Elastoplastic Finite Element Analysis of Soil Problems with Implicit Standard Material Constitutive Laws

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ABSTRACT. *A new class of materials called Implicit Standard Materials is proposed. It allows to generalize Fenchel's inequality, and then to recover flow rule normality, for non-standard materials. We can show that implicit standard materials method describes several behaviours with simpler manner. We apply this approach to soils mechanics in order to build a non-associated constitutive law as the experience suggests. In term of FEM, an algorithm based on Newton's method is proposed. It allows to obtain a symmetric stiffness matrix in reverse to actual non-associated formulation. A bearing capacity problem is considered as numerical application. Some results and theorems are discussed.*

RÉSUMÉ. *Une nouvelle classe de matériaux appelée Matériaux Standards Implicites est proposée. Elle permet de généraliser l'inégalité de Fenchel, et ainsi de retrouver la règle de normalité, pour les matériaux non-standards. On montre que la méthode des matériaux standards implicites décrit plusieurs comportements d'une manière plus simple. Nous appliquons cette approche en mécanique des sols dans le but de construire une loi constitutive non associée comme l'exige l'expérience. Dans le cadre de la méthode des éléments finis, un nouvel algorithme basé sur la méthode de Newton est proposé. Il permet d'obtenir une matrice tangente symétrique à l'inverse de la formulation non associée actuelle. Le problème de la capacité portante est considéré comme application numérique à partir de laquelle nous discuterons quelques résultats et théorèmes relatifs à l'analyse limite.*

KEY WORDS : *elastoplasticity, soil mechanics, non associated law, limit analysis, finite element method.*

MOTS-CLÉS : *élastoplasticité, mécanique des sols, loi non associée, analyse limite, méthode des éléments finis.*

Notations

f	loading surface.
g	plastic potential.
b	bipotential.
$\mathbf{s}$	stress deviator.
s_m	hydrostatic pressure.
$\mathbf{u}$	displacements vector.
ε	strain tensor.
σ	stress tensor.
$\mathbf{t}$	surface traction.
q	uniform load.
$\mathbf{f}$	volumic forces .
e_m	trace of strains.
e_m^p	trace of plastic strains.
$\mathbf{e}$	strain deviator.
$\mathbf{e}^p$	plastic strain deviator.
c	cohesion.
θ	dilatancy angle.
φ	friction angle.
ρ	ratio θ/φ
$\mathbf{e}^e$	elastic strain deviator.
e_m^e	trace of elastic strains.
$\hat{\mathbf{n}}$	unit vector in deviator space.
$\mathbf{F}_e$	external forces.
$\mathbf{I}$	Kronecker tensor.
$\mathbf{k}_T$	tangent stiffness matrix.
$\lVert . \rVert$	Euclidian norm.
$\otimes$	tensorial product.
©	inf convolution product.
$(a)_+$	positive part symbol :
	$= a$ if $a \geq 0$.
	$= 0$ otherwize.

1. Introduction

The soil materials exhibit various properties that distinguish them from metals and makes them very difficult to provide accurate and realistic numerical solutions of boundary value problems. Among these properties, can be quoted the following ones [17] :

- the soils are heterogeneous materials, because of the presence of detritus such plants in many layers ;
- the soil layers are strongly influenced by water, the grains being often surrounded by a pellicle of water ;
- the soils are anisotropic materials because of the sedimentary structure ;
- the soils are non standard materials.

Practically, taking into account all of these properties is an impossible task and it is out of the scope of this paper to analyze all of them. The purpose of this work is to focus the attention on the last point. It is generally acknowledged that the plastic flow rule of the soil materials is not associated. In other words, the plastic strain rate vector is not normal to the yield locus. This property leads to various special effects such as presence of softening in the load displacement curve [36] and significative decreasings of the limit load with respect to the corresponding standard material in many applications.

The usual modelling of a non associative flow rule is based on a couple of stress functions, the yield function to define the yield locus and Melan's plastic potential to give the direction of the plastic strain rate vector. Unfortunately, such a formulation leads to the loss of the good properties of standard materials deriving from the key-concepts of convexity and normality [2,5,7,8,55]. In particular, one can cite the existence of functionals wich allow to apply the usual calculus of variation. From the theoretical point of view, this provides a simple method to prove the existence of solutions of the boundary value problem by means of the functional and convex analysis tools [1]. From the numerical point of view, the mathematical programming algorithm may be used to solve discretized problems. Another very important theoretical property of standard materials is the possibility to prove upper and lower bound theorems concerning the direct calculation of the limit load.

On the other hand, the non standard materials are unpleasant. Hence, all of the previous good features are lost. Because the normality and convexity are very convenient tools, the theory of the Implicit Standard Materials (ISM) was imagined in order to extend in a natural way the good properties of the usual standard materials to non associated flow rules [5,7,9,24-26]. How this result can be achieved?

In fact, the normality rule can be preserved but only in the weak form of an implicit relation (in the sense of the implicit function theorem). The new idea was initially applied successfully in unilateral contact problems with Coulomb's friction [24-26]. Morover, the new concept supplies a theoretical frame to model the constitutive law of soils. For the simplicity of the purpose, Rudnicki-Rice model

[13] is considered, wich involves Drucker-Prager yield locus, depending on two material parameters, the cohesion stress c and the friction angle φ. The non associativity of the flow rule is caracterized by a third parameter, the plastic dilatancy angle θ, lying within the range from 0 to φ. This simple model was used by several authors in the numerical applications and will be adopted in this work to analyse the non associativity, although a more realistic representation of the yield locus taking into account the influence of the third stress invariant was proposed in the experimental testing literature [33,34,53].

A particular attention is taken in the definition of the flow rule at the vertex of Drucker-Prager cone, but one of the originalities of the present paper is to prove Rudnicki-Rice material is an Implicit Standard one by introducing a suitable bipotential depending on both stress and plastic strain rate tensers. The properties of the so called bipotential are based on an extension of Fenchel's inequality [23] and allow to generalized Ziegler potential [6] and, in the frame of the Convex Analysis, Moreau superpotential [54].

One of the advantages of the new formulation is to extend the classical Calculus of Variations to non associated constitutive laws. In the theoretical frame of the Implicit Standard Materials, a new functional, called bifunctional, is introduced, depending on both the displacement and stress field. The exact solution of the Boundary Value Problem corresponds to the simultaneous minimization of the bifunctional, firstly with respect to kinematically admissible displacement fields, when the stress field is equal to the exact one, and secondly with respect to statically admissible stress fields, when the displacement field is the exact one. The two minimization problems are a direct extension of the dual variational principles of displacements and stresses.

The second part of the present paper is concerned with the numerical computation of the elastoplastic evolution problems. Concerning the soil materials, one can quote the work of Hoëg [42,45], Tang and Hoëg [44], Costet and Sanglerat [46], Schofield and Wroth [47]. Recently, a reference book related to this topic was published by Chen [60].

In the present paper, the finite element method is used to discretized the boundary value problem, in order to compute the elastoplastic evolution problem. The implicit scheme (Moreau's catching up algorithm [14-16]) is used for the time-integration of the differential constitutive law. This predictor-corrector scheme may be interpreted as an extension of Simo-Taylor radial return algorithm [10].

Using the finite element method, the unknowns of the problems are the nodal displacements and the local stresses at the integration points of the elements. In this space of unknowns, the solution of the boundary value problem lies at the intersection of the non linear subspace G defined by the implicit constitutive law and the linear subspace A_d of the statically admissible solutions. This suggests to use one of the key-idea of Ladeveze's LATIN method [29-31]. On this base, a new iterative algorithm is proposed to compute the step solution. The current iteration involves a local stage (upwards search direction) and a global one (downward search direction). In the global stage, the stresses are fixed and a new approximation of the nodal displacement increment is computed by solving the global equilibrium equation in the finite element sense. In the local stage, the displacement are fixed and a new approximation of the local stresses is computed by solving the implicit constitutive law at each integration point. Of course, the computation time of the local stage is negligible with respect to the one of the global stage.

Two strategies are proposed. In the first one, Newton's scheme is applied to the whole set of equations. The structural tangent stiffness matrix results unsymmetric and not necessary definite positive and consequently often ill-conditioned. This is the kind of tangent stiffness matrix generally proposed in the literature [10,36]. In the second one, Newton's scheme is separatly applied to the equilibrium equation and to the implicit constitutive law. The most relevant point is the symmetry and definite positivity of the tangent stiffness matrix. The numerical applications shown the latter strategy leads to a significative improvement with respect to the former one in the point of view of the computation time.

The main numerical difficulty in applying Newton's method is to estimate the step size in order to inforce the convergence of the iterative scheme. In the present paper, the adopted strategy is based on the continuation method [48-52].

The formulation of Implicit Standard Materials is a constructive method in the sence it allows to proposed new variational principles and numerical algorithms. It is an alternative theory to Panagiatopoulos hemivariational inequation approach [43] and Barros-Marques-Martins formulation [32]. The last one is very artful, but the authors of the present paper think the introduction of hardening is artificial and not necessary to understand the nature of the non associativity.

Historically, the rigid perfectly plastic material was extensively used in the soil problems to obtain analytical solutions, by using the slip line theory (Sokolovski [37]), the method of limit equilibrium (Fellenius [38], Terzaghi [39] and Taylor [40]), or limit analysis (Drucker [11], Prager [35], Schield [18], Chen and Davidson [17-41]). But, unfortunately, the limit analysis does not allow to take into account the important effect of lack of normality. In this frame, the Implicit Standard Material model provides a new point of view on the rather old question of extending the limit analysis to non standard behaviours. In a paper written in 1953, Drucker proves the limit load of any non standard material is less than the one of corresponding standard one [56]. His result is in fact a "minoration" property but not a bounding theorem in limit analysis sense. Other related results are discussed in the literature by Mroz [19-20], Palmer [21], De Jong [22], Dais [27,28] and Telega [57], but does not seem to provide decisive improvements. In a recent work [58], de Saxcé and Bousshine proposed an extension of the bound theorems of the limit analysis to the class of the Implicit Standard Materials. Some relevant features are pointed out, such as non uniquiness of the limit load, coupling of the lower and upper bound problems and that limit load can be reached after a softening zone. The present paper provides some numerical results concerning the last point and the assesment of the sensitivity of the limit load with respect to the lack of associativity of the plastic yielding load. Related results concerning the non associated soil material problems coupled with Coulomb's dry friction contact are presented recently by de Saxcé and Bousshine [59].

2. The Implicit Standard Material

In solid mechanics, a very large range of material behaviours can be represented by a relation of the following form :

$$\mathbf{x} \in \mathbf{A}(\mathbf{y}), \quad \mathbf{y} \in \mathbf{A}^{-1}(\mathbf{x}) \tag{2.1}$$

where **A** is a multivalued mapping, and **x** (resp. **y**) is a generalized strain (resp. stress) vector. The quantities **x** and **y** may be understood in a very large meaning : instantaneous values, velocities or finite increments. This definition is very large but generally gives very little relevant informations about the properties required for solving the boundary value problem, that is the existence of solutions. These difficulties can be reflected in the numerical implementation of the solving technique [10].

A more restricted range of material leading to " good properties " for the boundary value problem is one of the so-called standard materials. The existence of two *convex* potentials V(**x**) and W(**y**) is postulated. It is supposed they satisfy *Fenchel's inequality* [23] :

$$\forall\ (\mathbf{x}', \mathbf{y}'), \qquad V(\mathbf{x}') + W(\mathbf{y}') \geq \mathbf{x}' \cdot \mathbf{y}' \tag{2.2}$$

A couple (**x**, **y**) of strain and stress vectors is said *extremal* if the equality is achieved in (2. 2) :

$$V(\mathbf{x}) + W(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y} \tag{2.3}$$

From (2. 2) and (2. 3), it can be deduced that for any extremal couple (**x**, **y**), one has :

$$\begin{aligned} &\forall\ \mathbf{x}' \qquad V(\mathbf{x}') - V(\mathbf{x}) \geq \mathbf{y} \cdot (\mathbf{x}'-\mathbf{x}) \\ &\forall\ \mathbf{y}' \qquad W(\mathbf{y}') - W(\mathbf{y}) \geq \mathbf{x} \cdot (\mathbf{y}'-\mathbf{y}) \end{aligned} \tag{2.4}$$

So, **x** and **y** are related by subdifferential mappings (see the annex) :

$$\mathbf{y} \in \partial V(\mathbf{x}), \quad \mathbf{x} \in \partial W(\mathbf{y}) \tag{2.5}$$

The potential W is said the conjugate one of V.

This formalism allows to represent multivalued constitutive laws such as in plasticity or viscoplasticity. Neverthless, numerous behaviours are encountered in soil materials, which do not belongs to this very practical family of the standard materials, because of the non-associated flow rule.

A " good generalisation " of the standard materials was proposed in [24-25-26], which preserves the notion of extremal couple and the convexity assumptions. For this familly of materials, called *implicit standard materials*, the existence of a function b(**x**, **y**), *convex* with respect to **x**, when **y** remains constant, and *convex* with respect to **y**, when **x** remains constant, is postulated. The function b is said a *bipotential* if the following inequality is satisfied :

$$\forall\ (\mathbf{x}', \mathbf{y}'), \qquad b(\mathbf{x}', \mathbf{y}') \geq \mathbf{x}' \cdot \mathbf{y}' \tag{2.6}$$

A couple (**x**, **y**) is said *extremal* if the equality is achieved in (2.6) :

$$b(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} \tag{2.7}$$

Any extremal couple is characterized by the following relations :

$$\begin{aligned} &\forall\ \mathbf{x}', \qquad b(\mathbf{x}', \mathbf{y}) - b(\mathbf{x}, \mathbf{y}) \geq \mathbf{y} \cdot (\mathbf{x}'-\mathbf{x}) \\ &\forall\ \mathbf{y}', \qquad b(\mathbf{x}, \mathbf{y}') - b(\mathbf{x}, \mathbf{y}) \geq \mathbf{x} \cdot (\mathbf{y}'-\mathbf{y}) \end{aligned} \tag{2.8}$$

Therefore, **x** and **y** are related by subdifferential mappings :

$$\mathbf{y} \in \partial_x b(\mathbf{x}, \mathbf{y}), \qquad \mathbf{x} \in \partial_y b(\mathbf{x}, \mathbf{y}) \tag{2.9}$$

This relations allows to represent a multivalued constitutive law but the relationship between **x** and **y** is now implicit, in the sense of the implicit function theorem. Of course, explicit standard materials are particular cases of implicit standard ones with a separable bipotential :

$$b(\mathbf{x}, \mathbf{y}) = V(\mathbf{x}) + W(\mathbf{y}) \tag{2.10}$$

3. A non associated flow rule for soil materials

In this section, the behaviour of soil materials characterized by Drucker-Prager plastic yielding surface [17] is considered. For convenience, this following quantities are introduced :

- the hydrostatic pressure : $s_m = \frac{1}{3} Tr(\sigma)$

- the stress deviator : $\mathbf{s} = \sigma - s_m \mathbf{1}$

- the trace of strain tensor : $em = Tr(\varepsilon)$

- the strain deviator : $\mathbf{e} = \varepsilon - \frac{1}{3} e_m \mathbf{1}$

where convention signs are those of solids mechanics.(tensile stresses are positive).

In this representation , the plastic law can be modelized by the dual vectors of stresses $\sigma=(s_m,s)$ and of plastic strain rates $\dot{\varepsilon}^p = (\dot{e}^p_m, \dot{\mathbf{e}}^p)$.

The model is related to soil materials characterized by a Drucker-Prager plastic yielding surface. For these ones, the plastically admissible stresses belong to the following set (Fig.1) :

$$K_\sigma = \left\{ (s_m, s) \text{ such that } \frac{1}{k_d} ||s|| + s_m \, tg\varphi \le c \right\} \tag{3.1}$$

Where c is the cohesion, φ is the friction angle and k_d is a constant will be chosen so that the condition in (3.1) is reduced in plane strain to Coulomb's condition of this form [7] (Fig.1) :

$$|\tau| \le c + \sigma \, tg\varphi \tag{3.2}$$

For realistic materials, the flow rule is generally non-associated and characterized by a *dilatancy angle* θ within the range from 0 to φ. For any regular stress point of the plastic yielding surface, the plastic strain rate vector has a direction defined by θ. At the vertex ($s_m = c / tg\varphi$, $||\mathbf{s}|| = 0$), this vector belongs to the cone of vectors of orientations less than or equal to θ (Fig. 1).

In this paper, a new formulation of Rudnicki-Rice non-associated constitutive law for soil material [13] is proposed on the basis of the implicit standard material approach. With the usual notation of indicator function ψ_{K_σ} (see the annex), the following flow rule is introduced :

$$\left(\dot{e}^p_m + k_d \left(tg\varphi - tg\theta\right) ||\dot{e}^p||, \ \dot{e}^p\right) \in \partial\psi_{K_\sigma}(\sigma) \tag{3.3}$$

For the geometrical meaning of this law, it is conveniant to introduce the convex cone :

$$C_\mu = \left\{ (s_m, s) \text{ such that } ||s|| \le -\mu \, s_m \right\} \tag{3.4}$$

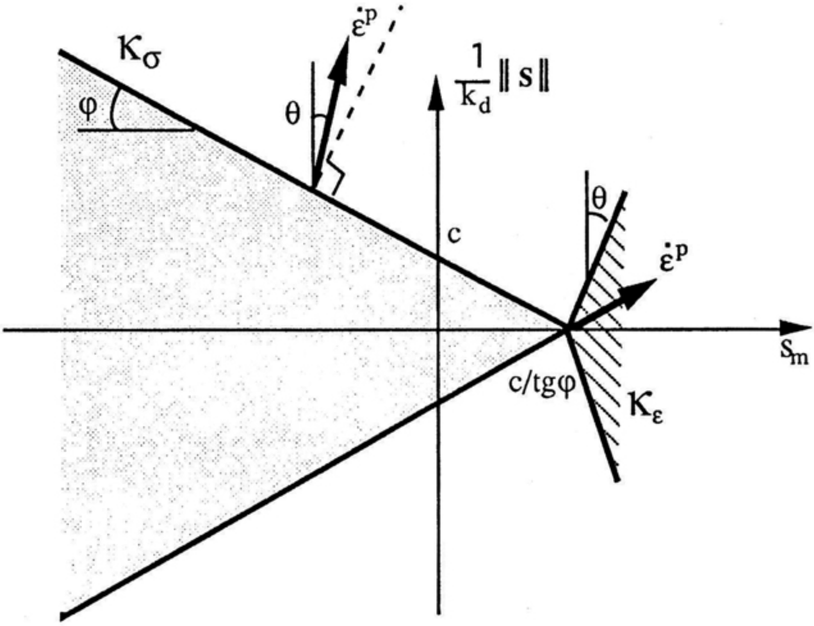

Figure 1. *A non associated flow rule*

and its dual :

$$C_\mu^* = \left\{ \left(\dot{e}_m^p, \dot{\mathbf{e}}^p\right) \text{ such that } \dot{e}_m^p \geq \mu \left\| \dot{\mathbf{e}}^p \right\| \right\} \tag{3.5}$$

So, the flow rule (3.3) becomes :

$$\left(\dot{e}_m^p + k_d \left(\mathrm{tg}\varphi - \mathrm{tg}\theta\right) \left\|\dot{\mathbf{e}}^p\right\| , \dot{\mathbf{e}}^p\right) \in \partial\psi_{C_\mu}\left(s_m - \frac{c}{\mathrm{tg}\varphi}, \mathbf{s}\right) \tag{3.6}$$

with

$$\mu = k_d \, \mathrm{tg}\varphi \tag{3.7}$$

Then, the plastic strain rate belongs to the following set of admissible plastic strain rates :

$$K_\varepsilon = \left\{ \left(\dot{e}_m^p, \dot{\mathbf{e}}^p\right) \text{ such that } \dot{e}_m^p \geq k_d \, \mathrm{tg}\theta \left\|\dot{\mathbf{e}}^p\right\| \right\} \tag{3.8}$$

If σ belongs to the interior of K_σ, plastic yielding does not occur ($\dot{\varepsilon}^p = 0$). Besides, at any regular stress point of the plastic yielding surface, the equality is reached in (3.8) and the direction of the plastic strain rate is equal to θ. On the other hand, for the vertex, the subdifferential in (3.3) is the whole dual cone (3.5) and any direction up to θ is allowed for the plastic strain rate. This discussion shows the flow rule (3.3) is convenient to represent typical soil plastic constitutive laws.

4. The implicit standard material form of this law

The plastic yielding law (3.3) involves the normality feature but is not strictly speaking an associated flow rule because of the additionnal term in $||\dot{e}^p||$ occuring in the hydrostatic component. Neverthless, it can be shown that this constitutive law can be considered as an implicit standard material law.

For this, the following function is proposed :

$$b_p\left(\dot{\varepsilon}^p, \sigma\right) = \psi_{K_\varepsilon}\left(\dot{\varepsilon}^p\right) + \frac{c\,\dot{e}_m^p}{tg\varphi} + \psi_{K_\sigma}(\sigma) + k_d\left(tg\theta - tg\varphi\right)\left(s_m - \frac{c}{tg\varphi}\right)||\dot{e}^p|| \qquad (4.1)$$

for which two propositions can be demonstrated.

Theorem 4.1 : *The function (4.1) is a bipotential.*

Proof : To sheck condition (2.6), it is sufficient to verify that

$$\forall\, \dot{\varepsilon}^p \in K_\varepsilon\,, \qquad \forall\, \sigma \in K_\sigma\,,$$

$$\frac{c\,\dot{e}_m^p}{tg\varphi} + k_d\left(tg\theta - tg\varphi\right)\left(s_m - \frac{c}{tg\varphi}\right)||\dot{e}^p|| \;\geq\; s_m\dot{e}_m^p + s\cdot\dot{e}^p \qquad (4.2)$$

Firstly, for any σ in K_σ, taking into account Cauchy-Schwartz inequality, it holds :

$$-k_d tg\varphi\left(s_m - \frac{c}{tg\varphi}\right)||\dot{e}^p|| \geq ||s|| \cdot ||\dot{e}^p|| \geq s\,.\dot{e}^p \qquad (4.3)$$

On the other hand, for any $\dot{\varepsilon}^p$ in K_ε and σ in K_σ, one has :

$$s_m \leq \frac{c}{tg\varphi} \quad \text{and} \quad \dot{e}_m^p \geq k_d tg\theta\,||\dot{e}^p|| \qquad (4.4)$$

Hence,

$$k_d tg\theta\left(s_m - \frac{c}{tg\varphi}\right)||\dot{e}^p|| \geq \left(s_m - \frac{c}{tg\varphi}\right)\dot{e}_m^p \qquad (4.5)$$

Then, condition (4.2) results from inequalities (4.3) and (4.5), and this achieves the proof.

***Theorem 4.2** : The extremal couples for the bipotential (4.1) satisfy the flow rule (3.3) and conversely.*

Proof : Applying (2.9a) to the bipotential (4.1), it can be deduced that the extremal couples satisfy the flow rule (3.3).

For the inverse proposition, a couple ($\dot{\varepsilon}^p$, σ) satisfying flow rule (3.3) is considered and the satisfaction of (2.7) has to be demonstrated. As shown in the previous section, $\dot{\varepsilon}^p$ and σ are admissible, and thus

$$b_p(\dot{\varepsilon}^p, \sigma) = \frac{c\, \dot{e}^p_m}{tg\varphi} + k_d\,(tg\theta - tg\varphi)\left(s_m - \frac{c}{tg\varphi}\right) ||\dot{e}^p|| \tag{4.6}$$

Or, after some algebraic manipulations :

$$b_p(\dot{\varepsilon}^p, \sigma) = \frac{c}{tg\varphi}\left(\dot{e}^p_m - k_d\, tg\theta\, ||\dot{e}^p||\right) + k_d\, tg\theta\, ||\dot{e}^p||\, s_m + k_d\,(c - tg\varphi\, s_m)\, ||\dot{e}^p|| \tag{4.7}$$

When the plastic strain rate vanishes (no plastic yielding), (2.7) is trivially fulfilled. Otherwise, the stress point is on the boundary of K_σ, and the equality is reached in (3.1) and (3.8), then

$$b_p(\dot{\varepsilon}^p, \sigma) = \dot{e}^p_m\, s_m + ||s|| \cdot ||\dot{e}^p|| \tag{4.8}$$

If σ is a regular point of the plastic yielding surface, flow rule (3.3) implies that vectors s and $\dot{e}^p$ are colinear, and (2.7) is satisfied. Finally, for the vertex, the deviatoric stress vanishes and (2.7) is again fulfilled, that achieves proof.

Of course this new flow rule may seem to be artificial in the absence of experimental validation. But the authors think it is a good extension of both the usual associative flow rule and the rule of the friction material. This extension is based on the concept of implicit standard material. This class of materials involves on one hand in a natural way the usual standard materials, and on the other hand, the material behaviour of surface contact with friction, bidimensional analogous of the friction material. Thus, this new concept supplies a theoretical frame to model constitutive laws of soils, radically different from the usual one. Now the law does not derive from a couple of stress functions, the plastic potential $g(\sigma)$ and the yield function $f(\sigma)$, but from a single function, the bipotential $b(\dot{\varepsilon}^p,\sigma)$, depending on both the stresses and the plastic strain rates. The authors believe that the new theoretical model of the implicit standard materials should be considered by experimentators to model soil behaviours. Of course, the presented formulation might be improved if necessary to involve for example a variable yielding angle.

Remark : it can be noted that the critical term in the bipotential (4.1) is the last one because it is responsible for the coupling between stresses and strain rates. Of course, the event $\theta = \varphi$ corresponds to the particular case of the associated flow rule for which the bipotential is separable :

$$b_p(\dot{\varepsilon}^p, \sigma) = V_p(\dot{\varepsilon}^p) + W_p(\sigma) \tag{4.9}$$

with

$$V_p(\dot{\varepsilon}^p) = \psi_{K_\varepsilon}(\dot{\varepsilon}^p) + \frac{c\,\dot{e}_m^p}{tg\varphi}, \qquad W_p(\sigma) = \psi_{K_\sigma}(\sigma) \tag{4.10}$$

5. Elastoplastic evolution problem and time integration scheme

Now, the classical hypothesis of strain decomposition in elastoplasticity is considered :

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p \tag{5.1}$$

For an isotropic material, Hooke's elastic law is given by :

$$\dot{s}_m = K_c\, \dot{e}_m^e \ , \quad \dot{s} = 2\mu\, \dot{e}^e \tag{5.2}$$

Where K_c is the bulk modulus and μ is Coulomb's shear modulus. For sake of clarity, this law can be written in a condensed linear form :

$$\dot{\sigma} = \mathbf{D}^e\, \dot{\varepsilon}^e \tag{5.3}$$

On the other hand, the flow rule of the previous section is given by an implicit normality law as follows :

$$\dot{\varepsilon}^p \in \partial_\sigma b_p(\dot{\varepsilon}^p, \sigma), \quad \sigma \in \partial_{\dot{\varepsilon}^p} b_p(\dot{\varepsilon}^p, \sigma) \tag{5.4}$$

So, the history of the couple (σ, ε^p) associated with the strain history $\varepsilon(t)$ is the solution $(\sigma(t), \varepsilon^p(t))$ of the following multivalued differential equation system of the first order :

$$\mathbf{D}^{e\,-1}\, \dot{\sigma} + \dot{\varepsilon}^p = \dot{\varepsilon}(t), \quad \dot{\varepsilon}^p \in \partial_\sigma b_p(\dot{\varepsilon}^p, \sigma) \tag{5.5}$$

For numerical application, our purpose here is to apply a time integration scheme leading to an incremental formulation. Because of convergence and stability requirements, the implicit scheme, suggested first by Moreau for elastoplasticity and known as the catching up algorithm, is considered [14-16].

Of course, one must take care of distinguishing the meaning of the term " implicit " when applied to material law or to time discretization scheme. To formulate the incremental law associated with the implicit scheme, let us consider the notations

$$\Delta\sigma = \sigma_1 - \sigma_0\,,\ \ \Delta\varepsilon = \varepsilon_1 - \varepsilon_0\,,\ \ \Delta\varepsilon^e = \varepsilon_1^e - \varepsilon_0^e\,,\ \Delta\varepsilon^p = \varepsilon_1^p - \varepsilon_0^p \qquad (5.6)$$

where the index 0 (resp. 1) is relative to the beginning (resp. the end) of the step. The implicit scheme gives :

$$\Delta\varepsilon^p = \Delta t\, \dot{\varepsilon}_1^p \qquad (5.7)$$

As the plastic rule is quasi-static and positively homogeneous, it is satisfied for the increment (5.7) if it is satisfied for the plastic strain rate. This fact can be checked immediatly for law (3.3). Then, the couple $(\Delta\varepsilon^p, \sigma_0+\Delta\sigma_0)$ is extremal for the bipotential b_p :

$$\Delta\varepsilon^p \in \partial_\sigma b_p\,(\Delta\varepsilon^p, \sigma_0+\Delta\sigma), \quad \Delta\sigma \in \partial_{\dot{\varepsilon}^p} b_p\,(\Delta\varepsilon^p, \sigma_0+\Delta\sigma) - \sigma_0 \qquad (5.8)$$

This suggests to introduce the function :

$$\Delta b_p\,(\Delta\varepsilon^p, \Delta\sigma) = b_p\,(\Delta\varepsilon^p, \sigma_0+\Delta\sigma) - \sigma_0\cdot\Delta\varepsilon^p \qquad (5.9)$$

It is easy to prove that, as b_p, Δb_p is a bipotential. Moreover, if (5.8) holds, the couple $(\Delta\varepsilon^p, \Delta\sigma)$ is extremal for the bipotential Δb_p.

On the other hand, the integration scheme gives for the elastic law :

$$\Delta\sigma = \mathbf{D}^e\,\Delta\varepsilon^e \qquad (5.10)$$

Introducing the incremental strain and complementary energy density :

$$\Delta V_e\,(\Delta\varepsilon^e) = \frac{1}{2}\,\Delta\varepsilon^e\,\mathbf{D}^e\,\Delta\varepsilon^e = \frac{K_c}{2}\,(\Delta e_m^e)^2 + \mu\,\|\Delta\mathbf{e}^e\|^2$$

$$\Delta W_e\,(\Delta\sigma) = \frac{1}{2}\,\Delta\sigma\,\mathbf{D}^{e^{-1}}\Delta\sigma = \frac{1}{2K_c}\,(\Delta s_m)^2 + \frac{1}{4\mu}\,\|\Delta s\|^2 \qquad (5.11)$$

The elastic law can be expressed as a normality law :

$$\Delta\varepsilon^{e} \in \partial\, \Delta W_e\, (\Delta\sigma), \qquad \Delta\sigma \in \partial\, \Delta V_e\, (\Delta\varepsilon^{e}) \quad (5.12)$$

Finally, the stress increment $\Delta\sigma$ associated to a given strain increment $\Delta\varepsilon$ is such that the couple $(\Delta\sigma, \Delta\varepsilon\text{-}\Delta\varepsilon^{p})$ is extremal for the elastic bipotential Δb_e and the couple $(\Delta\sigma, \Delta\varepsilon^{p})$ is extremal for the plastic bipotential Δb_p.

Remark : as noted at section 2, it is equivalent to say that the couple $(\Delta\varepsilon^{e}, \Delta\sigma)$ is extremal for the separable elastic bipotential

$$\Delta b_e\, (\Delta\varepsilon^{e}, \Delta\sigma) = \Delta V_e\, (\Delta\varepsilon^{e}) + \Delta W_e\, (\Delta\sigma) \tag{5.13}$$

6. The incremental elastoplastic bipotential

In this section, it will be proved that the incremental law linking $\Delta\varepsilon$ and $\Delta\sigma$ can be presented as an implicit standard material one. To this end, it is suggested here to extend the concept of inf-convolution (see annex) to the bipotential approach. Let $b_1(\mathbf{x}, \mathbf{y})$ and $b_2(\mathbf{x}, \mathbf{y})$ be two bipotentials. Then, we define a new function denoted

$$b = b_1 \copyright\!\!\!/_{\mathbf{x}}\, b_2 \tag{6.1}$$

called inf-convolution of b_1 and b_2 with respect to the variable $\mathbf{x}$ and defined by :

$$b(\mathbf{x}, \mathbf{y}) = \inf_{\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}} \left(b_1(\mathbf{x}_1, \mathbf{y}) + b_2(\mathbf{x}_2, \mathbf{y}) \right) \tag{6.2}$$

or equivalently

$$b(\mathbf{x}, \mathbf{y}) = \inf_{\mathbf{x}'} \left(b_1(\mathbf{x}-\mathbf{x}', \mathbf{y}) + b_2(\mathbf{x}', \mathbf{y}) \right) \tag{6.3}$$

***Theorem 6.1** : The inf-convolution of two bipotentials is a bipotential.*

Proof : It is easy to see that the convexity requirements are fulfilled. Moreover, because of condition (2.6) applied to b_1 and b_2, one has :

$$\forall\, \mathbf{x}, \mathbf{x}', \mathbf{y}, \quad b_1(\mathbf{x}-\mathbf{x}', \mathbf{y}) + b_2(\mathbf{x}', \mathbf{y}) \geq \mathbf{x} \cdot \mathbf{y} \tag{6.4}$$

Consequently :

$$\forall\ \mathbf{x},\ \mathbf{y},\quad \inf_{\mathbf{x}'}\ (\ b_1(\mathbf{x}\text{-}\mathbf{x}', \mathbf{y}) + b_2(\mathbf{x}', \mathbf{y})\ ) \geq \mathbf{x} \cdot \mathbf{y} \tag{6.5}$$

Thus, it is proved that the inf-convolution is a bipotential.

***Theorem 6.2** : if the couple (x-x*, y) is extremal for b_1 and the couple (x*, y) is extremal for b_2, then, the couple (x, y) is extremal for the inf-convolution b.*

Proof : The previous assumptions imply that

$$b_1(\mathbf{x}\text{-}\mathbf{x}^*, \mathbf{y}) = (\mathbf{x}\text{-}\mathbf{x}^*) \cdot \mathbf{y}\ ,\quad b_2(\mathbf{x}^*, \mathbf{y}) = \mathbf{x}^* \cdot \mathbf{y} \tag{6.6}$$

Then, it results that

$$b_1(\mathbf{x}\text{-}\mathbf{x}^*, \mathbf{y}) + b_2(\mathbf{x}^*, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} \tag{6.7}$$

To achieve the proof, it is sufficient to show that the minimum value in (6.3) is reached for $\mathbf{x}' = \mathbf{x}^*$. Applying condition (2.8a) to the considered extremal couples, it holds :

$$\begin{aligned} &\forall\ \mathbf{x}',\quad b_1(\mathbf{x}\text{-}\mathbf{x}', \mathbf{y}) - b_1(\mathbf{x}\text{-}\mathbf{x}^*, \mathbf{y}) \geq \mathbf{y} \cdot (\mathbf{x}^*\text{-}\mathbf{x}') \\ &\forall\ \mathbf{x}',\quad b_2(\mathbf{x}', \mathbf{y}) - b_2(\mathbf{x}^*, \mathbf{y}) \geq \mathbf{y} \cdot (\mathbf{x}'\text{-}\mathbf{x}^*) \end{aligned} \tag{6.8}$$

which implies that

$$\forall\ \mathbf{x}',\quad b_1(\mathbf{x}\text{-}\mathbf{x}', \mathbf{y}) + b_2(\mathbf{x}', \mathbf{y}) \geq b_1(\mathbf{x}\text{-}\mathbf{x}^*, \mathbf{y}) + b_2(\mathbf{x}^*, \mathbf{y}) \tag{6.9}$$

This proves the results.

The previous developments suggest to introduce the incremental elastoplastic bipotential :

$$\Delta b = \Delta b_e \ ©_{\Delta\varepsilon^p}\ \Delta b_p \tag{6.10}$$

So, in order to satisfy the implicit time integration scheme, it is seen that the couple $(\Delta\varepsilon, \Delta\sigma)$ must be extremal for the bipotential Δb :

$$\Delta b(\Delta\varepsilon, \Delta\sigma) = \Delta\varepsilon \cdot \Delta\sigma \tag{6.11}$$

$$\Delta\varepsilon \in \partial_{\Delta\sigma}\, \Delta b(\Delta\varepsilon, \Delta\sigma),\qquad \Delta\sigma \in \partial_{\Delta\varepsilon}\, \Delta b(\Delta\varepsilon, \Delta\sigma) \tag{6.12}$$

Remark : for the particular case of the associated flow rule, it can be noted that the bipotential Δb_p is separable :

$$\Delta b_p\ (\Delta\varepsilon^p, \Delta\sigma) = \Delta V_p\ (\Delta\varepsilon^p) + \Delta W_p\ (\Delta\sigma) \qquad (6.13)$$

where

$$\Delta V_p\ (\Delta\varepsilon^p) = V_p\ (\Delta\varepsilon^p) - \sigma_0 \cdot \Delta\varepsilon^p$$

$$(6.14)$$

$$\Delta W_p\ (\Delta\sigma) = W_p\ (\sigma_0 + \Delta\sigma)$$

Introducing (6.13) in (6.10), it is shown that Δb is separable

$$\Delta b = \Delta V + \Delta W \qquad (6.15)$$

where

$$\Delta V = (\Delta V_e \copyright \Delta V_p)\ (\Delta\varepsilon) = \operatorname*{Inf}_{\Delta\varepsilon^p}\ (\Delta V_e(\Delta\varepsilon - \Delta\varepsilon^p) + \Delta V_p\ (\Delta\varepsilon^p))$$

$$(6.16)$$

$$\Delta W = \Delta W_e + \Delta W_p$$

It can be shown as in [4] that ΔV is the very conjugate to ΔW in Fenchel sense.

7. The elastoplastic bipotential for soil material

Now, the implicit integration scheme is applied to a soil material with the non associated flow rule of section 4, following the method proposed at section 6. Combining the plastic bipotential (4.1) and the elastic potentials (5.11), and taking into account (5.9) and (5.13), the incremental elastoplastic bipotential (6.10) is equal to :

$$\Delta b(\Delta\varepsilon, \Delta\sigma) = \operatorname*{Inf}_{\Delta\varepsilon^p,\ k_d\, tg\theta \|\Delta e^p\| \le \Delta e^p_m} \{\mu\|\Delta \mathbf{e} - \Delta \mathbf{e}^p\|^2 + \frac{K_c}{2}(\Delta e_m - \Delta e^p_m)^2 +$$

$$\frac{c}{tg\varphi}\ \Delta e^p_m\ + k_d\ (tg\theta - tg\varphi)\ (s_{m0} + \Delta s_m - \frac{c}{tg\varphi})\ \|\Delta \mathbf{e}^p\| - s_{m0}\Delta e^p_m - s_0 \cdot \Delta \mathbf{e}^p\}$$

$$+ \frac{1}{4\mu}\ \|\Delta s\|^2 + \frac{1}{2K_c}\ (\Delta s_m)^2 + \psi_{\Delta K}\ (\Delta\sigma) \qquad (7.1)$$

where

$$\Delta\kappa = \{(\Delta s_m, \Delta s) \text{ such that } \|s_0+\Delta s\| \le k_d[c - tg\varphi(s_{m0}+\Delta s_m)]\} \qquad (7.2)$$

is the set of plastically admissible stress increments.

If the particular case of the vertex of (7.2) is not considered, it may be assumed that the equality is reached in the constraint :

$$\Delta e_m^p = k_d \, tg\theta \, \|\Delta e^p\| \qquad (7.3)$$

This allows to eliminate Δe_m^p from (7.1) :

$$\Delta b(\Delta\varepsilon, \Delta\sigma) = \inf_{\Delta\varepsilon^p} \Big\{ \mu \, \| \Delta \mathbf{e} - \Delta \mathbf{e}^p \|^2 + \frac{k_c}{2} (\Delta e_m - k_d \, tg\theta \, \| \Delta \mathbf{e}^p \|)^2 +$$

$$+ k_d \big[c - tg\varphi \, (s_{m0} + \Delta s_m) + \Delta s_m \, tg\theta \big] \, \| \Delta \mathbf{e}^p \| - \mathbf{s}_0 \, \Delta \mathbf{e}^p \Big\} +$$

$$+ \frac{1}{4\mu} \| \Delta \mathbf{s} \|^2 + \frac{1}{2k_c} (\Delta s_m)^2 + \psi_{\Delta\kappa} (\Delta s_m, \Delta \mathbf{s}) \qquad (7.4)$$

Now, only the non trivial case of plastic yielding with non vanishing $\Delta \mathbf{e}^p$ is considered. The function to minimize is differentiable. Thus, at the optimum, the following stationnarity condition must be fulfilled :

$$\Big\{ 2\mu + \frac{1}{\| \Delta \mathbf{e}^p \|} \big[k_c \, (k_d \, tg\theta \, \| \Delta \mathbf{e}^p \| - \Delta e_m) \, k_d \, tg\theta +$$

$$+ k_d \, (c - tg\varphi \, (s_{m0} + \Delta s_m) + \Delta s_m \, tg\theta) \big] \Big\} \, \Delta \mathbf{e}^p = 2\mu \, \Delta \mathbf{e} + \mathbf{s}_0 \qquad (7.5)$$

The vector $\Delta \mathbf{e}^p$ and $(2\mu\Delta \mathbf{e}+\mathbf{s}_0)$ are colinear. For convenience, the following notations are introduced :

$$\eta_0 = \frac{\mathbf{s}_0}{2\mu}, \qquad \widehat{\mathbf{n}} = \frac{\Delta \mathbf{e} + \eta_0}{\| \Delta \mathbf{e} + \eta_0 \|}, \qquad \varepsilon_c = \frac{K_c}{2\mu}, \qquad \varepsilon_d = \frac{k_d}{2\mu} \qquad (7.6)$$

So, there holds :

$$\Delta \mathbf{e}^p = \| \Delta \mathbf{e}^p \| \, \widehat{\mathbf{n}} \qquad (7.7)$$

Equating the norm of both hand sides of (7.5) and using the positive part symbol $(...)_+$, there holds :

$$\| \Delta e^p \| = \frac{1}{1 + \varepsilon_c k_d^2 \operatorname{tg}^2\theta} \Big\{ \| \Delta e + \eta_0 \| - \varepsilon_d \left[c - \operatorname{tg}\varphi \left(s_{m0} + \Delta s_m \right) + \operatorname{tg}\theta \left(\Delta s_m - K_c \Delta e_m \right) \right] \Big\}_+ \tag{7.8}$$

Indeed, equation (7.5) has a solution only if this condition is satisfied :

$$\| \Delta e + \eta_0 \| \geq \varepsilon_d \left[c - \operatorname{tg}\varphi \left(s_{m0} + \Delta s_m \right) + \operatorname{tg}\theta \left(\Delta s_m - K_c \Delta e_m \right) \right] \tag{7.9}$$

Otherwise, the trivial case of elastic loading occurs.

If (7.5) is fulfilled, the value of stress increments are determined by :

$$\Delta s = 2\mu \left(\Delta e - \Delta e^p \right), \qquad \Delta s_m = K_c \left(\Delta e_m - \Delta e_m^p \right) \tag{7.10}$$

Finally the elastoplastic law in implicit form is :

$$\Delta s_r = 2\mu \left\{ \| \Delta e + \eta_0 \| - \| \Delta e^p \| \right\} \hat{n} - (s_0 + \Delta s) = 0$$

$$\Delta s_{mr} = K_c \left(\Delta e_m - k_d \operatorname{tg}\theta \| \Delta e^p \| \right) - \Delta s_m = 0 \tag{7.11}$$

where the norm of Δe^p is given by (7.8). Computing the optimal value in (7.4) by virtue of (7.7-8) and using again the positive value symbol in order to involve the trivial case of elastic loading, the incremental elastoplastic bipotential is equal to :

$$\Delta b(\Delta\varepsilon, \Delta\sigma) = \frac{K_c}{2} (\Delta e_m)^2 + \frac{1}{4\mu} \| \Delta s \|^2 + \frac{1}{2K_c} (\Delta s_m)^2 + \psi_{\Delta K}(\Delta\sigma) + \mu \Big\{ \| \Delta e \|^2 - \frac{1}{1 + \varepsilon_c k_d^2 \operatorname{tg}^2\theta} \Big(\| \Delta e + \eta_0 \| - \varepsilon_d \left[c - \operatorname{tg}\varphi \left(s_{m0} + \Delta s_m \right) + \operatorname{tg}\theta \left(\Delta s_m - k_c \Delta e_m \right) \right] \Big)_+^2 \Big\} \tag{7.12}$$

This bipotential is differentiable with respect to strain increments. In fact, the elastoplastic bipotential regularizes the non-differentiable plastic potential b_p. So, the incremental law (6.12) can be written :

$$\Delta\sigma = \frac{\partial \Delta b(\Delta\varepsilon, \Delta\sigma)}{\partial \Delta\varepsilon} \tag{7.13}$$

Applied to (7.12), this equation gives the relation (7.11) again.

Remark : in the particular case of the associated flow rule, it can be noted that the bipotential (7.12) degenerates in the sum (6.15) of two conjugate potentials :

$$\Delta V(\Delta\varepsilon) = \frac{k_c}{2}(\Delta e_m)^2 + \mu \left\{ \| \Delta \mathbf{e} \|^2 - \frac{1}{1+\varepsilon_c k_d^2 \, tg^2\theta} \left(\| \Delta \mathbf{e} + \eta_0 \| - \right.\right.$$

$$\left.\left. \varepsilon_d \left[c - tg\varphi \, (k_c \Delta e_m + s_{m0}) \right] \right)_+^2 \right\} \quad (7.14)$$

$$\Delta W(\Delta\sigma) = \frac{1}{4\mu} \|\Delta \mathbf{s}\|^2 + \frac{1}{2K_c} (\Delta s_m)^2 + \psi_{\Delta K}(\Delta\sigma) \quad (7.15)$$

In this particular case, ΔV is in fact the regularization of the non-differentiable potential ΔV_p, by means of Moreau-Yoshida transform [4], because the elastic potential ΔV_e given by (5.11a) is the square of a norm in the elastic strain increment space.

8. Variational principles

Let Ω be a structure of boundary S, subjected during a time increment to imposed body forces $\Delta \mathbf{f}$, imposed surface traction increments $\Delta \mathbf{t}$ on the part S_1 of S, and imposed displacement increments $\Delta \bar{\mathbf{u}}$ on the remaining part $S_0 = S - S_1$ of the boundary.

A displacement increment field is kinematically admissible (K.A.) if the following compatibility conditions are fulfilled :

$$\Delta\varepsilon(\Delta \mathbf{u}^k) = \text{grad}_s \, \Delta \mathbf{u}^k \quad \text{in } \Omega, \quad \Delta \mathbf{u}^k = \Delta \bar{\mathbf{u}} \quad \text{on } S_0 \quad (8.1)$$

A stress increment field is said statically admissible (S.A.) if the following equilibrium equations are satisfied :

$$\text{div}(\Delta\sigma^s) + \Delta \mathbf{f} = 0 \quad \text{in } \Omega, \quad \Delta t(\Delta\sigma^s) = \Delta\sigma^s . \mathbf{n} = \Delta \bar{\mathbf{t}} \quad \text{on } S_1 \quad (8.2)$$

The aim of this section is to present a variational formulation for the implicit standard material behaviour, given by the bipotential (6.10). Hence, as proposed first in [24], the following new functional, called *bifunctional*, is introduced :

$$\Delta B(\Delta \mathbf{u}, \Delta\sigma) = \int_\Omega \Delta b(\Delta\varepsilon(\mathbf{u}), \Delta\sigma) \, d\Omega \; -$$

$$\int_{\Omega} \Delta\mathbf{f}\cdot\Delta\mathbf{u} \ d\Omega - \int_{S_1} \Delta\bar{\mathbf{t}}\cdot\Delta\mathbf{u} \ dS - \int_{S_0} \Delta\mathbf{t}(\Delta\sigma)\cdot \Delta\bar{\mathbf{u}} \, dS \tag{8.3}$$

This definition allows to extend the classical calculus of variation to the implicit standard material. Because the bifunctional cannot be split anymore, the displacement and stress problems are *coupled*. So, it can be proved that a field couple ($\Delta\mathbf{u}$, $\Delta\sigma$), exact solution of the boundary value problem defined by (8.1), (8.2) and the constitutive law (6.11-12), is simultaneously solution of the following variational principles :

$$\inf_{\Delta\mathbf{u}^k \text{ K.A.}} \Delta B(\Delta\mathbf{u}^k, \Delta\sigma) \quad \text{and} \quad \inf_{\Delta\sigma^s \text{ S.A.}} \Delta B(\Delta\mathbf{u}, \Delta\sigma^s) \tag{8.4}$$

For example, let us prove the displacement one. Due to the property (2.8a) of the extremal couple, one has :

$$\Delta B(\Delta\mathbf{u}^k,\Delta\sigma) - \Delta B(\Delta\mathbf{u},\Delta\sigma) \geq \int_{\Omega} \Delta\sigma\cdot\left(\Delta\varepsilon(\Delta\mathbf{u}^k)-\Delta\varepsilon(\Delta\mathbf{u})\right) d\Omega$$

$$- \int_{\Omega} \Delta\mathbf{f} \ . \left(\Delta\mathbf{u}^k-\Delta\mathbf{u}\right) d\Omega - \int_{S_f} \Delta\bar{\mathbf{t}} \ . \left(\Delta\mathbf{u}^k-\Delta\mathbf{u}\right) dS - \int_{S_u} \Delta\mathbf{t}(\Delta\sigma) \ . \left(\Delta\mathbf{u}^k-\Delta\bar{\mathbf{u}}\right) dS$$

$$\tag{8.5}$$

As an exact solution $\Delta\sigma$ is statically admissible, the minimum principle results from equilibrium equations (8.2) and Green's formula :

$$\Delta B(\Delta\mathbf{u}^k, \Delta\sigma) \geq \Delta B(\Delta\mathbf{u}, \Delta\sigma) \tag{8.6}$$

In a similar way, the stress principle can be deduced from compatibility conditions (8.1). Let us prove now the *existence* of the solution. For this sake, it is remarked that the solution can be obtained by successive approximation and combination of the two principles. Let ($\Delta\mathbf{u}_i$, $\Delta\sigma_i$) be the approximative solution at iteration i. Let $\Delta\sigma_{i+1}$ be a statically admissible stress field and $\Delta\mathbf{u}^k$ a kinematically admissible displacement field such that

$$\Delta\varepsilon \ (\Delta\mathbf{u}^k) \in \partial_{\Delta\sigma} \Delta b \ (\Delta\varepsilon \ (\Delta\mathbf{u}_i), \Delta\sigma_{i+1}) \tag{8.7}$$

Then, because of minimum principle (8.4b), one has :

$$\Delta B(\Delta\mathbf{u}_i, \Delta\sigma_i) \geq \Delta B(\Delta\mathbf{u}_i, \Delta\sigma_{i+1}) \tag{8.8}$$

Besides, let $\mathbf{u}_{i+1}$ be a kinematically admissible displacement field and $\Delta\sigma^s$ a statically admissible stress field such that :

$$\Delta\sigma^s \in \partial_{\Delta\varepsilon}\, \Delta b\,(\Delta\varepsilon\,(\Delta \mathbf{u}_{i+1}),\, \Delta\sigma_{i+1}) \tag{8.9}$$

From minimum principle (8.6), it results that :

$$\Delta B(\Delta \mathbf{u}_i,\, \Delta\sigma_{i+1}) \geq \Delta B(\Delta \mathbf{u}_{i+1},\, \Delta\sigma_{i+1}) \tag{8.10}$$

So, a minimizing sequence of ΔB is constructed.

It is now assumed that the couples belong to some reflexive Banach space **X** with norm $\|\cdot\|$. Then, the existence of an exact solution of (8.4) can be proved if the sequence $(\Delta \mathbf{u}_i, \Delta\sigma_i)$ is bounded by extracting a convergent subsequence [4]. The boundedness property may result from some adequate assumption or from the classical hypothesis of coercivity

$$\lim_{\|\Delta \mathbf{u}, \Delta\sigma\| \to +\infty} \Delta B(\Delta \mathbf{u}, \Delta\sigma) = +\infty \tag{8.11}$$

Let $(\Delta \mathbf{u}_{i'}, \Delta\sigma_{i'})$ be the bounded subsequence. Then

$$(\Delta \mathbf{u}_{i'}, \Delta\sigma_{i'}) \to (\Delta \mathbf{u}, \Delta\sigma) \quad \text{weakly in } \mathbf{X} \tag{8.12}$$

Remark : it may be noted that for the associated flow rule, with separable bipotential (6.15), the bifunctional is reduced to the sum :

$$\Delta B(\Delta \mathbf{u}, \Delta\sigma) = \Delta\Phi(\Delta \mathbf{u}) + \Delta\Pi(\Delta\sigma) \tag{8.13}$$

of the incremental total strain energy :

$$\Delta\Phi(\Delta \mathbf{u}) = \int_\Omega \Delta V(\Delta\varepsilon(\mathbf{u}))\, d\Omega - \int_\Omega \Delta \mathbf{f}\cdot\Delta \mathbf{u}\;\, d\Omega - \int_{S_1} \Delta\bar{\mathbf{t}}\cdot\Delta \mathbf{u}\;\, dS \tag{8.14}$$

and of the incremental total complementary energy :

$$\Delta\Pi(\Delta\sigma) = \int_\Omega \Delta W(\Delta\sigma)\, d\Omega - \int_{S_0} \Delta \mathbf{t}(\Delta\sigma)\cdot\Delta\bar{\mathbf{u}}\;\, dS \tag{8.15}$$

9. Finite element discretization

For numerical applications, the method of displacement finite elements is used. The approximation of the displacement increment field is defined by the relation

$$\Delta \mathbf{u}(\mathbf{x}) = \mathbf{N}(\mathbf{x})\, \Delta \mathbf{U} \tag{9.1}$$

where $\Delta \mathbf{U}$ is the unknown nodal displacement increment vector and $\mathrm{N}(\mathbf{x})$ is a matrix of polynomial shape functions. The boundary compatibility conditions (8.1b) on S_u are satisfied setting particular nodal displacement increments. The associated strain increment field is defined by considering the internal compatibility equations (8.1a) :

$$\Delta \varepsilon(\mathbf{x}) = \mathbf{B}(\mathbf{x})\, \Delta \mathrm{U} \tag{9.2}$$

with

$$\mathbf{B}(\mathbf{x}) = \mathrm{grad}_s\, \mathbf{N}(\mathbf{x}) \tag{9.3}$$

Introducing the generalized nodal force increment vector :

$$\Delta \mathbf{F} = \int_{\Omega} \mathbf{N}^T \Delta \mathbf{f}\, d\Omega + \int_{S_1} \mathbf{N}^T \Delta \mathbf{t}\, dS \tag{9.4}$$

the bifunctional (8. 3) has the following discretized form :

$$\Delta \mathrm{B}(\Delta \mathbf{U}, \Delta \sigma) = \int_{\Omega} \Delta \mathrm{b}(\mathbf{B} \Delta \mathbf{U}, \Delta \sigma)\, d\Omega - \Delta \mathbf{F}^T \Delta \mathbf{U} \tag{9.5}$$

Here, the local stress increments are not discretized as in the stress principle [30], but can be deduced from the nodal displacement increment value by solving the equation :

$$\Delta \sigma = \frac{\partial\, \Delta \mathrm{b}(\mathbf{B} \Delta \mathbf{U}, \Delta \sigma)}{\partial\, \Delta \varepsilon} \tag{9.6}$$

resulting from (7.13) combined with (9.2). So, only the displacement principle (8.4a) is considered.

As the bipotential is differentiable with respect to the strain increments, the bifunctional is differentiable with respect to the nodal increments. The minimum in (8.4a) is reached if the stationarity condition is fulfilled :

$$\int_{\Omega} \mathbf{B}^T \frac{\partial\, \Delta \mathrm{b}(\mathbf{B} \Delta \mathbf{U}, \Delta \sigma)}{\partial\, \Delta \varepsilon}\, d\Omega - \Delta \mathbf{F} = 0 \tag{9.7}$$

Combining the structural equilibrium equations (9.7) with the incremental law (9.6), it can be seen that the solution of the boundary value problem must verify the following equation system :

$$\Delta\sigma_r = \frac{\partial\, \Delta b(\mathbf{B}\Delta\mathbf{U}\,,\Delta\sigma)}{\partial\, \Delta\varepsilon} - \Delta\sigma = 0, \quad \Delta\mathbf{F}_r = \int_\Omega \mathbf{B}^T\, \Delta\sigma\, d\Omega\ - \Delta\mathbf{F}\ = 0 \qquad (9.8)$$

In principle, the local equation (9.8a) should be satisfied anywhere. In practical implementation, the integrals are computed numerically by Gauss integration scheme. Then the local equations are only considered at Gauss points. Of course, the stress increments does not fulfill the local equilibrium equations (8.2) but only the global ones (9.8b) in a mean sense with the weight functions N(**x**).

10. Two solution techniques based on Newton scheme

The equation system (9.8) is non linear. It can be solved by using Newton scheme which involves the computation of the local tangent matrix defined as the derivative of stress increments with respect to strain increments :

$$\mathbf{D} = \frac{\partial\Delta\sigma}{\partial\Delta\varepsilon} \qquad (10.1)$$

Because the incremental constitutive law (9.6) is an implicit relation between stress ans strain increments, the implicit function derivative theorem may be used :

$$\mathbf{D} = -\left(\frac{\partial\Delta\sigma_r}{\partial\Delta\sigma}\right)^{-1} \frac{\partial\Delta\sigma_r}{\partial\Delta\varepsilon} = \left(\mathbf{I} - \mathbf{D}_c\right)^{-1}\, \mathbf{D}_i \qquad (10.2)$$

where

$$\mathbf{D}_c = \frac{\partial^2\Delta b}{\partial\Delta\sigma\partial\Delta\varepsilon}, \quad \mathbf{D}_i = \frac{\partial^2\Delta b}{\partial\Delta\varepsilon^2} \qquad (10.3)$$

and **I** is the unit matrix.

It can be noted that generally $\mathbf{D}_c$ and consequently **D** are not symmetric matrices. The matrix $\mathbf{D}_i$ may be considered as the symmetric kernel of **D**. Of course, for explicit standard materials (associated flow rule), $\mathbf{D}_c$ vanishes and **D** equals $\mathbf{D}_i$.

Now depending on the normality form of the constitutive law being considered or not in the algorithm, we proposed two solution techniques based on Newton scheme.

11. Coupled solution technique

If Newton scheme is globally applied to the whole set of equations (9.8), one has :

$$\Delta\sigma_r^{k+1} = \Delta\sigma_r^k + \left(\mathbf{D}_c^k - \mathbf{I}\right)\left(\Delta\sigma^{k+1} - \Delta\sigma^k\right) + \mathbf{D}_i^k \mathbf{B} \left(\Delta\mathbf{U}^{k+1} - \Delta\mathbf{U}^k\right) = 0$$

$$\Delta\mathbf{F}_r^{k+1} = \Delta\mathbf{F}_r^k + \int_\Omega \mathbf{B}^T \left(\Delta\sigma^{k+1} - \Delta\sigma^k\right) d\Omega = 0 \tag{11.1}$$

The first equation can be solved with respect to $\Delta\sigma^{k+1}$. Then, the expression of $\Delta\sigma^{k+1}$ is introduced in the second equation which can be solved with respect to $\Delta\mathbf{U}^{k+1}$. Finally, Newton scheme leads to the iterative computation :

$$\Delta\mathbf{U}^{k+1} = \Delta\mathbf{U}^k - \left(\mathbf{K}_T^k\right)^{-1} \Delta\mathbf{R}^k$$

$$\Delta\sigma^{k+1} = \Delta\sigma^k + \mathbf{D}^k \mathbf{B} \left(\Delta\mathbf{U}^{k+1} - \Delta\mathbf{U}^k\right) + \left(\mathbf{I} - \mathbf{D}_c^k\right)^{-1} \Delta\sigma_r^k \tag{11.2}$$

with the tangent stiffness matrix is

$$\mathbf{K}_T = \int_\Omega \mathbf{B}^T \mathbf{D}\, \mathbf{B} \ \ d\Omega \tag{11.3}$$

where $\mathbf{D}$ computed by (10.2), and with the residual

$$\Delta\mathbf{R} = \Delta\mathbf{F}_r - \int_\Omega \mathbf{B}^T \left(\mathbf{I} - \mathbf{D}_c\right)^{-1} \Delta\sigma_r \, d\Omega \tag{11.4}$$

The stiffness matrix (11.3) is not symmetric, and this can be explained by the fact that direct application of Newton scheme to the whole set of equations (9.8) breaks the standard character of equations. In this algorithm, the residual (11.4) is the sum of the error on equilibrium equations and of the error on the constitutive law which is only satisfied at the limit.

12. Uncoupled solution technique

To introduce the second algorithm, it can be noted the similarity of this formulation with one of the key ideas of the large time increment or LATIN method proposed by Ladeveze [29-30-31]. Using the terminology of the latter approach, the couple ($\Delta \mathbf{U}$, $\Delta\sigma$), solution of the boundary value problem, is at the intersection of the non linear subspace Γ defined by the constitutive law (9.8a) and the linear subspace A_d of the statically admissible solutions, defined by Eq. (9.8b) (Fig. 2). The linearization of (9.8a) corresponds to the local stage, associated with an upwards search direction E^+, and the linearization of (9.8b) to the global stage, with a downwards search direction E^-.

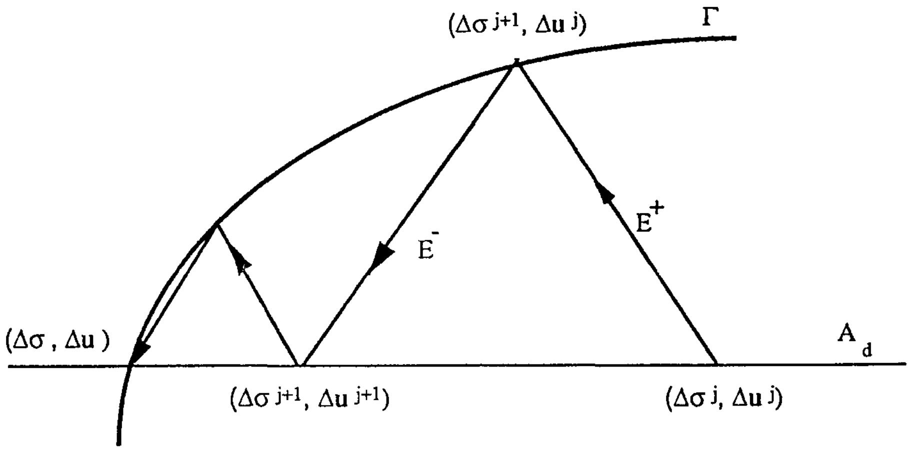

Figure 2. *The LATIN method*

Nevertheless, it must be noted that in the non standard solution technique, the local step is approximate, while in LATIN method, it is exactly solved. This remarque leads to a new solution technique with two stages (Fig. 2).

1) *local stage* : $\Delta \mathbf{U}^k$ is fixed and a new approximation of increment stresses $\Delta\hat{\sigma}$ is computed by solving :

$$\Delta\hat{\sigma}_r = \frac{\partial\, \Delta b(\mathbf{B}\Delta\mathbf{U}^k, \Delta\hat{\sigma})}{\partial\, \Delta\varepsilon} - \Delta\hat{\sigma} = 0 \tag{12.1}$$

Newton scheme leads to :

$$\Delta\hat{\sigma}^{j+1} = \Delta\hat{\sigma}^{j} + (\mathbf{I} - \mathbf{D}_c^j)^{-1}\, \Delta\hat{\sigma}_r^{j} \tag{12.2}$$

2) *global stage* : $\Delta\hat{\sigma}$ is fixed and a new approximation ΔU^{k+1} is computed by solving :

$$\Delta\sigma_r = \frac{\partial\, \Delta b(\mathbf{B}\Delta\mathbf{U}^{k+1}, \Delta\hat{\sigma})}{\partial\, \Delta\varepsilon} - \Delta\hat{\sigma} = 0\ , \qquad \Delta\mathbf{F}_r = \int_\Omega \mathbf{B}^T\Delta\hat{\sigma}\, d\Omega - \Delta\mathbf{F} = 0 \tag{12.3}$$

Newton sheme leads to :

$$\Delta\mathbf{U}^{k+1} = \Delta\mathbf{U}^{k+1} - \left(\mathbf{K}_i^k\right)^{-1}\Delta\mathbf{F}_r^k \tag{12.4}$$

where the stiffness matrix

$$\mathbf{K}_i = \int_\Omega \mathbf{B}^T\,\mathbf{D}_i\mathbf{B}d\Omega \tag{12.5}$$

is symmetric (conversely to $\mathbf{K_T}$) and the residual is the one of classical step-by-step method.

13. Local tangent matrix for soil material

For the incremental law presented at section 7, it can be noted that the requirement for Newton scheme of Δb being twice differentiable with respect to strain increment is not satisfied at the onset of plastic yielding, i.e when equality is reached in condition (7.9). Neverthless, this does not generally lead to difficulty in numerical applications.

From (7.9), we deduce

$$\frac{\partial\|\Delta\mathbf{e}^p\|}{\partial\Delta\mathbf{e}} = \frac{\hat{\mathbf{n}}}{1+\varepsilon_c k_d^2 tg^2\theta}\ , \quad \frac{\partial\|\Delta\mathbf{e}^p\|}{\partial\Delta e_m} = \frac{\varepsilon_c k_d tg\theta}{1+\varepsilon_c k_d^2 tg^2\theta}\ , \quad \frac{\partial\|\Delta\mathbf{e}^p\|}{\partial\Delta s_m} = \frac{\varepsilon_d\left(tg\varphi - tg\theta\right)}{1+\varepsilon_c k_d^2 tg^2\theta} \tag{13.1}$$

Putting

$$\alpha = 2\mu\,(1 - \frac{\|\Delta\mathbf{e}^p\|}{\|\Delta\mathbf{e}+\eta_0\|})\ , \qquad \beta = \frac{K_c k_d}{1+\varepsilon_c k_d^2 tg^2\theta}$$

$$\gamma = \frac{\beta^2 tg\theta}{K_c}\left(\varepsilon_c k_d^2 tg^3\theta + tg\varphi\right) \tag{13.2}$$

$$\delta = k_d \, \beta \, tg^2\theta - \alpha \, ; \qquad \omega = \gamma - \alpha$$

After some computation, the following expressions of the matrices of section 10 are deduced by derivation of (7.11) :

$$\mathbf{D}_i = \frac{\beta}{k_d} 1\otimes 1 + \alpha \left(\mathbf{I} - \frac{1}{3} 1\otimes 1\right) + \delta \, \hat{\mathbf{n}}\otimes\hat{\mathbf{n}} - \beta \, tg\theta \left(1\otimes\hat{\mathbf{n}} + \hat{\mathbf{n}}\otimes 1\right) \tag{13.3}$$

$$\mathbf{D}_c = \frac{1}{3}\frac{\beta}{k_c} \left(tg\theta - tg\varphi\right) \hat{\mathbf{n}}\otimes 1 \tag{13.4}$$

Taking into account

$$\left(\mathbf{I} - \mathbf{D}_c\right)^{-1} = \mathbf{I} + \frac{1}{3}\frac{\beta}{k_c}\left(tg\theta - tg\varphi\right) \hat{\mathbf{n}}\otimes 1 \tag{13.5}$$

Owing to (10.2), the (non symmetric) tangent matrix is equal to :

$$\mathbf{D} = \frac{\beta}{k_d} 1\otimes 1 + \alpha \left(\mathbf{I} - \frac{1}{3} 1\otimes 1\right) + \omega \, \hat{\mathbf{n}}\otimes\hat{\mathbf{n}} - \beta \, tg\theta \left(1\otimes\hat{\mathbf{n}} + \hat{\mathbf{n}}\otimes 1\right) + \frac{\beta^2}{k_c \, k_d}\left(tg\theta - tg\varphi\right) \hat{\mathbf{n}}\otimes 1 \tag{13.6}$$

<u>Remark</u> : for a soil material with surface friction, the plastic yielding angle θ vanishes and the plastic strain is incompressible. The norm of plastic strain increments (7.8) is reduced to :

$$\|\Delta \mathbf{e}^p\| = \left(\|\Delta \mathbf{e} + \eta_0\| - \varepsilon_d \left[c - tg\varphi \left(s_{m0} + \Delta s_m\right) \right] \right)_+ \tag{13.7}$$

and the stress increment is given by :

$$\Delta \mathbf{s}_r = k_d \left[c - tg\varphi \left(s_{m0} + \Delta s_m\right)\right] \hat{\mathbf{n}} - \left(\mathbf{s}_0 + \Delta \mathbf{s}\right) = 0$$

$$\Delta s_{mr} = K_c \Delta e_m - \Delta s_m = 0 \tag{13.8}$$

The bipotential (7.12) is reduced to

$$\Delta b\left(\Delta\varepsilon, \Delta\sigma\right) = \frac{K_c}{2}\left(\Delta e_m\right)^2 + \frac{1}{4\mu}\|\Delta \mathbf{s}\|^2 + \frac{1}{2K_c}\left(\Delta s_m\right)^2 + \psi_{\Delta K}\left(\Delta\sigma\right) +$$

$$\mu \left\{ \|\Delta \mathbf{e}\|^2 - \left(\|\Delta \mathbf{e} + \eta_0\| - \varepsilon_d \left[c - tg\varphi \left(s_{m0} + \Delta s_m\right)\right] \right)_+^2 \right\} \tag{13.9}$$

The matrix (13.3-5) are reduced to :

$$\mathbf{D}_i = k_c\ 1\otimes 1 + \alpha\ (\mathbf{I}-\frac{1}{3}\ 1\otimes 1) - \alpha\ \hat{\mathbf{n}}\otimes\hat{\mathbf{n}} \tag{13.10}$$

$$\mathbf{D}_c = -\frac{1}{3}\ k_d\ \mathrm{tg}\varphi\hat{\mathbf{n}}\otimes 1 \tag{13.11}$$

$$(\mathbf{I}-\mathbf{D}_c)^{-1} = \mathbf{I} - \frac{1}{3}\ k_d\ \mathrm{tg}\varphi\ \mathbf{n}\otimes 1 \tag{13.12}$$

The tangent matrix is equal to :

$$\mathbf{D} = k_c\ 1\otimes 1 + \alpha\ (\mathbf{I}-\frac{1}{3}\ 1\otimes 1) - \alpha\ \hat{\mathbf{n}}\otimes\hat{\mathbf{n}} - k_c\ k_d\ \mathrm{tg}\varphi\ \hat{\mathbf{n}}\otimes 1 \tag{13.13}$$

which is identical to the one obtained by Simo-Taylor in [10].

14. Numerical applications

14.1 Rectangular soil sample

The goal of this example is to compare a numerical solution with a complete analytical exact solution, in order to validate the program.

A rectangular soil sample is subjected to an uniform load q_0=4 MN/m (figure 3). The values of the soil properties are the following ones :

- Young's modulus : $E = 5.\ 10^4$ Mpa.
- Poisson's coefficient : $\nu = 0.33$
- cohesion stress : $c = 30$ MPa
- friction angle : $\varphi = 40°$

The analytical limit load obtained in [7] is given by :

$$\sigma_y^l = \frac{c}{\mathrm{tg}\varphi}\ \frac{2}{1 \pm \dfrac{(2-\rho\psi)}{\sqrt{3\psi\,(2-\rho^2\psi)}}} \tag{14.1}$$

where ± is related to traction (+) or compression (-). Expression above gives the exact limit load because it can be obtained by both upper and lower bound theorems [7].

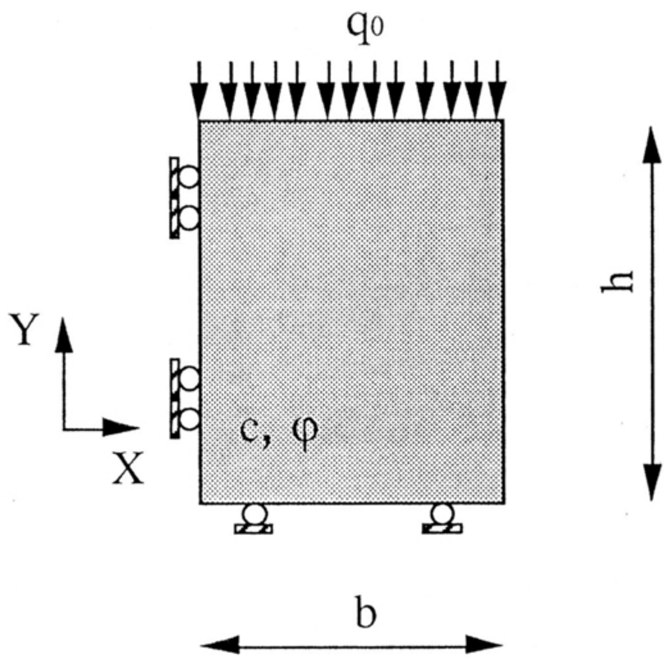

Figure 3. *Rectangular sample*

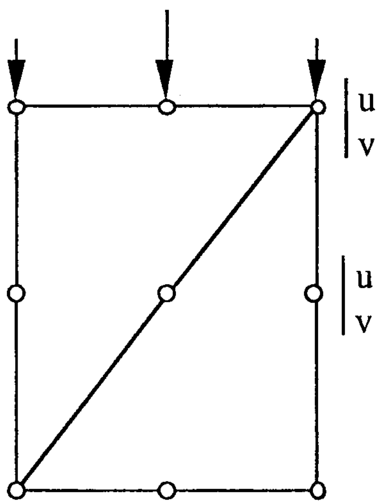

Figure 4. *Meshe with 2 T6 elements*

The structure is discretized with two parabolic triangular T6 elements (figure 4) and the loading is controled by displacement. The numerical values of the limit stress denoted a are given in Table 1 for different values of the non associativity indicator $\rho = tg\theta/tg\varphi$.

	Compression		Traction	
ρ	σ_y^l	α^l	σ_y^l	α^l
1.0	-128.67	- 32.167	27.978	6.994
0.5	-122.63	- 30.657	27.682	6.920
0.0	-108.44	- 27.110	26.888	6.721

Table 1. *Limit stresses and limit multipliers*

Concordance between exact and numerical solution is satisfactory. It can be remarqued that any non associated limit stress is always lower than the associated limit stress, that is in agreement with Drucker's theorem [11] (figures 5,6).

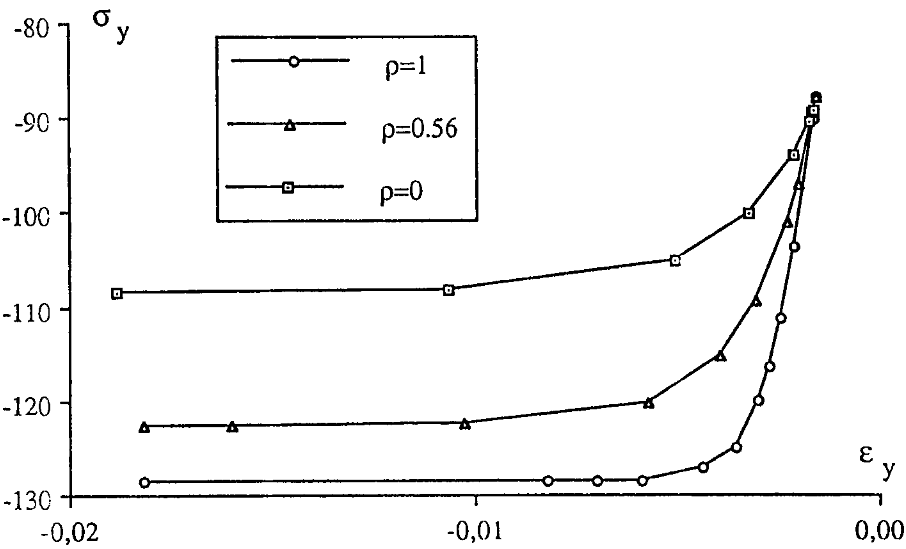

Figure 5. *Rectangular sample : traction loading*

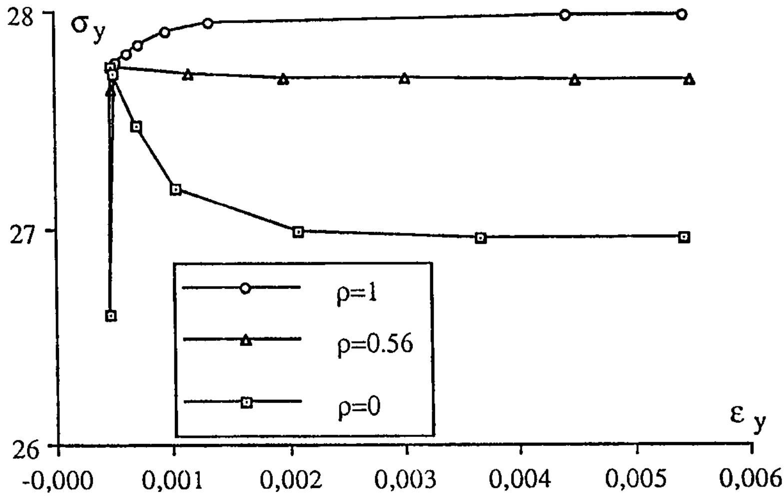

Figure 6. *Rectangular sample : traction loading*

On the other hand, in the traction case, the elastic stress is greater than the non associated limit stress. This can be explained by the appearance of a post-peak softening zone due to the non standard behaviour of soil in traction.

14.2 Shallow strip footing

The purpose of this example is the study of the limit load sensitivity to non associativity. We consider a rigid shallow strip footing subjected to an uniform load. The soil stratum is supposed homogeneous and isotropic. Friction soil-footing is not considered. The soil stratum and boundary conditions are shown in figure 7. The mesh involving 102 T6 elements and 233 nodes is shown in figure 8.The soil properties are :

$E = 0.3x10^5$ kN/m^2	$\varphi = 20°$.
$\nu = 0.3$	$\gamma = 0$.
$c = 10$ kN/m^2	$\rho = 1, 0.5$ ou 0.

The numerical values of the limit pressure corresponding to different values of the no associativity factor ρ are given in table 2. The corresponding load-displacement curves are given at Figure 9. In these curves, U_c represents the displacement of the right hand upper corner node, on the symmetry axis.

For the standard material (ρ=1), the numerical limit load is very close to the exact limit load corresponding to Prandtl mechanism, obtained with Mohr-Coulomb criterion. The sensitivity of the results to the non associativity factor ρ is significant. The velocity field of the failure mechanism for the standard material and φ=20° is shown at figure 10. The corresponding ones for the same friction angle and dilatancy angle θ equal to 10° and 0° are respectively given at figures 11 and 12. Significative modifications of the failure mechanism when compared with the standard case can be remarked.

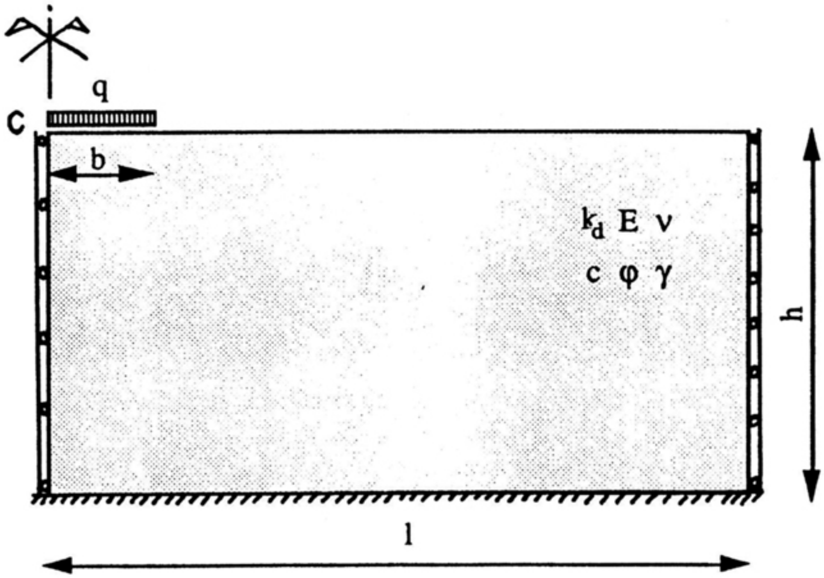

Figure 7. *Shallow strip footing*

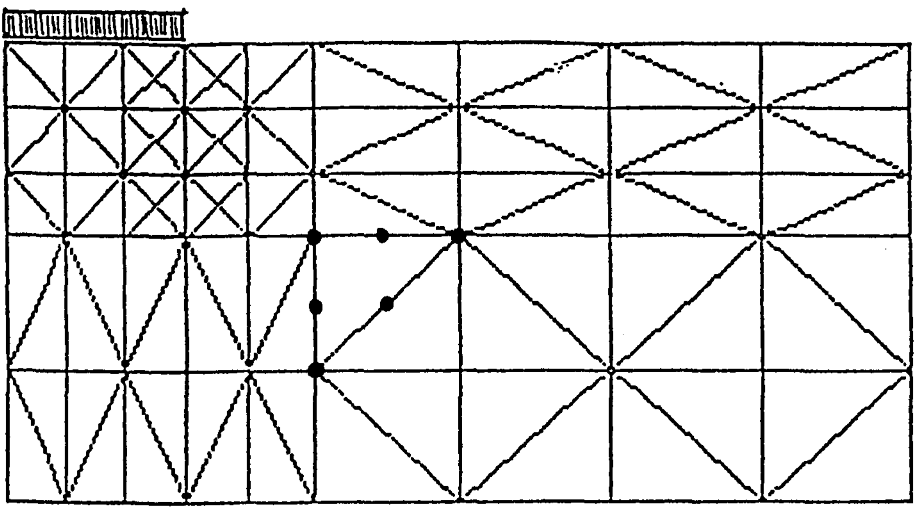

Figure 8. *Meshe for the shallow strip footing*

ρ	α
1.	14.94
0.5	14.79
0.	13.98

Table 2. *Limit multiplier respect to the no associativity factor ρ*

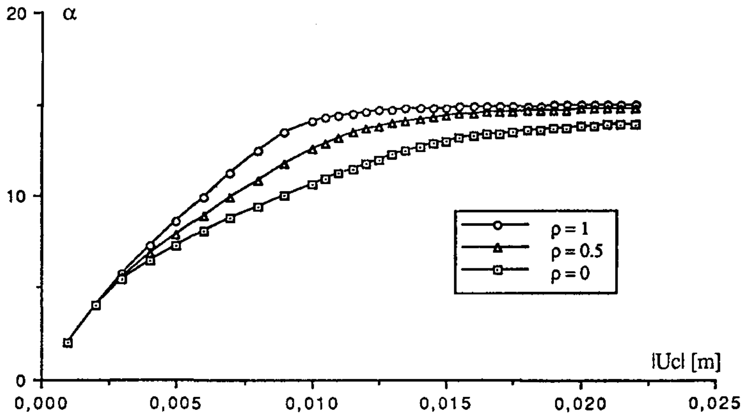

Figure 9. *Load-displacement curves for shallow strip footing*

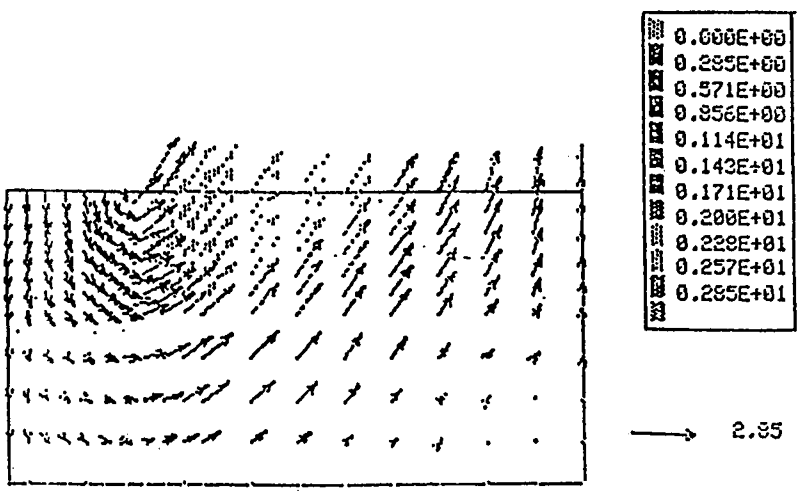

Figure 10. *Failure mechanism for $\varphi = 20°$ and standard material*

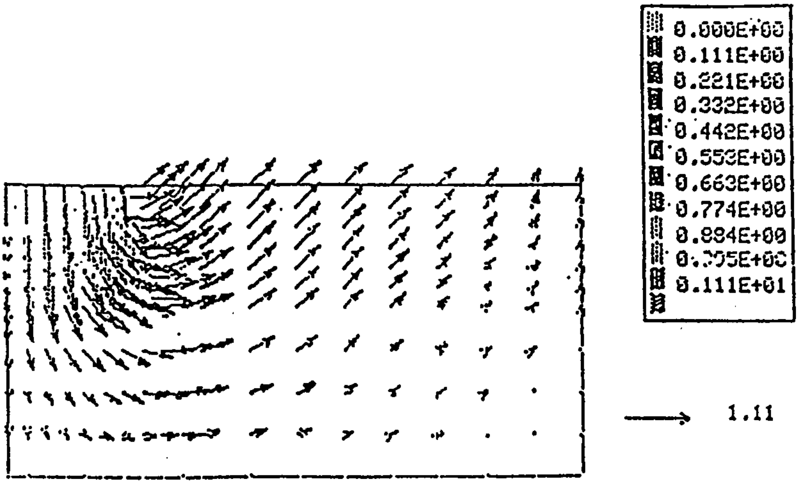

Figure 11. *Failure mechanism for $\varphi = 20°$ and $\theta = 10°$*

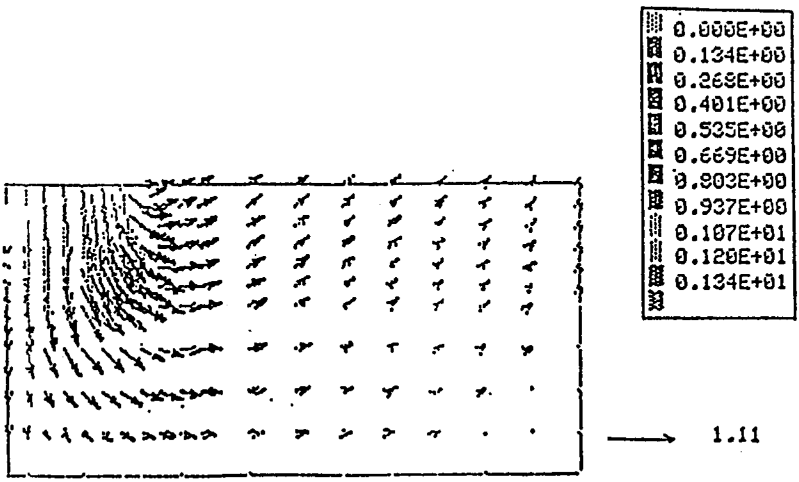

Figure 12. *Failure mechanism for $\varphi = 20°$ and $\theta = 0°$*

14.3 Performances of the proposed algorithm

The aim of this section is to test the proposed algorithm in non associated case with symmetric tangent stiffness matrix. The following data are considered :

$\varphi = 20°$
$\gamma = 0$
$\rho = 0.5$

The computation times are summarized at table 3. The result obtained by the classical algorithm with no symmetric stiffness matrix and the new one are identical. Nevertheless, cpu time is reduced of 30%, for the present example, when using the algorithm with symmetric operator. Moreover, gain in memory space is significative because only half of tangent stiffness matrix is stored.

tangente stifness matrix	cpu-time(1)
symmetric	69mn
no symmetric	98mn

(1) on VAX VS-3100 computer

Table 3. *Computation times*

14.4 Deep footing

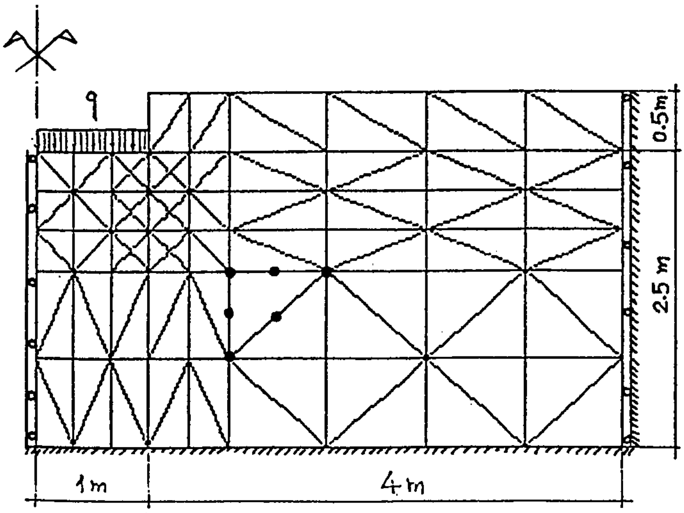

Figure 13. *Meshe for the deep footing*

The case of deep footing is traited. The mesh includes 114 T6 elements and 259 nodes (figure 13). The reference load is q_0=10 kN/m. The soil properties are :

$E = 0.3x10^5$ kN/m^2 $\qquad \varphi = 20°$

$\nu = 0.3$ $\qquad \gamma = 16$ kN/m^3

$c = 10\ kN/m^2$ $\rho = 1, 0.5$ ou 0.

The numerical results are summarized in table 4 and figure 14. The mechanisms near limit state are shown in figures 15 and 16.

ρ	α
1.	32.32
0.5	31.47
0.	28.46

Table 4. *Limit factors for the deep footing*

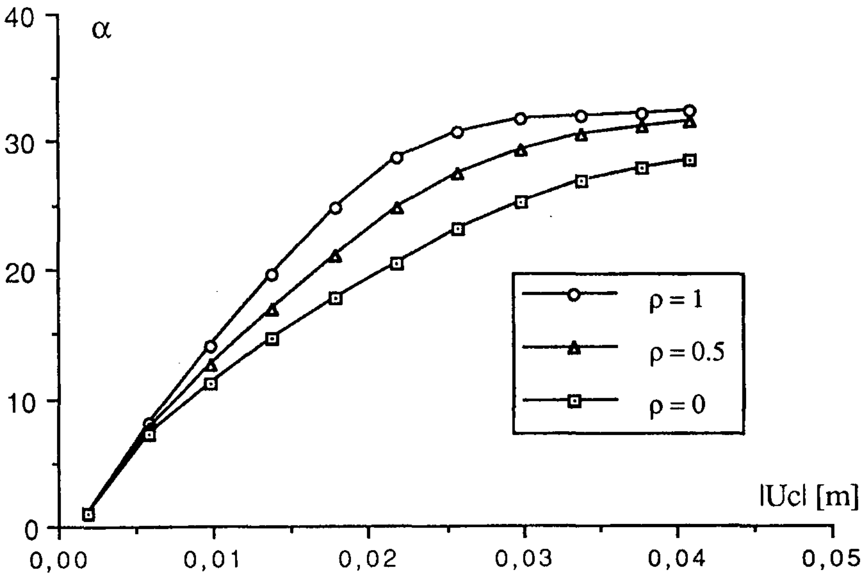

Figure 14. *Load-displacement curves for the deep footing*

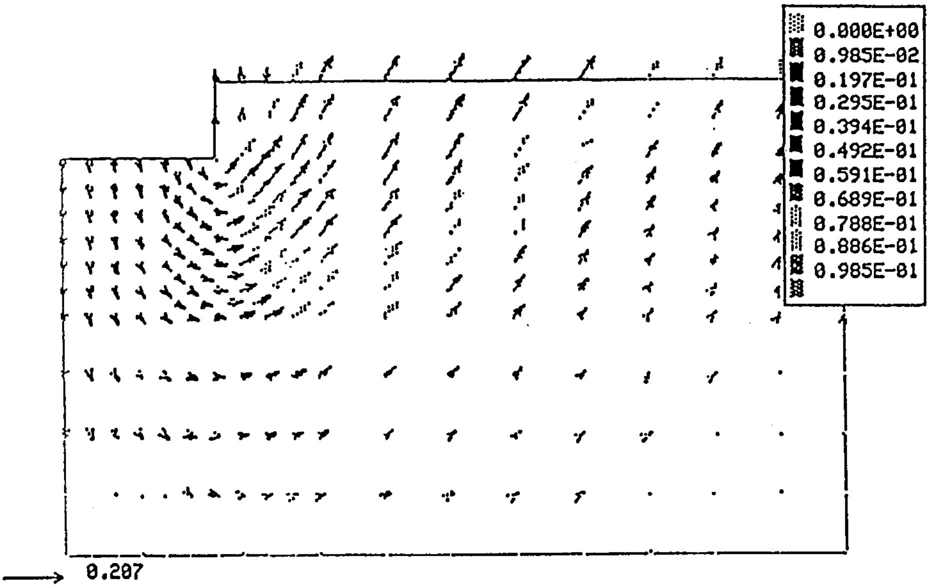

Figure 15. *Failure mechanism for $\varphi = 20°$ and $\theta = 20°$*

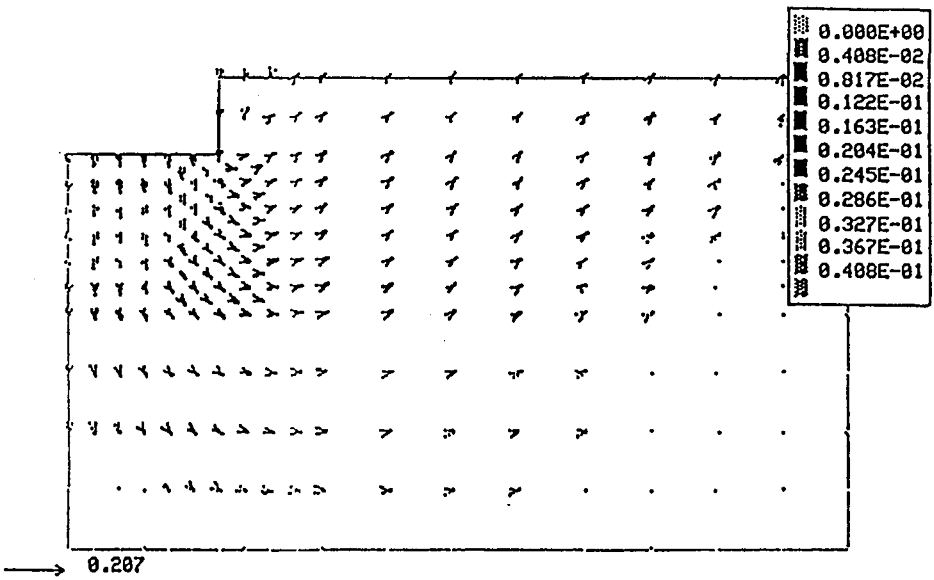

Figure 16. *Failure mechanism for $\varphi = 20°$ and $\theta = 10°$*

14.5 Circular footing

The same mesh as at figure 8 is considered, with a reference load $q_0=10$ kN/m^2. The soil properties are :

$E = 0.3x10^5\ kN/m^2$ $\varphi = 10°$
$\nu = 0.3$ $\gamma = 0.$
$c = 10\ kN/m^2$ $\rho = 1,\ 0.5$ ou 0.

Results are summarized in table 5 and figure 17. Figures 18 and 19 show mechanism when reaching the limit state. Conversely to the previous problems, the sensitivity to the non associativity is small.

ρ	α
1.	8.99
0.5	8.89
0.	8.65

Table 5. *Limit factors for the circular footing*

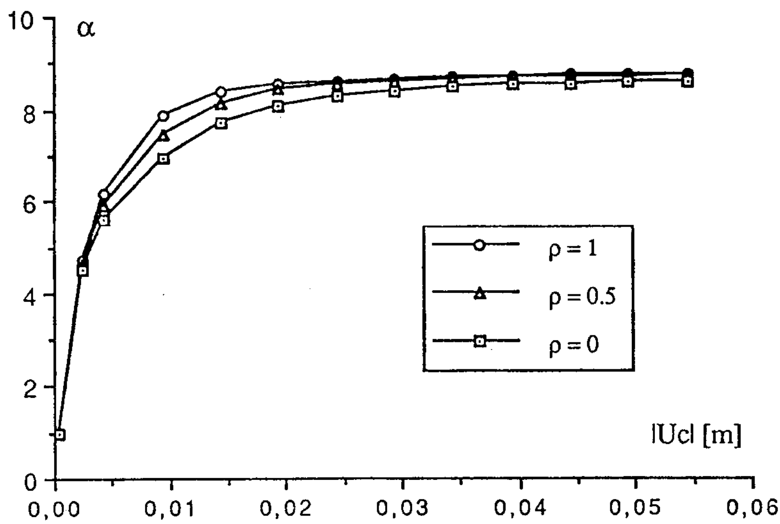

Figure 17. *Load-displacement curves for the circular footing*

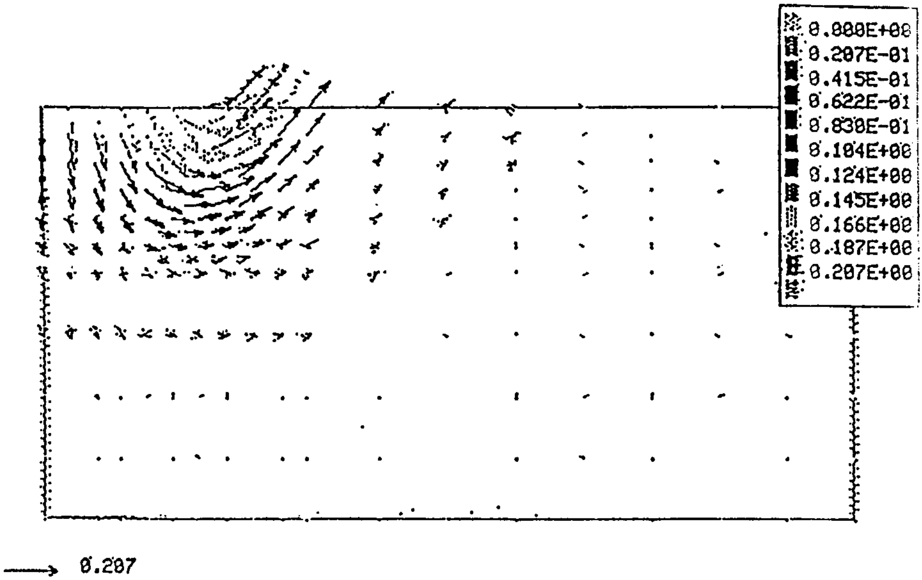

Figure 18. *Failure mechanism for $\varphi = 20°$ and $\theta = 20°$*

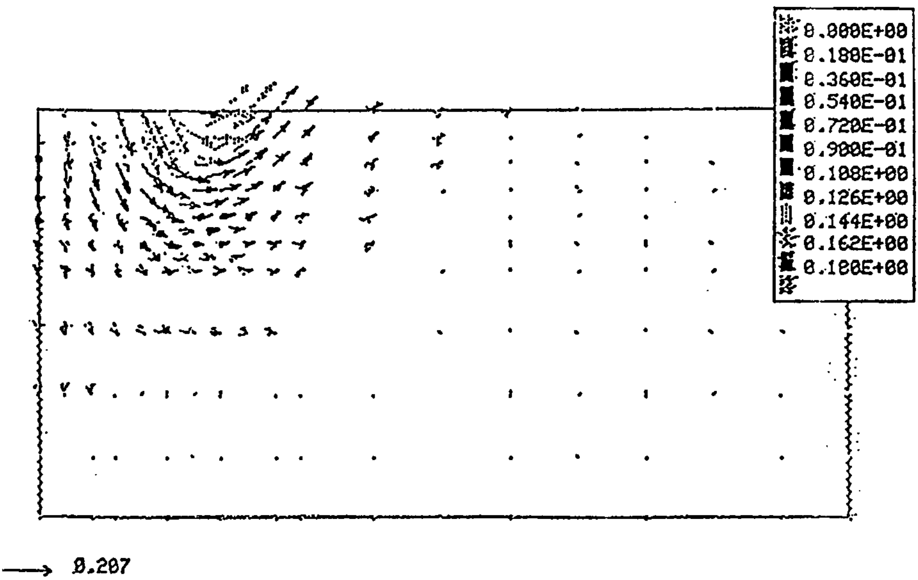

Figure 19. *Failure mechanism for $\varphi = 20°$ and $\theta = 10°$*

14.6 Tunnel stability

The last example is concerned by the modelization of the ground surrounding a tunnel with circular cross-section. 160 T6 elements and 359 nodes are used in the mesh represented at figure 20. The reference load is q_0=10 kN/m. The soil properties are :

$E = 0.3 \times 10^5$ kN/m^2 $\varphi = 20°$
$\nu = 0.3$ $\gamma = 0.$
$c = 10$ kN/m^2 $\rho = 1, 0.5$ ou 0.

The results are given at table 6 , figures 21, 22 and 23.

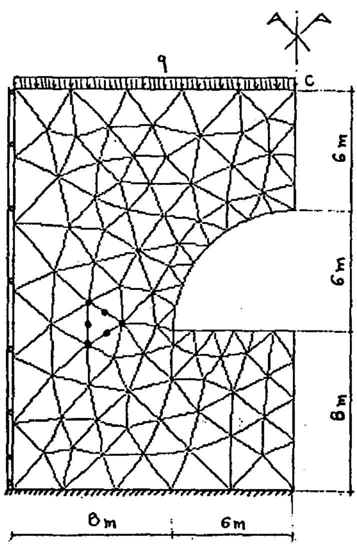

Figure 20. *Meshe for the stability tunnel problem*

ρ	α
1.	3.29
0.5	3.20
0.	2.98

Table 6. *Limit factors for the stability tunnel problem*

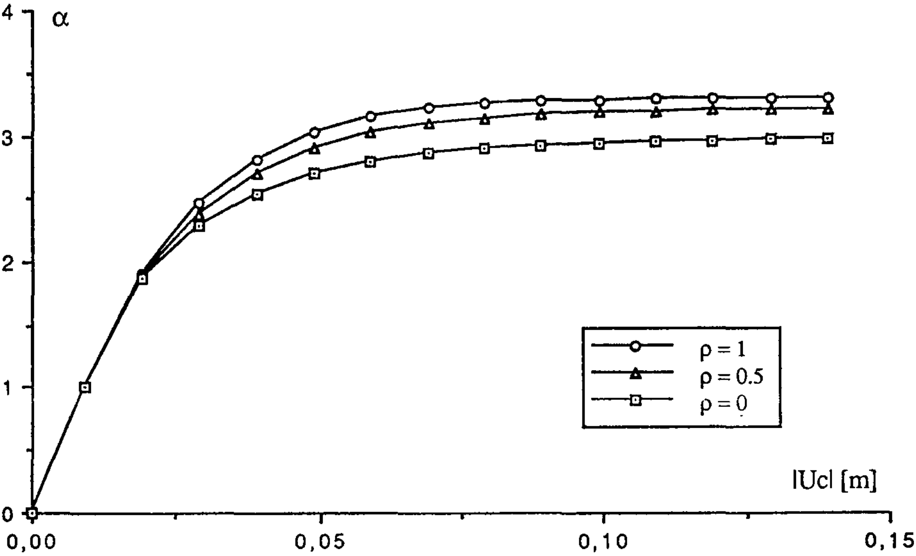

Figure 21. *Load-displacement curves for the stability tunnel problem*

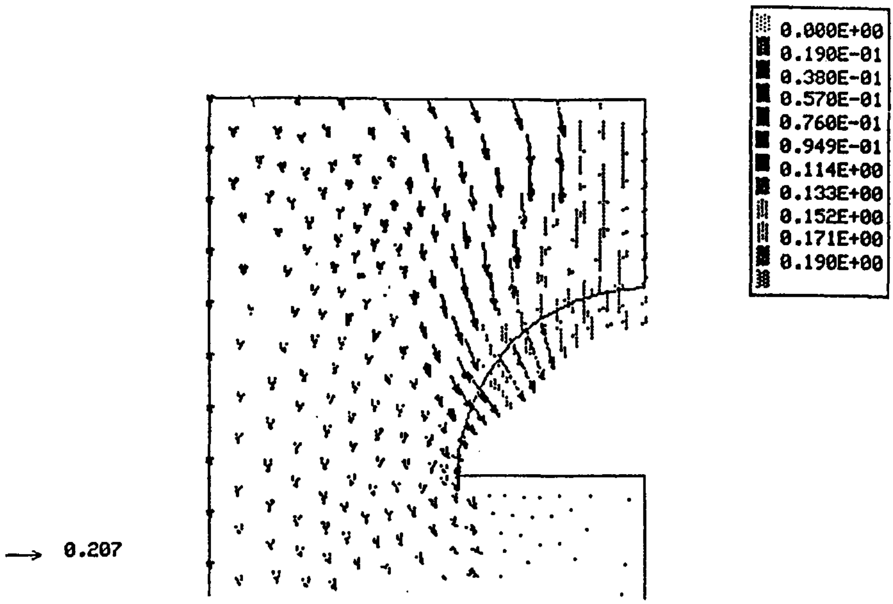

Figure 22. *Failure mechanism for $\varphi = 20°$ and $\theta = 20°$*

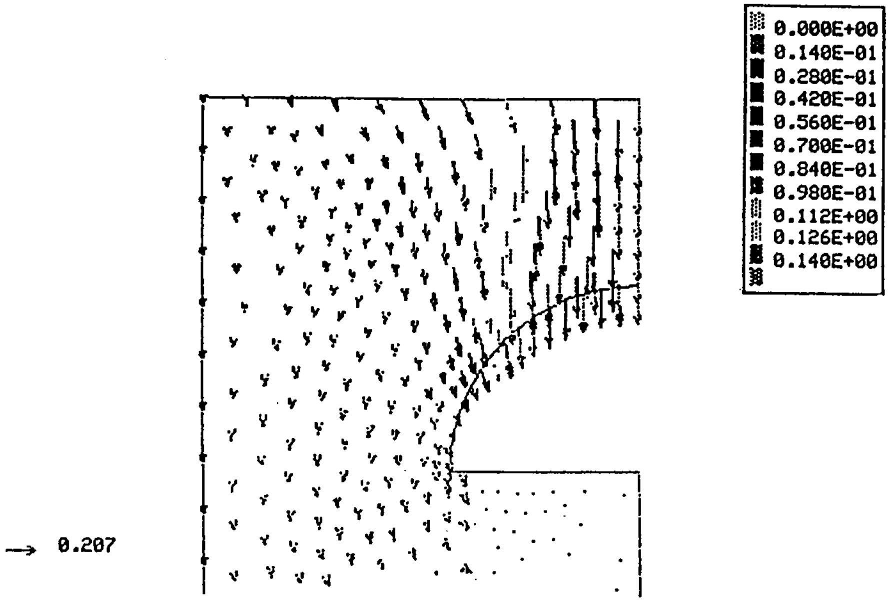

Figure 23. *Failure mechanism for $\varphi = 20°$ and $\theta = 10°$*

15. Conclusion

On the basis of the Implicit Standard Materials, a new method to modelize the non associated flow rules of soils and to state related variational principles is proposed. This constructive method suggests an iterative algorithm based on Newton's scheme to compute the elastoplastic evolution problems. The analysis of the equation structure from the point of view of the Implicit Standard Materials allows to obtain a symmetric and positive definite tangent stiffness matrix, and to reduce significantly the computation time with respect to usual Newton's scheme leading to a not symmetric, not positive definite and often ill-conditioned tangent stiffness matrix.

The new algorithm involving a symmetric tangent stiffness matrix based on the implicit standard material approach was shown to be stable and to lead to a significant reduction of the computation time. Comparison with known analytical solutions in both standard and non standard cases proves the program to be valid. Various other problems without available reference solution were considered in order to assess the sensitivity to the non associativity.

The various numerical tests seen above show that the limit state sensitivity to non associativity is variable but generally significant and must be taken into account in the computations. This suggests to develop experimental programs in order to test the plastic dilatancy angle θ.

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Annex

Let Y be a locally convex separated topological vector space and Y' its dual. The duality is denoted $< \mathbf{x}, \mathbf{y}' >$ for $\mathbf{x} \in Y$, $\mathbf{y}' \in Y'$. Let F be a convex function defined on Y with values in $\mathbf{R} = [-\infty, +\infty]$. The vector $\mathbf{y}'$ is a subgradient of F at the point $\mathbf{x}$ if $\mathbf{y}'$ is the slope of an affine minorant of F exact at the point $\mathbf{x}$. The set of the subgradients of F at $\mathbf{x}$ is called the subdifferential :

$$\partial F(\mathbf{x}) = \{\mathbf{y}' \in Y' \text{ such that } \forall \mathbf{u} \in Y, F(\mathbf{u}) - F(\mathbf{x}) \geq <\mathbf{y}', \mathbf{u} - \mathbf{x}>\}$$

If F is differentiable, then $\partial F(\mathbf{x}) = \{F'(\mathbf{x})\}$.

Let $K \subset Y$ be a closed convex set. The indicator function of the convex K is denoted $\Psi_K(\mathbf{x})$ and defined by :

$$\Psi_K(\mathbf{x}) = 0 \quad \text{if } \mathbf{x} \in K$$

$$\Psi_K(\mathbf{x}) = +\infty \quad \text{if } \mathbf{x} \notin K$$

Then, for $\mathbf{x} \in K$, $\partial\Psi_K(\mathbf{x}) = \{\mathbf{y}' \in Y' \text{ such that } \forall \mathbf{u} \in K, <\mathbf{y}', \mathbf{u} - \mathbf{x}> \leq 0\}$
If $\mathbf{x}$ belongs to the interior of K, then $\partial\Psi_K(\mathbf{x}) = \{0\}$.
If $\mathbf{x}$ is on the boundary of K, $\partial\Psi_K(\mathbf{x})$ is the outward normal set to K at the point $\mathbf{x}$.
The conjugate functional F* is defined on Y' by the Fenchel transform

$$F^*(\mathbf{y}') = \sup_{\mathbf{u} \in Y}(<\mathbf{y}', \mathbf{u}> - F(\mathbf{u}))$$

Hence $F^*(\mathbf{y}') = <\mathbf{y}', \mathbf{x}> - F(\mathbf{x})$

when $\mathbf{y}' \in \partial F(\mathbf{x})$.

Besides $\quad \mathbf{x} \in \partial F^*(\mathbf{y}')$.