The asymptotic-numerical method : an efficient perturbation technique for nonlinear structural mechanics

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ABSTRACT. *Perturbation techniques (asymptotic expansions) have been widely used in many engineering fields for solving nonlinear problems. However, the solution is often represented by the first few terms of a perturbation expansion, which leads to a qualitative approximation rather than a quantitative one. Our aim is to show that a perturbation technique can also lead to a powerfull numerical method for some classes of structural problems, provided that it is combined with a finite element method to account for complex geometries, and that a large number of terms of expansions are determined.*

RÉSUMÉ. *Les méthodes de perturbations (développements asymptotiques) sont utilisées depuis fort longtemps pour la résolution des problèmes non linéaires dans de nombreux domaines scientifiques. Cependant, elles sont bien souvent appliquées dans un cadre purement analytique, pour construire des solutions approchées à l'aide de seulement deux ou trois termes. Notre objectif est de montrer qu'une technique de perturbation peut aussi conduire à une méthode numérique extrêmement efficace pour certaines classes de problèmes non linéaires de structures, si on la combine avec une méthode d'éléments finis pour pouvoir traiter des structures à géométries complexes, et si on est capable de calculer un grand nombre de termes de la série.*

KEY WORDS : *nonlinear computation, perturbation techniques, finite elements, geometric nonlinearity.*
MOT-CLÉS : *calculs non-linéaires, méthode de perturbation, éléments finis, non-linéarité géométrique.*

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1. Introduction

Nonlinear structural problems are generally solved using predictor-corrector methods, such as the very standard Newton-Raphson scheme (Riks ; 1972 ; 1984) ; (Ramm ; 1981) ; (Crisfield ; 1983 ; 1991). Such algorithms are successfull for determining nonlinear solution branches. However, the computing time is usually large as compared to a linear problem solution, and, the automatisation of the continuation process is not always robust.

A family of "Asymptotic-Numerical-Method" based on perturbation techniques and finite element methods have been developed for nonlinear elastic structures. They have been proposed by Damil and Potier-Ferry (1990) for computing perturbed bifurcations, and applied in (Azrar, Cochelin, Damil, Potier-Ferry ; 1993) for computing the postbuckling behaviour of elastic plates and shells. Next, they have been extended to any nonlinear elastic solutions (Cochelin, Damil, Potier-Ferry ; 1994). In contrast to predictor-corrector algorithms, the nonlinear equilibrium pathes are determined by mean of asymptotic expansions : the unknown U and the parameter λ are represented by power series expansions with respect to a control parameter "a". By introducing the expansions into the equilibrium equation, the nonlinear problem is transformed into a sequence of linear problems . The principle of the asymptotic-numerical-method is to build up these linear problems in a recurrent manner and to solve them by a very classical finite element method. Hence, a large number of terms of the series can be numerically computed. Because all the linear problems have the same stiffness matrix, the method requires only one matrix triangulation. One gets a continuous analytic representation of the branch which differs from the "point by point" representation of predictor-corrector algorithms. These asymptotic-numerical methods fall into the category of perturbation techniques that have been already addressed by Masur and Schreyer (1967), Thompson and Walker (1968), Connor and Morin (1971), Gallagher (1975), Glaum, Belytschko, and Masur (1975), Noor and Peters (1980).

The asymptotic expansions have generally a finite radius of convergence, which limits the validity of the polynomial representation to a neighbourhood of the starting point (U_o,λ_o). We have established that, in some cases, the transformation of the polynomial approximations into asymptotically equivalent rational fractions, called Padé approximants (Baker, Graves-Morris ; 1981), can significantly improve the domain of convergence (Cochelin, Damil, Potier-Ferry ; 1994). Another way of improving the domain of validity of these polynomial representations is to apply the reduced basis technique presented and tested by Noor and coll. (Noor, Peters ; 1980 ; 1981 ; 1983), (Noor ; 1981), (Noor, Anderssen ; 1991). The principle is to apply a Rayleigh-Ritz reduction technique to the original nonlinear problem, using the first vectors Ui of the series as a Ritz basis. Because, the ANM is cheap and completely automatic, it is efficient for generating the reduced basis (Cochelin, Damil, Potier-Ferry ; 1993 ; 1994).

When these techniques are applied in a step by step manner, they become powerfull numerical methods which permits to compute the continuation of complex

nonlinear solution branches (Cochelin, Damil, Potier-Ferry ; 1994), (Cochelin, 1994).

The main objective of this paper is to show that a large number of terms of the series can be easily determined with cheap computations, and by using very standard finite elements. The efficiency is due to the choice of a quadratic framework for the expansion process (Damil & Potier-Ferry ; 1990). More precisely, there are two main parts in an Asymptotic-Numerical Method (ANM). The first one is the perturbation technique, or the expansion process, which consists in generating the sequence of linear problems satisfied by the terms of the series $\mathbf{U}_i$ and λ_i. The second one is the numerical solving of these linear problems by FEM. Concerning the first part, it is more advantageous to use a mixed formulation, ie, using both the displacements and the stresses as unknowns, because the governing equations are only quadratic in that case. A pure displacement formulation would involve a cubic nonlinearity and the expansion process would be more intricate. Concerning the second part, it is preferable to have a displacement formulation, on which most existing FE software are based. Hence, we shall start with a mixed formulation when generating the linear problems. Next, each linear mixed problem will be transformed into a displacement problem and a constitutive law, so that it can be solved using very standard FEM. The efficiency of the present methods is partially due to the use of a mixed formulation to generate the linear problems and of a displacement formulation for their solutions.

For simplicity, we present only one example to test the performance of the method. Others ones can be found in previous references.

2. Formulation of the elasticity problem

We consider an elastic body occupying a volume Ω_0 bounded by $\partial\Omega_0$. The displacement field is denoted by $\mathbf{u}$, and the Green-Lagrange strain tensor is given by

$$\gamma(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + {}^t\nabla \mathbf{u}\right) + \frac{1}{2}\left({}^t\nabla \mathbf{u} \,.\nabla \mathbf{u}\right) = \gamma^{l}(\mathbf{u}) + \gamma^{nl}(\mathbf{u},\mathbf{u}) \qquad [1]$$

The corresponding Piola-Kirchhoff stress tensor of the second kind is denoted by S. There are various ways of formulating the equilibrium problem for the body. This depends on the choice of the unknowns and on the choice between local equations and variational formulation. For instance, the unknowns can be chosen as the displacement $\mathbf{u}$ alone, or the mixed variables $(\mathbf{u},S)$ or even the three field variables $(\mathbf{u},\gamma,S)$. In general, this choice depends on the method used for solving the problem. For example, the displacement variational formulation is well adapted for the finite element calculation.

In that paper, where we shall use a perburbation technique , the mixed Hellinger-Reissner formulation is more convenient. Indeed, it gives a quadratic

nonlinearity with respect to the mixed unknown (**u**,**S**). Hence, the asymptotic expansions will be very simple.

Remarks :
- A displacement formulation provides a cubic nonlinearity, which leads to a more intricate expansion procedure.
- Eventhough we are using a mixed formulation at this stage, we will eliminate the stresses later on, in order to use classical displacement finite elements.

For elastic bodies the governing equations can be stated from the mixed Hellinger-Reissner functional :

$$HR(\mathbf{u},\mathbf{S},\lambda) = \int_{\Omega_0} \left(\mathbf{S}:\gamma - \tfrac{1}{2}\,\mathbf{S}:\mathbf{D}^{-1}:\mathbf{S} \right) dv - \lambda P_e(\mathbf{u}) \qquad [2]$$

$$P_e(\mathbf{u}) = \int_{\Omega_0} \rho\,\mathbf{b}.\mathbf{u}\; dv + \int_{\partial\Omega_f} \mathbf{t}.\mathbf{u}\; d\Gamma \qquad [3]$$

where $\rho\mathbf{b}$ are the body forces , $\mathbf{t}$ describes the tractions acting on $\partial\Omega_f$ and D the tensor of elastic moduli. The applied body and surface forces are assumed to be proportional to a scalar loading parameter λ. The variation of [2] yields the governing equations :

$$\int_{\Omega_0} \left(\mathbf{S}:\delta\gamma + \delta\mathbf{S}:\gamma - \mathbf{S}:\mathbf{D}^{-1}:\delta\mathbf{S} \right) dv - \lambda P_e(\delta\mathbf{u}) = 0 \qquad [4]$$

where

$$\delta(\gamma(\mathbf{u})) = \gamma^{l}(\delta\mathbf{u}) + 2\gamma^{nl}(\mathbf{u},\delta\mathbf{u}) \qquad [5]$$

The equation [4] represents both the equilibrium and the constitutive equations. If we introduce the mixed variable

$$\mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{S} \end{pmatrix}$$

the nonlinear problem [4] involves a linear operator **L**, a quadratic operator **Q**, and a given vector **F**, in the following form :

$$\mathbf{L}(\mathbf{U}) + \mathbf{Q}(\mathbf{U}\,,\,\mathbf{U}) = \lambda\mathbf{F} \qquad [6]$$

where

$$\langle \mathbf{L}(\mathbf{U}) , \delta\mathbf{U} \rangle = \int_{\Omega_0} \left(\mathbf{S}:\gamma^{l}(\delta\mathbf{u}) + \delta\mathbf{S}:\left(\gamma^{l}(\mathbf{u}) - \mathbf{D}^{-1}:\mathbf{S}\right) \right) dv \qquad [7]$$

$$\langle \mathbf{Q}(\mathbf{U},\mathbf{U}) , \delta\mathbf{U} \rangle = \int_{\Omega_0} \left(\mathbf{S}:2\gamma^{nl}(\mathbf{u},\delta\mathbf{u}) + \delta\mathbf{S}:\gamma^{nl}(\mathbf{u},\mathbf{u}) \right) dv \qquad [8]$$

$$\langle \mathbf{F} , \delta\mathbf{U} \rangle = \int_{\Omega_0} \rho\, \mathbf{b}.\delta\mathbf{u}\, dv + \int_{\partial\Omega_t} \mathbf{t}.\delta\mathbf{u}\, d\Gamma \qquad [9]$$

We shall use the same notation for the quadratic operator $\mathbf{Q}(\mathbf{U},\mathbf{U})$ and the associated bilinear quadratic form $\mathbf{Q}(\mathbf{U}_1,\mathbf{U}_2)$ that is given by :

$$\langle \mathbf{Q}(\mathbf{U}_1,\mathbf{U}_2) , \delta\mathbf{U} \rangle = \int_{\Omega_0} \left(\mathbf{S}_1:\gamma^{nl}(\mathbf{u}_2,\delta\mathbf{u}) + \mathbf{S}_2:\gamma^{nl}(\mathbf{u}_1,\delta\mathbf{u}) + \delta\mathbf{S}:\gamma^{nl}(\mathbf{u}_1,\mathbf{u}_2) \right) dv \qquad [10]$$

3. Method for solving the nonlinear problem

We shall use two techniques for solving the nonlinear problem [6] : a perturbation technique and a classical displacement finite element method. The perturbation technique transforms the initial nonlinear problem into a sequence of linear problems with the same operator. The finite element technique transforms the continuous problem into a discrete one, to be solved on a computer. One can distinguish two different approaches depending on which order these techniques are applied :

1- discretise the nonlinear problem first, then apply the perturbation technique.
2- apply the perturbation technique on the nonlinear continuous problem first, then discretise the linear problems.

These two approaches give the same final result, however, we shall use the second one which is much more convenient than the first one, as will be discussed later.

3.1. *The perturbation technique*

Let us suppose that there exists an initial solution point of the nonlinear problem [6] that we denote by $\mathbf{U}_0$, λ_0 . We assume that the equilibrium solution path $\mathbf{U}$, λ can be expanded, in the neighbourhood of this point solution, in terms of a parameter "a ", in the form

$$\begin{aligned} \mathbf{U}(a) &= \mathbf{U}_0 + a\,\mathbf{U}_1 + a^2\,\mathbf{U}_2 + \ldots \\ \lambda(a) &= \lambda_0 + a\,\lambda_1 + a^2\,\lambda_2 + \ldots \end{aligned} \qquad [11]$$

A good choice for the control parameter "a" is the linearized arc-length parameter defined by [12]. It corresponds to the projection of the pair $(\mathbf{u}-\mathbf{u}_0,\lambda-\lambda_0)$, ie, the displacement and the load parameter, on the tangent direction $\mathbf{u}_1$, λ_1 :

$$a = \langle \mathbf{u} - \mathbf{u}_0 , \mathbf{u}_1 \rangle + (\lambda - \lambda_0)\lambda_1 \qquad [12]$$

Other types of parameter "a" can be chosen : a displacement control type or a load control type. But they fail in the presence of limit points.

If we substitute eqn [11] into eqns [6] and [12] , and we equate terms with the same power of a, we obtain the following sequence of linear problems:

order 1

$$\mathbf{L}_t(\mathbf{U}_1) = \lambda_1 \mathbf{F} \qquad \langle \mathbf{u}_1, \mathbf{u}_1 \rangle + \lambda_1\lambda_1 = 1 \qquad [13]$$

order 2

$$\mathbf{L}_t(\mathbf{U}_2) = \lambda_2 \mathbf{F} - \mathbf{Q}(\mathbf{U}_1, \mathbf{U}_1) \qquad \langle \mathbf{u}_2, \mathbf{u}_1 \rangle + \lambda_2\lambda_1 = 0 \qquad [14]$$

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order p

$$\mathbf{L}_t(\mathbf{U}_p) = \lambda_p \mathbf{F} - \sum_{r=1}^{p-1} \mathbf{Q}(\mathbf{U}_r, \mathbf{U}_{p-r}) \qquad \langle \mathbf{u}_p, \mathbf{u}_1 \rangle + \lambda_p\lambda_1 = 0 \qquad [15]$$

where the tangent operator $\mathbf{L}_t$ is defined by

$$\mathbf{L}_t(\,\cdot\,) = \mathbf{L}(\,\cdot\,) + 2\mathbf{Q}(\mathbf{U}_0\,,\,\cdot\,)$$

and $\mathbf{Q}(.\,,\,.)$ is the quadratic operator defined in eqn [10]. Solving these linear problems, we get the vectors $\mathbf{U}_i=(\mathbf{u}_i,\mathbf{S}_i)^t$ and the coefficients λ_i. The problem at order p [15] is similar to an elastic linear problem on Ω_0. It will be solved by a classical displacement finite element method. Since all these linear problems have the same linear operator $\mathbf{L}_t$, after discretization, we shall need only one matrix triangulation.

3.2. *Return to a displacement formulation*

Let us rewrite the mixed linear problem at the order p given by eqn [15]. With the notation [7]-[8]-[9], the mixed unknown $(\mathbf{u}_p,\mathbf{S}_p)$ satisfies the following equation :

$$\int_{\Omega_0}\left(\begin{array}{l}\mathbf{S}_p:\left(\gamma^l(\delta\mathbf{u})+2\gamma^{nl}(\mathbf{u}_0,\delta\mathbf{u})\right)+\mathbf{S}_0:2\gamma^{nl}(\mathbf{u}_p,\delta\mathbf{u})\\ +\delta\mathbf{S}:\left(\gamma^l(\mathbf{u}_p)+2\gamma^{nl}(\mathbf{u}_0,\mathbf{u}_p)-\mathbf{D}^{-1}:\mathbf{S}_p\right)\end{array}\right)dv$$
$$=\lambda_p\mathbf{P}_e(\delta\mathbf{u})-\int_{\Omega_0}\sum_{r=1}^{p-1}\left(\mathbf{S}_r:2\gamma^{nl}(\mathbf{u}_{p-r},\delta\mathbf{u})+\delta\mathbf{S}:\gamma^{nl}(\mathbf{u}_r,\mathbf{u}_{p-r})\right)dv \qquad [16]$$

In order to use a classical displacement finite element method, we transform this mixed linear problem into a displacement problem and a constitutive equation. We obtain the following constitutive equations:

$$\mathbf{S}_p = \mathbf{D}:\left(\gamma^l(\mathbf{u}_p)+2\gamma^{nl}(\mathbf{u}_0,\mathbf{u}_p)+\sum_{r=1}^{p-1}\gamma^{nl}(\mathbf{u}_r,\mathbf{u}_{p-r})\right) \qquad [17]$$

and the following equilbrium equations:

$$\int_{\Omega_0}\left(\left(\gamma^l(\delta\mathbf{u})+2\gamma^{nl}(\mathbf{u}_0,\delta\mathbf{u})\right):\mathbf{D}:\left(\gamma^l(\mathbf{u}_p)+2\gamma^{nl}(\mathbf{u}_0,\mathbf{u}_p)\right)+\mathbf{S}_0:2\gamma^{nl}(\mathbf{u}_p,\delta\mathbf{u})\right)dv$$
$$=\lambda_p\mathbf{P}_e(\delta\mathbf{u})-\int_{\Omega_0}\left(\begin{array}{l}\left(\sum_{r=1}^{p-1}\mathbf{S}_r:2\gamma^{nl}(\mathbf{u}_{p-r},\delta\mathbf{u})\right)+\\ \left(\sum_{r=1}^{p-1}\gamma^{nl}(\mathbf{u}_r,\mathbf{u}_{p-r})\right):\mathbf{D}:\left(\gamma^l(\delta\mathbf{u})+2\gamma^{nl}(\mathbf{u}_0,\delta\mathbf{u})\right)\end{array}\right)dv \qquad [18]$$

The equilibrium equation [18] and the condition [15-b] yield a well-posed problem for the displacement $\mathbf{u}_p$ and the load parameter λ_p. The stress $\mathbf{S}_p$ at order p is defined by equation [17].

3.3. *Discretization by finite element*

The displacements $\mathbf{u}$ and their virtual counterparts $\delta\mathbf{u}$ are related to nodal displacements v and nodal virtual displacement δv via

$$[\mathbf{u}]=[\mathbf{N}]\,[\mathbf{v}] \qquad [\delta\mathbf{u}]=[\mathbf{N}]\,[\delta\mathbf{v}] \qquad [19]$$

where [N] is the shape function matrix . The strain displacement relationship [1] is expressed in a matricial form as,

$$[\gamma] = \left[\gamma^l\right]+\left[\gamma^{nl}\right] = \left[\gamma^l\right]+\frac{1}{2}[\mathbf{A}(\mathbf{v})]\,[\theta(\mathbf{v})]$$
$$[\theta(\mathbf{v})] = [\mathbf{G}]\,[\mathbf{v}] \qquad [20]$$

The nonlinear part of the strain is conveniently written using the product of a matrix [A(v)] and a vector [θ(v)] which are both linear in v (Zienkiewicz, Taylor ; 1991), (Criesfield ; 1991). The matrix [G] relates the nodal displavement v and the displacement gradient vector [θ(v)]. Similary, equation [5] is expressed as

$$\begin{aligned}[\delta\gamma] &= \left[\delta\gamma^{l}\right]+\left[\delta\gamma^{nl}\right] \quad = [\mathbf{B}(v)]\,[\delta v] \\ &= \left([\mathbf{B}_l]+[\mathbf{B}_{nl}]\right)[\delta v] = \left([\mathbf{B}_l]+[\mathbf{A}(v)]\,[\mathbf{G}]\right)[\delta v]\end{aligned} \qquad [21]$$

where $[B_l]$ is the classical strain matrix and the matrix $[B_{nl}]$ is linear in v. The stress-strain relationship is in the form

$$[\mathbf{S}] = [\mathbf{D}]\,[\gamma] \qquad [22]$$

Substituting [19]-[22] into [17] and [18] yields the constitutive equations

$$\left[\mathbf{S}_p\right] = [\mathbf{D}]\left\{\left[\mathbf{B}(v_0)\right]\left[\mathbf{v}_p\right] + \sum_{r=1}^{p-1}\frac{1}{2}\left[\mathbf{A}(v_r)\right]\left[\theta\left(\mathbf{v}_{p-r}\right)\right]\right\} \qquad [23]$$

and the displacement problems

$$\left[\mathbf{K}_T\right]\left[\mathbf{v}_p\right] = \lambda_p[\mathbf{F}] + \left[\mathbf{F}_p^{nl}\right] \qquad [24]$$

where $[\mathbf{K}_T]$ is the classical tangent stiffness matrix evaluated at $\mathbf{U}_0$, λ_0

$$\left[\mathbf{K}_T\right] = \int\left(\left[\mathbf{B}(v_0)\right]^t[\mathbf{D}]\left[\mathbf{B}(v_0)\right] + [\mathbf{G}]^t\left[\tilde{\mathbf{S}}_0\right][\mathbf{G}]\right)dv. \qquad [25]$$

In [25], $\left[\tilde{S}_0\right]$ is a symmetric matrix that contains the initial stresses $\mathbf{S}_0$.
The right hand side of [24] is the sum of the nodal forces [F] and of the vector

$$\left[\mathbf{F}_p^{nl}\right] = -\int[\mathbf{G}]^t\left(\sum_{r=1}^{p-1}\left[\mathbf{A}\left(\mathbf{v}_{p-r}\right)\right]^t[\mathbf{S}_r]\right) + \left[\mathbf{B}(v_0)\right]^t[\mathbf{D}]\left(\sum_{r=1}^{p-1}\frac{1}{2}\left[\mathbf{A}\left(\mathbf{v}_{p-r}\right)\right]\left[\theta(\mathbf{v}_r)\right]\right)dv \qquad [26]$$

which depends on the solutions until order p-1. The additional condition in [15] can be written, in matrix notation, in the form

$$\left[\mathbf{v}_p\right]^t\left[\mathbf{v}_1\right] + \lambda_p\lambda_1 = 0 \qquad [27]$$

Finally the unknowns $\mathbf{v}_p$, and λ_p are determined by solving numerically the linear problems [24] including the additional conditions [27], the stresses $\mathbf{S}_p$ being given by [23]. The key elements of the computational algorithm are :

a- Evaluation of the solution at order 1

Solve : $\left[\hat{\mathbf{v}}\right] = \left[\mathbf{K_T}\right]^{-1}\left[\mathbf{F}\right]$

The unknowns at order 1 follow from :

$$\lambda_1 = \frac{1}{\sqrt{1+\left[\hat{\mathbf{v}}\right]^t\left[\hat{\mathbf{v}}\right]}}$$

$$\left[\mathbf{v}_1\right] = \lambda_1\left[\hat{\mathbf{v}}\right]$$

$$\left[\mathbf{S}_1\right] = \left[\mathbf{D}\right]\left[\mathbf{B}(\mathbf{v}_0)\right]\left[\mathbf{v}_1\right]$$

b- Evaluation of the solution at order p

At each Gauss point, evaluate,

$$\left[\mathbf{S}_p^{nl}\right] = \left[\mathbf{D}\right]\sum_{r=1}^{p-1}\frac{1}{2}\left[\mathbf{A}(\mathbf{v}_r)\right]\left[\theta\left(\mathbf{v}_{p-r}\right)\right]$$

$$\left[\mathbf{S}_p^{*}\right] = \sum_{r=1}^{p-1}\left[\mathbf{A}\left(\mathbf{v}_{p-r}\right)\right]^t\left[\mathbf{S}_r\right]$$

then evaluate the force vector,

$$\left[\mathbf{F}_p^{nl}\right] = -\int\left(\left[\mathbf{G}\right]^t\left[\mathbf{S}_p^{*}\right] + \left[\mathbf{B}(\mathbf{v}_0)\right]^t\left[\mathbf{S}_p^{nl}\right]\right)dv$$

and solve

$$\left[\mathbf{v}_p^{nl}\right] = \left[\mathbf{K_T}\right]^{-1}\left[\mathbf{F}_p^{nl}\right]$$

The unknowns at order p follow from :

$$\lambda_p = -\lambda_1\left[\mathbf{v}_p^{nl}\right]^t\left[\mathbf{v}_1\right]$$

$$\left[\mathbf{v}_p\right] = \frac{\lambda_p}{\lambda_1}\left[\mathbf{v}_1\right] + \left[\mathbf{v}_p^{nl}\right]$$

$$\left[\mathbf{S}_p\right] = \left[\mathbf{D}\right]\left[\mathbf{B}(\mathbf{v}_0)\right]\left[\mathbf{v}_p\right] + \left[\mathbf{S}_p^{nl}\right]$$

3.4. *Discussion*

In order to get a simple algebra in our Asymptotic-Numerical-Method (ANM), we have first used a mixed Hellinger-Reissner variational formulation, which leads to a quadratic nonlinearity. After the perturbation technique, the continuous mixed linear problems have been transformed into displacement ones, in order to use a classical finite element method. If we had discretised the nonlinear problem before applying the perturbation technique, we would have been faced with the following difficulties :

- the use of a mixed formulation which provides a simple algebra, involves the use of mixed finite elements with stress discretization.
- the use of a displacement formulation which avoids to discretise the stress, involves a cubic nonlinearity and a more complex algebra.

That is why we have applied the perturbation technique before discretization.

Computing time and implementation

In our implementation, the computation of the series [11] until order N requires :

- One global tangent stiffness matrix evaluation and triangulation
- N evaluations of right-hand-side vectors $[F^{nl}]$
- N+1 backsubstitution

Hence, the computing time is almost the same as for a single step of the modified Newton-Raphson procedure with N+1 iterations. The evaluation of the nonlinear forces vectors $[F^{nl}]$ is very similar to the evaluation of a residual vector, and it requires almost the same computing time (see example in next section). In one hand, a few more time is needed because of the summations in the calculation of the stresses $[S^{nl}][S^*]$. On the other hand, some saving is obtained because the same matrix $[B(v_0)]$ is used for all the $[F^{nl}]$.

The implementation of the asymptotic-numerical method in a finite element computer program that is designed for geometric nonlinear analysis is rather straighforward. The only non-standard routines are the ones that compute the stresses $[S^{nl}]$ and $[S^*]$ and that evaluate the nonlinear forces $[F^{nl}]$. For an efficient implementation, the following two remarks should be considered :

- the stresses $[S^{nl}]$ used for the evaluation of $[F^{nl}]$ should be stored since they appear again in the calculation of $[S_p]$.
- The minimum storing requirement is to store the displacement $[V_p]$ and the stresses $[S_p]$ at each order. However, it is also recommended to store the vectors of displacement gradient $[\theta_p]$ at each order. This permits a direct evaluation of the stresses $[S^{nl}][S^*]$ whithout recomputing numerous [G][v] products.

4. Numerical example

In order to illustrate the performance of the ANM we consider a classical test problem already discussed in (Riks ; 1984), (Almroth, Stehlin, Brogan ; 1981) and (Noor ; 1981). A cylindrical shell with two diametrically opposite cut out is loaded by a uniform axial compression. The geometry and the material characteritics are given in Figure 1. For symmetry reasons, one eight of the shell has been discretised with a regular mesh involving 1800 triangular DKT element. The total number of degrees of freedom is 5796. For evaluating the accuracy of approximate solutions, we shall compute the norm of the residual vector, the norm being taken as the greatest absolute value of the vector components.

4.1 Analysis of the asymptotic solution

The figure 1 and the table 1 show the asymptotic solutions with 10 and 20 terms of expansion. It is clear that the series have a finite radius of convergence around a=12 (λ=1.6) and that we get only the beginning of the nonlinear response. It is also clear, when looking at the residual norm, that we get a very accurate solution inside the radius of convergence. For example, with a maximun allowance of 0.1 for the residual norm, the asymptotic solution is acceptable until a=8 (λ=1.25) at order 10, and until a=11 (λ=1.5) at order 20. So, there is no need of a corrector step.

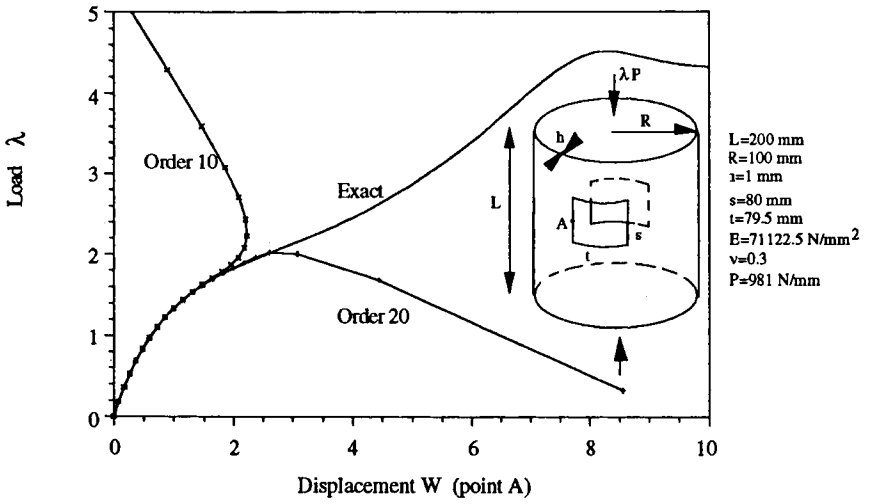

Fig.1 : Cutout cylinder test. The asymptotic-numerical solutions at order 10 and 20 are compared to the exact solution.

	Order 10			Order 20		
Path parameter	W	λ	Residual norm	W	λ	Residual norm
a=1	0.0839664	0.1819137	0.853E-11	0.0839664	0.1819137	0.925E-11
a=2	0.1731853	0.3561970	0.890E-08	0.1731853	0.3561970	0.197E-10
a=3	0.2682802	0.5225942	0.777E-06	0.2682802	0.5225942	0.287E-10
a=4	0.3699208	0.6807915	0.185E-04	0.3699208	0.6807915	0.319E-10
a=5	0.4787978	0.8304294	0.216E-03	0.4787981	0.8304293	0.137E-08
a=6	0.5955797	0.9711271	0.161E-02	0.5955822	0.9711257	0.657E-07
a=7	0.7208452	1.1025229	0.870E-02	0.7208589	1.1025143	0.173E-05
a=8	0.8549838	1.2243376	0.373E-01	0.8550454	1.2242981	0.295E-04
a=9	0.9980596	1.3364660	0.133	0.9982916	1.3363165	0.361E-03
a=10	1.1496317	1.4391036	0.413	1.1503829	1.4386154	0.351E-02
a=11				1.3106679	1.5315104	0.273E-01
a=12				1.4780231	1.6156329	0.179
a=13				1.6508368	1.6919829	1.002

Table 1 : Evolution of the norm of the residual vector with the control parameter "a", for the asymptotic-numerical solutions at order 10 and 20. For comparison, when λ=1, the norm of the applied force vector is 38.5.

Length of the tangent prediction	Number of corrector iterations to achieve : residual norm < 0.1	Solution	
		W	λ
a=1	1	0.0822896	0.1800354
a=2	2	0.1702890	0.3503044
a=3	3	0.2633346	0.5159976
a=4	5	0.3635113	0.6721766
a=5	8	0.4712724	0.8201659
a=6	13	0.5865320	0.9613998
a=7	No convergence after 25 iterations		

Table 2 : Convergence of the corrector for the modified Newton-Raphson algorithm.

The table 2 reports the result of a modified Newton-Raphson process, which involves almost the same computing effort. The domain of convergence of the corrector is only around a=6 (λ=1). This comparison shows that the radius of convergence of perturbation series is not necessarily small as it is often thought. Notice that the ANM converges exactly to the same solution as the Newton-Raphson process do, because the same residual vector is vanished in the two approaches.

4.2 Continuation procedure

An efficient continuation procedure can be obtained by applying the ANM in a step by step manner. It suffice to define a new starting point inside the radius of convergence and to reapply the ANM from that new point to progress along the

branch. Because of the analytic representation of the nonlinear branch, it is easy to analyse the domain of convergence and to define these new starting points. The results of such continuation for the test problem are shown in figures 2 and 3.

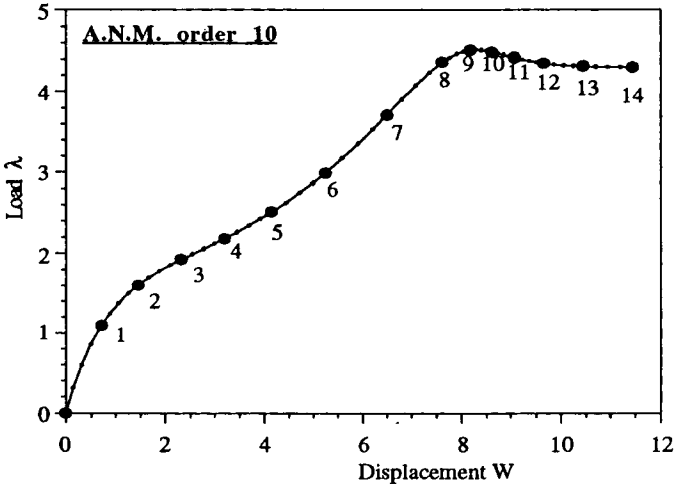

Fig.2 : Continuation process with the asymptotic-numerical method. With 10 terms of expansions, the solution is advanced until w=11.5 in 14 steps with a maximum error of 7.7 10^{-4}.

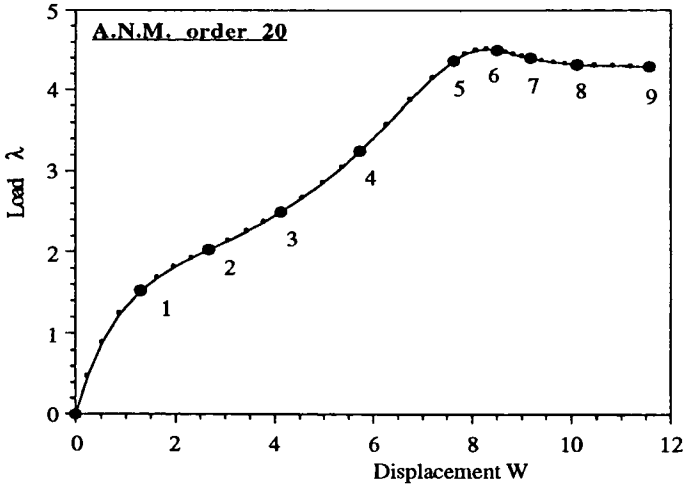

Fig.3 : Continuation process with the asymptotic-numerical method. With 20 terms of expansions, the same solution is obtained in 9 steps with a maximum error of 4.4 10^{-3}.

The starting points have been determined automatically using the following heuristic criteria (Cochelin, 1994), with an ε value of 10^{-5}.

$$a_{max} = \left(\varepsilon \frac{\|u_1\|}{\|u_N\|} \right)^{1/N-1} \qquad [28]$$

At order 10, 14 steps are needed to advance the solution until w=11.5. At order 20, the number of steps is reduced to 9. The maximum error, defined as the ratio of the norm of the residual vector and of the applied force vector, was found to be 7.7 10^{-4} at order 10, and 4.4 10^{-3} at order 20. The accuracy of the solution could be easily improved if desired by reducing the step length, ie, taking a smaller ε.

4.3. Computing time of the ANM

	A.N.M. Order 10 14 steps			A.N.M Order 20 9 steps		
	number	time (sec.)	%	number	time (sec.)	%
Total time		2107	(100)		1877	(100)
K_t evaluation	14	512	(24.3)	9	329	(17.5)
F^{nl} evaluation	14x10=140	532	(25.2)	9x20=180	714	(38.0)
K_t triangulation	14	775	(36.8)	9	523	(27.9)
Backsubstitution	14x11=154	178	(8.4)	9x21=189	222	(11.8)
Residual vector evaluation	14	69	(3.3)	9	44	(2.3)

Table 3 : Computing times of the asymptotic-numerical method.

The computing times of the asymptotic-numerical method are given with some details in table 3. The following comments can be done :

- at order 10, the evaluation of 10 r.h.s. [F^{nl}] requires the same effort as for one tangent stiffness matrix.
- For this test problem, it is more efficient to use 20 terms of the series than only 10. The extra computing time spent for evaluation of the numerous [F^{nl}] is compensated by less [Kt] evaluations and triangulations.
- The evolution of these computing times for larger numbers of d.o.f. can be anticipated as follows : the evaluation of [Kt] and [F^{nl}] increases linearly with the number of d.o.f., but not the triangulation. For very large systems, the evaluation of 20 [F^{nl}] may become very small as compared to the [Kt] triangulation.

For comparison, the computing time for 15 steps of the modified Newton-Raphson method, involving 15 [Kt] triangulations and 89 iterations, is 1950 secondes within our program. We do not want to discuss if 15 steps are sufficient to

obtain the solution until w=11.5. This depends strongly on the strategy used in for the choice of the step length, and the discussion is left to the reader. However, our experience in solving nonlinear problems reveals that the ANM is faster than classical predictor-corrector algorithms.

4.4 Discussion

We have shown that the ANM permits to determine a local representation of the solution branch with short computing time. We have underlined that the polynomial approximation is so accurate inside the radius of convergence that there is no need of a corrector step. Also, the domain of convergence was found to be rather large. When applied in a step by step manner, the ANM becomes a powerfull method for computing complex nonlinear solution branches.

Thanks to the analytical representation of the branch at each step, the continuation process is very efficient, and at the same time, very easy to automatise. Indeed, the domain of convergence does not need to be a-priori estimated as in a classical predictor-corrector algorithm. It is determined a-posteriori at each step, by analysing the radius of convergence of the series. As a consequence, the size of the steps are optimal all along the branch, while the continuation process is kept very robust. We want to insist on the fact that, the steplength is determined from the convergence properties of the current step, and not from that of the previous steps. As a consequence, a sudden curvature of the solution path is easily anticipated within the present method.

More than the short computing time, the very advantage of the ANM is the great facility of use for a non-expert user. Because the stepsize control is naturally managed by the algorithm itself, the only parameter provided by the user in our program is the number of steps. This is very convenient when treating a new problem without knowing the main characteristics of the solution branch. The nonlinear response is generally obtained with very few runs of the program.

5. Conclusions

In this paper, we have reviewed the asymptotic-numerical method and we have given the major technical details for its implementation in a finite element program. The numerical efficiency for geometric nonlinear problems has been demonstrated by mean of a numerical example. Because of the limited space, we have only dealed with the computation of a generic nonlinear branch starting from regular points U_0, λ_0. However, the ANM can also be efficiently used for the computation of a bifurcating branch, starting from a bifurcation point which is a singular point. In that case, the tangent stiffness matrix is not invertible and a special procedure is required for solving the linear problems (Azrar, Cochelin, Damil, Potier-Ferry ; 1993). The detection of bifurcation points on linear or nonlinear branches has also been considered by mean of asymptotic-numerical methods in (Boutyour, Cochelin,

Potier-Ferry ; 1993). Also, related topics such as the improvement of the polynomial series by Padé approximants, or the technique of reduced basis can be found in the references by the authors.

Now, we are extending the ANM for nonlinear dynamics, for finite rotation of shells, and also for instabilities in fluid-structure interaction. For strong nonlinearities such as plasticity, the ANM will also permit to generate a Ritz basis (Damil, Potier-Ferry, Braikat, 1994)

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