On a mixed finite element formulation involving large rotations for geometrically nonlinear elasticity

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ABSTRACT. *A geometrically nonlinear extension of a mixed finite element formulation based on a complementary energy functional involving rotations and nonsymetric stresses is investigated. The resulting nonlinear strain measure involves an independent rotation tensor and resembles a micropolar approach. As a point of departure, the hybrid mixed functional is derived and the stationarity conditions together with their linearization are given with respect to the reference and the spatial configuration. A possible application of well balanced approximation spaces to the geometrically nonlinear case is discussed.*

RÉSUMÉ. *Nous nous intéressons à une formulation mixte de l'élasticité géométriquement non linéaire, basée sur une fonctionnelle de l'énergie complémentaire, qui fait intervenir les rotations et les contraintes non symétriques. La mesure de déformation non linéaire résultante fait intervenir un tenseur de rotations indépendantes et elle ressemble à l'approche micro-polaire. Comme le point du départ, la fonctionnelle hybride-mixte est proposée, ainsi que ses formes linéarisées en version matérielle et spatiale. Un choix d'espace d'approximation bien balancé est présenté pour le cas géométriquement non linéaire.*

KEY WORDS : *mixed finite element formulation, complementary energy functional, large strains, large rotations.*
MOTS-CLÉS : *formulation mixte des éléments finis, fonctionnelle de l'énergie complémentaire, grandes déformations, grandes rotations.*

1 Introduction

In the classical finite element approach, the displacements are the only unknowns. Unfortunately, for low order approximations this formulation exhibits a poor bending performance and breaks down for incompressible material. A remedy is to employ mixed finite elements which have some other quantities besides the displacements as independent unknowns, e.g. the stresses as primary unknowns with higher continuity requirements imposed than for the displacements. Moreover, mixed finite elements have become an important tool in computational mechanics in order to achieve more accurate stress approximations.

For the *linear elasticity problem* a variety of mixed finite element formulations based on a complementary energy functional has been proposed for which the analysis of *stability* and *convergence* is well understood, see e.g. ARNOLD, BREZZI & DOUGLAS [ABD84], STENBERG [STE88] or STEIN & ROLFES [STR90] for a mechanical interpretation of the abstract mathematical results. Moreover, recent *algorithmic treatments* of the STENBERG [STE88] approach for linear elasticity problems, partially coupled with the boundary element method, and extensive comparisons with other competitive elements are very promising, see BRAESS, KLAAS, NIEKAMP, STEIN & WOBSCHAL [BKW95], BRINK, CARSTENSEN & STEIN [BCS95], BRINK, KLAAS, NIEKAMP & STEIN [BKS95], KLAAS, SCHRÖDER, STEIN & MIEHE [KSS95]. Therefore, the *geometrically nonlinear extension* of the underlying concepts, applied e.g. by STENBERG [STE88], seems very tempting.

As a particular interesting and challenging aspect, the geometrically nonlinear extension of the considered mixed finite element formulation will include besides large strains as well *finite rotations*. Alternative approaches to solid and membrane finite element formulations based on different approximation spaces and involving independent drilling degrees of freedom at finite rotations were advocated by IBRAHIMBEGOVIĆ, TAYLOR & WILSON [ITW90], IBRAHIMBEGOVIĆ [IBR93] and IBRAHIMBEGOVIĆ & FREY [IBF93].

The underlying geometrically linear formulation, which motivates the attempt towards a geometrically nonlinear extension persued in this work, is based on the complementary HELLINGER-REISSNER principle and the achievement of traction continuity by LAGRANGE parameters. It is wellknown that the construction of stable mixed finite element spaces is far from trivial. Early investigations on this topic within the geometrically linear case include the papers by AMARA & THOMAS [ATH79], the PEERS element advocated by ARNOLD, BREZZI & DOUGLAS [ABD84], and the work by BREZZI, DOUGLAS & MARINI [BDM85]. The particular element considered in this contribution is proposed by STENBERG [STE88] and involves three independent fields as unknowns: the stress tensor, which is not *a priori* assumed to be symmetric, the continuum rotation, which acts as a LAGRANGE multiplier to enforce the symmetry of the stress tensor in a weak form, a procedure often credited to FRAEIJS DE VEUBEKE [FDV75], and the displacement vector. Thereby, the stresses are required to satisfy $\bar{\sigma} \in [L_2(\mathcal{B})]^{n_{dim} \times n_{dim}}$ and $\mathrm{div}(\bar{\sigma}) \in [L_2(\mathcal{B})]^{n_{dim}}$ while the rotation vector and the displacements are sought in $\boldsymbol{\omega} \in [L_2(\mathcal{B})]^{n_{dim}}$ and $\boldsymbol{u} \in [L_2(\mathcal{B})]^{n_{dim}}$. Hence, merely the discrete tractions across interelement boundaries are continuous. Finally, to ease the rather cumbersome construction of finite element spaces with continuous interelement tractions, the continuity is enforced by additional LAGRANGE multipliers. This results in discontinuous trial functions across interelement boundaries for all fields except for the last mentioned LAGRANGE multipliers. Thus most element degrees of freedom can be eliminated at the element level before assembling the global system of equations.

The objective of this work is to investigate the possibility of extending the approach of STENBERG [STE88] to the geometrically nonlinear case. In particular it is interesting how the symmetry condition for the stresses is incorporated for the case of large rotations. To this end, we first recall a sequence of steps, starting from the DIRICHLET principle and leading to the desired hybrid mixed functional of the geometrically linear case and its discretization. Subsequently, we follow exactly these steps for the construction of the appropriate geometrically nonlinear functional. Different types of geometrically nonlinear complementary energy functionals have been discussed in the early work by FRAEIJS DE VEUBEKE [FDV72]. For the approach persued in the present work it turns out that the resulting formulation on the one hand very much resembles a *micropolar* approach, compare e.g. STEINMANN [STE94], with only the couple stresses missing. On the other hand, the spatial description of the stationarity condition and its linearization may be recast into a structure almost identical to the geometrically linear case if we introduce the concepts of LIE-type and *nominal* derivatives. Finally, as the main thrust of this work it can be demonstrated that the STENBERG approximations may as well be applied to geometrically nonlinear elasticity problems. Thereby, the unknown fields are the *micropolar* and thus *nonsymmetric* BIOT stress, the *micropolar* and thus *independent* rotation with associated spatial spin acting as a LAGRANGE multiplier to enforce the symmetry of the *micropolar* CAUCHY stress tensor in a weak form and the spatial placement.

2 Geometrically Linear Formulation

To set the stage, we first derive the variational background for the considered mixed element systematically for the geometrically linear case. Clarifying each step that leads to the final formulation will then provide a transparent and sound guideline for the geometrically nonlinear case. Thereby, we assume throughout isotropic *hyperelastic* material behaviour characterized by a, possibly nonlinear, *stored energy function* W or, equivalently, by a *complementary stored energy function* W^*.

Kinematics Let $\mathcal{B} \subset \mathrm{IR}^3$ denote the domain occupied by a material body with particles labeled by $\boldsymbol{x}$. The boundary $\partial\mathcal{B}$ is subdivided into disjoint parts $\partial\mathcal{B} = \partial\mathcal{B}^\sigma \cup \partial\mathcal{B}^u$ and $\partial\mathcal{B}^\sigma \cap \partial\mathcal{B}^u = \emptyset$ with either NEUMANN or DIRICHLET boundary conditions $\boldsymbol{t} = \boldsymbol{t}^p$ or $\boldsymbol{u} = \boldsymbol{u}^p$ given. Deformations are described by the standard displacement field $\boldsymbol{u} : \mathcal{B} \rightarrow \mathrm{IR}^3$ with *continuum* rotation vector $\boldsymbol{\varphi} = \mathrm{axl}(\nabla_x^{skw}\boldsymbol{u}) = \frac{1}{2}\mathrm{crl}(\boldsymbol{u}) : \mathcal{B} \rightarrow \mathrm{IR}^3$. In addition an *independent* rotation field with rotation vector $\boldsymbol{\omega} : \mathcal{B} \rightarrow \mathrm{IR}^3$ is introduced for later use. The corresponding spin tensors are denoted by

$$\boldsymbol{\Phi} = \nabla_x^{skw}\boldsymbol{u} = \mathrm{spn}(\boldsymbol{\varphi}) \quad \text{and} \quad \boldsymbol{\Omega} = \mathrm{spn}(\boldsymbol{\omega}), \tag{1}$$

respectively. Here, spn(•) and axl(•) are defined as

$$\mathrm{spn}(\bullet) = -\overset{3}{e}\cdot(\bullet) \in \mathrm{so}(3) \quad \text{and} \quad \mathrm{axl}(\bullet) = -\frac{1}{2}\overset{3}{e}:(\bullet) \in \mathrm{IR}^3 \tag{2}$$

with $\overset{3}{e}$ the third order RICCI or permutation tensor. Moreover, crl(•) denotes the curl of (•)

$$\mathrm{crl}(\bullet) = -\overset{3}{e}: \nabla_x(\bullet) \in \mathrm{IR}^3. \tag{3}$$

Step 1 Standard displacement based finite element expansions are discretizations of the DIRICHLET principle. Thereby, the functional Π depends merely on the displacement field via the assumed dependence of the stored energy function $W(\epsilon)$ on the symmetric strain measure $\epsilon = \nabla_x^{sym}\boldsymbol{u} = \nabla_x\boldsymbol{u} - \boldsymbol{\Phi}$

$$\Pi(\boldsymbol{u}) = \int_{\mathcal{B}} W(\epsilon)\,\mathrm{d}V - \int_{\partial\mathcal{B}^\sigma} \boldsymbol{u}\cdot\boldsymbol{t}^p\,\mathrm{d}A \rightarrow \min \tag{4}$$

Here, body forces were neglected for the definition of the external potential energy Π^{ext}. In its discretized version this principle typically renders a lower rate of convergence for the stress approximations than for the displacement expansions, which have to satisfy $\boldsymbol{u}^h \in [H^1(\mathcal{B})]^{n_{dim}}$, i.e. $\nabla_x\boldsymbol{u}^h \in [L_2(\mathcal{B})]^{n_{dim}\times n_{dim}}$, thus $\boldsymbol{u}^h$ are required to be continuous.

Step 2 A two field formulation may be obtained by introducing an *independent* symmetric stress field $\boldsymbol{\sigma}$ into the formulation by the means of a LEGENDRE transformation

$$W(\epsilon) = \boldsymbol{\sigma}^t : \epsilon - W^*(\boldsymbol{\sigma}) = \boldsymbol{\sigma}^t : \nabla_x^{sym}\boldsymbol{u} - W^*(\boldsymbol{\sigma}). \tag{5}$$

Incorporating the complementary stored energy function $W^*(\boldsymbol{\sigma})$ into Eq. 4 by this classical construction we end up with the PRANGE-HELLINGER-REISSNER functional Π^{phr}

$$\begin{aligned} \Pi^{phr}(\boldsymbol{u},\boldsymbol{\sigma}) &= \int_{\mathcal{B}} \left[\boldsymbol{\sigma}^t : \nabla_x^{sym}\boldsymbol{u} - W^*(\boldsymbol{\sigma})\right]\,\mathrm{d}V \\ &\quad - \int_{\partial\mathcal{B}^\sigma} \boldsymbol{u}\cdot\boldsymbol{t}^p\,\mathrm{d}A. \end{aligned} \tag{6}$$

Clearly, an equivalent representation of the PRANGE-HELLINGER-REISSNER functional Π^{phr} with reduced regularity requirements for the displacement field $\boldsymbol{u}^h \in [L_2(\mathcal{B})]^{n_{dim}}$ is obtained by partial integration and application of the GAUSS theorem to render

$$\begin{aligned}\Pi^{phr}(\boldsymbol{u}, \boldsymbol{\sigma}) &= \int_{\mathcal{B}} [-\mathrm{div}\boldsymbol{\sigma}^t \cdot \boldsymbol{u} - W^*(\boldsymbol{\sigma})] \,\mathrm{d}V \\ &+ \int_{\partial\mathcal{B}^\sigma} [\boldsymbol{t} - \boldsymbol{t}^p] \cdot \boldsymbol{u} \,\mathrm{d}A + \int_{\partial\mathcal{B}^u} \boldsymbol{t} \cdot \boldsymbol{u}^p \,\mathrm{d}A.\end{aligned} \tag{7}$$

On the other hand, the latter representation imposes the requirement $\boldsymbol{\sigma}^h \in [L_2(\mathcal{B})]^{n_{dim} \times n_{dim}}$ and $\mathrm{div}(\boldsymbol{\sigma}^h) \in [L_2(\mathcal{B})]^{n_{dim}}$ on the discretized stresses abbreviated into $\boldsymbol{\sigma}^h \in [H_{div}(\mathcal{B})]^{n_{dim} \times n_{dim}}$. This is a weaker requirement than $\boldsymbol{\sigma}^h \in [H^1(\mathcal{B})]^{n_{dim} \times n_{dim}}$ but still synonymous with the cumbersome reciprocity condition of continuous discrete tractions $\boldsymbol{t}^h$ across interelement boundaries.

In the sequel, we will continue to consider representations according to Eq. 6 and Eq. 7 in parallel, since this procedure makes the derivations in the geometrically nonlinear case more transparent.

Step 3 As the next step, for a *hybrid method* we introduce a LAGRANGE multiplier $\boldsymbol{\lambda}$ with $\boldsymbol{u} = \boldsymbol{\lambda}$ on $\partial\mathcal{B}^\sigma$ and $\boldsymbol{u}^p = \boldsymbol{\lambda}$ on $\partial\mathcal{B}^u$. Then, on the one hand, the condition $\boldsymbol{u} = \boldsymbol{\lambda}$ is enforced in Eq. 6 via the stress vector $\boldsymbol{t} = \boldsymbol{\sigma}^t \cdot \boldsymbol{n}$ acting as a LAGRANGE multiplier. The resulting hybrid mixed functional Π^{hyb} follows as

$$\begin{aligned}\Pi^{hyb}(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda}) &= \int_{\mathcal{B}} [\boldsymbol{\sigma}^t : \nabla_x^{sym}\boldsymbol{u} - W^*(\boldsymbol{\sigma})] \,\mathrm{d}V \\ &+ \int_{\partial\mathcal{B}^\sigma} [[\boldsymbol{\lambda} - \boldsymbol{u}] \cdot \boldsymbol{t} - \boldsymbol{\lambda} \cdot \boldsymbol{t}^p] \,\mathrm{d}A.\end{aligned} \tag{8}$$

On the other hand, the condition $\boldsymbol{t} = \boldsymbol{t}^p$ is enforced in Eq. 7 via $\boldsymbol{\lambda}$ acting as a LAGRANGE multiplier to give

$$\begin{aligned}\Pi^{hyb}(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda}) &= \int_{\mathcal{B}} [-\mathrm{div}\boldsymbol{\sigma}^t \cdot \boldsymbol{u} - W^*(\boldsymbol{\sigma})] \,\mathrm{d}V \\ &+ \int_{\partial\mathcal{B}^\sigma} [\boldsymbol{t} - \boldsymbol{t}^p] \cdot \boldsymbol{\lambda} \,\mathrm{d}A + \int_{\partial\mathcal{B}^u} \boldsymbol{t} \cdot \boldsymbol{u}^p \,\mathrm{d}A.\end{aligned} \tag{9}$$

Obviously, the latter representation allows for a relaxation of the interelement traction continuity requirement.

Step 4 Finally, the condition $\nabla_x^{sym}\boldsymbol{u} = \nabla_x\boldsymbol{u} - \boldsymbol{\Phi}$ is relaxed by introducing the independent rotation $\boldsymbol{\Omega} \in \mathrm{so}(3)$. Then, on the one hand, condition $\boldsymbol{\Phi} = \boldsymbol{\Omega}$ is enforced in Eq. 8 by skewsymmetric stresses $\bar{\boldsymbol{\sigma}}^{skw} \in \mathrm{so}(3)$ which act as a LAGRANGE multiplier. The resulting functional Π^* is then expressed in terms of the sum of $\boldsymbol{\sigma}$ and $\bar{\boldsymbol{\sigma}}^{skw}$, i.e. the nonsymmetric stress $\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \bar{\boldsymbol{\sigma}}^{skw}$

$$\begin{aligned}\Pi^*(\boldsymbol{u}, \bar{\boldsymbol{\sigma}}, \boldsymbol{\lambda}, \boldsymbol{\omega}) &= \int_{\mathcal{B}} \left[\bar{\boldsymbol{\sigma}}^t : [\nabla_x \boldsymbol{u} - \boldsymbol{\Omega}] - W^*(\boldsymbol{\sigma})\right] \mathrm{d}V \\ &+ \int_{\partial\mathcal{B}^\sigma} [[\boldsymbol{\lambda} - \boldsymbol{u}] \cdot \boldsymbol{t} - \boldsymbol{\lambda} \cdot \boldsymbol{t}^p] \, \mathrm{d}A.\end{aligned} \tag{10}$$

In fact, the variational point of departure for the geometrically linear mixed element considered in this work is obtained by partial integration and application of the GAUSS theorem to Eq. 10. Recall, that this allows to lower the regularity requirements for the displacement field $\boldsymbol{u}^h \in [L_2(\mathcal{B})]^{n_{dim}}$ and to increase these requirements for the stress field $\bar{\boldsymbol{\sigma}}^h \in [H_{div}(\mathcal{B})]^{n_{dim} \times n_{dim}}$ with interelement continuity of the tractions $\boldsymbol{t}^h$ enforced by the multiplier $\boldsymbol{\lambda}^h$. Moreover, the introduction of nonsymmetric stresses opens the door to a slightly modified application of the approaches to scalar valued second order elliptic problems advocated by RAVIART & THOMAS [RTH77] and BREZZI, DOUGLAS & MARINI [BDM85]. Thereby, the extended Eq. 9 with BOLTZMANN condition $\bar{\boldsymbol{\sigma}}^{skw} = \mathbf{o}$ enforced by $\boldsymbol{\Omega}$ results in the functional Π^*

$$\begin{aligned}\Pi^*(\boldsymbol{u}, \bar{\boldsymbol{\sigma}}, \boldsymbol{\lambda}, \boldsymbol{\omega}) &= \int_{\mathcal{B}} \left[-\mathrm{div}\bar{\boldsymbol{\sigma}}^t \cdot \boldsymbol{u} - \bar{\boldsymbol{\sigma}}^t : \boldsymbol{\Omega} - W^*(\boldsymbol{\sigma})\right] \mathrm{d}V \\ &+ \int_{\partial\mathcal{B}^\sigma} [\boldsymbol{t} - \boldsymbol{t}^p] \cdot \boldsymbol{\lambda} \, \mathrm{d}A + \int_{\partial\mathcal{B}^u} \boldsymbol{t} \cdot \boldsymbol{u}^p \, \mathrm{d}A.\end{aligned} \tag{11}$$

The latter representation allows for an optimal order of convergence for both, the stress and the displacement approximations.

Stationarity Condition Variation of the functional Π^* renders as EULER-LAGRANGE equations (i) the balance of linear momentum, (ii) the complementary constitutive law for the strains, (iii) the hybrid regularization on $\partial\mathcal{B}^\sigma$, (iv) the NEUMANN boundary conditions and (v) the BOLTZMANN condition of vanishing skewsymmetric stresses. Due to the last condition, which is the result of Step 4, essentially both, the balance of linear momentum as well as the balance of angular momentum enter the variational principle in a weak form.

Based on the representation in Eq. 10 the weak form of this set of equations is given by

$$\begin{aligned}\delta\Pi^* &= \int_{\mathcal{B}} \nabla_x \delta\boldsymbol{u} : \bar{\boldsymbol{\sigma}}^t \, \mathrm{d}V - \int_{\partial\mathcal{B}^\sigma} \delta\boldsymbol{u} \cdot \boldsymbol{t} \, \mathrm{d}A \\ &+ \int_{\mathcal{B}} \delta\bar{\boldsymbol{\sigma}} : \left[\nabla_x^t \boldsymbol{u} - \boldsymbol{\Omega}^t - \partial_{\boldsymbol{\sigma}} W^*\right] \mathrm{d}V \\ &+ \int_{\partial\mathcal{B}^\sigma} \delta\boldsymbol{t} \cdot [\boldsymbol{\lambda} - \boldsymbol{u}] \, \mathrm{d}A \\ &+ \int_{\partial\mathcal{B}^\sigma} \delta\boldsymbol{\lambda} \cdot [\boldsymbol{t} - \boldsymbol{t}^p] \, \mathrm{d}A \\ &- \int_{\mathcal{B}} \delta\boldsymbol{\Omega} : \bar{\boldsymbol{\sigma}}^t \, \mathrm{d}V.\end{aligned} \tag{12}$$

Accordingly, based on the representation in Eq. 11 the weak form of this set of equations transforms into

$$\begin{aligned}\delta\Pi^* = & - \int_{\mathcal{B}} \delta\boldsymbol{u} \cdot \mathrm{div}\bar{\boldsymbol{\sigma}}^t \,\mathrm{d}V \\ & - \int_{\mathcal{B}} \left[\mathrm{div}\delta\bar{\boldsymbol{\sigma}}^t \cdot \boldsymbol{u} + \delta\bar{\boldsymbol{\sigma}} : \left[\boldsymbol{\Omega}^t + \partial_{\boldsymbol{\sigma}} W^*\right]\right] \mathrm{d}V \\ & + \int_{\partial\mathcal{B}^\sigma} \delta\boldsymbol{t} \cdot \boldsymbol{\lambda} \,\mathrm{d}A + \int_{\partial\mathcal{B}^u} \delta\boldsymbol{t} \cdot \boldsymbol{u}^p \,\mathrm{d}A \\ & + \int_{\partial\mathcal{B}^\sigma} \delta\boldsymbol{\lambda} \cdot [\boldsymbol{t} - \boldsymbol{t}_p] \,\mathrm{d}A \\ & - \int_{\mathcal{B}} \delta\boldsymbol{\Omega} : \bar{\boldsymbol{\sigma}}^t \,\mathrm{d}V. \end{aligned} \tag{13}$$

Linearization The second variation renders the linearization of the weak form or rather the HESSE matrix of the mixed hybrid principle. For a possible nonlinear dependence of W^* on $\boldsymbol{\sigma}$ the proper evaluation of the HESSE matrix leads to the optimal convergence within an iterative NEWTON solution scheme for the discretized system of equations.

Thereby, on the one hand linearization of Eq. 12 results in

$$\begin{aligned}\Delta\delta\Pi^* = & \int_{\mathcal{B}} [\nabla_x \delta\boldsymbol{u} - \delta\boldsymbol{\Omega}] : \Delta\bar{\boldsymbol{\sigma}}^t \,\mathrm{d}V \\ & + \int_{\mathcal{B}} [\nabla_x \Delta\boldsymbol{u} - \Delta\boldsymbol{\Omega}] : \delta\bar{\boldsymbol{\sigma}}^t \,\mathrm{d}V \\ & - \int_{\mathcal{B}} \delta\bar{\boldsymbol{\sigma}} : \partial^2_{\boldsymbol{\sigma}\boldsymbol{\sigma}} W^* : \Delta\bar{\boldsymbol{\sigma}} \,\mathrm{d}V \\ & + \int_{\partial\mathcal{B}^\sigma} \delta\boldsymbol{t} \cdot [\Delta\boldsymbol{\lambda} - \Delta\boldsymbol{u}] \,\mathrm{d}A \\ & + \int_{\partial\mathcal{B}^\sigma} \Delta\boldsymbol{t} \cdot [\delta\boldsymbol{\lambda} - \delta\boldsymbol{u}] \,\mathrm{d}A. \end{aligned} \tag{14}$$

Equivalently, on the other hand linearization of Eq. 13 renders

$$\begin{aligned}\Delta\delta\Pi^* = & - \int_{\mathcal{B}} \left[\delta\boldsymbol{u} \cdot \mathrm{div}\Delta\bar{\boldsymbol{\sigma}}^t + \delta\boldsymbol{\Omega} : \Delta\bar{\boldsymbol{\sigma}}^t\right] \mathrm{d}V \\ & - \int_{\mathcal{B}} \left[\Delta\boldsymbol{u} \cdot \mathrm{div}\delta\bar{\boldsymbol{\sigma}}^t + \Delta\boldsymbol{\Omega} : \delta\bar{\boldsymbol{\sigma}}^t\right] \mathrm{d}V \\ & - \int_{\mathcal{B}} \left[\delta\bar{\boldsymbol{\sigma}} : \partial^2_{\boldsymbol{\sigma}\boldsymbol{\sigma}} W^* : \Delta\bar{\boldsymbol{\sigma}}\right] \mathrm{d}V \\ & + \int_{\partial\mathcal{B}^\sigma} [\delta\boldsymbol{\lambda} \cdot \Delta\boldsymbol{t} + \Delta\boldsymbol{\lambda} \cdot \delta\boldsymbol{t}] \,\mathrm{d}A. \end{aligned} \tag{15}$$

In the mathematical literature on the considered mixed formulation Eq. 15 constitutes the variational point of departure within the framework of linear elasticity.

3 Discretization

In the sequel we review for simplicity a possible *plane* triangle discretization based on a modification of the lowest order BREZZI, DOUGLAS & MARINI [BDM85] spaces which has recently become quite popular, see STENBERG [STE88]. All expansions will be parametrized in triangular or rather barycentric coordinates λ_k, $k = 1, 2, 3$. Then shape functions for linear polynomials and the bubble function on a triangle follow as

$$N^k = \lambda_k \quad \text{for} \quad k = 1, 2, 3 \quad \text{and} \quad N^4 = \lambda_1 \lambda_2 \lambda_3. \tag{16}$$

For a triangle with straight edges the geometry is parametrized in terms of the corner coordinates $\boldsymbol{x}_k$

$$\boldsymbol{x}^h = \sum_{k=1}^{3} N^k \boldsymbol{x}_k. \tag{17}$$

Note that a linearized rigid body mode $\boldsymbol{u}^{rbm}$ with $\nabla_x^{sym}(\boldsymbol{u}^{rbm}) = \mathbf{o}$ has the representation

$$\boldsymbol{u}^{rbm} = \boldsymbol{c} + \text{spn}(\theta) \cdot \sum_{k=1}^{3} N^k \boldsymbol{x}_k \quad \text{with} \quad \text{spn}(\theta) = \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{18}$$

In addition, the curl in $\mathbb{R}^2$ of the scalar bubble function is defined as

$$\text{crl}(N^4) = \begin{bmatrix} N^4_{,x_2} \\ -N^4_{,x_1} \end{bmatrix} \in \mathbb{R}^2. \tag{19}$$

With these preliminaries at hand the following elementwise approximation for the nonsymmetric stress tensor $\bar{\boldsymbol{\sigma}}$ is introduced

$$\bar{\boldsymbol{\sigma}}^{h,t} = \sum_{k=1}^{3} N^k \bar{\boldsymbol{\sigma}}_k + N^4 \text{spn}(\bar{\boldsymbol{\sigma}}_4) + \bar{\boldsymbol{\sigma}}_5 \otimes \text{crl}(N^4). \tag{20}$$

Thereby, the stress is parametrized in terms of the 15 unknowns

$$\bar{\boldsymbol{\sigma}}_k = \begin{bmatrix} \sigma_k^1 & \sigma_k^2 \\ \sigma_k^3 & \sigma_k^4 \end{bmatrix}, \quad \text{spn}(\bar{\boldsymbol{\sigma}}_4) = \sigma_4 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \bar{\boldsymbol{\sigma}}_5 = \begin{bmatrix} \sigma_5^1 \\ \sigma_5^2 \end{bmatrix}. \tag{21}$$

Since by construction div $\text{crl}(N^4) = 0$, the divergence of the stress expansion is here given by

$$\text{div}\bar{\boldsymbol{\sigma}}^{h,t} = \sum_{k=1}^{3} \bar{\boldsymbol{\sigma}}_k \cdot \nabla_x N^k + \text{spn}(\bar{\boldsymbol{\sigma}}_4) \cdot \nabla_x N^4. \tag{22}$$

Moreover, it is easily verified that the discretized stress vector $\boldsymbol{t}^h = \bar{\boldsymbol{\sigma}}^{h,t} \cdot \boldsymbol{n}$ is a linear polynomial and follows at e.g. the edge opposite to the corner k in terms of the unknowns contained in the $\bar{\boldsymbol{\sigma}}_k \cdot \boldsymbol{n}$

$$\boldsymbol{t}_k^h = N^i \bar{\boldsymbol{\sigma}}_i \cdot \boldsymbol{n} + N^j \bar{\boldsymbol{\sigma}}_j \cdot \boldsymbol{n} \quad \text{with} \quad i, j, k \quad \text{cyclic permutation.} \tag{23}$$

For the displacements the minimum possible expansion, i.e. the linearized rigid body modes

$$\boldsymbol{u}^h = \boldsymbol{u}_1 + \operatorname{spn}(\boldsymbol{u}_2) \cdot \sum_{k=1}^{3} N^k \boldsymbol{x}_k \tag{24}$$

is elementwise chosen in terms of the 3 unknowns contained in

$$\boldsymbol{u}_1 = \begin{bmatrix} u_1^1 \\ u_1^2 \end{bmatrix} \quad \text{and} \quad \operatorname{spn}(\boldsymbol{u}_2) = u_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{25}$$

The axial scalar ω of the independent rotation $\boldsymbol{\Omega} \in \mathrm{so}(2)$ is elementwise approximated by a linear polynomial with 3 unknowns

$$\omega^h = \sum_{k=1}^{3} N^k \omega_k \quad \rightsquigarrow \quad \boldsymbol{\Omega}^h = \sum_{k=1}^{3} N^k \omega_k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{26}$$

Finally, as the only continuous quantity, the boundary or edge displacements $\boldsymbol{\lambda}$ are chosen for e.g. the edge opposite to node k as linear polynomials with 4 unknowns

$$\boldsymbol{\lambda}_k^h = N^i \boldsymbol{\lambda}_i + N^j \boldsymbol{\lambda}_j \quad \text{with} \quad i, j, k \quad \text{cyclic permutation.} \tag{27}$$

4 Geometrically Nonlinear Extension

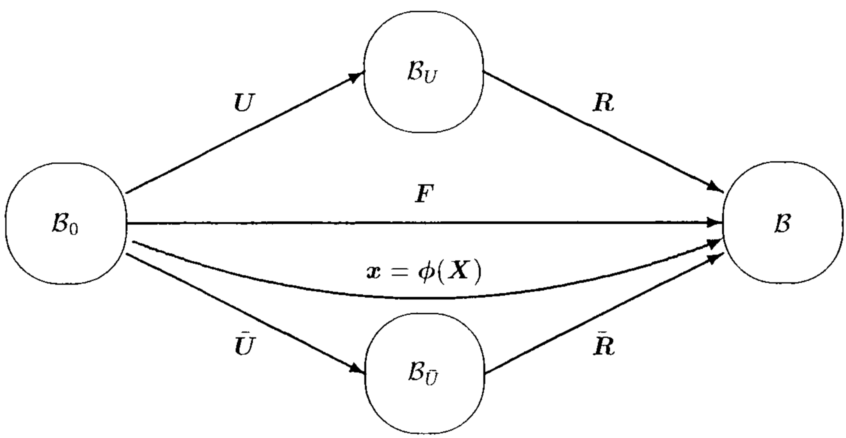

Figure 1: Nonlinear Deformation Map & Multiplicative Decompositions of the Deformation Gradient

In this section we extend the underlying variational formulation for the considered mixed element to the geometrically nonlinear case. At first glance, this appears to be no trivial task. Therefore, we follow the same sequence of steps as in the geometrically linear case in order to arrive in the most transparent way at the desired hybrid mixed functional Π^*. To this end, we first recall briefly the essential kinematics of a geometrically nonlinear continuum description.

Kinematics Let $\mathcal{B}_0 \subset \mathbb{R}^3$ denote the domain occupied by a material body with particles labeled by $\boldsymbol{X}$ in the reference configuration. Thereby, the boundary $\partial\mathcal{B}_0$ is subdivided into disjoint parts $\partial\mathcal{B}_0 = \partial\mathcal{B}_0^\sigma \cup \partial\mathcal{B}_0^u$ and $\partial\mathcal{B}_0^\sigma \cap \partial\mathcal{B}_0^u = \emptyset$ with either NEUMANN or DIRICHLET boundary conditions $\boldsymbol{T} = \boldsymbol{T}^p$ or $\boldsymbol{x} = \boldsymbol{x}^p$ given. Deformations are described by the standard nonlinear map $\boldsymbol{\phi}(\boldsymbol{X}) : \mathcal{B}_0 \longrightarrow \mathcal{B}$ taking particles labeled by their position $\boldsymbol{X}$ in the reference configuration $\mathcal{B}_0$ to their placement $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{X})$ in the spatial configuration $\mathcal{B}$. Then the associated linear tangent map follows as the deformation gradient $\boldsymbol{F} = \nabla_X \boldsymbol{\phi} : T\mathcal{B}_0 \rightarrow T\mathcal{B}$ with $\det \boldsymbol{F} > 0$ and *continuum* rotation $\boldsymbol{R} \in \mathrm{SO}(3)$. In addition an *independent* rotation field with rotation tensor $\bar{\boldsymbol{R}} \in \mathrm{SO}(3)$ is introduced for later use, see Fig. 1. Variation of these rotation tensors defines the corresponding spatial spin tensors

$$\boldsymbol{\Phi}_\delta = \delta\boldsymbol{R} \cdot \boldsymbol{R}^t = \mathrm{spn}(\boldsymbol{\varphi}_\delta) \quad \text{and} \quad \boldsymbol{\Omega}_\delta = \delta\bar{\boldsymbol{R}} \cdot \bar{\boldsymbol{R}}^t = \mathrm{spn}(\boldsymbol{\omega}_\delta). \tag{28}$$

Next, we define the *symmetric* right *continuum* stretch tensor $\boldsymbol{U}$ with $\det \boldsymbol{U} > 0$ as the primary strain measures and for later use the *nonsymmetric* right *micropolar* stretch tensor $\bar{\boldsymbol{U}}$ with $\det \bar{\boldsymbol{U}} > 0$

$$\boldsymbol{U} = \boldsymbol{R}^t \cdot \boldsymbol{F} \in \mathrm{S}_+(3) \quad \text{and} \quad \bar{\boldsymbol{U}} = \bar{\boldsymbol{R}}^t \cdot \boldsymbol{F} \in \mathrm{GL}_+(3). \tag{29}$$

Based on these definitions we will extend the results obtained for the geometrically linear case to the geometrically nonlinear case involving large strains and large rotations. Stress measures defined on the different configurations follow from the choice of the corresponding conjugate strain measures and will be introduced when the need arises.

Step 1 To start with, the DIRICHLET principle is given in accordance with the principle of objectivity with the stored energy function $W(\boldsymbol{U})$ depending on the *symmetric* right stretch tensor $\boldsymbol{U} = \sqrt{\boldsymbol{F}^t \cdot \boldsymbol{F}}$. Then the functional Π merely depends on the placement $\boldsymbol{x}$

$$\Pi(\boldsymbol{x}) = \int_{\mathcal{B}_0} W(\boldsymbol{U})\, \mathrm{d}V - \int_{\partial\mathcal{B}_0^\sigma} \boldsymbol{x} \cdot \boldsymbol{T}^p \, \mathrm{d}A. \tag{30}$$

Here, body forces were neglected and deadloading surface tractions were assumed for the definition of the external contribution Π^{ext} to the functional Π. The choice of $\boldsymbol{U}$ as argument of the stored energy function appears to be arbitrary, e.g. the right CAUCHY-GREEN tensor $\boldsymbol{C} = \boldsymbol{U}^2$ would serve the same purpose. Nevertheless, the choice of the right stretch tensor enables the most natural incorporation of an *independent* rotation field into the variational formulation, as will be seen later in Step 4.

Step 2 In analogy to the procedure for the geometrically linear case an independent *symmetric* BIOT stress field $\boldsymbol{\Sigma}_b$ is introduced into the formulation by the means of a LEGENDRE transformation

$$W(\boldsymbol{U}) = \boldsymbol{\Sigma}_b^t : \boldsymbol{U} - W^*(\boldsymbol{\Sigma}_b) = \left[\boldsymbol{R}\cdot\boldsymbol{\Sigma}_b^t\right] : \boldsymbol{F} - W^*(\boldsymbol{\Sigma}_b). \quad (31)$$

Incorporating $W^*(\boldsymbol{\Sigma}_b)$ by the LEGENDRE transformation in Eq. 31 into Eq. 30 we end up with the geometrically nonlinear PRANGE-HELLINGER-REISSNER functional Π^{phr}

$$\Pi^{phr}(\boldsymbol{x},\boldsymbol{\Sigma}_b) = \int_{\mathcal{B}_0} \left[\boldsymbol{\Sigma}_b^t : \boldsymbol{U} - W^*(\boldsymbol{\Sigma}_b)\right] \mathrm{d}V - \int_{\partial\mathcal{B}_0^\sigma} \boldsymbol{x}\cdot\boldsymbol{T}^p \,\mathrm{d}A. \quad (32)$$

Integration by parts and application of the GAUSS theorem together with the introduction of the stress vector $\boldsymbol{T} = [\boldsymbol{R}\cdot\boldsymbol{\Sigma}_b^t]\cdot\boldsymbol{N}$ refering to $\partial\mathcal{B}_0$, where the *continuum* rotation $\boldsymbol{R}$ follows from the polar decomposition of $\boldsymbol{F}$, allows for the equivalent representation

$$\begin{aligned}\Pi^{phr}(\boldsymbol{x},\boldsymbol{\Sigma}_b) &= \int_{\mathcal{B}_0} \left[-\mathrm{Div}(\boldsymbol{R}\cdot\boldsymbol{\Sigma}_b^t)\cdot\boldsymbol{x} - W^*(\boldsymbol{\Sigma}_b)\right] \mathrm{d}V \\ &+ \int_{\partial\mathcal{B}_0^\sigma} [\boldsymbol{T}-\boldsymbol{T}^p]\cdot\boldsymbol{x}\,\mathrm{d}A + \int_{\partial\mathcal{B}_0^u} \boldsymbol{T}\cdot\boldsymbol{x}^p\,\mathrm{d}A.\end{aligned} \quad (33)$$

Remark Before we proceed some words on the complementary stored energy function $W^*(\boldsymbol{\Sigma}_b)$ in Eq. 31 are in order. Firstly, for isotropic *hyperelasticity* we conclude from Eq. 31 that $\boldsymbol{U}$ and $\boldsymbol{\Sigma}_b$ are related by

$$\boldsymbol{\Sigma}_b^t = \partial_{\boldsymbol{U}} W(\boldsymbol{U}) \quad \text{and} \quad \boldsymbol{U}^t = \partial_{\boldsymbol{\Sigma}_b} W^*(\boldsymbol{\Sigma}_b). \quad (34)$$

Here, we tolerate the notational redundancy of appending the transpose $(\bullet)^t$ to a symmetric tensor as an anticipation of the later derivations.

Moreover, it is pointed out in OGDEN [OGD84] that it is always possible to define *uniquely* a complementary stored energy function $W^*(\boldsymbol{\Sigma}_b)$ under satisfaction of certain physically motivated conditions. This is in contrast to the case of a complementary stored energy function which is formulated in terms of the 1. PIOLA-KIRCHHOFF stress tensor $W^*(\boldsymbol{\Sigma}_1)$ with $\boldsymbol{\Sigma}_1^t = \boldsymbol{R}\cdot\boldsymbol{\Sigma}_b^t$ since the inversion of the constitutive relation $\boldsymbol{\Sigma}_1^t = \partial_{\boldsymbol{F}} W(\boldsymbol{F})$ is not uniquely defined.

Example A prominent and simple example for a stored energy function in the spirit of the ST. VENANT material is provided by

$$W(\boldsymbol{U}) = \frac{1}{2}\lambda\left[\boldsymbol{U}:\boldsymbol{I}-3\right]^2 + \mu\left[\boldsymbol{U}^2:\boldsymbol{I} - 2\boldsymbol{U}:\boldsymbol{I}+3\right]. \quad (35)$$

The constitutive law for the BIOT stress follows as

$$\boldsymbol{\Sigma}_b^t = \lambda\left[\boldsymbol{U}:\boldsymbol{I}-3\right]\boldsymbol{I} + 2\mu\left[\boldsymbol{U}-\boldsymbol{I}\right] \quad (36)$$

and is easily invertible to render

$$\boldsymbol{U}^t = \frac{1}{2\mu}\boldsymbol{\Sigma}_b - \frac{\lambda}{2\mu[2\mu+3\lambda]}[\boldsymbol{\Sigma}_b : \boldsymbol{I}]\,\boldsymbol{I} + \boldsymbol{I}. \tag{37}$$

Obviously, the corresponding *complementary* stored energy function follows as

$$W^*(\boldsymbol{\Sigma}_b) = \frac{1}{4\mu}\boldsymbol{\Sigma}_b^2 : \boldsymbol{I} - \frac{\lambda}{4\mu[2\mu+3\lambda]}[\boldsymbol{\Sigma}_b : \boldsymbol{I}]^2 + \boldsymbol{\Sigma}_b : \boldsymbol{I}. \tag{38}$$

Step 3 Next, we introduce a hybrid regularization with $\boldsymbol{x} = \boldsymbol{\lambda}$ on the boundary $\partial\mathcal{B}_0^\sigma$. On the one hand, the condition $\boldsymbol{x} = \boldsymbol{\lambda}$ is enforced in Eq. 32 via the stress vector $\boldsymbol{T}$ acting as a LAGRANGE multiplier to render the hybrid mixed functional Π^{hyb}

$$\begin{aligned}\Pi^{hyb}(\boldsymbol{x}, \boldsymbol{\Sigma}_b, \boldsymbol{\lambda}) &= \int_{\mathcal{B}_0} \left[\boldsymbol{\Sigma}_b^t : \boldsymbol{U} - W^*(\boldsymbol{\Sigma}_b)\right] \mathrm{d}V \\ &+ \int_{\partial\mathcal{B}_0^\sigma} [[\boldsymbol{\lambda} - \boldsymbol{x}]\cdot\boldsymbol{T} - \boldsymbol{\lambda}\cdot\boldsymbol{T}^p]\, \mathrm{d}A.\end{aligned} \tag{39}$$

Equivalently, on the other hand, the condition $\boldsymbol{T} = \boldsymbol{T}^p$ on $\partial\mathcal{B}_0^\sigma$ is enforced in Eq. 33 via the LAGRANGE multiplier $\boldsymbol{\lambda}$ to yield

$$\begin{aligned}\Pi^{hyb}(\boldsymbol{x}, \boldsymbol{\Sigma}_b, \boldsymbol{\lambda}) &= \int_{\mathcal{B}_0} \left[-\mathrm{Div}(\boldsymbol{R}\cdot\boldsymbol{\Sigma}_b^t)\cdot\boldsymbol{x} - W^*(\boldsymbol{\Sigma}_b)\right] \mathrm{d}V \\ &+ \int_{\partial\mathcal{B}_0^\sigma} [\boldsymbol{T} - \boldsymbol{T}^p]\cdot\boldsymbol{\lambda}\,\mathrm{d}A + \int_{\partial\mathcal{B}_0^u} \boldsymbol{T}\cdot\boldsymbol{x}^p\,\mathrm{d}A.\end{aligned} \tag{40}$$

Step 4 Finally, the condition $\boldsymbol{U} = \boldsymbol{R}^t \cdot \boldsymbol{F}$ is relaxed by introducing the independent rotation field $\bar{\boldsymbol{R}} \in \mathrm{SO}(3)$ with associated spatial spin $\boldsymbol{\Omega}_\delta \in \mathrm{so}(3)$. To this end, the nonsymmetric stretch $\bar{\boldsymbol{U}} = \bar{\boldsymbol{R}}^t \cdot \boldsymbol{F}$ is introduced and the condition $[\bar{\boldsymbol{U}}]^{skw} = \mathbf{o}$ is enforced in Eq. 39 by skewsymmetric BIOT like stresses $\bar{\boldsymbol{\Sigma}}_b^{skw} \in \mathrm{so}(3)$ which act as a LAGRANGE multiplier. The resulting functional Π^* is then expressed in terms of the sum of $\boldsymbol{\Sigma}_b$ and $\bar{\boldsymbol{\Sigma}}_b^{skw}$, i.e. the nonsymmetric BIOT like stress $\bar{\boldsymbol{\Sigma}}_b = \boldsymbol{\Sigma}_b + \bar{\boldsymbol{\Sigma}}_b^{skw}$

$$\begin{aligned}\Pi^*(\boldsymbol{x}, \bar{\boldsymbol{\Sigma}}_b, \boldsymbol{\lambda}, \bar{\boldsymbol{R}}) &= \int_{\mathcal{B}_0} \left[\bar{\boldsymbol{\Sigma}}_b^t : \bar{\boldsymbol{U}} - W^*(\boldsymbol{\Sigma}_b)\right] \mathrm{d}V \\ &+ \int_{\partial\mathcal{B}_0^\sigma} \left[[\boldsymbol{\lambda} - \boldsymbol{x}]\cdot\bar{\boldsymbol{T}} - \boldsymbol{\lambda}\cdot\boldsymbol{T}^p\right] \mathrm{d}A.\end{aligned} \tag{41}$$

Here the stress vector $\bar{\boldsymbol{T}} = \bar{\boldsymbol{\Sigma}}_1^t \cdot \boldsymbol{N}$ is formally defined with the *micropolar*-type 1.PIOLA-KIRCHHOFF stress tensor $\bar{\boldsymbol{\Sigma}}_1^t = \bar{\boldsymbol{R}} \cdot \bar{\boldsymbol{\Sigma}}_b^t$.

Since the point of departure for the geometrically linear mixed element of the STENBERG type was Eq. 11, we extend Eq. 40 to take into account the *independent* $\bar{\boldsymbol{R}}$ and the *nonsymmetric* $\bar{\boldsymbol{\Sigma}}_b$ which implicitly enforces the symmetry condition $[\boldsymbol{F}\cdot\boldsymbol{\Sigma}_b\cdot\bar{\boldsymbol{R}}^t]^{skw}=\mathbf{0}$ on the spatial *micropolar* KIRCHHOFF stress

$$\Pi^*(\boldsymbol{x},\bar{\boldsymbol{\Sigma}}_b,\boldsymbol{\lambda},\bar{\boldsymbol{R}}) = \int_{\mathcal{B}_0}\left[-\mathrm{Div}(\bar{\boldsymbol{R}}\cdot\bar{\boldsymbol{\Sigma}}_b^t)\cdot\boldsymbol{x}-W^*(\boldsymbol{\Sigma}_b)\right]\mathrm{d}V + \int_{\partial\mathcal{B}_0^\sigma}\left[\bar{\boldsymbol{T}}-\boldsymbol{T}^p\right]\cdot\boldsymbol{\lambda}\,\mathrm{d}A+\int_{\partial\mathcal{B}_0^u}\boldsymbol{T}\cdot\boldsymbol{x}^p\,\mathrm{d}A. \tag{42}$$

Please observe that this formulation demands the product of the *independent* rotation $\bar{\boldsymbol{R}}^h$ and the stress $\bar{\boldsymbol{\Sigma}}_b^{t,h}$ to satisfy $[\bar{\boldsymbol{R}}\cdot\bar{\boldsymbol{\Sigma}}_b^t]^h\in[H_{div}(\mathcal{B}_0)]^{n_{dim}\times n_{dim}}$ which is in contrast to the geometrically linear case where only the stress is required to fullfill $\bar{\boldsymbol{\sigma}}^h\in[H_{div}(\mathcal{B})]^{n_{dim}\times n_{dim}}$. Nevertheless, the relaxation of the traction continuity requirement via the multiplier $\boldsymbol{\lambda}$ refers as well to $\bar{\boldsymbol{T}}=[\bar{\boldsymbol{R}}\cdot\bar{\boldsymbol{\Sigma}}_b^t]\cdot\boldsymbol{N}$ and thus allows the expansions for both, $\bar{\boldsymbol{R}}^h$ and $\bar{\boldsymbol{\Sigma}}_b^h$ to be discontinuous along interelement boundaries.

Stationarity Condition in $\mathcal{B}_0$ Variation of the hybrid mixed functional Π^* renders as EULER-LAGRANGE equations (i) the balance of linear momentum, (ii) the complementary constitutive law for the strains, (iii) the hybrid regularization on $\partial\mathcal{B}_0^\sigma$, (iv) the NEUMANN boundary conditions and (v) the BOLTZMANN condition of vanishing skewsymmetric stresses as manifestation of the balance of angular momentum.

The stationarity condition for the functional in Eq. 41 is expressed with respect to the reference configuration

$$\begin{aligned}\delta\Pi^* &= \int_{\mathcal{B}_0}\nabla_X\delta\boldsymbol{x}:\left[\bar{\boldsymbol{R}}\cdot\bar{\boldsymbol{\Sigma}}_b^t\right]\mathrm{d}V-\int_{\partial\mathcal{B}_0^\sigma}\delta\boldsymbol{x}\cdot\bar{\boldsymbol{T}}\,\mathrm{d}A\\ &+ \int_{\mathcal{B}_0}\delta\bar{\boldsymbol{\Sigma}}_b:\left[\bar{\boldsymbol{U}}^t-\partial_{\boldsymbol{\Sigma}_b}W^*\right]\mathrm{d}V\\ &+ \int_{\partial\mathcal{B}_0^\sigma}\delta\bar{\boldsymbol{T}}\cdot[\boldsymbol{\lambda}-\boldsymbol{x}]\,\mathrm{d}A\\ &+ \int_{\partial\mathcal{B}_0^\sigma}\delta\boldsymbol{\lambda}\cdot\left[\bar{\boldsymbol{T}}-\boldsymbol{T}^p\right]\mathrm{d}A\\ &- \int_{\mathcal{B}_0}\boldsymbol{\Omega}_\delta:\left[\bar{\boldsymbol{R}}\cdot\bar{\boldsymbol{\Sigma}}_b^t\cdot\boldsymbol{F}^t\right]\mathrm{d}V.\end{aligned} \tag{43}$$

Taking into account the definition of the *micropolar*-type 1.PIOLA-KIRCHHOFF stress tensor $\bar{\boldsymbol{\Sigma}}_1^t=\bar{\boldsymbol{R}}\cdot\bar{\boldsymbol{\Sigma}}_b^t$, the variation of the stress vector $\bar{\boldsymbol{T}}$ is abbreviated into $\delta\bar{\boldsymbol{T}}=\delta\boldsymbol{\Sigma}_1^t\cdot\boldsymbol{N}$ with

$$\delta\boldsymbol{\Sigma}_1^t=\boldsymbol{\Omega}_\delta\cdot\bar{\boldsymbol{\Sigma}}_1^t+\bar{\boldsymbol{R}}\cdot\delta\bar{\boldsymbol{\Sigma}}_b^t. \tag{44}$$

Likewise, the third and last term in Eq. 43 might be combined into

$$\delta\bar{\boldsymbol{\Sigma}}_b^t:\left[\bar{\boldsymbol{R}}^t\cdot\boldsymbol{F}\right]+\boldsymbol{\Omega}_\delta:\left[\boldsymbol{F}\cdot\bar{\boldsymbol{\Sigma}}_1\right]\Rightarrow\delta\boldsymbol{\Sigma}_1^t:\boldsymbol{F}. \tag{45}$$

Alternatively, as the variational point of departure for the geometrically nonlinear mixed element, partial integration and application of the GAUSS theorem renders the following representation for the EULER-LAGRANGE equations of the functional in Eq. 42

$$
\begin{aligned}
\delta\Pi^* \;=\; & -\int_{\mathcal{B}_0} \delta\boldsymbol{x} \cdot \mathrm{Div}(\bar{\boldsymbol{R}} \cdot \bar{\boldsymbol{\Sigma}}_b^t)\,\mathrm{d}V \\
& -\int_{\mathcal{B}_0} \left[\mathrm{Div}(\delta\bar{\boldsymbol{\Sigma}}_1^t) \cdot \boldsymbol{x} + \delta\bar{\boldsymbol{\Sigma}}_b : \partial_{\boldsymbol{\Sigma}_b} W^*\right] \mathrm{d}V \\
& + \int_{\partial\mathcal{B}_0^u} \delta\bar{\boldsymbol{T}} \cdot \boldsymbol{x}^p \,\mathrm{d}A + \int_{\partial\mathcal{B}_0^\sigma} \delta\bar{\boldsymbol{T}} \cdot \boldsymbol{\lambda}\,\mathrm{d}A \\
& + \int_{\partial\mathcal{B}_0^\sigma} \delta\boldsymbol{\lambda} \cdot \left[\bar{\boldsymbol{T}} - \boldsymbol{T}^p\right] \mathrm{d}A
\end{aligned}
\qquad (46)
$$

It is interesting to note that in this representation the symmetry requirement on $\bar{\boldsymbol{R}} \cdot \bar{\boldsymbol{\Sigma}}_b^t \cdot \boldsymbol{F}^t$ does not show up explicitly but is rather implicitly hidden in the $\mathrm{Div}(\delta\bar{\boldsymbol{\Sigma}}_1^t) \cdot \boldsymbol{x}$ term.

Stationarity Condition in $\mathcal{B}$ Geometrically nonlinear variational principles usually take a more concise structure if expressed in terms of spatial quantities like e.g. the spatial velocity gradient $\boldsymbol{l}_\delta = \nabla_x \delta\boldsymbol{x} = \nabla_X \delta\boldsymbol{x} \cdot \boldsymbol{F}^{-1}$. Due to the independent rotation, *push-forward* of the BIOT stress and its variation renders the *micropolar*-type CAUCHY stress tensor and its LIE-type derivative

$$\bar{\boldsymbol{\sigma}}^t = J^{-1}\bar{\boldsymbol{R}} \cdot \bar{\boldsymbol{\Sigma}}_b^t \cdot \boldsymbol{F}^t \quad \text{and} \quad \mathcal{L}_\delta(\bar{\boldsymbol{\sigma}}^t) = J^{-1}\bar{\boldsymbol{R}} \cdot \delta\bar{\boldsymbol{\Sigma}}_b^t \cdot \boldsymbol{F}^t. \qquad (47)$$

Moreover, we introduce the *nominal* derivative of *micropolar*-type CAUCHY stress tensor by the relation

$$\bar{\boldsymbol{\sigma}}^t = J^{-1}\bar{\boldsymbol{\Sigma}}_1^t \cdot \boldsymbol{F}^t \quad \leadsto \quad \mathcal{N}_\delta(\bar{\boldsymbol{\sigma}}^t) = J^{-1}\delta\bar{\boldsymbol{\Sigma}}_1^t \cdot \boldsymbol{F}^t. \qquad (48)$$

It is easily verified that $\mathcal{N}_\delta(\bar{\boldsymbol{\sigma}}^t)$ and $\mathcal{L}_\delta(\bar{\boldsymbol{\sigma}}^t)$ are connected via

$$\mathcal{N}_\delta(\bar{\boldsymbol{\sigma}}^t) = \mathcal{L}_\delta(\bar{\boldsymbol{\sigma}}^t) + \boldsymbol{\Omega}_\delta \cdot \bar{\boldsymbol{\sigma}}^t. \qquad (49)$$

Taking into account the NANSON formula $J\boldsymbol{N}\,\mathrm{d}A = \boldsymbol{F}^t \cdot \boldsymbol{n}\,\mathrm{d}a$, the spatial traction vector and its its *nominal* derivative follow as

$$\bar{\boldsymbol{t}} = \bar{\boldsymbol{\sigma}}^t \cdot \boldsymbol{n} \quad \text{and} \quad \mathcal{N}_\delta(\bar{\boldsymbol{t}}) = \mathcal{N}_\delta(\bar{\boldsymbol{\sigma}}^t) \cdot \boldsymbol{n}. \qquad (50)$$

Equivalently, *push-forward* of the right stretch tensor and its constitutive pendant leads to the *micropolar*-type spatial metric and the corresponding constitutive expression

$$\bar{g} = \bar{\boldsymbol{R}} \cdot \bar{\boldsymbol{U}} \cdot \boldsymbol{F}^{-1} \quad \text{and} \quad \partial_{\bar{\tau}} W^* = \boldsymbol{F}^{-t} \cdot \partial_{\boldsymbol{\Sigma}_b} W^* \cdot \bar{\boldsymbol{R}}^t. \qquad (51)$$

We are now in the position to recast Eq. 43 in the *spatial* setting

$$\begin{aligned}\delta\Pi^* &= \int_{\mathcal{B}} \boldsymbol{l}_\delta : \bar{\boldsymbol{\sigma}}\,\mathrm{d}v - \int_{\partial\mathcal{B}^\sigma} \delta\boldsymbol{x}\cdot\bar{\boldsymbol{t}}\,\mathrm{d}a \\ &+ \int_{\mathcal{B}} \mathcal{L}_\delta(\bar{\boldsymbol{\sigma}}) : \left[\bar{\boldsymbol{g}}^t - \partial_{\bar{\boldsymbol{\tau}}} W^*\right]\,\mathrm{d}v \\ &+ \int_{\partial\mathcal{B}^\sigma} \mathcal{N}_\delta(\bar{\boldsymbol{t}})\cdot[\boldsymbol{\lambda}-\boldsymbol{x}]\,\mathrm{d}a \\ &+ \int_{\partial\mathcal{B}^\sigma} \delta\boldsymbol{\lambda}\cdot[\bar{\boldsymbol{t}}-\boldsymbol{t}^p]\,\mathrm{d}a \\ &- \int_{\mathcal{B}} \boldsymbol{\Omega}_\delta : \bar{\boldsymbol{\sigma}}\,\mathrm{d}v.\end{aligned} \tag{52}$$

Finally, with the transformation of the material divergence operation into its spatial counterpart

$$\mathrm{Div}(\bullet) = J\,\mathrm{div}\left(J^{-1}(\bullet)\cdot\boldsymbol{F}^t\right) \tag{53}$$

the stationarity condition in Eq. 46 is rewritten in the spatial setting as

$$\begin{aligned}\delta\Pi^* &= -\int_{\mathcal{B}} \delta\boldsymbol{x}\cdot\mathrm{div}(\bar{\boldsymbol{\sigma}}^t)\,\mathrm{d}v \\ &-\int_{\mathcal{B}} \left[\mathrm{div}\left(\mathcal{N}_\delta(\bar{\boldsymbol{\sigma}}^t)\right)\cdot\boldsymbol{x} + \mathcal{L}_\delta(\bar{\boldsymbol{\sigma}}) : \partial_{\bar{\boldsymbol{\tau}}} W^*\right]\,\mathrm{d}v \\ &+\int_{\partial\mathcal{B}^u} \mathcal{N}_\delta(\bar{\boldsymbol{t}})\cdot\boldsymbol{x}^p\,\mathrm{d}a + \int_{\partial\mathcal{B}^\sigma} \mathcal{N}_\delta(\bar{\boldsymbol{t}})\cdot\boldsymbol{\lambda}\,\mathrm{d}a \\ &+\int_{\partial\mathcal{B}^\sigma} \delta\boldsymbol{\lambda}\cdot[\bar{\boldsymbol{t}}-\boldsymbol{t}^p]\,\mathrm{d}a\end{aligned} \tag{54}$$

Linearization in $\mathcal{B}$ The second variation renders the *symmetric* HESSE matrix of the mixed hybrid principle or equivalently the linearization of the corresponding stationarity condition. In a numerical setting the HESSE matrix constitutes the basis for the tangential stiffness matrix of the discretized functional which ensures optimal convergence of the NEWTON-RAPHSON scheme. To this end, we denote the linearization of $(\bullet)$ by $\Delta(\bullet)$ and assign the associated meaning to the spatial rates $\mathcal{L}_\Delta(\bullet)$ and $\mathcal{N}_\Delta(\bullet)$. Moreover, we introduce the spatial backrotated derivative $\mathcal{R}_\Delta(\bullet)$ defined by

$$\mathcal{R}_\Delta(\bullet) = \left[\Delta\left[(\bullet)\cdot\bar{\boldsymbol{R}}\right]\right]\cdot\bar{\boldsymbol{R}}^t. \tag{55}$$

Additionally, taking the second variation of $\bar{\boldsymbol{R}}$ into account we set

$$\Delta\boldsymbol{\Omega}_\delta = -[\boldsymbol{\Omega}_\delta\cdot\boldsymbol{\Omega}_\Delta]^{skw}. \tag{56}$$

Finally, the spatial compliance tensor follows in a straightforward manner by *push-forward* which is given in symbolic notation

$$\partial^2_{\bar{\boldsymbol{\tau}}\bar{\boldsymbol{\tau}}} W^* = [\boldsymbol{F}^{-1}\otimes\bar{\boldsymbol{R}}^t] : \partial^2_{\boldsymbol{\Sigma}_b\boldsymbol{\Sigma}_b} W^* : [\boldsymbol{F}^{-1}\otimes\bar{\boldsymbol{R}}^t]. \tag{57}$$

With these preliminaries at hand, the linearization of the stationarity condition in Eq. 43 or Eq. 52 follows after lengthly calculations in the *spatial* setting in a remarkably concise format which resembles the *structure* of the geometrically linear theory in Eq. 14

$$
\begin{aligned}
\Delta\delta\Pi^* &= \int_{\mathcal{B}} [\boldsymbol{l}_\delta - \boldsymbol{\Omega}_\delta] : \mathcal{N}_\Delta(\bar{\boldsymbol{\sigma}}^t)\,\mathrm{d}v \\
&+ \int_{\mathcal{B}} [\boldsymbol{l}_\Delta - \boldsymbol{\Omega}_\Delta] : \mathcal{N}_\delta(\bar{\boldsymbol{\sigma}}^t)\,\mathrm{d}v \\
&- \int_{\mathcal{B}} J\mathcal{L}_\delta(\bar{\boldsymbol{\sigma}}) : \partial_{\bar{\mathcal{T}}\bar{\mathcal{T}}} W^* : \mathcal{L}_\Delta(\bar{\boldsymbol{\sigma}})\,\mathrm{d}v \\
&+ \int_{\partial\mathcal{B}^\sigma} \mathcal{R}_\Delta(\boldsymbol{\lambda} - \boldsymbol{x}) \cdot \mathcal{N}_\delta(\bar{\boldsymbol{t}})\,\mathrm{d}a \\
&+ \int_{\partial\mathcal{B}^\sigma} \mathcal{R}_\delta(\boldsymbol{\lambda} - \boldsymbol{x}) \cdot \mathcal{N}_\Delta(\bar{\boldsymbol{t}})\,\mathrm{d}a \\
\star\quad &- \int_{\mathcal{B}} [\boldsymbol{\Omega}_\delta \cdot \boldsymbol{\Omega}_\Delta]^{sym} : \bar{\boldsymbol{\sigma}}^t\,\mathrm{d}v \\
\star\quad &- \int_{\partial\mathcal{B}^\sigma} [\boldsymbol{\lambda} - \boldsymbol{x}] \cdot [\boldsymbol{\Omega}_\delta \cdot \boldsymbol{\Omega}_\Delta]^{sym} \cdot \bar{\boldsymbol{t}}\,\mathrm{d}a
\end{aligned} \tag{58}
$$

Clearly, the *geometric* stiffness contributions typical for all geometrically nonlinear variational principles are implicitly contained in the nominal rates $\mathcal{N}_\delta(\bullet)$ and $\mathcal{N}_\Delta(\bullet)$. Moreover, the two integrals marked with a $\star$ denote additional *geometric* stiffness terms which stem from the *independent* rotation $\bar{\boldsymbol{R}}$.

Equivalently, the linearization of the stationarity condition in Eq. 46 or Eq. 54 expressed in the *spatial* setting may be recast into an intriguing concise format which resembles the *structure* of the geometrically linear theory in Eq. 15

$$
\begin{aligned}
\Delta\delta\Pi^* &= -\int_{\mathcal{B}} \left[\delta\boldsymbol{x} \cdot \operatorname{div}\left(\mathcal{N}_\Delta(\bar{\boldsymbol{\sigma}}^t)\right) + \boldsymbol{\Omega}_\delta : \mathcal{N}_\Delta(\bar{\boldsymbol{\sigma}}^t)\right]\,\mathrm{d}v \\
&\quad -\int_{\mathcal{B}} \left[\Delta\boldsymbol{x} \cdot \operatorname{div}\left(\mathcal{N}_\delta(\bar{\boldsymbol{\sigma}}^t)\right) + \boldsymbol{\Omega}_\Delta : \mathcal{N}_\delta(\bar{\boldsymbol{\sigma}}^t)\right]\,\mathrm{d}v \\
&\quad -\int_{\mathcal{B}} J\mathcal{L}_\delta(\bar{\boldsymbol{\sigma}}) : \partial_{\bar{\mathcal{T}}\bar{\mathcal{T}}} W^* : \mathcal{L}_\Delta(\bar{\boldsymbol{\sigma}})\,\mathrm{d}v \\
&\quad +\int_{\partial\mathcal{B}^\sigma} [\mathcal{R}_\Delta(\boldsymbol{\lambda}) \cdot \mathcal{N}_\delta(\bar{\boldsymbol{t}}) + \mathcal{R}_\delta(\boldsymbol{\lambda}) \cdot \mathcal{N}_\Delta(\bar{\boldsymbol{t}})]\,\mathrm{d}a \\
\star\quad &\quad -\int_{\partial\mathcal{B}^\sigma} \boldsymbol{x} \cdot [\boldsymbol{\Omega}_\Delta \cdot \mathcal{N}_\delta(\bar{\boldsymbol{t}}) + \boldsymbol{\Omega}_\delta \cdot \mathcal{N}_\Delta(\bar{\boldsymbol{t}})]\,\mathrm{d}a \\
\star\quad &\quad -\int_{\mathcal{B}} [\boldsymbol{\Omega}_\delta \cdot \boldsymbol{\Omega}_\Delta]^{sym} : \bar{\boldsymbol{\sigma}}^t\,\mathrm{d}v \\
\star\quad &\quad -\int_{\partial\mathcal{B}^\sigma} [\boldsymbol{\lambda} - \boldsymbol{x}] \cdot [\boldsymbol{\Omega}_\delta \cdot \boldsymbol{\Omega}_\Delta]^{sym} \cdot \bar{\boldsymbol{t}}\,\mathrm{d}a
\end{aligned} \tag{59}
$$

5 Discretization & Update Procedure

Finally, we briefly elaborate on a possible plane triangle discretization with the lowest order BREZZI, DOUGLAS & MARINI [BDM85] spaces applied to the geometrically nonlinear case. To this end, all approximations will refer to variations and iterative increments with the same expansions in the spirit of a GALERKIN-BUBNOV method. Then the total quantities follow from a configurational update.

Discretization Firstly, the elementwise approximation for the increment of the nonsymmetric BIOT stress tensor $\Delta\bar{\Sigma}_b$ is introduced

$$\Delta\bar{\Sigma}_b^{h,t} = \sum_{k=1}^{3} N^k \Delta\bar{\Sigma}_k + N^4 \mathrm{spn}(\Delta\bar{\Sigma}_4) + \Delta\bar{\Sigma}_5 \otimes \mathrm{Crl}(N^4). \tag{60}$$

Here, the derivatives in the material curl Crl(•) refer to the reference configuration $\mathcal{B}_0$ and the stress is parametrized in terms of the 15 unknowns contained in

$$\Delta\bar{\Sigma}_k \in \mathrm{IR}^{2\times 2}, \qquad \mathrm{spn}(\Delta\bar{\Sigma}_4) \in \mathrm{so}(2), \qquad \Delta\bar{\Sigma}_5 \in \mathrm{IR}^2. \tag{61}$$

Secondly, for the increments in the placement the minimum possible expansion, i.e. the linearized rigid body modes

$$\Delta \boldsymbol{x}^h = \Delta\boldsymbol{x}_1 + \mathrm{spn}(\Delta\boldsymbol{x}_2) \cdot \sum_{k=1}^{3} N^k \boldsymbol{X}_k \tag{62}$$

is elementwise chosen in terms of the 3 unknowns contained in

$$\Delta\boldsymbol{x}_1 \in \mathrm{IR}^2 \quad \text{and} \quad \Delta\mathrm{spn}(\boldsymbol{x}_2) \in \mathrm{so}(2). \tag{63}$$

Thirdly, the angular increment $\omega_\Delta \in \mathrm{IR}$ or equivalently the spatial spin $\boldsymbol{\Omega}_\Delta \in \mathrm{so}(2)$ of the independent rotation $\bar{\boldsymbol{R}} \in \mathrm{SO}(2)$ is elementwise approximated by a linear polynomial with 3 unknowns

$$\omega_\Delta^h = \sum_{k=1}^{3} N^k \omega_k \quad \rightsquigarrow \quad \boldsymbol{\Omega}_\Delta^h = \sum_{k=1}^{3} N^k \omega_k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{64}$$

Finally, the only continuous quantity, the boundary or edge placements $\boldsymbol{\lambda}$, may be chosen as in the geometrically linear case.

Update Procedure Typically, the nonlinear set of equations resulting from the discretized weak form in Eq. 46 is solved iteratively via the NEWTON method by a sequence of n_{it} linearized problems within a finite time step

$$\Delta t = {}^{n+1}t - {}^{n}t = \bigcup_{i=1}^{n_{it}} {}^{i+1}t - {}^{i}t. \tag{65}$$

To this end, the residuum is linearized about the known configuration at ${}^{i}t$ with ${}^{i=1}t = {}^{n}t$ to yield iterative increments, i.e.

$$\text{Given:} \quad {}^{i}\bar{\boldsymbol{\Sigma}}_b^h \quad {}^{i}\boldsymbol{x}^h \quad {}^{i}\bar{\boldsymbol{R}}^h \quad \rightsquigarrow \quad \Delta\bar{\boldsymbol{\Sigma}}_b^h \quad \Delta\boldsymbol{x}^h \quad \boldsymbol{\Omega}_\Delta^h. \tag{66}$$

Then, the BIOT stress and the placement are updated *additively*

$${}^{i+1}\bar{\boldsymbol{\Sigma}}_b^h = {}^{i}\bar{\boldsymbol{\Sigma}}_b^h + \Delta\bar{\boldsymbol{\Sigma}}_b^h \quad \text{and} \quad {}^{i+1}\boldsymbol{x}^h = {}^{i}\boldsymbol{x}^h + \Delta\boldsymbol{x}^h, \tag{67}$$

while the independent rotation is updated *multiplicatively*

$${}^{i+1}\bar{\boldsymbol{R}}^h = \exp(\boldsymbol{\Omega}_\Delta^h) \cdot {}^{i}\bar{\boldsymbol{R}}^h. \tag{68}$$

In $\mathbb{R}^3$, the exponential $\exp(\circ) \in SO(3)$ of a skewsymmetric tensor $(\circ) = \text{spn}(\bullet) \in so(3)$ with axial vector $(\bullet) \in \mathbb{R}^3$ is given by the EULER-RODRIGUES formula

$$\exp(\circ) = \cos\|\bullet\|\boldsymbol{I} + \frac{\sin\|\bullet\|}{\|\bullet\|}(\circ) + \frac{1-\cos\|\bullet\|}{\|\bullet\|^2}(\bullet)\otimes(\bullet). \tag{69}$$

For the present case of a planar discretization with the cartesian basis $\boldsymbol{E}_I$ in $\mathbb{R}^2$ we merely consider in plane rotations about the out of plane $\boldsymbol{E}_3$-axis. This results in a simple representation for the *pseudo* vector of rotation $\boldsymbol{\theta}^h = \log(\bar{\boldsymbol{R}}^h) = \theta^h \boldsymbol{E}_3$ and the angular increment $\boldsymbol{\omega}_\Delta^h = \omega_\Delta^h \boldsymbol{E}_3$. Finally, due to standard trigonometric relations, the update for $\bar{\boldsymbol{R}}^h$ at ${}^{i+1}t$ boils down to the simple matrix representation relative to $\boldsymbol{E}_I$

$${}^{i+1}\bar{\boldsymbol{R}}^h = \begin{bmatrix} \cos({}^{i}\theta^h + \omega_\Delta^h) & -\sin({}^{i}\theta^h + \omega_\Delta^h) \\ \sin({}^{i}\theta^h + \omega_\Delta^h) & \cos({}^{i}\theta^h + \omega_\Delta^h) \end{bmatrix}. \tag{70}$$

Moreover, the weak form in Eq. 46 contains expressions which in $\mathbb{R}^3$ involve a second order curvature tensor $\boldsymbol{\Gamma} = \text{axl}(\nabla_X \bar{\boldsymbol{R}} \cdot \bar{\boldsymbol{R}}^t)$

$$\text{Div}(\bar{\boldsymbol{R}} \cdot \bar{\boldsymbol{\Sigma}}_b^t) = \bar{\boldsymbol{R}} \cdot \text{Div}(\bar{\boldsymbol{\Sigma}}_b^t) + 2\text{axl}([\bar{\boldsymbol{\Sigma}}_1^t \cdot \boldsymbol{\Gamma}^t]^{skw}). \tag{71}$$

Thereby, the second order curvature tensor $\boldsymbol{\Gamma}$ is updated at ${}^{i+1}t$ by

$${}^{i+1}\boldsymbol{\Gamma} = \boldsymbol{\kappa}(\boldsymbol{\Omega}_\Delta^h) \cdot \nabla_X \boldsymbol{\omega}_\Delta^h + \exp(\boldsymbol{\Omega}_\Delta^h) \cdot {}^{i}\boldsymbol{\Gamma}, \tag{72}$$

with second order update tensor $\boldsymbol{\kappa}(\circ) = \boldsymbol{\kappa}(\text{spn}(\bullet))$ defined as

$$\boldsymbol{\kappa}(\circ) = \frac{\sin\|\bullet\|}{\|\bullet\|}\boldsymbol{I} + \frac{1-\cos\|\bullet\|}{\|\bullet\|^2}(\circ) + \frac{\|\bullet\|-\sin\|\bullet\|}{\|\bullet\|^3}(\bullet)\otimes(\bullet). \tag{73}$$

To proof of these results please refer to STEINMANN [STE94]. For a planar application $\boldsymbol{\Gamma}$ boils down to the simple representation involving the vector $\boldsymbol{\gamma} \in \mathbb{R}^3$ with $\boldsymbol{\gamma} \cdot \boldsymbol{E}_3 = 0$

$$\boldsymbol{\Gamma} = \boldsymbol{E}_3 \otimes \boldsymbol{\gamma} \quad \rightsquigarrow \quad 2\text{axl}([\bar{\boldsymbol{\Sigma}}_1^t \cdot \boldsymbol{\Gamma}^t]^{skw}) = \boldsymbol{E}_3 \times [\bar{\boldsymbol{\Sigma}}_1^t \cdot \boldsymbol{\gamma}], \tag{74}$$

with update for $\boldsymbol{\gamma}$ at ${}^{i+1}t$ given by

$${}^{i+1}\boldsymbol{\gamma} = \nabla_X({}^{i}\theta^h + \omega_\Delta^h). \tag{75}$$

The configurational update procedure completes the description of the possible extension of the STENBERG [STE88] approach to the geometrically nonlinear case.

6 Conclusion

The possibility of extending mixed finite element methods to the geometrically nonlinear case has been investigated. In fact, the extension of the considered mixed finite element formulation includes as a particular interesting and challenging aspect besides large strains as well *finite rotations*. To provide a guideline, we first reviewed the sequence of steps, starting from the DIRICHLET principle and leading to the desired hybrid mixed functional of the geometrically linear case and its discretization. For the construction of the appropriate geometrically nonlinear functional we then persued exactly these steps. Thereby, it turns out that the resulting formulation resembles very much a *micropolar* approach with only the couple stresses missing. Moreover, the spatial description of the stationarity condition and its linearization were recast into a structure almost identical to the geometrically linear case by the aid of LIE-type and *nominal* derivatives. Finally, as the main result of this work, it was examined how to apply the STENBERG expansions to geometrically nonlinear elasticity problems. Thereby, we proposed to discretize the *micropolar* BIOT stress, the *micropolar* rotation in terms of its associated spatial spin and the spatial placement. To complete the description we briefly discussed the update procedure for these fields within an iterative NEWTON-RAPHSON solution scheme. In summary we believe that the present work provides a sound variational basis with strong continuum mechanical impact to our future numerical implementation of the considered mixed finite element.

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