\frac{r_{31}}{r_{32}}\right) \qquad \longrightarrow \qquad 2 \text{ solutions } \psi_1, \psi_2$$

$$\begin{cases} \sin\theta = r_{31}\sin\phi + r_{32}\cos\phi \\ \cos\theta = r_{33} \end{cases} \tag{279}$$

$$\begin{cases} \cos\psi = r_{11}\cos\phi - r_{12}\sin\phi \\ \sin\phi = r_{21}\cos\phi - r_{22}\sin\phi \end{cases}$$

The computation of angular velocities may also result from geometric reasoning. They are obtained from the sum of three elementary contributions : a spin $\dot{\psi}$ about Ox_3, a spin $\dot{\theta}$ about Ox_1' and a spin $\dot{\phi}$ about Ox_3''.

The resulting spatial angular velocity vector can thus be expressed in the form

$$\omega = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + \mathbf{R}(X_3, \psi) \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} + \mathbf{R}(X_3, \psi)\mathbf{R}(X_1, \theta) \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} \tag{280}$$

hence the result

$$\omega = \begin{bmatrix} 0 & \cos\psi & \sin\psi\sin\theta \\ 0 & \sin\psi & -\cos\psi\sin\theta \\ 1 & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} \tag{281}$$

The material angular velocity can be put in a similar form

$$\Omega = \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} + \mathbf{R}^T(X_3, \phi) \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} + \mathbf{R}^T(X_3, \phi)\mathbf{R}^T(X_1, \theta) \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \tag{282}$$

and has for expression

$$\Omega = \begin{bmatrix} \sin\phi \sin\theta & \cos\phi & 0 \\ \cos\phi \sin\theta & -\sin\phi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} \tag{283}$$

10.2. Bryant angles

Likewise, Bryant angles provide a system of three independent parameters which consists of expressing the transformation from spatial frame $\mathbf{E}_1\ \mathbf{E}_2\ \mathbf{E}_3$ to material frame $\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3$ as a sequence of three elementary rotations about *successive material axes* (figure 4). The choice is now

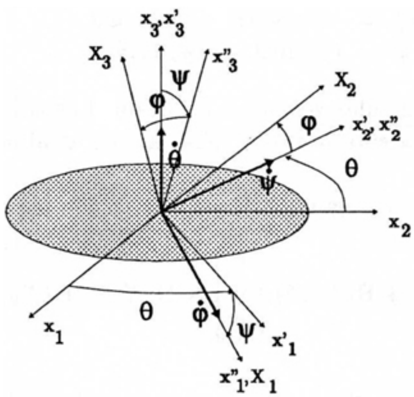

Figure 4: *Finite rotation in terms of Bryant angles*

- a θ rotation about Ox_3: $\mathbf{R}(X_3, \theta)$
- a ψ rotation about Ox_2': $\mathbf{R}(X_2, \psi)$
- a ϕ rotation about Ox_1'': $\mathbf{R}(X_1, \phi)$

The resulting frame transformation is

$$\mathbf{x} = \mathbf{R}(X_3, \theta)\ \mathbf{R}(X_2, \psi)\ \mathbf{R}(X_1, \phi)\ \mathbf{X} = \mathbf{R}\ \mathbf{X} \tag{284}$$

The rotation operator is thus obtained as the result of three elementary rotations

$$\boxed{\mathbf{R} = \mathbf{R}(X_3, \theta)\ \mathbf{R}(X_2, \psi)\ \mathbf{R}(X_1, \phi)} \tag{285}$$

It has for explicit expression

$$\mathbf{R} = \begin{bmatrix} \cos\theta\cos\psi & -\sin\theta\cos\phi + \cos\theta\sin\psi\sin\phi & \sin\theta\sin\phi + \cos\theta\sin\psi\cos\phi \\ \sin\theta\cos\psi & \cos\theta\cos\phi + \sin\theta\sin\psi\sin\phi & -\cos\theta\sin\phi + \sin\theta\sin\psi\cos\phi \\ -\sin\psi & \cos\psi\sin\phi & \cos\psi\cos\phi \end{bmatrix} \quad (286)$$

Singularities may also occur in the transformation since colinearity of rotation axes arises when

$$\psi = \pm\pi$$

A satisfactory solution to the kinematic inversion is obtained when using the function $\tan^{-1}$

$$\theta = \tan^{-1}(\frac{r_{21}}{r_{11}}) \qquad \longrightarrow \qquad 2 \text{ solutions } \theta_1, \theta_2$$

$$\begin{cases} \cos\psi = r_{21}\sin\theta + r_{11}\cos\theta \\ \sin\psi = -r_{31} \end{cases} \quad (287)$$

$$\begin{cases} \cos\phi = -r_{12}\sin\theta + r_{22}\cos\theta \\ \sin\phi = r_{13}\sin\theta - r_{23}\cos\theta \end{cases}$$

The computation of angular velocities is obtained as the sum of three elementary contributions : a spin $\dot{\theta}$ about Ox_3, a spin $\dot{\psi}$ about Ox_2' and a spin $\dot{\phi}$ about Ox_1''.

The resulting spatial angular velocity vector can takes the form

$$\omega = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} + \mathbf{R}(X_3, \theta) \begin{bmatrix} 0 \\ \dot{\psi} \\ 0 \end{bmatrix} + \mathbf{R}(X_3, \theta)\mathbf{R}(X_2, \psi) \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} \quad (288)$$

hence the result

$$\omega = \begin{bmatrix} 0 & -\sin\theta & \cos\psi\cos\theta \\ 0 & \cos\theta & \cos\psi\sin\theta \\ 1 & 0 & -\sin\psi \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\psi} \\ \dot{\phi} \end{bmatrix} \quad (289)$$

The material angular velocity can be put in similar form

$$\boldsymbol{\Omega} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{R}^T(X_1, \phi) \begin{bmatrix} 0 \\ \dot{\psi} \\ 0 \end{bmatrix} + \mathbf{R}^T(X_1, \phi)\mathbf{R}^T(X_2, \psi) \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \quad (290)$$

and has for expression

$$\boldsymbol{\Omega} = \begin{bmatrix} -\sin\psi & 0 & 1 \\ \cos\psi\sin\phi & \cos\phi & 0 \\ \cos\psi\cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\psi} \\ \dot{\phi} \end{bmatrix} \quad (291)$$

11. Application to rigid body dynamics : energy preserving time integration of top motion

An interesting application to parametrization of finite rotations in the context of computational dynamics is the time integration of the motion equations of a top in a gravity field. The top is a conservative system for which it is highly desirable to compute a long term response which exhibits the property of energy conservation. For sake of generality, it will be modelled here a general rigid body (6 generalized DOF) submitted to 3 kinematic constraints.

Following the work of J. Simo [SIM91] and O. Bauchau [BAU95], we apply to this constrained system the mid-point rule which consists to express dynamic equilibrium at mid-interval between sampling instants. This raises the problem of defining the rotation at half-time: it is observed in this case that describing the relative rotation from t_n to t_{n+1} in terms of Euler Parameters [GER94] provides the most efficient parametrization of the rotation at $t_{n+\frac{1}{2}}$

Attention is also paid to the expression of the attachment constraints: it is shown that in order to preserve energy, we have to inforce the first time derivative of the constraints rather than the position constraints themselves. Thanks to energy conservation, this weak inforcement of constraints does not produce significant drift in the displacement response of the system and thus, does not require any special stabilization procedure.

11.1. Motion equations of a top in descriptor form

Let us adopt the center of mass of the top as the origin of the material frame, and its attachment point as the origin of the spatial frame. The kinetic energy

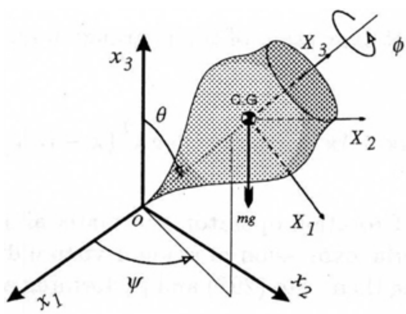

Figure 5: *Analysis of top motion*

of the top may then be split into translation and rotation contributions

$$\mathcal{K} = \frac{1}{2}\mathbf{\Omega}^T \mathbf{J}\mathbf{\Omega} + \frac{1}{2}m\dot{\mathbf{x}}^T\dot{\mathbf{x}} \qquad (292)$$

where m is the mass of the top, $\dot{\mathbf{x}}$ is the velocity vector of the center of mass expressed in the spatial frame, $\mathbf{J}$ is the inertia tensor of the top measured in material axes and $\boldsymbol{\Omega}$ is the material expression of the angular velocity vector (81).
Assuming that the reference for the potential energy is the origin of the spatial frame, the potential energy may be expressed in the form

$$\mathcal{V} = -m\mathbf{g}^T\mathbf{x} \tag{293}$$

where $\mathbf{g}$ is the acceleration vector. For example, if the gravity is acting along the negative x_3 direction, $g = [0 \quad 0 \quad g]^T$ and $\mathcal{V} = +mgx_3$.
Finally, let us denote by vector $-\mathbf{X}_g$ the location of the top attachment point in material coordinates. The center of mass is then constrained to verify at any time instant t the geometric relationship

$$\boldsymbol{\Phi} = \mathbf{x} - \mathbf{R}\mathbf{X}_g = 0 \tag{294}$$

The Lagrangian of the constrained system can be constructed in the form

$$\mathcal{L} = \mathcal{K} - \mathcal{V} + \lambda^T\boldsymbol{\Phi} \tag{295}$$

where λ is a vector of Lagrangian multipliers associated to the constraint. Its components may be interpreted as the reaction forces at the attachment point. The motion equations result from the application of Hamilton's principle

$$\delta \int_{t_1}^{t_2} \mathcal{L} dt = 0 \tag{296}$$

Substituting the explicit expression of the Lagrangian and performing the variation yields

$$\int_{t_1}^{t_2} \left[\delta\boldsymbol{\Omega}^T\mathbf{J}\boldsymbol{\Omega} + \delta\dot{\mathbf{x}}^T m\dot{\mathbf{x}} + \delta\mathbf{x}^T(\lambda + m\mathbf{g}) + \delta\lambda^T(\mathbf{x} - \mathbf{R}\mathbf{X}_g) - \lambda^T\delta\mathbf{R}\mathbf{X}_g\right] dt = 0 \tag{297}$$

where the variations of rotation operator and material angular velocities may be related to the material expression of angular virtual displacements by (142) and (150). Substituting them into (297) and performing an integration by parts yields

$$\begin{aligned} &\left[\delta\mathbf{x}^T m\dot{\mathbf{x}} + \delta\boldsymbol{\Theta}^T\mathbf{J}\boldsymbol{\Omega}\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[\delta\boldsymbol{\Theta}^T(-\mathbf{J}\dot{\boldsymbol{\Omega}} - \tilde{\boldsymbol{\Omega}}\mathbf{J}\boldsymbol{\Omega} - \tilde{\mathbf{X}}_g\mathbf{R}^T\lambda)\right. \\ &\left.+\delta\mathbf{x}^T(-m\ddot{\mathbf{x}} + \lambda + m\mathbf{g}) + \delta\lambda^T(\mathbf{x} - \mathbf{R}\mathbf{X}_g)\right] dt = 0 \end{aligned} \tag{298}$$

We get thus the motion equations in the differential-algebraic form

$$\begin{aligned} &m\ddot{\mathbf{x}} - \lambda = m\mathbf{g} \\ &\mathbf{J}\dot{\boldsymbol{\Omega}} + \tilde{\boldsymbol{\Omega}}\mathbf{J}\boldsymbol{\Omega} + \tilde{\mathbf{X}}_g\mathbf{R}^T\lambda = 0 \\ &-\mathbf{x} + \mathbf{R}\mathbf{X}_g = 0 \end{aligned} \tag{299}$$

It is of interest to note that by defining the spatial and linear expressions of linear and angular momenta

$$\mathbf{p} = m\dot{\mathbf{x}} \qquad\qquad \mathbf{h} = \mathbf{R}\mathbf{J}\boldsymbol{\Omega} \tag{300}$$

the first two equations (299) may still be rewritten in the simpler form

$$\dot{\mathbf{p}} - \boldsymbol{\lambda} = m\mathbf{g} \qquad\qquad \dot{\mathbf{h}} + \tilde{\mathbf{x}}\boldsymbol{\lambda} = 0 \tag{301}$$

11.2. Time discretization

11.2.1. The mid-point rule

The mid-point integration rule is based on the application of the mean value theorem which states that any continuous and derivable function $f(t)$ can be expressed at a time $t+h$ in the form

$$f(t+h) = f(t) + h\left.\frac{df}{dt}\right|_{(t+\alpha h)} \qquad \alpha \in [0,1] \tag{302}$$

When applied to the solution of a first order nonlinear differential system

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y},t) \tag{303}$$

it yields the second-order accurate difference formula

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(\mathbf{y}_{n+\alpha}, t_{n+\alpha}) + O(h^2) \qquad \text{with} \qquad t_{n+\alpha} = (1-\alpha)t_n + \alpha t_{n+1} \tag{304}$$

The mid-point rule is a particular case of (304)

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(\mathbf{y}_{n+\frac{1}{2}}, t_{n+\frac{1}{2}}) + O(h^2) \qquad \text{with} \qquad t_{n+\frac{1}{2}} = \frac{1}{2}(t_n + t_{n+1}) \tag{305}$$

which is equivalent to the trapezoidal rule in the linear case.

11.2.2. Application to the top equations

Both equilibrium equations (301) are discretized using the mid-point rule

$$\begin{aligned} \mathbf{p}_{n+1} &= \mathbf{p}_n + h(\boldsymbol{\lambda}_{n+\frac{1}{2}} + m\mathbf{g}) \\ \mathbf{h}_{n+1} &= \mathbf{h}_n + h\tilde{\mathbf{x}}_{n+\frac{1}{2}}\boldsymbol{\lambda}_{n+\frac{1}{2}} \end{aligned} \tag{306}$$

The special treatment to be applied to the constraint equation (299.c) will appear naturally later on from energy conservation considerations.

Discretization of linear momentum The linear momentum equation is next expressed at mid-point

$$\mathbf{p}_{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{p}_n + \mathbf{p}_{n+1}) = \frac{m}{h}(\mathbf{x}_{n+1} - \mathbf{x}_n) \tag{307}$$

giving

$$\mathbf{p}_{n+1} = \frac{2m}{h}(\mathbf{x}_{n+1} - \mathbf{x}_n) - \mathbf{p}_n \tag{308}$$

and through sustitution of (308) the discretized translation equilibrium equation (306.a) may be put in the final form

$$\frac{2m}{h^2}(\mathbf{x}_{n+1} - \mathbf{x}_n) - \frac{2}{h}\mathbf{p}_n - \lambda_{n+\frac{1}{2}} = m\mathbf{g} \tag{309}$$

Discretization of angular momentum The angular momentum equation is expressed likewise

$$\mathbf{h}_{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{h}_n + \mathbf{h}_{n+1}) = \mathbf{R}_{n+\frac{1}{2}}\mathbf{J}\boldsymbol{\Omega}_{n+\frac{1}{2}} \tag{310}$$

with

$$\tilde{\boldsymbol{\Omega}}_{n+\frac{1}{2}} = \mathbf{R}^T_{n+\frac{1}{2}}\dot{\mathbf{R}}_{n+\frac{1}{2}} = \frac{1}{h}\mathbf{R}^T_{n+\frac{1}{2}}(\mathbf{R}_{n+1} - \mathbf{R}_n) \quad \text{and} \quad \boldsymbol{\Omega}_{n+\frac{1}{2}} = \text{vect}(\tilde{\boldsymbol{\Omega}}_{n+\frac{1}{2}}) \tag{311}$$

and through substitution of (310) and (311) the discretized rotation equilibrium equation (306.b) takes the final form

$$\frac{2}{h^2}\mathbf{R}_{n+\frac{1}{2}}\mathbf{J}\text{vect}(\mathbf{R}^T_{n+\frac{1}{2}}(\mathbf{R}_{n+1} - \mathbf{R}_n)) - \frac{2}{h}\mathbf{h}_n + \tilde{\mathbf{x}}_{n+\frac{1}{2}}\lambda_{n+\frac{1}{2}} = 0 \tag{312}$$

11.3. Rotation parametrization

In order to define the configuration which is half-way between $\mathbf{R}_n$ and $\mathbf{R}_{n+1}$, let us decompose the rotation increment from $\mathbf{R}_n$ to $\mathbf{R}_{n+1}$ in the form of two successive equal rotations

$$\mathbf{R}_n^T\mathbf{R}_{n+1} = \mathbf{F}^2 \tag{313}$$

The resulting operator $\mathbf{F}$ is such that

$$\mathbf{R}_{n+\frac{1}{2}} = \mathbf{R}_n\mathbf{F} = \mathbf{R}_{n+1}\mathbf{F}^T \tag{314}$$

and verifies the orthonormality properties

$$\mathbf{F}\mathbf{F}^T = \mathbf{F}^T\mathbf{F} = \mathbf{I} \tag{315}$$

The matrix of angular velocities (311) may be put in the form

$$\tilde{\boldsymbol{\Omega}}_{n+\frac{1}{2}} = \frac{1}{h}\mathbf{F}^T\mathbf{R}_n^T(\mathbf{R}_n\mathbf{F}^2 - \mathbf{R}_n) = \frac{1}{h}(\mathbf{F} - \mathbf{F}^T) \tag{316}$$

so that the discretized angular velocities at mid-point are approximated by

$$\boldsymbol{\Omega}_{n+\frac{1}{2}} = \frac{1}{h}\text{vect}(\mathbf{F} - \mathbf{F}^T) = \frac{2}{h}\text{vect}(\mathbf{F}) \tag{317}$$

Since $\mathbf{F}$ is a rotation operator, vect($\mathbf{F}$) has the property

$$\mathbf{F}\text{vect}(\mathbf{F}) = \mathbf{F}^T\text{vect}(\mathbf{F}) = \text{vect}(\mathbf{F}) \tag{318}$$

which expresses the fact that the rotation direction

$$\mathbf{n} = \frac{\text{vect}(\mathbf{F})}{\|\text{vect}(\mathbf{F})\|} \tag{319}$$

remains invariant under the rotation.

By making use of the above property, the discretized equation of equilibrium can be rewritten in the form

$$\frac{4}{h^2}\mathbf{R}_n\mathbf{F}\mathbf{J}\mathbf{F}^T\text{vect}(\mathbf{F}) - \frac{2}{h}\mathbf{h}_n + \tilde{\mathbf{x}}_{n+\frac{1}{2}}\lambda_{n+\frac{1}{2}} = 0 \tag{320}$$

The matrix $\mathbf{F}$ describing the half rotation may be constructed in two alternative ways.

In terms of Euler parameters: Let us describe the relative rotation from $\mathbf{R}_n$ to $\mathbf{R}_{n+1}$ in terms of its invariants $\mathbf{n}$ and ϕ ($\mathbf{n}$ being the direction of the rotation axis in frame $\mathbf{R}_n$, and ϕ being the amplitude of the rotation) :

$$\mathbf{R}_n^T\mathbf{R}_{n+1} = \mathbf{R}(\mathbf{n}, \phi) \tag{321}$$

with the general expression (46) of the rotation operator.

Supposing that the direction of the rotation is kept constant, (321) may then be split in two equal rotations of the form

$$\mathbf{F} = \mathbf{R}(\mathbf{n}, \frac{1}{2}\phi) \tag{322}$$

From eqn (46) expressed for ($\mathbf{n}$, $\frac{1}{2}\phi$) we note that the vector part of $\mathbf{F}$ is nothing else than the vector part of Euler parameters

$$\text{vect}(\mathbf{F}) = \mathbf{n}\sin\frac{\phi}{2} = \mathbf{e} \tag{323}$$

Euler parameters may thus be used to parametrize matrix $\mathbf{F}$. Substituting eqn (323) in the explicit expression of $\mathbf{R}(\mathbf{n}, \frac{1}{2}\phi)$ yields

$$\mathbf{F} = \mathbf{R}(\mathbf{n}, \frac{1}{2}\phi) = e_0\mathbf{I} + \frac{1}{1+e_0}\mathbf{e}\mathbf{e}^T + \tilde{\mathbf{e}} \tag{324}$$

This representation involves only the three components e_1, e_2 and e_3, e_0 being computed by (196)

The property that the direction of the rotation remains unaffected by the half rotation can be written in the form

$$\mathbf{F}\mathbf{e} = \mathbf{F}^T\mathbf{e} = \mathbf{e} \tag{325}$$

Owing to (317) the angular velocities at mid point are very simply computed by

$$\mathbf{\Omega}_{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{\Omega}_n + \mathbf{\Omega}_{n+1}) = \frac{2}{h}\mathbf{e} = \frac{2}{h}\mathbf{F}\mathbf{e} = \frac{2}{h}\mathbf{F}^T\mathbf{e} \tag{326}$$

In terms of the conformal rotation vector An alternative and also quite elegant way to perform the decomposition of the relative rotation in two equal parts is to describe the rotation in terms of the CRV defined in terms of Euler parameters by (252) and in terms of the rotation invariants $\mathbf{n}$ and ϕ by (253). The scalar part c_0 is also expressed in terms of the modulus of the vector part $\mathbf{c}$ by (254). Equation (256) gives the corresponding expression of the rotation matrix.

It has already been demonstrated that the rotation matrix (321) can be split into two successive half rotations

$$\mathbf{R}(\mathbf{c}) = \mathbf{F}^2 \tag{327}$$

given by (258). The half rotation maintains the invariance of the rotation direction

$$\mathbf{F}\mathbf{c} = \mathbf{F}^T\mathbf{c} = \mathbf{c} \tag{328}$$

and the computation of its vector part yields

$$2\text{vect}(\mathbf{F}) = \text{vect}(\mathbf{F} - \mathbf{F}^T) = \frac{2}{(4 - c_0)}\mathbf{c} \tag{329}$$

Owing to (317) the velocities at mid-point are computed by

$$\boldsymbol{\Omega}_{n+\frac{1}{2}} = \frac{1}{2}(\boldsymbol{\Omega}_n + \boldsymbol{\Omega}_{n+1}) = \frac{2\mathbf{c}}{h(4 - c_0)} \tag{330}$$

Parametrized form of equilibrium equations The use of both Euler and conformal rotation parameters yields very similar expressions of the discretized equations of rotational equilibrium. Substituting (326) and (330) successively into (312) yields

$$\frac{4}{h^2}\mathbf{R}_n\mathbf{F}\mathbf{J}\mathbf{F}^T\mathbf{e} - \frac{2}{h}\mathbf{h}_n + \tilde{\mathbf{x}}_{n+\frac{1}{2}}\lambda_{n+\frac{1}{2}} = 0 \tag{331}$$

in terms of Euler parameters and

$$\frac{4}{(4 - c_0)h^2}\mathbf{R}_n\mathbf{F}\mathbf{J}\mathbf{F}^T\mathbf{c} - \frac{2}{h}\mathbf{h}_n + \tilde{\mathbf{x}}_{n+\frac{1}{2}}\lambda_{n+\frac{1}{2}} = 0 \tag{332}$$

in terms of CRV.

Because of their even greater simplicity to represent the mid-point rotation, the Euler parameters of the relative rotation have been prefered to the conformal rotation vector for the numerical implementation of the method which is presented hereafter.

11.4. Energy conservation

In order to express the balance of energy on one time step, let us multiply the translation equilibrium equation by $h\mathbf{v}^T_{n+\frac{1}{2}}$, the rotational equilibrium equation by $h\Omega^T_{n+\frac{1}{2}}\mathbf{R}^T_n = 2\text{vect}(\mathbf{F})^T\mathbf{R}^T_n$ and add both terms to compute the scalar quantity

$$\begin{aligned}\mathcal{A} = \; & h\mathbf{v}^T_{n+\frac{1}{2}}\left[\tfrac{2}{h^2}m(\mathbf{x}_{n+1}-\mathbf{x}_n) - \tfrac{2}{h}\mathbf{p}_n - \lambda_{n+\frac{1}{2}} - m\mathbf{g}\right] + \\ & h\Omega^T_{n+\frac{1}{2}}\mathbf{R}^T_n\left[\tfrac{4}{h^2}\mathbf{R}_n\mathbf{F}\mathbf{J}\mathbf{F}^T\text{vect}(\mathbf{F}) - \tfrac{2}{h}\mathbf{h}_n + \tilde{\mathbf{x}}_{n+\frac{1}{2}}\lambda_{n+\frac{1}{2}}\right] = 0\end{aligned} \tag{333}$$

Let us compute successively

$$\begin{aligned}\mathcal{A}_1 &= h\mathbf{v}^T_{n+\frac{1}{2}}\left(\tfrac{2}{h^2}m(\mathbf{x}_{n+1}-\mathbf{x}_n) - \tfrac{2}{h}\mathbf{p}_n - m\mathbf{g}\right) \\ &= \tfrac{h}{2}(\mathbf{v}_n + \mathbf{v}_{n+1})^T\left(\tfrac{1}{h}m(\mathbf{v}_{n+1}+\mathbf{v}_n) - \tfrac{2}{h}m\mathbf{v}_n\right) - (\mathbf{x}_{n+1}-\mathbf{x}_n)m\mathbf{g} \\ &= (\mathcal{K}_{n+1} - \mathcal{K}_n)_{tr} + (\mathcal{V}_{n+1} - \mathcal{V}_n)\end{aligned}$$

$$\begin{aligned}\mathcal{A}_2 &= h\Omega^T_{n+\frac{1}{2}}\mathbf{R}^T_n\left(\tfrac{4}{h^2}\mathbf{R}_n\mathbf{F}\mathbf{J}\mathbf{F}^T\text{vect}(\mathbf{F}) - \tfrac{2}{h}\mathbf{h}_n\right) \\ &= 2\Omega^T_{n+\frac{1}{2}}\left(\mathbf{F}\mathbf{J}\mathbf{F}^T\Omega_{n+\frac{1}{2}} - \mathbf{R}^T_n\mathbf{h}_n\right) \\ &= \tfrac{1}{2}(\Omega_n + \Omega_{n+1})^T\mathbf{J}(\Omega_{n+1}+\Omega_n) - (\Omega_{n+1}+\Omega_n)^T\mathbf{J}\Omega_n \\ &= (\mathcal{K}_{n+1} - \mathcal{K}_n)_{rot}\end{aligned} \tag{334}$$

and therefore

$$\mathcal{A}_1 + \mathcal{A}_2 = \mathcal{K}_{n+1} - \mathcal{K}_n + \mathcal{V}_{n+1} - \mathcal{V}_n \tag{335}$$

Equation (335) expresses the conservation of energy over one time step provided that we have

$$\mathcal{A}_3 = -h\mathbf{v}^T_{n+\frac{1}{2}}\lambda_{n+\frac{1}{2}} + h\Omega^T_{n+\frac{1}{2}}\mathbf{F}^T\mathbf{R}^T_n\tilde{\mathbf{x}}_{n+\frac{1}{2}}\lambda_{n+\frac{1}{2}} = 0 \tag{336}$$

Since

$$\tilde{\mathbf{x}}_{n+\frac{1}{2}} = \mathbf{R}_{n+\frac{1}{2}}\tilde{\mathbf{X}}_g\mathbf{R}^T_{n+\frac{1}{2}} \tag{337}$$

we get the condition

$$(-\mathbf{v}^T_{n+\frac{1}{2}} + \Omega^T_{n+\frac{1}{2}}\tilde{\mathbf{X}}_g\mathbf{R}^T_{n+\frac{1}{2}})\lambda_{n+\frac{1}{2}} = 0 \tag{338}$$

which is fulfilled if

$$\mathbf{v}_{n+\frac{1}{2}} = -\mathbf{R}_{n+\frac{1}{2}}\tilde{\mathbf{X}}_g\Omega_{n+\frac{1}{2}} = +\mathbf{R}_{n+\frac{1}{2}}\tilde{\Omega}_{n+\frac{1}{2}}\mathbf{X}_g \tag{339}$$

The condition (339) corresponds to the time derivative of the initial constraint expressed at mid-point. Owing to (317), it can still be rewritten

$$-\mathbf{x}_{n+1} + \mathbf{x}_n - 2\mathbf{R}_n\mathbf{F}\tilde{\mathbf{X}}_g\mathbf{F}^T\mathbf{e} = 0 \tag{340}$$

11.5. Nonlinear solution

In view of their numerical solution, the motion equations at $t_{n+\frac{1}{2}}$ are written in residual form

$$\mathbf{r} = \begin{bmatrix}\mathbf{r}_x \\ \mathbf{r}_e \\ \mathbf{r}_\lambda\end{bmatrix} \begin{bmatrix}\mathbf{f}_{ext} + \lambda + \frac{2}{h}\mathbf{p}_n - \frac{2m}{h^2}(\mathbf{x}_{n+1}-\mathbf{x}_n) \\ -\tilde{\mathbf{x}}_{n+\frac{1}{2}}\lambda + \frac{2}{h}\mathbf{h}_n - \frac{4}{h^2}\mathbf{R}_n\mathbf{F}\mathbf{J}\mathbf{e} \\ \mathbf{x}_{n+1} - \mathbf{x}_n + 2\mathbf{R}_n\mathbf{F}\tilde{\mathbf{X}}_g\mathbf{e}\end{bmatrix} = 0 \tag{341}$$

with

$$\tilde{\mathbf{x}}_{n+\frac{1}{2}} = \mathbf{R}_{n+\frac{1}{2}} \tilde{\mathbf{X}}_g \mathbf{R}^T_{n+\frac{1}{2}} \tag{342}$$

The tangent iteration matrix is the jacobian matrix of (341)

$$\mathbf{S} = -\frac{\partial(\mathbf{r}_x,\ \mathbf{r}_e,\ \mathbf{r}_\lambda)}{\partial(\mathbf{x},\ \mathbf{e},\ \lambda)} = \begin{bmatrix} \mathbf{S}_{xx} & 0 & \mathbf{S}_{x\lambda} \\ 0 & \mathbf{S}_{ee} & \mathbf{S}_{e\lambda} \\ \mathbf{S}_{\lambda x} & \mathbf{S}_{\lambda e} & 0 \end{bmatrix} \tag{343}$$

It is not symmetric, its different terms being given by

$$\begin{aligned}
\mathbf{S}_{xx} &= -\frac{\partial \mathbf{r}_x}{\partial \mathbf{x}} = \frac{2m}{h^2}\mathbf{I} \\
\mathbf{S}_{x\lambda} &= -\frac{\partial \mathbf{r}_x}{\partial \lambda} = -\mathbf{I} \\
\mathbf{S}_{ee} &= -\frac{\partial \mathbf{r}_e}{\partial \mathbf{e}} = \mathbf{R}_n \left\{ \frac{4}{h^2}\mathbf{F}\mathbf{J} + \sum_i \left[\frac{4}{h^2}\frac{\partial \mathbf{F}}{\partial \mathbf{e}_i}\mathbf{J} + \left(\frac{\partial \mathbf{F}}{\partial \mathbf{e}_i}\tilde{\mathbf{X}}_g \mathbf{F}^T + \mathbf{F}\tilde{\mathbf{X}}_g \frac{\partial \mathbf{F}^T}{\partial \mathbf{e}_i} \right) \mathbf{R}_n^T \lambda \right] \mathbf{k}_i^T \right\} \\
\mathbf{S}_{e\lambda} &= -\frac{\partial \mathbf{r}_e}{\partial \lambda} = \tilde{\mathbf{x}}_{n+\frac{1}{2}} \\
\mathbf{S}_{\lambda x} &= -\frac{\partial \mathbf{r}_\lambda}{\partial \mathbf{x}} = -\mathbf{I} \\
\mathbf{S}_{\lambda e} &= -\frac{\partial \mathbf{r}_\lambda}{\partial \mathbf{e}} = -2\mathbf{R}_n \left[\mathbf{F}\tilde{\mathbf{X}}_g + \sum_i \frac{\partial \mathbf{F}}{\partial \mathbf{e}_i}\tilde{\mathbf{X}}_g \mathbf{e}\mathbf{k}_i^T \right]
\end{aligned} \tag{344}$$

and $\mathbf{k}_i$ being the unit vector along direction i.
The time integration procedure is then as follows :

1. *Initialization*

   Given : $t_0 = 0,\ \mathbf{R}_0,\ \mathbf{\Omega}_0$

   Compute : $\mathbf{x}_0 = \mathbf{R}_0\mathbf{X}, \quad \mathbf{v}_0 = \mathbf{R}_0\mathbf{\Omega}_0\mathbf{X}$

   $\mathbf{p}_0 = m\mathbf{v}_0, \quad \mathbf{h}_0 = \mathbf{R}_0\mathbf{J}\mathbf{\Omega}_0$

2. *Time integration : while* $t_n < h$ *do*

   (i) *increment time :*

   $$t_{n+1} = t_n + h$$

   (ii) *predict new solution :*

   $$\mathbf{x}_{n+1} = \mathbf{x}_n + \frac{1}{2}h\mathbf{v}_n\ , \qquad \mathbf{e}_{n+1} = \mathbf{e}_n\ , \qquad \lambda = 0$$

   (iii) *iterate :*

   while $(\|\mathbf{r}_x\| > \epsilon_x \text{ .or. } \|\mathbf{r}_e\| > \epsilon_e \text{ .or. } \|\mathbf{r}_\lambda\| > \epsilon_\lambda)$ *do*

   – *Displacement and rotation correction at* $t_{n+\frac{1}{2}}$ *:*

   $$\mathbf{R}_{n+\frac{1}{2}} = \mathbf{R}_n\mathbf{F}\ , \qquad \mathbf{x}_{n+\frac{1}{2}} = \mathbf{R}_{n+\frac{1}{2}}\tilde{\mathbf{X}}_g\mathbf{R}^T_{n+\frac{1}{2}}$$

   – *residual evaluation :* $\mathbf{r}_x,\ \mathbf{r}_e,\ \mathbf{r}_\lambda$

   – *linear solution :* $\mathbf{S}d\mathbf{q} = \mathbf{r} \quad \Rightarrow \quad d\mathbf{x},\ d\mathbf{e},\ d\lambda.$

   – *incrementation :*

   $$\mathbf{x}_{n+1} = \mathbf{x}_{n+1} + d\mathbf{x}\ , \qquad \mathbf{e}_{n+1} = \mathbf{e}_{n+1} + d\mathbf{e}\ , \qquad \lambda = \lambda + d\lambda$$

   end

(iv) *solution updating at* $t_{n+\frac{1}{2}}$:

$$
\begin{aligned}
\mathbf{R}_{n+1} &= \mathbf{R}_{n+\frac{1}{2}}\mathbf{F} \\
\mathbf{v}_{n+1} &= \tfrac{2}{h}(\mathbf{x}_{n+1} - \mathbf{x}_n) - \mathbf{v}_n \ , \ \mathbf{p}_{n+1} = m\mathbf{v}_{n+1} \\
\mathbf{h}_{n+1} &= \tfrac{4}{h}\mathbf{R}_{n+\frac{1}{2}}\mathbf{J}\mathbf{e}_{n+1} - \mathbf{h}_n \ , \ \boldsymbol{\Omega}_{n+1} = (\mathbf{R}_{n+1}\mathbf{J})^{-1}\mathbf{h}_{n+1}
\end{aligned}
$$

end.

11.6. Numerical application

In order to demonstrate the numerical properties of the energy conserving methodology and algorithm, let us consider the problem of determining the trajectory of a symmetrical top in a gravity field.
It is classical to describe the instantaneous motion of the top in terms of Euler angles (figure 5) as described in section10.1..
Angle ϕ corresponds to the spin of the top about its rotation axis; θ gives the inclination of the top axis with respect to the vertical, and ψ gives the azimutal position of the top axis in the horizontal plane Ox_1x_2, describing thus the precession motion.
The rotation operator in terms of Euler angles and its inverse transformation (becoming singular when $\theta = 0$ or π) are respectively given by (278) and (279).

The time derivatives of Euler angles are related to the material angular velocities by the inverse relationship to (282)

$$
\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \frac{1}{\sin\theta} \begin{bmatrix} \sin\phi & \cos\phi & 0 \\ \cos\phi\sin\theta & -\sin\phi\sin\theta & 0 \\ -\sin\phi\cos\theta & -\cos\phi\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} \tag{345}
$$

Let us consider a symmetrical top with the following properties. Mass m : 5 kg, moments of inertia : $J_{11} = J_{22} = 0.8\ Kgm^2$, $J_{33} = 1.8\ kgm^2$, distance from CG to origin (attachment point) : $L = 1.3m$, gravity : $g = 9.81m/s^2$ (along negative x_3 axis). The initial position of the top is described in terms of Euler angles : $\phi_0 = 0$, $\theta_0 = \frac{\pi}{9}rad$, $\psi_0 = 0$.
Two response cases have been considered. In case 1, the top is simply dropped from its initial position with a spin velocity $\dot{\phi}_0 = 50rad/s$. Both other angular velocities are zero ($\dot{\psi}_0 = \dot{\theta}_0 = 0$). In case 2, the top is thrown from its initial position with a spin velocity $\dot{\phi}_0 = 50rad/s$, a precession angular velocity of $\dot{\psi}_0 = -10rad/s$ and a zero nutation angular velocity ($\dot{\theta}_0 = 0$).
Figures (6) and (7) display both computed responses in various forms and in terms of different kinematic and kinetic quantities, namely : (a) vertical displacement versus time, (b) three-dimensional trajectory of the CG, (c) Euler angle θ versus time, (d) Euler angle ϕ versus time, (e) time evolution of the kinematic constraints, (f) phase diagram of the non cyclic variable θ, (g) relative energy variation $E/E_0 - 1$ versus time, (h) transverse angular velocity Ω_1 versus time. The total energy of the system is obviously conserved during the period of observation of the motion as it could be expected from the very design of the integration algorithm. All the time evolutions demonstrate also that the

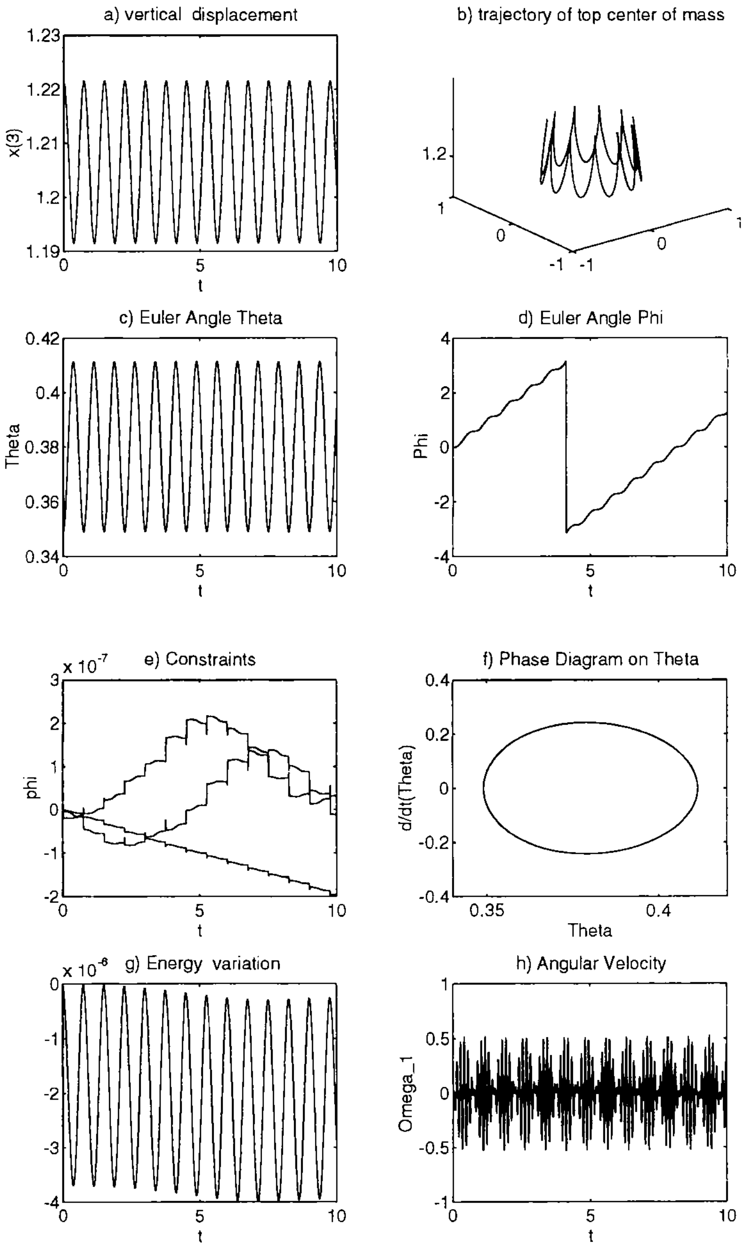

Figure 6: *Response of symmetrical top - case 1 :* $\dot{\theta}_0 = 0,\ \dot{\psi}_0 = 0,\ \dot{\phi}_0 = 50\ rad/s$

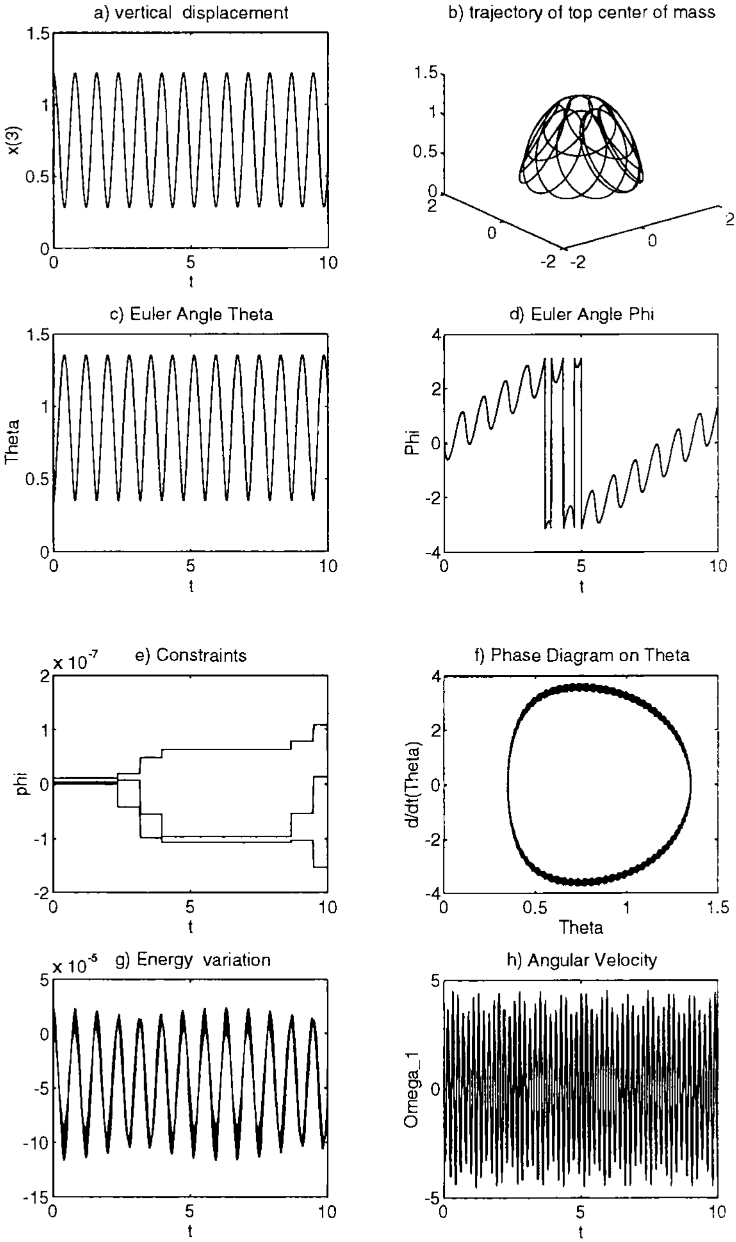

Figure 7: *Response of symmetrical top - case 2 :* $\dot{\theta}_0 = 0,\ \dot{\psi}_0 = -10,\ \dot{\phi}_0 = 50\ rad/s$

periodic character of the motion is perfectly preserved, which can be regarded as a direct consequence of the energy conservation property. Of interest also is the time evolution of the kinematic constraints : despite of the fact that they are satisfied only in weak form, their drift remains extremely small ($\leq 2.E-7$), observation which is a further consequence of energy conservation.

Conclusion

The present review of finite motion kinematics is far from being complete. It has voluntarily been limited to concepts and parametrization choices which are useful in the context of computational dynamics.

For sake of clarity, a purely algebraic point of view has been adopted rather than the more abstract and more general point of view where finite rotations are regarded as nonlinear objects belonging to the special orthogonal Lie group.

The application has been restricted to rigid body dynamics: the topic of finite motion parametrization in the context of continuum mechanics for modelling flexible bodies such as beams and shells can be found in other contributions to this special issue.

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A Matrix representation of vector operations

Vector

A vector $\vec{\mathbf{u}}$ of the Euclidian space $\mathcal{E}^3$ is represented by a column matrix collecting its Cartesian components, and its transpose is a row matrix

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{u}^T = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \tag{346}$$

Dot product

Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be two vectors of the Euclidian space $\mathcal{E}^3$. In matrix form, the dot product $\mathbf{u} \cdot \mathbf{v}$ will be achieved by performing the inner product between the column vectors $\mathbf{u}$ and $\mathbf{v}$, and therefore

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \mathbf{u}^T \mathbf{v} \tag{347}$$

where $(\cdot)^T$ denotes the transposition operation.

Linear transformation and second-order Cartesian tensor
A second-order Cartesian tensor a_{ij} is represented by a (3×3) transformation matrix $\mathbf{A}$, and a linear transformation in $\mathcal{E}^3$ is represented by the associated matrix-vector product. The image of vector $\mathbf{u}$ under transformation $\mathbf{A}$ is

$$\mathbf{v} = \mathbf{A}\mathbf{u} \tag{348}$$

If $\mathbf{A}$ is a one-to-one transformation, $\mathbf{A}^{-1}$ exists and

$$\mathbf{u} = \mathbf{A}^{-1}\mathbf{v} \tag{349}$$

The identity transformation is represented by $\mathbf{I}$:

$$\mathbf{u} = \mathbf{I}\mathbf{u} \tag{350}$$

Eigenvalue properties of a linear transformation
Provided that they are distinct, the eigensolutions of the homogeneous problem

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{351}$$

associated to the (3×3) linear transformation may be collected in

$$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1 \ \lambda_2 \ \lambda_3) \qquad \mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3] \tag{352}$$

They are such that

$$\mathbf{A} = \mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1} \tag{353}$$

In particular, we have

$$\mathbf{X}\mathbf{X}^* = \mathbf{X}^*\mathbf{X} = \mathbf{I} \qquad (\mathbf{X} \text{ unitary}) \tag{354}$$

under the condition of $\mathbf{A}$ being normal ($\mathbf{A}\mathbf{A}^* = \mathbf{A}^*\mathbf{A}$). The superscript * denotes the conjugate transposition operation.

Invariance properties of a linear transformation
Among the possible invariants of a linear transformation $\mathbf{A}$, let us mention

- the determinant of the transformation

$$\det \mathbf{A} = \lambda_1\lambda_2\lambda_3 \tag{355}$$

- the traces of the powers of $\mathbf{A}$

$$\mathrm{tr}(\mathbf{A}^k) = \lambda_1^k + \lambda_2^k + \lambda_3^k \tag{356}$$

- the vector part $\mathrm{vect}(\mathbf{A})$, defined in index notation as

$$[\mathrm{vect}(\mathbf{A})]_i = \frac{1}{2}\epsilon_{ijk}a_{kj} \tag{357}$$

where ϵ_{ijk} is the alternating tensor.

Symmetric and skew-symmetric transformations
A transformation matrix $\mathbf{A}$ is symmetric if $\mathbf{A} = \mathbf{A}^T$ and skew-symmetric if $\mathbf{A} = -\mathbf{A}^T$.
A general transformation $\mathbf{A}$ may thus be decomposed into its symmetric and skew-symmetric parts

$$\mathbf{A}_s = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) \qquad \text{and} \qquad \mathbf{A}_{ss} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T) \tag{358}$$

Clearly,

$$\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}_s)\ , \qquad \mathrm{vect}(\mathbf{A}) = \mathrm{vect}(\mathbf{A}_{ss}) \tag{359}$$

and thus, the trace of a skew-symmetric matrix and the vector part of a symmetric one both vanish.

Cross product and orthogonal projection
Let us define a (3×3) linear transformation $\tilde{\mathbf{u}}$ associated to a vector $\mathbf{u}$ such as

$$\mathbf{u} = \mathrm{vect}(\tilde{\mathbf{u}}) \qquad \Rightarrow \qquad \tilde{\mathbf{u}} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} = \mathrm{spin}(\mathbf{u}) \tag{360}$$

It has only rank 2 since the property $\tilde{\mathbf{u}}\mathbf{u} = 0$ implies that it has one zero eigenvalue associated to the eigenvector $\mathbf{u}$. The cross product $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ may be achieved in matrix form by premultiplying vector $\mathbf{v}$ by the linear transformation $\tilde{\mathbf{u}}$

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \tilde{\mathbf{u}}\mathbf{v} \tag{361}$$

The expression of the double vector product in terms of dot products

$$\vec{\mathbf{u}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \cdot \vec{\mathbf{w}})\vec{\mathbf{v}} - (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})\vec{\mathbf{w}} \tag{362}$$

may be written in matrix form

$$\tilde{\mathbf{u}}\tilde{\mathbf{v}}\mathbf{w} = \mathbf{v}\ (\mathbf{u}^T\mathbf{w}) - (\mathbf{u}^T\mathbf{v})\ \mathbf{w} \tag{363}$$

Since it is valid for arbitrary vector $\mathbf{w}$, we have the following matrix identity

$$\tilde{\mathbf{u}}\tilde{\mathbf{v}} = \mathbf{v}\mathbf{u}^T - \mathbf{u}^T\mathbf{v}\ \mathbf{I} \tag{364}$$

which, for a unit vector $\mathbf{n}$, yields

$$\tilde{\mathbf{n}}\tilde{\mathbf{n}} = \mathbf{n}\mathbf{n}^T - \mathbf{I} \tag{365}$$

The linear transformation

$$\boxed{\mathbf{P}_n = \tilde{\mathbf{n}}^T\tilde{\mathbf{n}} = \mathbf{I} - \mathbf{n}\mathbf{n}^T} \tag{366}$$

can be regarded as an *orthogonal projection operator* which projects an arbitrary vector onto the plane orthogonal to the direction $\mathbf{n}$.

Solution of a linear system
Let us consider a linear problem of the form

$$\mathbf{Ax} = \mathbf{b} \tag{367}$$

such that

$$\begin{cases} \mathbf{A} \in \mathcal{R}^{m\times n} & \text{with} \quad m > n \\ \text{rank}(\mathbf{A}) = r \leq m \end{cases} \tag{368}$$

The compatibility condition $\mathbf{AA}^+\mathbf{b} = \mathbf{b}$ between $\mathbf{A}$ and its pseudo-inverse $\mathbf{A}^+$ is not necessarily satisfied. The problem has thus generally no solution. This is why we transform it into a *minimum norm problem* like

$$\boxed{\min_{\mathbf{x}} ||\mathbf{Ax} - \mathbf{b}||^2} \tag{369}$$

The resulting advantage is that the transformed problem always has a solution

$$\mathbf{x} = \mathbf{A}^+\mathbf{b} \tag{370}$$

This solution is unique in two cases :

1. matrix $\mathbf{A}$ has full rank ($\text{rank}(\mathbf{A}) = n$) : the pseudo-inverse in then the Moore-Penrose inverse

$$\mathbf{A}^+ = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T \tag{371}$$

2. matrix $\mathbf{A}$ is also square ($m = n = \text{rank}(\mathbf{A})$) :

$$\mathbf{A}^+ = \mathbf{A}^{-1} \tag{372}$$

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