Consistent tangent matrices of curved contact operators involving anisotropic friction

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ABSTRACT. *Contact problems involving friction are difficult to solve because they are governed by multivalued tribological laws. Using a mixed penalty-duality formulation, a generalized Newton method has proved to be efficient. In order to introduce new non-linearities like anisotropic friction and curvature of the contact surface, appropriate tangent matrices have to be derived. The additional terms are discussed and algorithms are proposed when analytical expressions are not available. Two numerical tests are presented to show the performance of the generalized Newton method.*

RÉSUMÉ. *Les problèmes de contact avec frottement sont difficiles à résoudre à cause du caractère multivoque des lois envisagées. A partir d'une formulation mixte de type pénalité-dualité, une méthode de Newton généralisée s'est avérée très efficace. Pour prendre en compte de nouvelles non-linéarités telles l'anisotropie du frottement et la courbure des surfaces de contact, il est nécessaire de calculer correctement les matrices tangentes. Nous présentons les nouveaux termes ainsi que les algorithmes quand on ne peut recourir à des expressions analytiques. Deux tests numériques démontrent les possibilités de cette approche.*

KEY WORDS : *unilateral contact, friction, mixed formulation, Newton method, finite contact element.*

MOTS-CLÉS : *contact unilatéral, frottement, formulation mixte, méthode de Newton, élément fini contact.*

1. Introduction

For the last few years, we have developed a numerical approach to treat contact and friction like nearly ordinary non-linearities. Indeed many practical problems such as metal forming processes involve different and severe non-linearities as well as large deformations or plasticity. To solve each non-linearity, there exists a specific efficient numerical method. But the implementation of all techniques in the same computer code does not necessarily lead to an efficient and robust simulation tool for realistic problems. Moreover this implementation is not easy. Following a strategy developed by many authors and specially by A. Curnier in TACT program[CUR 87], we chose the Newton algorithm as the standard non-linear solution method which is often available in general Finite Element programs.

The major difficulty in applying a Newton type algorithm is due to the non-differentiability or the multivalued character of unilateral contact and friction laws. To circumvent this difficulty, it is either used a *penalty method* [CUR 88], or a *mixed formulation* inspired from an augmented Lagrangian functional [ALA 91]. In both cases the resulting frictional contact operator is continuous but not everywhere differentiable. However a generalised Newton method may be applied and some stability results were proved in simple cases [ALA 91].

For easy implementation, the frictional contact behaviour of the boundary of a deformable body is modelized by boundary *finite contact elements*. We emphazise that the terminology finite contact element is to be understood only in terms of implementation facilities; a contact element like a finite element has to provide elementary contributions to the internal forces and to the tangent stiffness matrix. The analogy between contact and finite element does not concern mathematical tools such as approximation spaces or interpolation procedures. Besides, recent studies show that the choice of concentrated forces at the nodes as it is done in this paper, corresponds to a consistent approximation scheme for hybrid formulations [AGO 93].

The aim of this paper is to show how to insert additional non-linearities related to contact conditions without modifying the code structure based on a single Newton iteration loop. In order to take into account drastic changes of curvature between contacting bodies or frictional anisotropy in a fully implicit scheme, it is necessary that the contact elements provide consistent tangent matrices. This study is comparable to the work realized in solid mechanics to introduce complex elastic plastic behaviour laws admitting anisotropic plasticity, kinematic and isotropic hardening [RAK 91, SIM 85].

In Section 2, a contact mechanics summary gives the necessary background. Section 3 presents the contact operator in a synthetic form obtained either by the penalty method or by the mixed formulation . The new non-linearities introduced in this paper do not modify the expression of the operator but imply additional terms in the tangent matrices which are discussed in sections 4 and 5. Some remarks are detailed in section 6 concerning the implementation of the approach. We show how to use a symbolic computation software to facilitate and partially to automate the implementation. We present two numerical tests in section 7 to prove the efficiency of this technique.

2. Contact mechanics summary

We shall restrict our attention to the large deformation contact friction problem between a deformable body and a rigid obstacle with sharp corners. Such problems arise in metal forming processes which justifies that the obstacle is often called tool in the next sections. Mechanical laws and their numerical treatment are written either for a particle (if a continuous medium is considered) or for a node (if the medium is discretized) with the same notations.

2.1. *Unilateral contact law*

In the following, we shall consider a single particle α (or node) of the (discretized) deformable body in the neighbourhood of the tool boundary. We note X its initial position, u its displacement and x=X+u its current position. In classical approaches, contact and friction laws are written with respect to a fixed local frame attached to a particle of the obstacle, chosen currently as the orthogonal projection of α. This is straightforward for flat contact problems, but in the case of curved contact, particulary if large slip increments occur on strongly curved tools, this strategy is not efficient. In this paper, the local frame is considered as unknown at the beginning of each loading step.

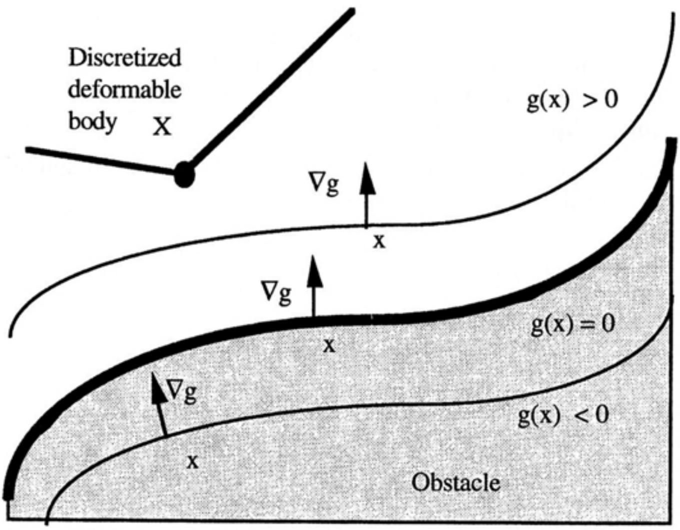


Figure 1. *Contact geometry described by an implicit function*

We assume the existence of *a twice continuously differentiable function g(x)* such that the inequality $g(x) \geq 0$ defines the feasible region that is the exterior of the tool. By this way, an analytical evaluation of the *normal unit vector* at any point on the boundary may be extended *inside* or *outside* the obstacle (*Figure 1*),

$$n(x) = \nabla g(x) / \|\nabla g(x)\|. \quad [1]$$

If such a function g is not available — for example if tools are described by polynomial parametric surface in Computer Aided Geometrical Design (CAGD) — we resort to numerical methods to compute orthogonal projection point and its associated normal vector [HEE 92a, HEE 95]. In this paper, we exclude this case because the resulting tangent matrices cannot be splitted into different meaningful parts due to their numerical evaluation.

Using the formalism of Convex Analysis like Moreau [MOR 79], the unilateral contact law can be written as

$$\lambda_n \in \partial\Psi_{R+}\{\ g(X+u)\ \}, \quad [2]$$

where λ_n is the normal contact stress (or force for a node of a discretized body), and $\partial\Psi_{R+}$ is the sub-differential of the indicator function of the positive half line.

2.2. *Incremental friction law*

Friction laws involve the relative tangential velocity between the two bodies. In the case of an incremental solution of quasi-static field problems, this velocity can be replaced directly by a *tangential slip increment* [CUR 88, ALA 91, HEE 95]. But the tangential slip increment (*Figure 2*), noted δ_t, must be evaluated with respect to a reference position on the obstacle noted x^{ref}, which is computed at the beginning of each loading step,

$$\delta_t\,(u) = (\,I - n(x)n(x)^T\,)(x - x^{ref}\,) = (\,I - n(x)n(x)^T\,)\,\delta. \quad [3]$$

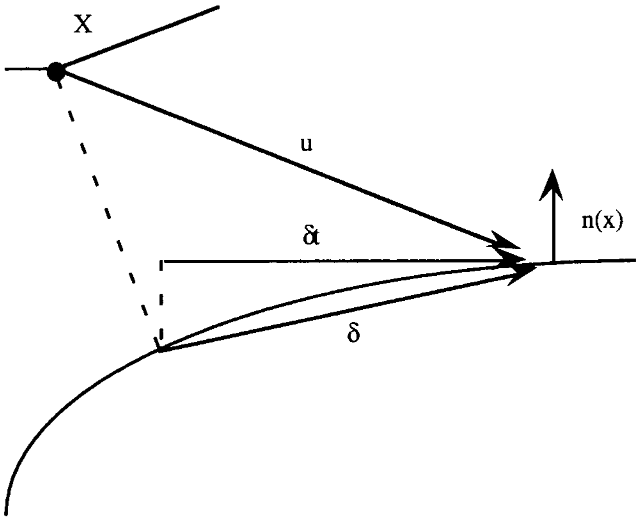


Figure 2. *Tangential slip increment on curved surface*

However, it is emphasized that the tangential slip increment is expressed in the local frame associated to x. Then δ_t becomes a *non-linear* function of u through the normal unit vector n(x). It is recalled that, in classical approaches, the local frame is evaluated at the reference position and is fixed over the loading step. Consequently, δ_t is a linear function of u. The reference position is the position of the node on the tool at the end of the previous step if the node was already in contact. Otherwise, the eventual impact point must be predicted to define a reference position. Different predictions can be used from the explicit projection on the obstacle to the implicit one [ALA 91, HEE 95].

The friction criterion and the slip rule of the Coulomb's classical law can be summarized in the inclusion [MOR 79],

$$\delta_t (u) \in \partial\Psi_{C(\lambda_n)}\{ \lambda_t \}. \qquad [4]$$

The *contact stress* (or force) is splitted into its normal and tangential components : $\lambda = \lambda_n n + \lambda_t$ (i.e. $\lambda_t = (I\text{-}nn^T)\lambda$). Consequently, λ_t denotes a vector but λ_n is a scalar. This remark holds for δ and σ in the following. The friction criterion $C(\lambda_n)$ depends on the pressure and the isotropic friction law consists in taking for criterion the closed disc centered at the origin with radius equal to the product of $(-\lambda_n)$ by the friction coefficient μ ,

$$C(\lambda_n) = \{ \lambda_t : \|\lambda_t\| \leq -\mu\lambda_n\} \qquad (\lambda_n \leq 0). \qquad [5]$$

An anisotropic law can be accomodated by taking an elliptical disc instead of a circular one [HE 93]. More generally, we assume the existence of a convex function h such that $C(\lambda_n)$ is written as follows,

$$C(\lambda_n) = \{ \lambda_t \text{ such that } h(\tau) \leq 0 \text{ where } \tau = \frac{\lambda_t}{-\lambda_n} \}. \qquad [6]$$

In the following τ is called the *normalized tangential stress* (or force if a node is concerned).

3. Frictional contact operator

3.1. *Quasi-optimisation problem*

The following formulation is based on the non-differentiable functional expressions [2] and [4] of the contact and friction laws. For convenience, we will also assume the existence of a differentiable strain energy functional ϕ, to characterize the elastic response of the deformable body, but it is emphasized that following developments will not depend on this assumption. Accordingly, the equilibrium of the discrete body with eventual contact is written as,

$$u \in \operatorname{argmin} \{ \phi(v) + \Psi_{R+}[g(X+v)] + \Psi^{*}_{C(\mathbf{u})}[\delta_{t(\mathbf{u})} (v)] \}. \qquad [7]$$

In this expression, we note discrete nodal variables by the same letters as the corresponding continuous variables. So, u (resp. v) is the generalized (resp. virtual) displacement. g(X+v) is a p-dimensional vector where p is the number of nodes candidate to contact. R+ is then the non negative cone in R^p. In the same way, C(u) denotes the cartesian product of the p local convex sets C. Ψ^{*}_{C} is the conjugate function of Ψ_C.

It must be stressed that the above problem is not a standard optimisation problem but only a quasi-optimisation one, because the convex set C and the tangential slip increment depend on the solution u. This solution is now characterized by quasi-variational inequalities (QVI)[CAP 79]. We coined the term quasi-optimisation by analogy : C depends on the solution u through the normal contact force and thus reflects the non-associated character of the slip rule in a coupled contact-friction formulation. By taking into account the curvature of the obstacle, the tangential slip increment δ_t is a non-linear function of u through the normal vector n(X+u).

3.2. *Mixed frictional contact operator*

The purpose of this section is to specify the form of the mixed frictional contact operator when new non linearities are taken into account without considering the method to obtain it. For complete information, readers should consult Alart and Curnier [ALA 91] where an approach inspired from the augmented lagrangian method is presented. Some extensions concerning curved contact have been developed by Heege and Alart [HEE 95] and Heegard and Curnier [HEE 93]. For mathematical aspects of the augmented Lagrangian technique, refer to several authors [ROC 76, FOR 76, FOR 83].

Finally, the equilibrium is characterized by the following system, the unknowns of which are the displacement u and the contact force λ,

$$\begin{cases} F_{int}(u) - F_{ext} + F(u,\lambda) = 0 \\ -\frac{1}{r}\left(\lambda - F(u,\lambda) \right) = 0 \end{cases} . \qquad [8]$$

Fint(u) - Fext — internal forces minus external ones — is the gradient of ϕ at the solution u and λ. $F(u,\lambda)$ defines a continuous *frictional contact operator* as follows,

$$F(u,\lambda) = \sigma_n^- \; n + \mathrm{proj}_{C(\sigma_n^-)}[\,\sigma_t\,], \qquad [9]$$

where $\sigma_n^- = \min(0, \sigma_n)$. By extension, we coin the term 'augmented' multipliers to specify σ_n and σ_t :

$$\sigma_n = \lambda_n + r\, g(X+u) \quad , \quad \sigma_t = \lambda_t + r\, \delta_t(u) \qquad [10]$$

The factor r is a positive real number. With a good but not too constraining choice of it [ALA 91], numerical instabilities are avoided and the fast rate of convergence of Newton's method is preserved. It is mentioned that when λ is zero vector, one recovers a penalty method and the system [8] can be reduced to the first equation; but the factor r must be large enough for a fair approximation of the contact and friction laws.

The operator $F(u,\lambda)$ is written for one node considered as a contact finite element according to the terminology defined in [ALA 91, HEE 95]. The expression [9] shows a projection on an 'augmented' convex set $C(\sigma_n^-)$ which is a crucial point to ensure the continuity of the operator. For flat contact (fixed local frame) and isotropic friction, this operator is conewise linear in 2D and raywise linear in 3D discretisation, relevant notions to study uniqueness conditions [ALA 93]. These properties are lost for anisotropic frictional curved contact.

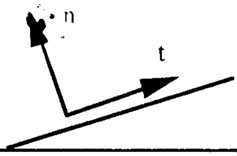
Status 	2D isotropic frictional flat contact operator ($g = \delta_n$ and (t,n) direct local frame)
Gap if $\sigma_n \geq 0$	$F_{gap}(u,\lambda) = 0$
Stick if $\sigma_n < 0$ and $\|\sigma_t\| + \mu\sigma_n < 0$	$F_{stick}(u,\lambda) = \sigma_n\, n + \sigma_t = \hat{\sigma}$ $\equiv \lambda + r\delta$
Slip ε if $\sigma_n < 0$ and $\varepsilon t^T \sigma_t + \mu\sigma_n \geq 0$	$F_{slip\varepsilon}(u,\lambda) = \sigma_n\,(n - \varepsilon\mu t)$ $= n^T(\lambda + r\delta)\,(n - \varepsilon\mu t)$

Table 1. *2D isotropic frictional flat contact operator*

For a better understanding, the 2D isotropic frictional flat contact is expressed according to the respective status in the *Table 1*. In 2D discretisation, the local frame is (t,n) where t is obtained from n by a local rotation of $-\pi/2$ radians. In the flat case, for 2D or 3D discretisation, this frame does not depend on the variables u and λ. Finally g may be replaced by the normal component of the relative displacement defined in [3] : $g = \delta_n$. Backward and forward slip status are distinguished by a ε factor ($\varepsilon=\pm1$) for isotropic friction. To introduce the anisotropic friction in 2D consists to distinguish the backward friction coefficient from the forward one. Consequently, the frictional contact operator remains conewise linear and is derived easily.

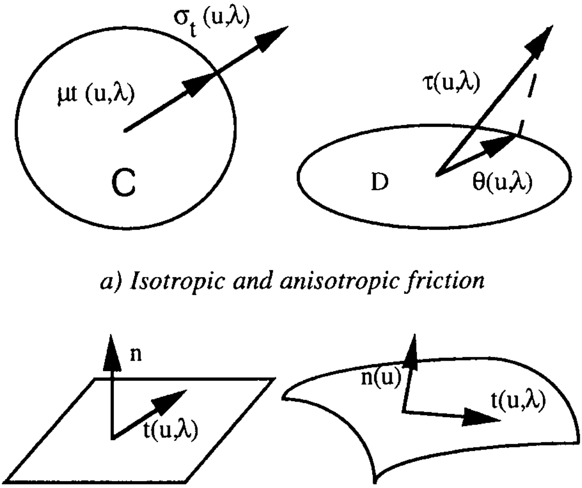
In 3D discretisation the operator is not piecewise linear. But only the slip status leads to a non-linear expression. However, for 3D flat contact case with isotropic friction, we can still define analytically the slip direction unit vector t which is a function of u and λ through σ_t. Consequently, for the slip status we then get,

$$F_{slip}(u,\lambda) = \sigma_n(u,\lambda)\,(\,n - \varepsilon\mu t\,(u,\lambda)\,) \quad \text{with } t = \frac{\sigma_t}{\|\sigma_t\|}. \qquad [11]$$

For 3D anisotropic frictional contact case, the previous expression holds by replacing $\varepsilon\mu t$ by θ which is the projection of the normalized tangential force τ on the *normalized section of the friction cone* noted D. Remark that θ is now defined with respect to the augmented multipliers σ_t and σ_n and not directly to the initial force λ as in [6],

$$\theta(u,\lambda) = \mathrm{proj}_{D(X+u)}\,\tau\,(u,\lambda) \quad \text{with} \quad \tau = \frac{\sigma_t}{-\sigma_n}. \qquad [12]$$

Typical situations are summarized in *Figure 3*.



a) Isotropic and anisotropic friction

b) Flat and curved contact

Figure 3. *Non linear functions due to anisotropy or curvature*

Moreover, in order to take into account the eventual heterogeneity of the friction law, the normalized section D is assumed to depend on the location of the contact node on the contact area. By convenience, the function h introduced in [6], may be defined for all positions x, whether x is on the obstacle ($g(x) = 0$) or not ($g(x) \neq 0$),

$$D(x) = D(X+u) = \{\theta : h(x,\theta) \leq 0 \}. \qquad [13]$$

4. Stick operator and its associated jacobian matrix

In this section we discuss briefly the jacobian matrix associated to the stick status. In this case, the anisotropy does not imply modifications for the tangent matrix because the projection on D is equal to identity. Only the curvature induces additional terms. We can rewrite the stick operator as follows,

$$F_{stick}(u,\lambda) = \sigma_n\, n + \sigma_t \;=\; \lambda + r\,(\,\delta + (g - \delta_n)n\,). \qquad [14]$$

It is useful to split the tangent matrix into derivatives with respect to λ and u. The first one is obvious according to the previous remark on the anisotropy,

$$\partial_\lambda F = I, \qquad [15]$$

$$\partial_u F = r\,I + r(\|\nabla g\| - 1)nn^T + r(g - \delta^T n - n\delta^T)\,\partial_u n$$
$$= r\,I + r(\|\nabla g\| - 1)nn^T + r\|\nabla g\|^{-1}[g(I - nn^T) - \delta^T nI - n\delta^T]\,Hg. \qquad [16]$$

The derivative of the normal vector n with respect to u depends on the hessian matrix of g noted Hg,

$$\partial_u n = \|\nabla g\|^{-1}\,(I - nn^T)\,Hg. \qquad [17]$$

The third term in [16] is typically a *curvature term* which vanishes for flat obstacles because then Hg equals to zero. For flat contact, it is convenient to choose g such that ∇g is a constant unit vector (equal to n). Then the second term disappears. For the curved case, in order to keep well conditionned matrices, it is interesting to take g such that $\|\nabla g\|$ is not too different from one. For example, a sphere centered at the point x° of radius R may be described by the following function g,

$$g(x) = \frac{1}{2R}\,(\,\|x - x°\|^2 - R\,). \qquad [18]$$

From practical point of view, the curvature terms can be omitted, provided that the expression of the stick operator is changed. In this case, the expression

$$\delta + (g - \delta_n)n = 0, \qquad [19]$$

involved in the second line of the equation [8] is equivalent to $\delta = 0$. To obtain this last equation, we can postulate a simplified expression for F used in the two lines of the system [8],

$$F_{stick}(u,\lambda) = \lambda + r\,\delta \equiv \hat{\sigma}. \qquad [20]$$

As a consequence, the derivatives are very simple : $\partial_\lambda F = I$ and $\partial_u F = r\,I$. Note that the continuity of the operator [9] is lost which is unfriendly from a mathematical point of view but it is practically efficient.

5. Slip jacobian matrix

In the more general case, we recall the form of the contact operator for the slip status,

$$F_{slip}(u,\lambda) = \sigma_n(u,\lambda)\,(\,n - \theta(u,\lambda)\,) \quad \text{with } \theta \text{ defined in [12].}$$

In order to operate a fully implicit scheme providing the quadratic convergence's rate of a Newton method, the tangent matrix has to be accurately computed. In this section we discuss the new terms introduced by anisotropic and heterogeneous friction or by a curvature of the obstacle. These modifications are important for the slip jacobian matrix. We examine the more general case of curved contact with heterogeneous and anisotropic friction. At first, we can remark that the derivative of F with respect to λ is not modified by the curvature,

$$\partial_\lambda F = (n - \theta)n^T + \partial_\tau \theta \left\{ I - (n - \tau)n^T \right\}. \quad [21]$$

The first term is the *basic term* present in the 2D flat contact case. The second one is related to the projection on the friction criterion. The derivative of F with respect to u is more complicated,

$$\partial_u F = r \, \|\nabla g\| \, (n - \theta)n^T + \left\{ \sigma_n I + (n - \theta)\lambda^T \right\} \partial_u n - \sigma_n \partial_u \theta. \quad [22]$$

The second term is an *intrinsic curvature term* although the third one includes all non-linearities. To resolve it we can express θ as a function $\hat{\theta}$ of u , λ and x,

$$\theta(u,\lambda) = \mathrm{proj}_{D(x)} \, \tau(u,\lambda) \,\big|_{x=X+u} = \hat{\theta}(x,u,\lambda) \,\big|_{x=X+u} . \quad [23]$$

Then $\partial_u \theta = \partial_x \hat{\theta} + \partial_\tau \theta \, \partial_u \tau$. The derivative with respect to u can be splitted into five parts ($\hat{\sigma}$ is defined in [20]),

$$\begin{aligned} \partial_u F = {} & r \, \|\nabla g\| \, (n - \theta)n^T && [24a] \\ & + r \, \partial_\tau \theta \left\{ I - (n - \|\nabla g\| \tau)n^T \right\} && [24b] \\ & + \left\{ \sigma_n I + (n - \theta)\lambda^T \right\} \partial_u n && [24c] \\ & - \partial_\tau \theta \{ \hat{\sigma}_n I + n \, \hat{\sigma}^T + \tau \lambda^T \} \partial_u n && [24d] \\ & - \sigma_n \partial_x \hat{\theta}. && [24e] \end{aligned}$$

In this expression, we distinguish successively the terms related to *anisotropy* (b), *curvature* (c), *coupling anisotropy and curvature* (d), *heterogeneity* (e). Moreover, the expression (b) is to compare to the second term in [21]. There are equal for the flat contact case ($\|\nabla g\| = \|n\| = 1$) . In the next section, anisotropy of the friction law and curvature of the contact boundary are discussed separately.

5.1. *Anisotropy and heterogeneity*

For convenience, the contact area is assumed to be flat. We can summarize the situation by the following expression which gives the derivatives with respect to u and λ,

$$\partial_u F = r\, \partial_\lambda F - \sigma_n \partial_x \hat{\theta} = r \left\{ \left[(n - \theta) n^T + \partial_\tau \theta \left\{ I - (n - \tau) n^T \right] \right\} - \sigma_n \partial_x \hat{\theta}. \qquad [25]$$

The difficulty is that θ, $\partial_\tau \theta$ and $\partial_x \hat{\theta}$ can be obtained only numerically. Thus a non-linear system in θ and ξ involved in the saddle-point problem associated to the Lagrangian,

$$L(\theta,\xi) = \frac{1}{2} \| \theta - \tau \|^2 + \xi \max(0, h(x,\theta)), \qquad [26]$$

has to be solved. The stationarity condition gives the following problem,

$$\text{find } \tilde{\theta} = (\theta, \xi) \in \mathsf{R}^2 * \mathsf{R}+ \text{ such that } \left\{ \begin{array}{ll} \theta + \xi \nabla_\theta h(x,\theta) - \tau = & 0 \\ h(x,\theta) \quad = & 0 \end{array} \right. . \qquad [27]$$

This system is noted synthetically as follows : $G_x(\tilde{\theta}) = 0$. Applying the Newton method yields,

$$[\partial_{\tilde{\theta}} G_x(\tilde{\theta}_i)] \{\tilde{\theta}_{i+1} - \tilde{\theta}_i\} = - \{ G_x(\tilde{\theta}_i) \}. \qquad [28]$$

Taking into account the special form of the jacobian matrix and its inverse,

$$\partial_{\tilde{\theta}} G_x = \begin{bmatrix} I + \xi \partial_\theta^2 h & \nabla_\theta h \\ (\nabla_\theta h)^T & 0 \end{bmatrix},$$

$$[\partial_{\tilde{\theta}} G_x]^{-1} = \frac{1}{f^T D f} \begin{bmatrix} (f^T D f) D - D f f^T D & D f \\ f^T D & -1 \end{bmatrix}, \qquad [29]$$

(where $f = \nabla_\theta h(x,\theta)$ and $D = [I + \xi \partial_\theta^2 h]^{-1}$), we obtain the more simple recursion formulae,

$$\xi_{i+1} = \frac{1}{g_i^T D_i g_i} \{ h(x,\theta_i) + g_i^T D_i (\tau - \theta_i) \}, \qquad [30a]$$

$$\theta_{i+1} = D_i \{ \tau - (\xi_{i+1} - \xi_i) g_i \}. \qquad [30b]$$

We can thus write $\partial_\tau\theta$ and $\partial_x\hat{\theta}$ by using the implicit function theorem. The simplest matrix is $\partial_\tau\theta$ which is a submatrix of $[\partial_{\tilde{\theta}} G_x]^{-1}$ evaluated at the convergence of the Newton's method indicated by the index c,

$$\partial_\tau\theta = D_c - \frac{D_c f_c f_c^T D_c}{f_c^T D_c f_c}. \qquad [31]$$

The second tangent matrix characteristic of the heterogeneity, $\partial_x\hat{\theta}$, is more complicated. By applying again the implicit function theorem to the system [27] rewritten as $G(x,\tilde{\theta}(x)) = 0$, we obtain,

$$\partial_x\tilde{\theta} = -[\partial_{\tilde{\theta}} G]^{-1}[\partial_x G] \quad \text{where} \quad [\partial_x G] = \begin{bmatrix} \xi\partial^2_{\theta x}h \\ \partial_x h \end{bmatrix}. \qquad [32]$$

We get $\partial_x\hat{\theta}$ by extracting the two first rows and columns of $\partial_x\tilde{\theta}$. To discuss the additional term in the tangent matrix in [25], it is useful to split $\partial_\lambda F$ into three parts,

$$\partial_\lambda F = (n - \theta)n^T + \partial_\tau\theta\,(I - nn^T) + \partial_\tau\theta\,\tau n^T. \qquad [33]$$

The first term is then a *basic term* which is present for 2D discretisation and isotropic friction. The third one is a *specific anisotropic expression* because it vanishes for the isotropic case. Indeed, if isotropy and homogeneity are assumed, the system [27] can be solved analytically, and the tangent matrix is easily derived,

$$\partial_\tau\theta = \partial_\tau(\mu t) = -\mu\sigma_n\,\partial_\tau\left(\frac{\sigma_t}{\|\sigma_t\|}\right) = \frac{-\mu\sigma_n}{\|\sigma_t\|}(I - tt^T) = \rho\,(I - tt^T). \qquad [34]$$

The expression of [33] is then simplified as follows,

$$\partial_\lambda F = (n - \mu t)n^T + \rho\,(I - tt^T)(I - nn^T) + \rho\,(\tau n^T - tt^T\tau n^T), \qquad [35]$$

$$\partial_\lambda F = (n - \mu t)n^T + \rho\,(I - nn^T - tt^T)\ . \qquad [36]$$

It is obvious that the third term in [35] is equal to zero. Finally the second part of [36] is a *3D term* which disappears in 2D discretisation.

5.2. *Curved contact (isotropy assumption)*

The contributions of geometric terms arising from unit normal vector changes at contact points are very complicated. In this context, we restrict our attention to the isotropic case. We resolve $\partial_u F$ in different parts, and we recall that $\partial_\lambda F$ is not modified by an eventual curvature [21],

$$\partial_u F = r\, M_u + N\, \partial_u n. \qquad [37]$$

From [24c] [24d] and [34] we deduce the expression of N which is developed in the following,

$$N = \sigma_n I + (n - \mu t)\lambda^T - \rho\, (I - tt^T)(\, \hat{\sigma}_n I + n\hat{\sigma}^T + \tau\lambda^T), \qquad [38]$$

$$N = \sigma_n I + (n - \mu t)\lambda^T - \rho\, (\, \hat{\sigma}_n I + n\hat{\sigma}^T + \tau\lambda^T - \hat{\sigma}_n\, tt^T - tt^T n\hat{\sigma}^T - tt^T\tau\lambda^T), \qquad [39]$$

$$N = \sigma_n I + (n - \mu t)\lambda^T - \rho\, [\, \hat{\sigma}_n (I - tt^T) + n\hat{\sigma}^T]. \qquad [40]$$

We recover the formulae presented in [ALA 92, HEE 95]. If the contact surfaces are defined in terms of parametric polynomial surfaces patches, a numerical Newton type method is necessary to reach $\partial_u n$, following the same idea applied to the anisotropic frictional non-linearity. Readers interested in details can refer to [HEE 92b]. If, like here, the surface is defined by an implicit function g, the gradient $\partial_u n$ can be analytically computed [17]. Then $N\partial_u n$ can be splitted into two meaningful terms, the first one [41a] *related only to the curvature* and the second one [41b] *coupling curvature and 3D discretisation effects*,

$$N\partial_u n = \|\nabla g\|^{-1}[\, \sigma_n (I - nn^T + \mu n t^T) + (n - \mu t)\lambda_t^T\,]\, Hg \qquad [41a]$$

$$+ \|\nabla g\|^{-1} \rho\, \hat{\sigma}_n [\, I - nn^T - tt^T]\, Hg. \qquad [41b]$$

Indeed the last expression vanishes in 2D discretisation.

6. Implementation aspects

In several programs the surfaces are generated by Computer Aided Geometrical Design techniques in terms of parametrized polynomial surface patches. We propose here an alternative approach by using an implicit function g. This assumption avoids to use numerical algorithms to compute reference position, normal distance and its first and second derivatives [HEE 95]. Indeed additional numerical techniques may lead to new numerical instabilities. But the difficulty is to find an implicit function for a given geometry and to compute its gradient and its hessian matrix.

Frictional contact is implemented in the TACT program [CUR 87] in terms of *contact elements*. The assembling, linear and non-linear solution algorithms of a standard finite element code have not to be modified. Thus a contact element, associated to a node of the potential contact area of the body, has to contribute an elemental non-linear system of equation which are assembled to the global one. Reminding that we use the Newton method, this elemental contribution consists in a local contact operator for the right hand side of the global system — similar to the elemental internal forces for a finite element — and the corresponding tangent matrix — similar to the elemental stiffness matrix.

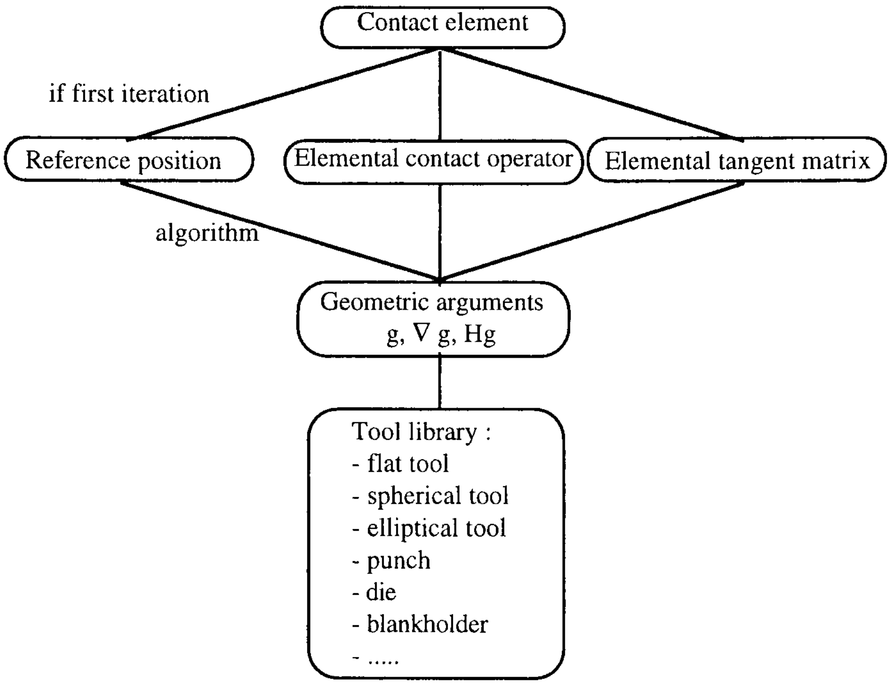


Figure 4. *Contact element organisation*

Finally a curved contact element consists in different tasks or subroutines for implementation (*Figure 4*). In order to take into account complex geometries, the user must insert and enrich a *tool library*. The main problem is to dispose of a function g and its first and second derivatives. For that, we used a *symbolic computation software* (MAPLE) which, from a given function, supplies with its gradient, its hessian matrix and the resulting Fortran instructions. Some tools may be built with other simple tools like the punch for example which is splitted into three parts : flat, cylindrical and toric. The last one requires a symbolic computation. In appendix we give a MAPLE procedure called Gfunc to construct a function g associated to a torus. The three last lines write the Fortran instructions evaluating g, its gradient and its hessian matrix. We recover these instructions in the subroutine describing a punch tool for its toric part (cf. the appendix).

7. Numerical tests

7.1. *Punch problems*

The contact between a deformable cylindrical punch and a rigid foundation presents an appropriate (benchmark) problem to illustrate the performance of the enriched formulation. To this end, the foundation can be assumed to be flat with isotropic friction, or flat with anisotropic friction, or curved (spherical or ellipsoidal). In the first case, we nearly recover the classical benchmark problem presented in [CUR 89]. The cylindrical punch is pressed against the obstacle, the upper boundary of the punch being uniformly moved. Loading and unloading are investigated. The mechanical and material data are specified in *Figure 5*. For the anisotropic case, the function h defining $C(\lambda_n)$ is

$$h(\theta) = \frac{\theta_x}{\mu_x^2} + \frac{\theta_y}{\mu_y^2} - 1. \qquad [42]$$

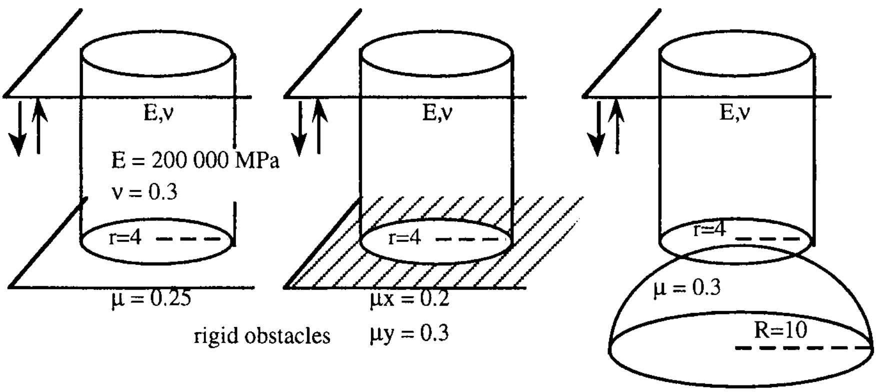


Figure 5. *Punch problems*

The punch is discretized by only two layers of 8-node finite elements, but the total number of 289 contact nodes proves adequate to accurately reach the complexity of the contact status upon unloading. The loading is obtained by 5 increments, the unloading by 10 decrements.

The main objectives are :

- to illustrate the convergence behaviour of the algorithm in the different cases,
- to compare the distribution of stick and slip areas upon unloading.

The logarithm of the convergence norm is plotted with respect to the iteration number in the *Figures 6, 7 and 8*.

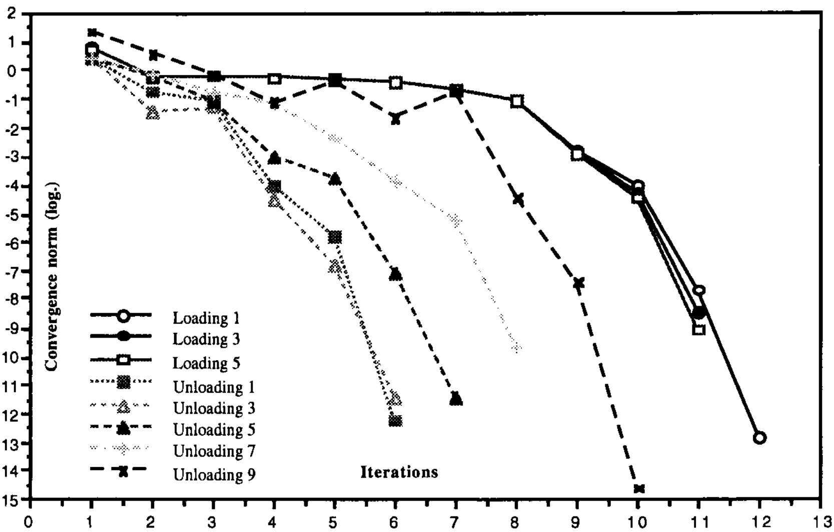


Figure 6. *Convergence behaviour — Flat isotropic case*

During loading, for flat contact, the rate of convergence does not depend on the step number. Indeed, since the status remains unchanged from one step to another, the system of equations keeps the same structure. The typical convergence behaviour shows two stages. During the first iterations, the algorithm searches the true contact status of each node, so that the convergence norm shows no valuable decrease. In a second stage, the convergence rate becomes quadratic.

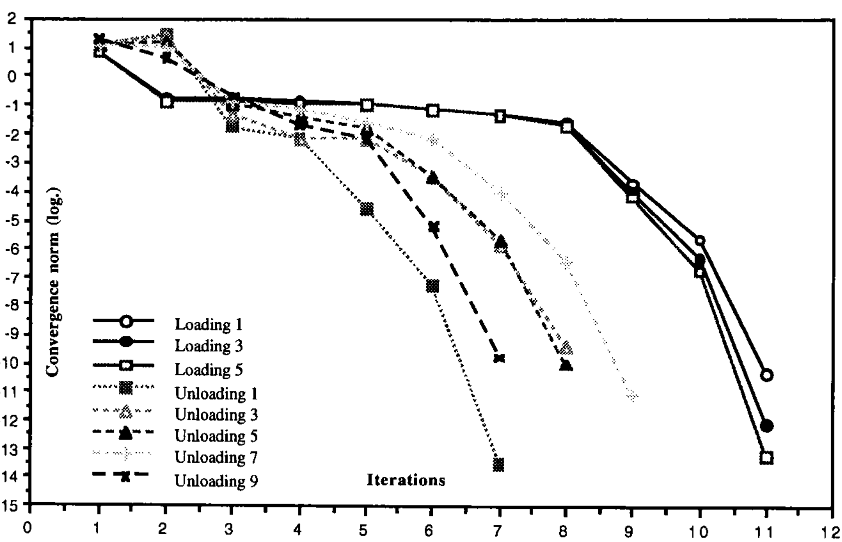


Figure 7. *Convergence behaviour — Flat anisotropic case*

Unlike flat cases, for the curved case, the number of contact node increases at each step as the convergence rate decreases.

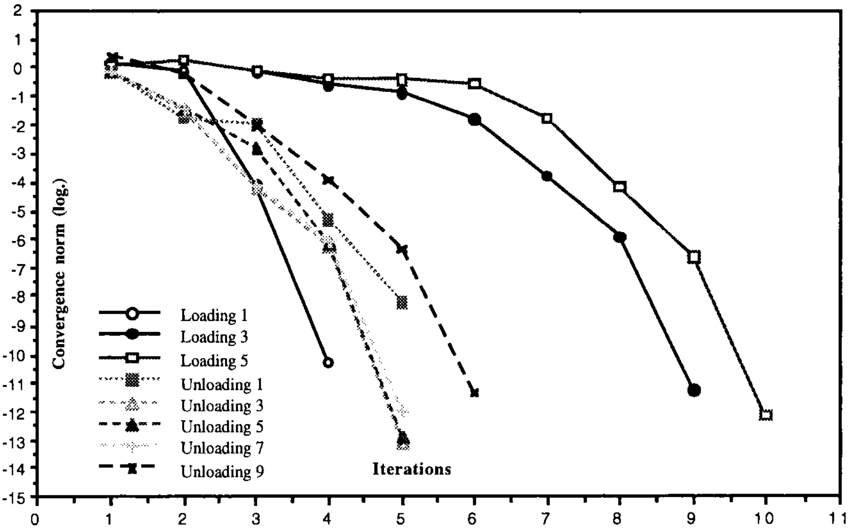


Figure 8. *Convergence behaviour of the generalized Newton method — Curved case*

Paradoxically, the convergence is better during unloading than during loading. However the contact area is more complicated, decomposed in many stick or slip regions.The convergence spoils as soon as the inward slip appears on the edge of the contact area : step 5 in isotropic case, step 3 in anisotropic one (*Figure 9*). In the curved case, this behaviour disappears because the number of contact nodes decreases simultaneously. The stick/slip annuli become nearly elliptical in anisotropic case (*Figure 9*)). Moreover, the inward slip region is not connex at step 8 before vanishing at step 10. In other words, some contact nodes keep slipping outward in the direction of a higher friction coefficient.

7.2. *Deep drawing simulation*

Metal forming processes involve tools with complex geometries. In several industrial programmes the surfaces are generated by Computer Aided Geometrical Design techniques in terms of parametrized polynomial surface patches. Using an alternative approach, simple tools are considered. The only aim of this example is to show the ability to solve within a single loop the non-linearities due to contact and to large elastic-plastic deformations.

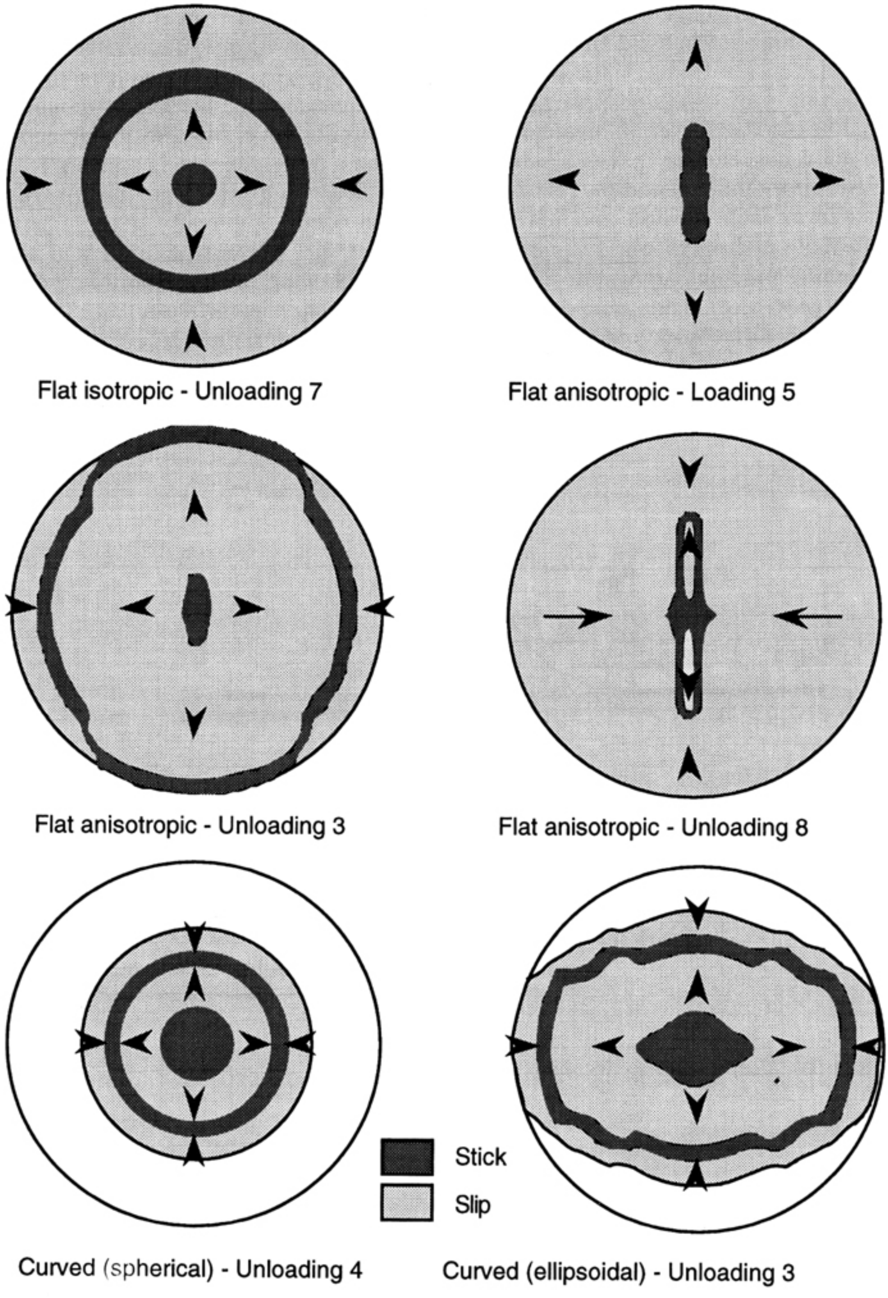


Figure 9. *Slip and Stick regions*

Taking advantage of the symmetries, only a quarter of the sheet is discretized into 450 eight-node solid finite element with the following features : Lagrangian formulation for finite elastic-plastic strains, Von Mises criterion with isotropic hardening and associated flow rule. The elastic limit and the hardening modulus must be divided by ten to have a realistic problem but the elastic-plastic elements are not robust enough in TACT programm. Indeed a deformation ϕ is admissible if it preserves the orientation ($\det\nabla\phi > 0$). This last condition is not necessarily assured at each Newton iteration. This situation occurs especially if the plastic deformation through one step is very important, i.e. the elastic limit and the hardening modulus are small. Consequently, the Newton method can diverge or converge to a non admissible deformation. To overcome this difficulty, an updated lagrangian formulation can be used [HEE95] and the constraint ($\det\nabla\phi > 0$) must be introduced in the elastic-plastic element. For more realistic simulations of deep drawing refer to [HEE 95]. We have 256 contact elements on the upper and lower boundaries of the sheet. The geometry of the tools is given on *Figure 10.*

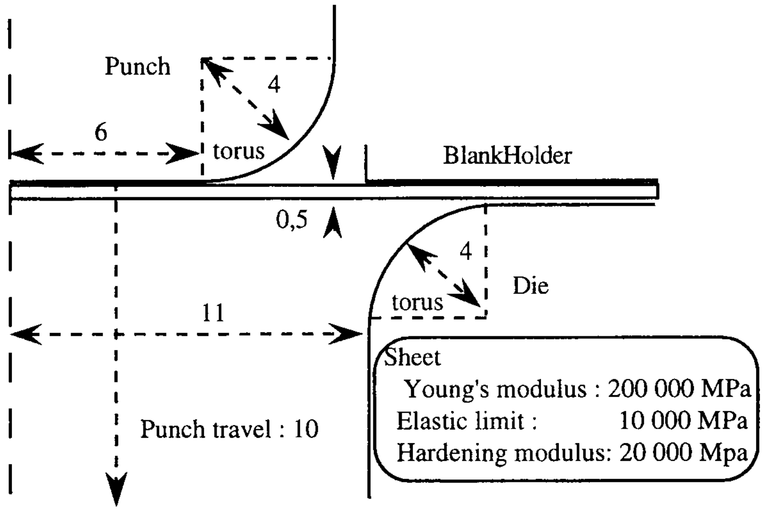


Figure 10. *Deep drawing process*

The process is divided in four phases. During the locking phase the blankholder advances against the sheet until a prescribed value. Thus the blankholder is controlled by kinematic shedule instead of a prescribed global restraining force. The pulling phase consists in the moving of the punch against the metal sheet. In the third phase the punch returns to its initial position. In the fourth phase, the blankholder is taken off.

The whole process is performed within 71 loading steps (phase 1 : 1; phase 2 : 40; phase3 : 10; phase 4 : 20). For the first phase, fifteen iterations are necessary. The second phase requires an average of twelve iterations by steps; we have simultaneously large plastic deformations and evolutive contact surfaces. In the third phase, a first springback stage occurs along the two first steps (10 iterations by steps), after that there is no contact between the sheet and the punch (only one iteration). The fourth phase is difficult to control in a quasi-static simulation. A large number of steps (17 iterations by steps on a average) is then necessary to find the instant when the blankholder looses contact with the sheet. When the blankholder is free, a dynamic phenomenon should be considered. *Figure 11* shows the geometry of a section at the beginning of the process, before and after the springback. The springback effect is emphasized due to the underestimated hardening modulus [CAO 89].

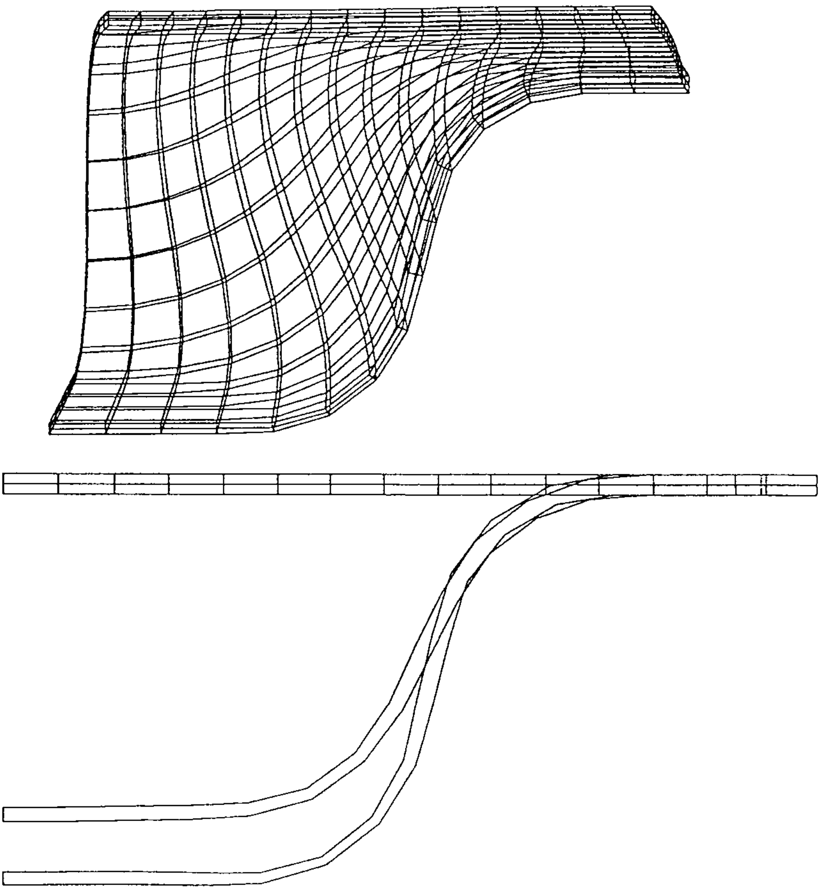


Figure 11. *Deformed mesh and sections before and after the springback.*

8. Conclusion

In this paper, we have shown that anisotropic friction and curved contact may be easily implemented in a fully implicit scheme using a generalized Newton method as a standard non linear solver. If the consistent matrices are used, we recover the typical convergence behaviour of the Newton algorithm. The performance of the method is not much modified when other non-linearities due to the deformable body behaviour are solved within the same iteration loop.

The methods presented in this paper seem to be interesting for contact with a rigid obstacle but they are not easy to extend to the contact between two deformable bodies. But, if the obstacle presents strongly curved surface this strategy leads to accurate results and stable algorithms.

References

[AGO 93] AGOUZAL A., Analyse numérique de méthodes de décomposition de domaines, methodes des domaines fictifs avec multiplicateurs de Lagrange, Université de Pau et des Pays de l'Adour, PhD Thesis, France, 1993.

[ALA 91] ALART P., CURNIER A., "A mixed formulation for frictional contact problems prone to Newton like solution methods", *Comp. Meth. in Appl. Mec. & Eng.*, 92, 3, 353-375, 1991.

[ALA 92] ALART P., "A simple contact algorithm applied to large sliding and anisotropic friction", *Proc. Contact Mechanics International Symposium*, (A.Curnier ed.), Presses Polytechniques Romandes, Lausanne, 1992.

[ALA 93] ALART P., "Critères d'injectivité et de surjectivité pour certaines applications de R^n dans lui-même : application à la mécanique du contact", *RAIRO Modélisation Mathématique et Analyse Numérique*, 27, 2, 203-222, 1993.

[CAO 89] CAO H.L., TEODOSIU C., "Finite element calculation of springback effects and residual stresses after 2D deep drawing", *Proc. 2nd Int. Conf. on Computational plasticity*, (Owen-Hinton-Onate ed.), Barcelona, 1989.

[CAP 79] CAPUZZO DOLCETTA I., MOSCO U., "Implicit complementarity problems and quasi-variational inequalities", *Variational Inequalities and Complementarity Problems*, Wiley, New York, 75-88, 1979.

[CUR 87] CURNIER A., *TACT : A contact analysis program, Proceedings of 3rd meeting on Unilateral Problems in Structural Analysis*, (Del Piero-Maceri ed.), Springer-Verlag, Wien, New-York,1987.

[CUR 88] CURNIER A., ALART P.A., "Generalized Newton Method for Contact Problems with Friction", *Journal de Mécanique Théorique et Appliquée, Special Issue : Numerical Method in Mechanics of Contact Involving Friction*, 67-82, 1988.

[FOR 76] FORTIN M., "Minimization of some non-differentiable functionals by augmented Lagrangian method of Hestenes and Powell", *Appl. Math. Opt.,* 2, 4, 1976.

[FOR 82] FORTIN M. GLOWINSKI, *Méthodes de Lagrangien augmenté*, Collection Méthodes Mathématiques de l'Informatique, Dunod, Paris, 236-250, 1982.

[HE 93] HE Q.C., CURNIER A., "Anisotropic dry friction between two orthotropic surface undergoing large displacements", *Eur. J. of Mech. A/Solid*, 12, 5, 631-666, 1993.

[HEE 92a] HEEGE A., ALART P., "On an implicit contact-friction algorithm dedicated to 3D sheet forming simulation", *Proceedings of NUMIFORM 92*, Valbonne, Balkema, Rotterdam, 479-484, 1992.
[HEE 92b] HEEGE A., Simulation numerique 3D du contact avec frottement et application a la mise en forme, Institut National Polytechnique de Grenoble, PhD Thesis, Grenoble, 1992.
[HEE 93] HEEGARD J.C., CURNIER A., "An augmented Lagrangian method for discrete large slip problems", *Int. J. Num. Meth. Eng.*, 36, 569-593, 1993.
[HEE 95] HEEGE A., ALART P., "A frictional contact element for strongly curved contact problems", submitted to *Int. J. Num. Meth. Eng.*
[MOR 79] MOREAU J.J., "Application of convex analysis to some problems of dry friction", *Trends of Pure Mathematics Applied to Mechanics*, (Zorski ed.), Pittman Publishing Ltd, London, 263-280, 1979.
[RAK 91] RAKOTOMANANA R.L., CURNIER A., LEYRAZ P.F., "An objective anisotropic elastic plastic model and algorithm applicable to bone mechanics", *Eur. J. of Mech. A/Solid*, 10, 3, 327-342, 1991.
[ROC 76] ROCKAFELLAR R.T., "Augmented Lagrangians and applications of the proximal point algorithm in convex programming", *Math. Oper. Res.*, 1, 2, 97-116, 1976.
[SIM 85] SIMO J.C. & TAYLOR R.L., "Consistent tangent operator for rate-independent elastoplasticity", *Comp. Meth. in Appl. Mec. & Eng.*, 48, 101-118, 1985.

9. Appendix

9.1. *About some formulae*

* Using the preliminary computation, we obtain [16] :

$$\partial_u \delta = I \quad ; \quad \partial_u g = \|\nabla g\|\, n^T \quad ; \quad \partial_u \delta_n = n^T + \delta^T \partial_u n.$$

* To recover [21], it is necessary to evaluate the following expressions :

$$\partial_\lambda F = (n - \theta)\partial_\lambda \sigma_n - \sigma_n\, \partial_\tau \theta\, \partial_\lambda \tau \quad \text{with}$$

$$\partial_\lambda \tau = \partial_\lambda\left(\frac{\sigma_t}{-\sigma_n}\right) = \frac{-1}{\sigma_n}\partial_\lambda \sigma_t + \frac{\sigma_t}{\sigma_n^2}\partial_\lambda \sigma_n \quad ; \quad \partial_\lambda \sigma_n = n^T \quad ; \quad \partial_\lambda \sigma_t = I - nn^T.$$

* Equation [22] is obvious by considering :

$$\partial_u F = (n - \theta)\partial_u \sigma_n + \sigma_n \partial_u n - \sigma_n \partial_u \theta) \text{ with } \partial_u \sigma_n = \partial_u(\lambda_n + rg) = \lambda^T \partial_u n + r\|\nabla g\|\, n^T.$$

* Equation [24] is more complicated to get. The difficulty is to derive θ with respect to u. We have successively :

$$\partial_u\lambda_t = \partial_u\{(I - nn^T)\lambda\} = -\partial_u(nn^T\lambda) = -\lambda_n\partial_u n - n\lambda^T\partial_u n,$$
$$\partial_u\delta_t = \partial_u\{(I - nn^T)\delta\} = (I - nn^T) - (\delta_n I + n\delta^T)\partial_u n,$$
$$\partial_u\sigma_t = \partial_u(\lambda_t + r\delta_t) = r(I - nn^T) - (\hat{\sigma}_n I + n\hat{\sigma}^T)\partial_u n,$$
$$\partial_u\tau = \frac{\sigma_t}{\sigma_n^2}\partial_u\sigma_n - \frac{1}{\sigma_n}\partial_u\sigma_t = \frac{-1}{\sigma_n}\{\tau\partial_u\sigma_n + \partial_u\sigma_t\},$$
$$= \frac{-1}{\sigma_n}\{\tau\lambda^T\partial_u n + r\|\nabla g\|\tau n^T + r(I - nn^T) - (\hat{\sigma}_n I + n\hat{\sigma}^T)\partial_u n\}.$$

To get [24b,d,e], this last expression has to be replaced in :

$$\partial_u\theta = \partial_x\hat{\theta} + \partial_\tau\theta\,\partial_u\tau\,.$$

9.2. *MAPLE programm and resulting FORTRAN instructions*

MAPLE Programm to describe a torus (function g, its gradient and its hessian matrix).

```
Gfunc := proc()
options trace;
   x1 := array([Z1,Z2,Z3]);
   P := array([[1,0,0],[0,1,0],[0,0,0]]);
   x2 := linalg[multiply](P,x1);
   xno := sqrt(linalg[dotprod](x2,x2));
   exp := (RG+RP)/xno;
   xc := -exp*x2;
   d1 := linalg[add](x1,xc);
   g1 := linalg[dotprod](d1,d1);
   G := 1/2*g1-1/2*RP^2;
   readlib(fortran);
   GGL := linalg[grad](G,x1);
   HGL := linalg[hessian](G,x1);
   fortran(G,optimized);
   fortran(GGL,optimized);
   fortran(HGL,optimized)
end
```

FORTRAN Programm to describe a cylindrical punch decomposed in three elemental parts : flat part, cylinder and torus.

```
      SUBROUTINE E7PUNC (ZL,CT,NDIM,
                 G,GGL,HGL)
      IMPLICIT REAL*8(A-H,O-Z)
      DIM; ZL(3),CT(16),HGL(3,3),GGL(3)
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C PUNCH TOOL SPLITTED INTO 3 PARTS :
C    CYLINDER, FLAT PART, TORUS.
C G    : FUNCTION DEFINING
C          THE OBSTACLE (G(X) < 0)
C GGL  : GRADIENT OF G
C HGL  : HESSIAN MATRIX OF G
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      ONE = 1.D0
      ZERO = 0.D0
      G = ZERO
      DO 20 I=1,NDIM
       GGL(I) = ZERO
       DO 10 J=1,NDIM
        HGL(I,J) = ZERO
 10    CONTINUE
 20  CONTINUE
      Z1 = ZL(1)
      Z2 = ZL(2)
      Z3 = ZL(3)
      Z12NOR = Z1*Z1 + Z2*Z2
      RP = CT(4)
      RG = CT(3)
      RGG = RG + RP
      IF(Z3.LE.ZERO) THEN
c
C     Cylinder
c
       G = ( Z12NOR - RGG*RGG )/RGG/2.D0
       GGL(1) = Z1/RGG
       GGL(2) = Z2/RGG
       HGL(1,1) = ONE/RGG/RGG
       HGL(2,2) = HGL(1,1)
      ELSE
       IF(Z12NOR.LE.RG*RG)THEN
c
C     Flat
c
        G = Z3 - RP
        GGL(3) = ONE
       ELSE
C
C      Torus ( by MAPLE software)
c
      t1 = RG
      t2 = Z1**2
      t3 = Z2**2
      t4 = t2+t3
      t5 = DSQRT(t4)
      t6 = 1/t5
      t11 = (Z1-t1*t6*Z1)**2
      t17 = (Z2-t1*t6*Z2)**2
      t19 = Z3**2
      t21 = RP**2
      G = (t11+t17+t19-t21)/2.D0
c
      t10 = Z1-t1*t6*Z1
      t11 = t4**2
      t13 = t5/t11
      t17 = -t1*t6
      t23 = Z2-t1*t6*Z2
      t26 = t1*t13*Z2*Z1
      GGL(1)=
t10*(ONE+t1*t13*t2+t17)+t23*t26
      GGL(2)=
t10*t26+t23*(ONE+t1*t13*t3+t17)
      GGL(3) = Z3
c
      t6 = t4**2
      t8 = t5/t6
      t11 = 1/t5
      t13 = -t1*t11
      t14 = 1+t1*t8*t2+t13
      t18 = t14*t1*t8*Z2*Z1
      t22 = Z1-t1*t11*Z1
      t24 = ONE/t6/t4
      t25 = t5*t24
      t28 = t1*t25*t2*Z2
      t32 = t1*t8*Z2
      t36 = ONE+t1*t8*t3+t13
      t40 = t1*t8*Z2*Z1*t36
      t44 = Z2-t1*t11*Z2
      t47 = t1*t25*t3*Z1
      t50 = t1*t8*Z1
      t61 = t1**2
      t64 = t61*t24*t3*t2
      t68 = t36**2
      t77 = t14**2
      HGL(1,2) = t18-3.D0*t22*t28+t22*t32
     1        +t40+t44*(-3.D0*t47+t50)
      HGL(3,3) = ONE
      HGL(1,3) = ZERO
      HGL(2,1) = HGL(1,2)
      HGL(2,2) = t64-3.D0*t22*t47+t22*t50+t68
     1      +t44*(-3.D0*t1*t25*t3*Z2+3*t32)
      HGL(2,3) = ZERO
      HGL(3,1) = ZERO
      HGL(1,1) = t77+t22*(-3.D0*t1*t25*t2*Z1
     1  +3.D0*t50)+t64-3.D0*t44*t28+t44*t32
      HGL(3,2) = ZERO
c
       ENDIF
      ENDIF
      RETURN
      END
```

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