Prediction of Anelastic Flow Localization in Finite Elastoplasticity with Damage

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ABSTRACT. *In this paper a energy-based coupled theory of Continuum Damage Mechanics (CDM) proposed by [SAA 92] and [BEN 92] is extended to the finite strain plasticity formulated in the eulerian reference system. The classical framework of irreversible thermodynamics with internal state variables is used to describe both the continuous damage and the non-linear hardening (isotropic and kinematic). To ensure the objectivity of the coupled constitutive equations a Rotating Frame Formulation (RFF) is used. The associated computational framework divorces the "geometrical" non-linearities (large streches and rotations) from the physical "material" ones (hardening and damage). Details of the numerical algorithm are provided including the effect of the coupling between classical elastoplasticity and the CDM. Finally a simple numerically oriented treatment of the localized damaged plastic flow is presented.*

RÉSUMÉ. *Dans cet article une théorie de la plasticité endommageable basée sur l'équivalence en énergie proposée dans [SAA 92] et [BEN 92] est étendue à la plasticité en transformation finie utilisant des variables eulériennes. Le cadre standard de la thermodynamique des processus irréversibles avec variables internes est utilisé pour modéliser les effets de l'endommagement continu et ceux de l'écrouissage non linéaire (isotrope et cinématique). Une formulation dans un référentiel tournant a été utilisée. Elle permet d'assurer l'objectivité des relations constitutives tout en séparant les non-linéarités physiques et géométriques. Un schéma de résolution numérique du problème couplé bien adapté au régime postcritique a été développé. La prévision de la formation des bandes de cisaillement dans une plaque en déformation plane est traitée en détail.*

KEY WORDS : *finite plasticity, isotropic damage, numerical integration, shear band initiation.*

MOTS-CLÉS : *plasticité, transformation finie, endommagement isotrope, intégration numérique, bande de cisaillement.*

1. Introduction

It is now clearly established, from the physical point of view, that the ductile fracture process of a wide class of materials is a consequence of large anelastic (plastic or viscoplastic) deformations and rotations giving rise to initiation of many micro-defects and their growth until final fracture. This legitimates the modeling of this kind of ductile fracture by coupling the elastoplastic behavior with some suitable description and measure of the damage under the finite transformations hypothesis. To achieve this an efficient description of the damage process and it is coupling with the elastoplastic behavior is needed from both the theoretical and numerical point of view. Many works in this field under the small strains and rotations hypothesis can be found in the litterature : see for example [LEC 74], [BUI 81], [BEN 85], [DRA 85], [SAA 85], [TVE 86], [LEM 86], [CHO 87], [DEB 87], [MAZ 89] and references given there. However much fewer works in the same field but under the large anelastic deformations and rotations hypothesis including damage effect can be found in the litterature as in [ROU 80], [TVE 82], [COR 82], [GEL 85], [SIM 87], [JU 90], [LEH 91] and [VOY 92].

When coupled with damage the constitutive equations exhibit some softening which increases with increasing damage. Hence the main difficulty encountered when using any coupled constitutive equations is the localization of anelastic flow associated with a change in the character of the incremental equations governing the equilibrium of the body. For the quasistatic loading condition the time independent equilibrium problem loses ellipticity, while under dynamic loading conditions wave speed becomes imaginary. As a consequence numerical solutions to this class of problems obtained by finite elements approximation show an inherent mesh dependency. These theoretical aspects, related to the localization of anelastic flow, have been observed by many authors in a variety of contexts and discussed from divers perpectives [RUD 75], [TVE 81], [SAN 83], [CHA 84], [SCH 86], [KLE 86], [NGU 84], [ORT 87], [PIG 87], [BIL 87], [SAA 87], [BEN 89] and many other works. Also many papers concerned with the practical and mathematical aspects of the stability, bifurcation and localization in continuum mechanics can be found in the recent collective books edited by [BAL 88] and [MAZ 89].

An outline of this paper is as follows. Some basic concepts of large deformations and rotations, needed for subsequent developments, are reviewed. Particularly the so called "Rotating Frame Formulation" (RFF) associated with the objectivity requirements is presented. In the second part, a continuum energy based damage model coupled with elastoplasticity at finite strains is proposed within the framework of irreversible thermodynamics with internal variables. The isotropic and kinematic hardening phenomena together with the isotropic damage are represented by some local internal variables. The coupling between the elastoplastic behavior and the isotropic damage is made by using the classical effective state variables with the hypothesis of the total energy equivalence. Within the framework of the RFF an objective set of constitutive equations accounting for both the positive hardening and damage softening is proposed as a natural extension of the classical infinitesimal coupled damage elastoplasticity. The third section of this work is concerned with the computational aspects of the coupled evolution problem. First the global equilibrium equations are written in the framework of classical Virtual Power Principle (VPP) with the eulerian variables. Then an implicit algorithm is used for an "efficient" integration of coupled constitutive equations. This integration scheme is carefully discussed on the light of the numerical stability and accuracy in post bifurcation (softening) stage. Finally the shear band problem is treated with both initiation and growth stages.

2. Kinematic preliminaries of large deformations

Let a continuum represented in its initial (undeformed) configuration C_0 by the cartesian position coordinates $\mathbf{X} = (X1, X2, X3)$. After some loading the body takes a new current (deformed) configuration C_t represented by the cartesian position coordinates $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$. The transformation of the body (pure deformation and rotation) between C_0 and C_t is then described by the transformation gradient $\mathbb{F}$ given by :

$$\mathbb{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \qquad [1]$$

The commonly used way to describe anelastic components of strain is the multiplicative decomposition of the gradient $\mathbb{F}$ into elastic and plastic parts $\mathbb{F}^e$ and $\mathbb{F}^p$ as initially proposed by [LEE 69] and widely used in the litterature :

$$\mathbb{F} = \mathbb{F}^e\,\mathbb{F}^p = \mathbb{R}\mathbb{U} = \mathbb{V}\mathbb{R} \qquad [2]$$

where use has been made of the classical polar decomposition theorem with $\mathbb{R}$ being the rotation (orthogonal) tensor. $\mathbb{U}$ and $\mathbb{V}$ respectively denote the left and right strech tensors.

Differentiating eqn. [2] and using the usual definition of the spatial velocity gradient $\mathbb{L}$ leads to the following decomposition of it into a pure strain rate tensor $\mathbb{D}$ and a spin rate tensor $\mathbb{W}$:

$$\mathbb{L} = \dot{\mathbb{F}}\,\mathbb{F}^{-1} = \mathbb{D} + \mathbb{W} = \dot{\mathbb{R}}\,\mathbb{R}^T + \mathbb{R}\,\dot{\mathbb{U}}\,\mathbb{U}^{-1}\,\mathbb{R}^T \qquad [3.a]$$

$$\mathbb{D} = \frac{1}{2}(\mathbb{L} + \mathbb{L}^T) = \frac{1}{2}\mathbb{R}\,(\dot{\mathbb{U}}\,\mathbb{U}^{-1} + \mathbb{U}^{-1}\,\dot{\mathbb{U}})\,\mathbb{R}^T \qquad [3.b]$$

$$\mathbb{W} = \frac{1}{2}(\mathbb{L} - \mathbb{L}^T) = \frac{1}{2}\mathbb{R}\,(\dot{\mathbb{U}}\,\mathbb{U}^{-1} - \mathbb{U}^{-1}\,\dot{\mathbb{U}})\,\mathbb{R}^T + \dot{\mathbb{R}}\,\mathbb{R}^T \qquad [3.c]$$

where the superimposed dot ($\dot{}$) denotes the usual material derivative.

From eqn. [3.c] the spin rate $\mathbb{W}$ is the sum of two terms. The first one represents the non triaxial character of the pure deformation (antisymmetric part of $\dot{\mathbb{U}}\, \mathbb{U}^{-1}$).The second term is the rate of rigid body rotation and is noted $\Omega = \dot{\mathbb{R}}\, \mathbb{R}^T$. It is clear that the first term of eqn. [3.c] is zero if the tensors $\mathbb{U}$ and $\dot{\mathbb{U}}$ have the same principal axes (or the principal axes of $\mathbb{D}$ coincide with those of the stretch tensor $\mathbb{V}$). In that case the spin rates $\mathbb{W}$ and Ω are identical.

The decomposition in eqn. [2] implies the concept of the virtual unloaded intermediate configuration C_i which is defined with an arbitrary rotation. To satisfy the objectivity requirements (frame indifference) there are two possible methods :

a. The undetermination of the rotation of the intermediate configuration is assumed and some invariance conditions are explicitely imposed to the constitutive equations. This means that the rotation of C_i does not have any effect in the constitutive equations. This approach of invariance is called "the intermediate configuration" approach ([LEE 69], [SID 73] and others...).

b. The orientation of the intermediate configuration is fixed by an adequate "director frame" related to the microstructure of the Representative Volume Element (RVE) and governed by an appropriate constitutive equations ([MAN 73], [DAF 79], [STO 82]). This approach is called the "natural local configuration".

In the present work the intermediate configuration approach is used and the objectivity requirements are ensured by using the RFF concept. This approach widely used in the recent litterature ([LAD 80], [REE 83)], [LOR 83], [DAF 87], [DOG 89], [HEA 92)],...) uses a rotational transport between the configuration C_0 and C_t with the rotation tensor $\mathbb{Q}$ rather then the classical convective transport with the gradient $\mathbb{F}$. The rotation tensor $\mathbb{Q}$ has the same nature then $\mathbb{R}$ given by eqn. [2] and usually $\mathbb{Q}$ is taken identical to $\mathbb{R}$.

In this work $\mathbb{Q}$ is defined as a rotation between the Global Frame (GF) and a given Rotating Frame (RF) attached to the RVE by an appropriate manner. It takes into account not only the rotation between the frames but also the material rotation due to the deformation of the body. Hence $\mathbb{Q}$ appears as an "internal variable" related to the orientation of the deformed configuration C_t. The evolution law (constitutive equation) governing this "geometrical" internal variable $\mathbb{Q}$ is postulated as in [DOG 89] :

$$\dot{\mathbb{Q}}\, \mathbb{Q}^T = \mathbb{W}_Q \qquad \text{with } \mathbb{Q}(t_0) = \mathbb{1} \qquad [4]$$

where $\mathbb{W}_Q$ ($\mathbb{W}_Q^T = - \mathbb{W}_Q$) is the spin rate of the RF. It is worth noting that if $\mathbb{W}_Q = \mathbb{W}$ then the corotational frame is recovered, and if $\mathbb{Q} = \mathbb{R}$ and $\mathbb{W}_Q = \dot{\mathbb{R}}\, \mathbb{R}^T$, the proper rotational frame is recovered.

For any second order tensor $\mathbb{Z}$, the objective rotational derivative with respect to the RF is given by :

$$\frac{D_Q \mathbb{Z}}{D_Q t} = \overset{\blacklozenge}{\mathbb{Z}} = \dot{\mathbb{Z}} + \mathbb{Z}\, \mathbb{W}_Q - \mathbb{W}_Q\, \mathbb{Z} \qquad [5]$$

from which the classical Zaremba-Jaumann and the Green-Naghdi derivatives can be obtained as particular cases.

Using the early defined time dependant rotation $\mathbb{Q}$, the "rotated" second order tensor $\mathbb{Z}_Q$ is obtained by the tensor $\mathbb{Z}$ transforming according to :

$$\mathbb{Z}_Q = \mathbb{Q}^T\, \mathbb{Z}\, \mathbb{Q} \qquad [6]$$

and by using eqn. [4] to [6] we have a simple and useful relation between the derivatives $\dot{\mathbb{Z}}_Q$ and $\overset{\blacklozenge}{\mathbb{Z}}$:

$$\dot{\mathbb{Z}}_Q = \mathbb{Q}^T\, \overset{\blacklozenge}{\mathbb{Z}}\, \mathbb{Q} \qquad [7]$$

Hence the first main advantage of this RFF concept is that the constitutive equations involve only the classical material time derivatives of any objective tensorial state variables $\mathbb{Z}_j$ by using eqn. [4] to [7]. Another main advantage comes from the fact that within the RFF the coupled elastoplastic constitutive equations are simply formulated in a particular intermediate configuration C_Q obtained from C_i by a rotation $\mathbb{Q}$. Consequently the classical thermodynamics of irreversible processes with rotated internal variables can apply as in the small strain theory.

3. Finite elastoplastic-damage constitutive equations

In this section a natural extension of the infinitesimal damage elastoplasticity to the finite deformations and rotations framework is presented using the RFF concept. The starting points are the elastoplastic constitutive equations with isotropic and kinematic non linear hardening (see [CHA78] and [LEM 85]) together with the isotropic CDM theory. The formulation is made using the following hypotheses :

. H1 : the thermal effect is neglected and only the isothermal elastoplasticity is considered

. H2 : the elastic strains remains vanishingly small compared to the plastic strains. This is closely followed by ductile metals having an elastic modulus order of magnitude higher then the flow stress. Consequently the kinematical decomposition of the velocity gradient is considerably simplified and reduced to the additive form as in small strain theory. This can be summarized as following :

$$\mathbb{F}^e = \mathbb{V}^e \approx \mathbb{1} + \varepsilon^e \qquad \text{with} \qquad \varepsilon^e << \mathbb{1} \qquad [8.a]$$

$$\mathbb{L} \approx \mathbb{L}^e + \mathbb{L}^p \quad \rightarrow \quad \mathbb{D} \approx \mathbb{D}^e + \mathbb{D}^p = \dot{\varepsilon}^J + \mathbb{D}^p \qquad [8.b]$$

where ε^e is the classical elastic small strain tensor and $\dot{\varepsilon}^J$ its Jaumann derivative.

. H3 : the elastic behavior is supposed linear and free from the plastic hardening effect.

. H4 : the plastic flow is supposed isotropic of Mises type with non-linear isotropic and kinematic hardening.

. H5 : the plastic damage is supposed isotropic and represented by one scalar internal variable in the Kachanov's sense ([KAC 58]).

- H6 : at the advanced stage of straining and damage (post localization stage) the isotropy of the anelastic flow is kept as a first approximation.

3.1. *State variables*

The classical framework of thermodynamics of irreversible processes is used with only one Observable State Variable (OSV) and three Internal State Variable (ISV) :

* The strain tensor as a unique OSV which is represented by the strain rate tensor $\mathbb{D}$.

* The three ISV are :

— (r, R) as a scalar variable representing the isotropic hardening with R being the measure of the yield surface radius.

— (α, $\mathbb{X}$) as a second order (purely deviatoric) tensor representing the kinematic hardening, with $\mathbb{X}$ being the measure in the stress space of the displacement of the yield surface center.

— (D, Y) as a scalar variable representing the damage phenomena in the KACHANOV's sense ([KAC 58]). Y is the damage energy release rate [CHA 78].

According to the RFF concept the above tensorial state variables must be replaced by the following "rotated" state variables indiced Q :

$$\sigma_Q = Q^T \sigma Q \qquad \varepsilon^e{}_Q = Q^T \varepsilon^e Q \qquad \mathbb{D}_Q = Q^T \mathbb{D} Q$$
$$\alpha_Q = Q^T \alpha Q \qquad X_Q = Q^T X Q \qquad [9]$$

where Q is the rotation between the cartesian frame attached to the RVE and the cartesian fixed frame as defined in section one, (eqn. [4]).

3.2. *Damage effect : effectives variables*

According to the well known concept of the effective stress ([CHA 78], [LEM 85]) together with the hypothesis of elastic energy equivalence ([COR 79]) the following effective stress and elastic strain tensors are defined :

$$\tilde{\sigma}_Q = \frac{\sigma_Q}{(1-D)^{1/2}} \qquad \tilde{\varepsilon}^e_Q = (1-D)^{1/2}\,\varepsilon^e_Q \tag{10}$$

When used in the state potential these effective variables introduce the damage effect into the elastic behavior as can be shown later.

This elastic energy based concept is extended by [SAA 92] to the stored energy to define the effect of the damage in the hardening variables :

$$\tilde{X}_Q = \frac{X_Q}{h_1(D)} \qquad \tilde{\alpha}_Q = h_1(D)\,\alpha_Q$$
$$\tilde{R} = \frac{R}{h_2(D)} \qquad \tilde{r} = h_2(D)\,r \tag{11}$$

where $h_1(D)$ and $h_2(D)$ are positive scalar valued decreasing functions of the damage variable D. Their explicit form must be determined phenomenologically according to the available experimental data [SAA 88]. In this work, for the sake of simplicity, the functions $h_1(D)$ and $h_2(D)$ are taken identical :

$$h_1(D) = h_2(D) = (1-D)^{1/2} \tag{12}$$

So that the effect of the damage on the elastic and plastic behaviors is the same.

3.3. *Constitutives equations*

In the reference [SAA93], the authors have proposed a set of phenomonological constitutive equations based on the thermodynamics of irreversible processes with internal variables under the small strain hypothesis. These constitutive equations take into account the fully coupling between thermal effect, non-linear isotropic and kinematic hardenings, and the CDM represented by a single scalar variable as specified in §3.2. This model was found to be efficient for the description of the overall behavior of an anelastic material in both the hardening stage and the softening one induced by the damage growth.

By using the "rotated" tensorial state variables given by eqn. [9], these constitutive equations can be extended to the finite transformation hypothesis in the framework of the RFF [BEN 92].

For the sake of simplicity only the final form of these constitutive equations will be given here in the strain space and under isothermal conditions :

State laws

$$\sigma_Q = \tilde{\Lambda} : \varepsilon^e_Q \qquad [13]$$

$$X_Q = \frac{2}{3} \tilde{C}\, \alpha_Q \qquad [14]$$

$$R = \tilde{B}\, r \qquad [15]$$

$$Y = Y_\sigma + Y_X + Y_R \qquad [16]$$

$$- Y_\sigma = \tilde{\sigma}_Q : \tilde{A}_Q : \tilde{\sigma}_Q \qquad [17]$$

$$- Y_X = \frac{1}{2\tilde{C}} J_2^2 \, (\tilde{X}_Q) \qquad [18]$$

$$- Y_R = \frac{1}{2\tilde{B}} (\tilde{R})^2 \qquad [19]$$

with $\tilde{\Lambda}_Q = (1\text{-}D)\, \Lambda_Q$, $\tilde{A}_Q = \tilde{\Lambda}^{-1}_Q$, $\tilde{C} = (1\text{-}D)C$ and $\tilde{B} = (1\text{-}D)B$ are material properties of the damaged material.

Complementary laws

* If $\frac{\partial f}{\partial \sigma_Q} : \tilde{\Lambda}_Q : \mathbb{D}_Q > 0 \Rightarrow$ damage-plastic loading :

. Additive split of strain rates :

$$\mathbb{D}_Q = \mathbb{D}^e_Q + \mathbb{D}^e_Q \qquad [20]$$

. Small elastic strain ($\mathbb{D}^e_Q = \dot{\varepsilon}^e_Q$)

$$\dot{\varepsilon}^e_Q = \Lambda^{-1}_Q : \dot{\sigma}_Q + \dot{\tilde{\Lambda}}^{-1}_Q : \sigma_Q \qquad [21]$$

. Damage-plastic constitutive equations (large plastic strains)

$$\dot{\sigma}_Q = \mathbb{L}_\sigma : \mathbb{D}_Q \qquad [22\text{ a}]$$

$$\dot{\alpha}_Q = \mathbb{L}_\alpha : \mathbb{D}_Q \qquad [22\ b]$$

$$\dot{r} = \mathbb{L}_r : \mathbb{D}_Q \qquad [22\ c]$$

$$\dot{D} = \mathbb{L}_D : \mathbb{D}_Q \qquad [22\ d]$$

$$\mathbb{L}_\sigma = \tilde{\Lambda}_Q - \frac{1}{H}\left[\left(\tilde{\Lambda}_Q : \frac{\partial f}{\partial \sigma_Q}\right) \otimes \left(\frac{\partial f}{\partial \sigma_Q} : \tilde{\Lambda}_Q\right) - \frac{\partial F}{\partial Y}\left(\frac{\partial f}{\partial \sigma_Q} : \Lambda_Q\right) \otimes \sigma_Q\right] \qquad [23\ a]$$

$$\mathbb{L}_\alpha = -\frac{1}{H}\left(\frac{\partial f}{\partial \sigma_Q} : \tilde{\Lambda}_Q\right) \otimes \frac{\partial F}{\partial X_Q} \qquad [23\ b]$$

$$\mathbb{L}_r = -\frac{1}{H}\frac{\partial F}{\partial R}\left(\frac{\partial f}{\partial \sigma_Q} : \tilde{\Lambda}_Q\right) \qquad [24\ a]$$

$$\mathbb{L}_d = -\frac{1}{H}\frac{\partial F}{\partial Y}\left(\frac{\partial f}{\partial \sigma_Q} : \tilde{\Lambda}_Q\right) \qquad [24\ b]$$

* if $\frac{\partial f}{\partial \sigma_Q} < 0 \Rightarrow$ purely elastic behavior (small elastic strain)

$$\dot{\varepsilon}^e_Q = \tilde{\Lambda}_Q^{-1} : \dot{\sigma}_Q \qquad [25\ a]$$

$$\dot{\sigma}_Q = \mathbb{L}_\sigma : \mathbb{D}_Q \qquad [25\ b]$$

$$\mathbb{L}_\sigma = \tilde{\Lambda}_Q \qquad [26\ a]$$

$$\mathbb{L}_\alpha = \mathbb{L}_r = \mathbb{L}_d = 0 \qquad [26\ b]$$

Where the isotropic yield function f is given under the following form :

$$f = J_2(\tilde{\sigma}_Q - \tilde{X}_Q) - \tilde{R} - k \qquad [27\ a]$$

and the plastic potential F is written as :

$$F = f + \frac{1}{2}\frac{a}{C}J_2^2(\tilde{X}_Q) + \frac{1}{2}\frac{b}{B}\tilde{R}^2 + \frac{S}{s+1}\left\langle\frac{-Y}{S}\right\rangle^{s+1}\frac{1}{(1-D)^\alpha} \qquad [27\ b]$$

J_2^2 (Z) define, in the stress space, the following norm :

$$J_2^2(Z) = \frac{3}{2}Z^D : Z^D \qquad [28]$$

with $\mathbf{Z}^D = \mathbf{Z} - \frac{1}{3}\,tr(\mathbf{Z})\,\mathbb{1}$ is the deviatoric part of the tenseur $\mathbf{Z}$ ($\mathbf{Z} \in \{\sigma, \mathbf{X}\}$).

H >0 is the strictly positive damage elastoplastic modulus given by :

$$H = \frac{2}{3}\, C \left[(1\text{-}D)\frac{\partial F}{\partial X_Q} - \frac{\partial F}{\partial Y}\alpha_Q\right] : \frac{\partial f}{\partial X_Q} + B\left[(1\text{-}D)\frac{\partial F}{\partial R} - \frac{\partial F}{\partial Y}\, r\right]\frac{\partial f}{\partial R}$$
$$+ \frac{\partial f}{\partial D}\frac{\partial F}{\partial Y} + \frac{\partial f}{\partial \sigma_Q} : (\tilde{\Lambda}_Q : \frac{\partial F}{\partial \sigma_Q}) - \frac{\partial F}{\partial Y}\, \frac{\partial f}{\partial \sigma_Q} : \frac{\sigma_Q}{1\text{-}D} \qquad [29]$$

Note that $\mathbb{L}_\sigma$ (the tangent elastoplastic operator) and $\mathbb{L}_\alpha$ are both non-symmetric rank forth tensors while $\mathbb{L}_r$ and $\mathbb{L}_d$ are symmetric rank two tensors.

4. Numerical approximation of the coupled problem

In this part the numerical resolution scheme of the initial-boundary value problem associated to the quasistatic evolution of a structure submitted to external forces and displacements is presented.

4.1. *Position of the evolution problem*

Let us consider a structure S occupying at time t the domain Ω defining the current configuration C_t. This structure is submitted to body forces **f**, surface tractions **F** on a part $\partial\Omega_F$ of the boundary $\partial\Omega_F$ of Ω, and imposed displacements **U** on the part $\partial\Omega_u$ of $\partial\Omega$ ($\partial\Omega_u \cup \partial\Omega_F = \partial\Omega$; $\partial\Omega_u \cap \partial\Omega_F = \phi$).

Assuming the initial state of S is fully known, we wish to find, at each time of the observation period $[t_0, T]$, the mechanical fields as defined before i.e. for each particule of S, **x**(X, t), σ(x,t), r(x,t), α(x,t), D(x,t). These unknowns are related by a following equations :

* quasistatic equilibrium equations in the VPP form :

$$-\int_{\Omega(t)} \mathrm{tr}\,[\sigma : \mathbb{D}\,(\mathbf{V}^*)]\, dV + \int_{\Omega(t)} \mathbf{f}.\mathbf{V}^*\, dV + \int_{\partial\Omega_F(t)} \mathbf{F}\ \mathbf{V}^*\, dS = 0 \qquad [30]$$

for any virtual velocity field **V*** kinematically admissible. $\mathbb{D}$(**V***) is the virtual strain rate.

* constitutive relations in the rotated frame as defined throughout the equations [20] to [25] ; these relations can be written formally as :

$$y = g\,(y, t) \qquad [31]$$

where y is the vector of the main variables $\{\sigma_Q, r, \alpha_Q, D\}$ and g(y,t) contains the operators $\mathbb{L}_\sigma, \mathbb{L}_r, \mathbb{L}_\alpha, \mathbb{L}_d$ and $\mathbb{D}_Q$.

* Compatibility equations

$$v = \dot{x}\ (X,T) \qquad \mathbb{L} = \text{Grad } V \qquad \mathbb{D} = \frac{1}{2}(\mathbb{L} + \mathbb{L}^T) \qquad [32]$$

* boundary conditions

$$\mathbf{x} = \mathbf{X} + \mathbf{u} \quad \text{on } \partial\Omega u \qquad \mathbf{F} = \boldsymbol{\sigma}.\mathbf{n} \text{ on } \partial\Omega F \qquad [33]$$

* initial conditions

$$\mathbf{x}\ (\mathbf{X},0) = \mathbf{X} \quad \sigma\ (\mathbf{x}, 0) = 0 \quad r(\mathbf{x}, 0) = 0 \quad \alpha\ (\mathbf{x}, 0) = 0 \quad d(\mathbf{x}, 0) = 0 \qquad [34]$$

Eqns. [30] to [34] define a highly non-linear evolution problem wich can be solved numerically as will be summarized hereafter.

4.2. *Global resolution scheme*

The strong non-linearities of the problem and the differentiel form of the constitutive relations suggest to divide the time period $[t_0,T]$ into time intervals $[t_n, t_n+\Delta t_n]$ (step-by-step method). Assuming all the mechanical fields known at time t_n, we have to find these values at time $t_n + \Delta t_n$.

Using the finite element method with isoparametric elements and the galerkin approximation, the equilibrium equation (eqn. [30]) at time $t_n + \Delta t_n$ can be formally written :

$$\mathfrak{R}\ (\sigma\ (x_i,t_n + \Delta t_n)) = 0 \qquad [35]$$

where $\mathfrak{R}$ is the equilibrium residual, function of the stress tensor and the position of each particule. In fact $\mathfrak{R}$ is a function of the only nodal unknowns x_i.

To solve eqn. [35], a classical Newton-Raphson scheme is used, to build out successive approximations $x_i^k\ (t_n + \Delta t_n)$ of the solution $x_i\ (t_n + \Delta t_n)$ by :

$$\begin{aligned} &x_i^0 = x_i\ (t_n) \\ &x_i^{k+1}\ (t_n + \Delta t_n) = x_i^k\ (t_n + \Delta t_n) + \Delta x \\ &[K_T]x_i^k\ \Delta x = -\ \mathfrak{R}^k\ (x_i) \end{aligned} \qquad [36]$$

$[K_T]_{x_i^k} = [\frac{\partial \Re}{\partial x}]_{x_i^k}$ is the tangent stiffness matrix at point x_i^k.

The computation of $\Re$ and the matrix $[K_T]$ for each iteration requires the value of the stress tensor $\sigma(t_n+\Delta t_n)$ at each integration point. This is obtained by integrating eqn. [31] over the time interval as will be discussed hereafter.

4.3. *Local integration of the constitutive equations*

As discussed in section one, the constitutive relations take a simpler form in a rotated frame defined by the rotation $\mathbb{Q}$ and make use of the deformation rate tensor $\mathbb{D}$. In order to determine $\mathbb{Q}$ and $\mathbb{D}$, it is necessary to characterize the kinematics on the time interval, when the position at time t_n and some approximation of this position at time $t_n + \Delta t_n$ are known. This can be done in several manners, as discribed in [BRA 85]. In this work, a linear evolution of the transformation gradient $\mathbb{F}$ is assumed :

$$\mathbb{F}(\tau) = \frac{1}{\Delta t} [(t_n + \Delta t_n - \tau)\, \mathbb{F}(x_i(t_n)) + (\tau - t_n)\, \mathbb{F}(x_i(t_n + \Delta t_n))] \qquad [37]$$

from which the computation of the tensors $\mathbb{L}$, $\mathbb{D}$ and $\mathbb{W}$ is made possible using eqn. [3]. Moreover $\mathbb{D}$ is taken as a constant equal to $\mathbb{D}(t_n + \frac{\Delta t_n}{2})$, which ensures the incremental objectivity [HUG 80].

The computation of $\mathbb{Q}$ depends on the choice of the objective derivative :

— for the corotational derivative, $\mathbb{Q}$ is obtained by integrating eqn. [4] using the generalized middle point rule :

$$\mathbb{Q}(\tau) = [\mathbb{1} - \frac{1}{2}\mathbb{W}(t_n + \frac{\Delta t_n}{2})]^{-1} [\mathbb{1} + \frac{1}{2}\mathbb{W}(t_n + \frac{\Delta t_n}{2})]^{-1} \mathbb{Q}(t_n) \qquad [38]$$

— for the proper rotation derivative, $\mathbb{Q}$ reduces to the rotation $\mathbb{R}$ resulting from the polar decomposition of $\mathbb{F}$ (eqn. [2]).

In the RFF, the constitutive relations are given by eqn. [31]. To integrate them, a semi-implicit algorithm with an automatic time substep control is used. The algorithm developped by [DEB 88], for non-damaged elastoplasticity, was modified to increase the numerical efficiency : the substep size control has been simplified and a return radial procedure has been implemented to ensure the verification of the plastic criterion at the end of each substep, which is particularly necessary in the softening stage ([BEN 91], [BEN 92]). The box 1 gives the main steps of this algorithm.

For each substep $[t, t + \delta t]$ in the intervalle $[t_n, t_n + \Delta t_n]$

a) discretize eqn. [31] using an Euler scheme and the generalized trapeze method in :

$$G(y(t+\delta t)) = y\,(t+\delta t) - y(t) - \delta t\,[(1-\theta)\,g(y(t),t) + \theta\,g(y(t+\delta t), t+\delta t)] = 0$$

$$\theta \in [0,1]$$

b) compute $y\,(t + \delta t)$ by and Newton-Raphson (NR) scheme such that :

* $|\,G\,(y\,(t + \delta t)\,| < \varepsilon_{NR}$

and

*number of iterations of NR scheme < NITERMAX

ε_{NR} is the required accuracy and NITERMAX the maximal number of iterations for the NR scheme

If both conditions are fullfilled

c) compute $y^*\,(t + \delta t)$, with a radial return method such that :

$f\,[y^*\,(t + \delta t] < \varepsilon_y$

ε_y is the required accuracy for the yield function .

go to a) for a next substep of same size

else

c) reduce δt into :

$$\delta t' = \frac{\varepsilon_{NR}}{|G\,(t+\delta t)|}\,\delta t$$

go to a)

Box 1. *Main steps of the proposed algorithm*

Since the unknowns at time $t_n + \Delta t_n$ are computed in the RF, they are obtained in the global frame by using the inverse transformation with Q^T.

4.4. *Numerical test*

Before the implementation of the above discussed algorithm in a standard general purpose finite element code some numerical patch tests were brought out for the validation of the algorithm. The details of this numerical study can be found in [BEN 91] and [BEN 92]. Here we limit ourselves to the integration of the coupled constitutive equations (eqn. [20] to [25]) in a given RVE (material point) using the following parameters values ([BEN 92]) :

— Isotropic elasticity :	E = 200 000.0 MPa ; ν = 0.33
— Yield limit :	k = 263.7 MPa
— Isotropic hardening :	Q = 3 003.4 MPa
	b = 15.9
— Kinematic hardening :	C = 10 000.0 MPa
	a = 20.0
— Isotropic damage :	S = 0.8 MPa
	s = 1.0
	α = 1.0

The RVE is subjected to a simple elongation (plane strain) defined by the transformation gradient :

$$\mathbb{F} = \begin{bmatrix} 1+\lambda(t) & 0 & 0 \\ 0 & 1/(1+\lambda(t)) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [39]$$

where $\lambda(t)$ is some time dependent loading parameter. The results are summarized in the figures 1 to 5 (see figures at the end of this paper, § 7) where both the coupled and uncoupled cases are compared. The figure 1 shows the variation of $J_2(\sigma_Q)$ versus the accumulated plastic strain defined by :

$$\dot{p}^2 = \frac{2}{3}\, \mathbb{D}^p_Q : \mathbb{D}^p_Q \qquad [40]$$

For the coupled case the stress starting from the yield stress k = 263.7 MPa, increases with increasing hardening, reaches a maximum values $J_2(\sigma_Q)_{max} \sim 325.0$ MPa and goes to zero when damage approches D_{cr} = 1. Figures 2 and 3 show the same behavior for the hardening stresses $J_2(\mathbb{X}_Q)$ and R. Note that this is not the case in the theory used by [LEM 85] and by many other authors where the internal stresses $\mathbb{X}$ and R remain unaffected by the damage.

The figure 4 shows the variation of the damage D versus p. The damage is overestimed in the uncoupled case. Finally the figure 5 shows the variation of the modulus H versus p. Clearely, H is always positive and is a decreasing function of the hardening variable. But in the coupled case H reaches some minimum value as described in the same figure [SAA 93].

5. Application to the flow localization prediction

The above developped model was implemented in the general purpose finite element code SIC available at the university of Compiègne ([AUN 90-1], [AUN 90-2]). Many numerical examples were treated as can be found in [BEN 91] or [BEN 92].

Since the main purpose of the present paper is to study the anelastic flow localization in finite elastoplasticity with damage, we limit ourselves to the well known shear band problem. This example is provided to show the capability of the proposed model (fully coupled damage-elastoplasticity at finite strain) to predict the shear band initiation and growth.

Consider the initially homogeneous plane strain plate represented in figure 6a.

This plate is submitted to the constant displacement rate $\dot{U} = 0.001$ mm/s as shown in the figure 6b. All the numerical calculations are made using the following material parameters corresponding to a mild steel [BEN 91] :

— Isotropic elasticity :	$E = 200\ 000.0$ MPa ; $\nu = 0.3$
— Yield limit :	$k = 400.0$ MPa
— Isotropic hardening :	$Q = 1\ 000.0$ MPa
	$b = 15.0$
— Isotropic damage :	$S = 1.0$ MPa
	$s = 1.0$
	$\alpha = 1.0$

Also the corotational derivative is used.

Because of the suspected dependance of the shear band width on the initial mesh size (see [BAZ 87], [SAA 87], [ORT 87], [BILL 87], etc.), different meshes involving 8 nodes quadrilateral elements were used as presented in the figures 7. Six nodes triangular elements were also used and gave the same results concerning the initiation and the growth as well as the orientation of the shear band [BEN 92].

A first trial of computations was performed using homogeneous boundary conditions as shown in the figure 8 : only the displacements in the y-direction are imposed. In this case the localization of the damaged-plastic flow seems to be mainly governed by the numerical accuracy of the integration sheme. In fact, many numerical tests show that the localization is due to the very small variation of the Young's modulus, itself caused by the small damage gradient [SAA 88]. Some typical results representing a possible localization mode are given in the figures 9, in the case of the mesh 1. At the early stage of loading (fig. 9a), the damage localizes along the line ABCDE, while near the final fracture a high damage localization is observed in the lowest band AB (fig. 9b). This gives a more pronounced necking inside the band AB as shown by the corresponding deformed configuration (fig. 10). On the figure 11 is reported the global force-displacement curve where the symbol (♦) indicate the situation of the figures 9a and 9b.

A second type of numerical simulations is concerned with the same plate but the displacements in the x-direction are imposed to zero. This corresponds to the experimental conditions of the tensile test. Some typical results are summarized in the figures 12 to 16. As a first observation we can note that the necking appears and grows in the middle of the specimen similarly for the three meshes as proved in the figures 12.

The isovalues of the damage and the accumulated plastic strain are presented for the three meshes just before the final fracture (fig. 13, 14, 15). It is worth noting that the localization occurs along two crossing bands AB and CD oriented with approximatively 45° from the y-direction, and independantly of the mesh size. The high localization of damage rising to a macroscopic crack is mesh-dependant : it takes place along the bands AB for the mesh 1, CD for the mesh 2 and CB for the mesh 3.

The global force-displacement curves are found to be similar for the three meshes. An example corresponding to the mesh 3 is given in the figure 16. It is worth noting that the final fracture occurs much earlier in this case (U_{max} = 0.153 mm) then in the previous one (U_{max} = 0.277 mm) while the maximum reached forces are the same. The figure 17 shows a typical local response (stress versus accumulated plastic strain p) for two different points P1 and P2. As depicted in figure 12.a the point P2 is located inside the localized zone while P1 is located outside. Before localization the points P1 and P2 have the same local responses. However, after localization P2 is in plastic-damaged loading (softening due to damage growth) while P1 is elastically unloaded. The figure 18 illustrates the evolution of local damage in point P2. The damage in point P1 is negligeable equal to 5.10-3 at the incepient of the unloading.

6. Conclusion

A new energy based coupled elastoplastic damage constitutive equation developped in [SAA 93] is generalized to the large strains and rotations using the RFF framework. This allows to separate the geometrical non linearities (large strains and rotations) from the material ones (non linear hardening, damage...) and ensure the objectivity requirement. The formulation of the coupled constitutive equations is then made in the rotated frame using the thermodynamics of irreversible processes with internal state variables as under the classical small strains theory.

The associated variational formulation uses the updated langrangian setting with the eulerian stress and strain variables. The stiffness matrix is calculated numerically by using the numerical perturbation technique wich is independent from the used constitutive equations. The global solution is advanced step by step using a classical incremental iterative Newton-Raphson method.

The presented results show clearly both the ability of the proposed approach to trigger the initiation and the evolution of the shear band without any initial perturbation technique and the efficiency of the obtained numerical tool for the prediction of the damaged elastic flow localization. This can be used for entire structure life prediction (behavior, crack initiation and evolution) in the framework of an appropriate non local formulation as can be adressed later.

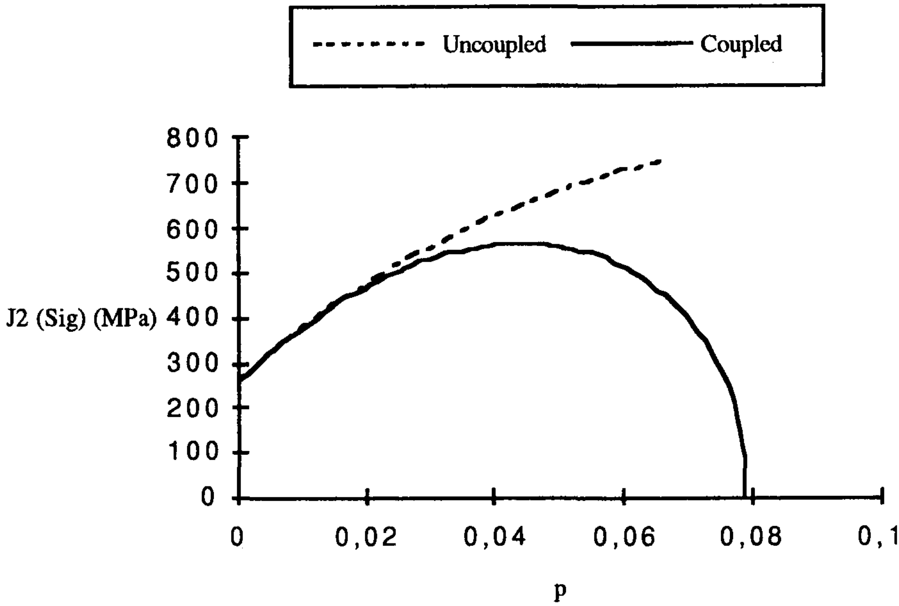


Figure 1. *Coupled and uncoupled stress-strain curve*

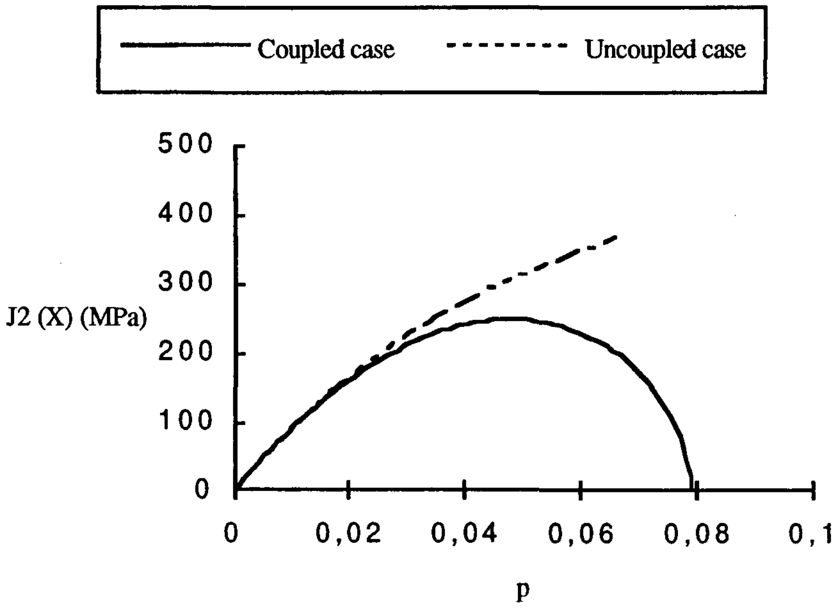


Figure 2. *Coupled and uncoupled kinematic hardening stress-strain curve*

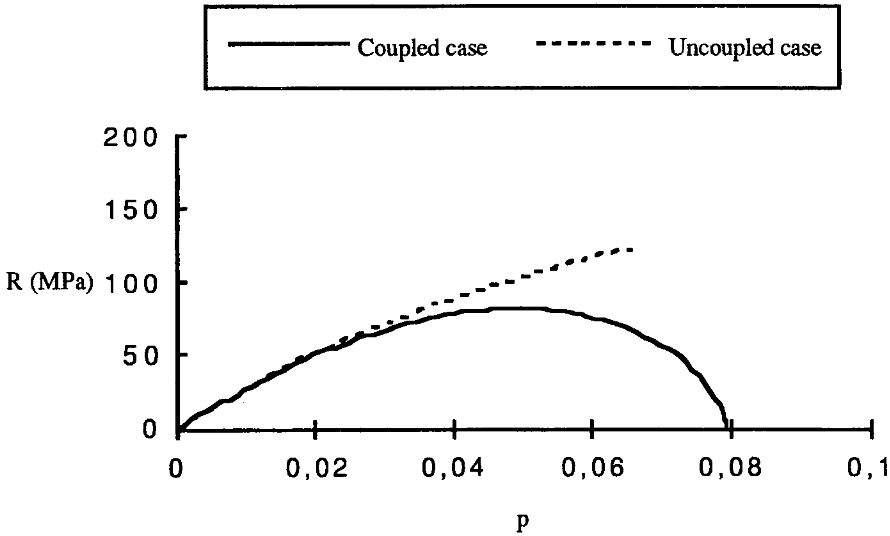


Figure 3. *Coupled and uncoupled isotropic hardening stress-strain curve*

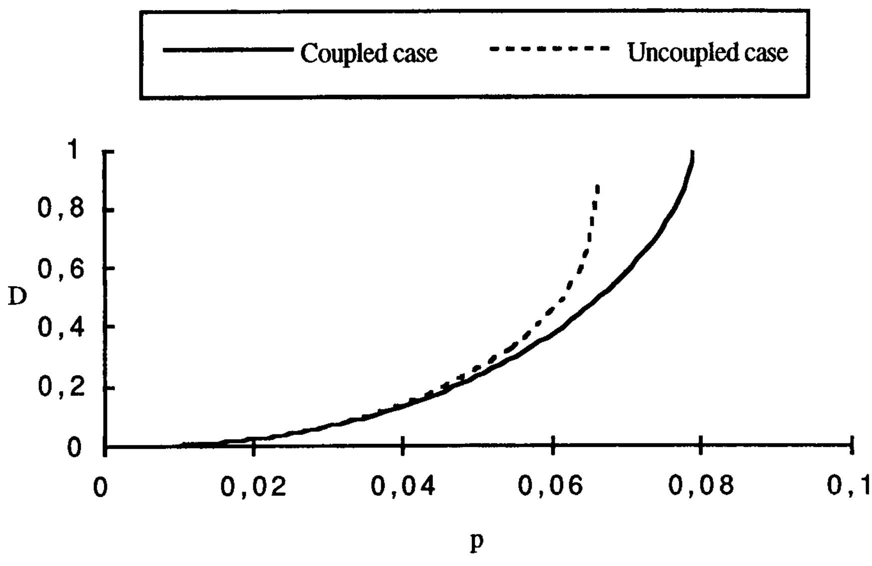


Figure 4. *Coupled and uncoupled damage-strain curve*

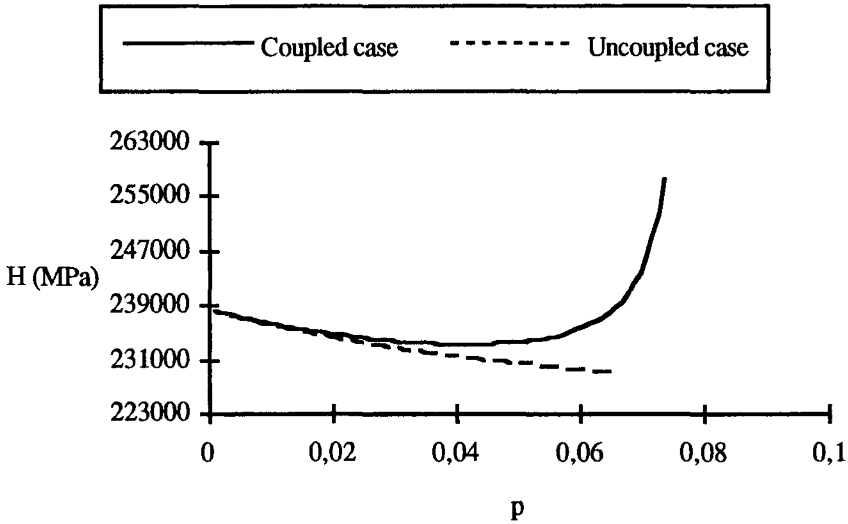
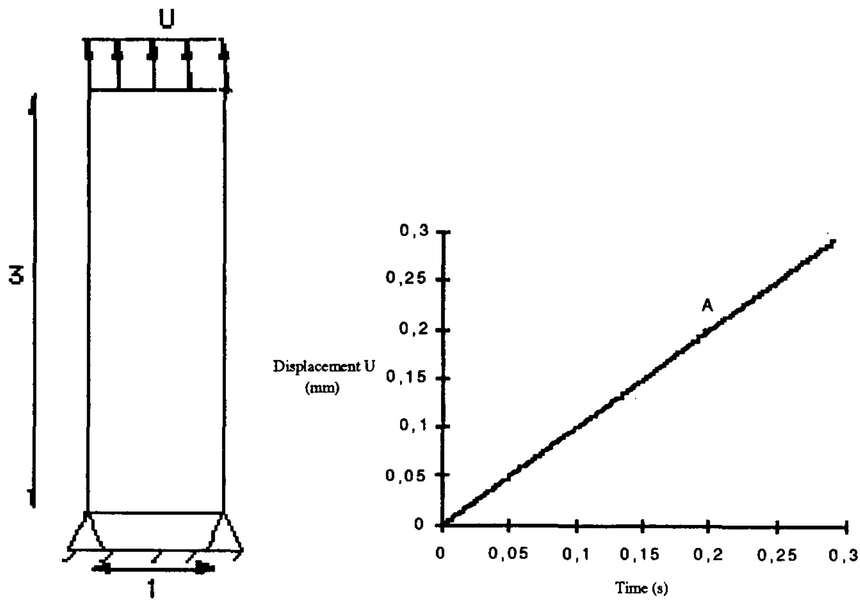


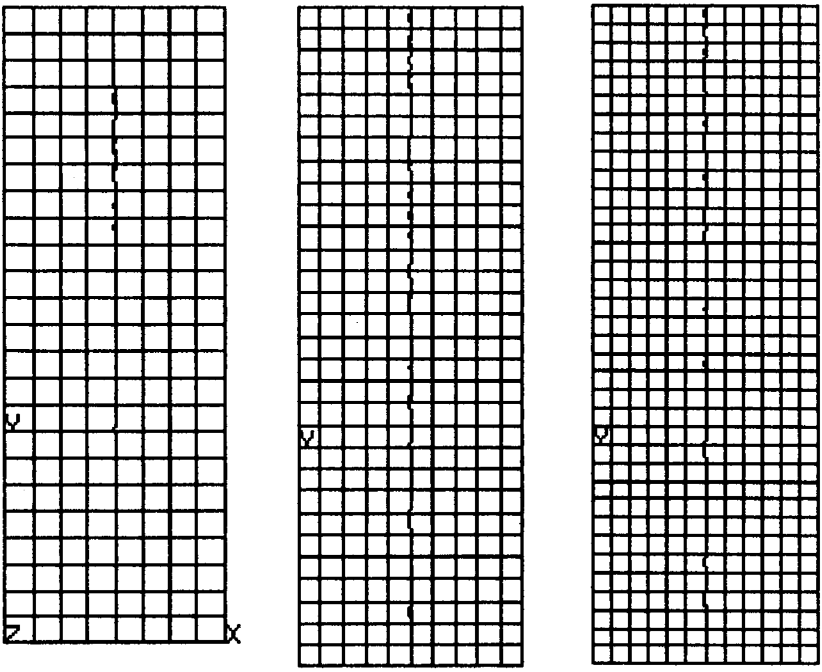
Figure 5. *Coupled and uncoupled elastoplastic modulus-strain curve*



(a) geometry of the plate

(b) imposed displacement

Figure 6. *Geometry and boundary conditions of the plate*



(a) Mesh 1 (192 elts.) (b) Mesh 2 (300 elts.) (c) Mesh 3 (432 elts.)

Figure 7. *Spatial discretisations of the plate*

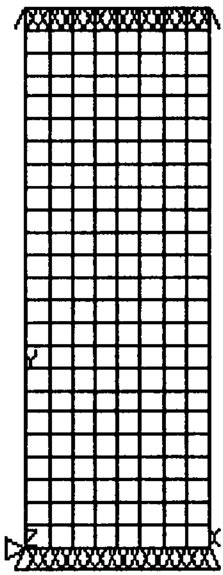
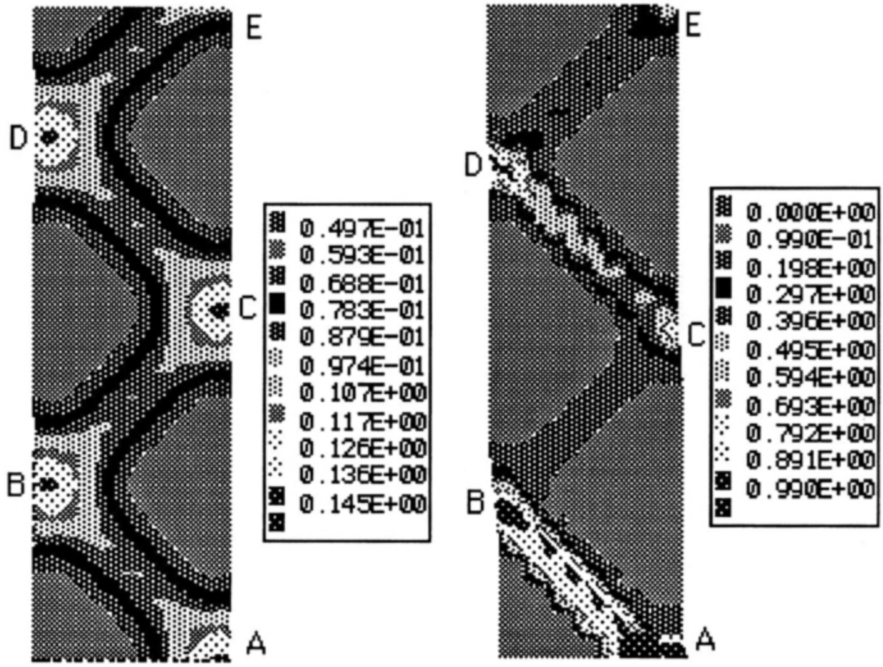


Figure 8. *"Homogeneous" boundary conditions*



(a) U = 0.196 mm *(b) U = 0.277 mm*

Figure 9. *Damage distribution for "homogeneous" boundary conditions*

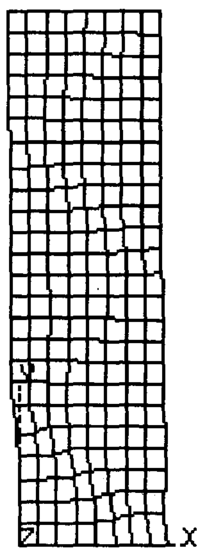


Figure 10. *Last deformed configuration for "homogeneous" boundary conditions*

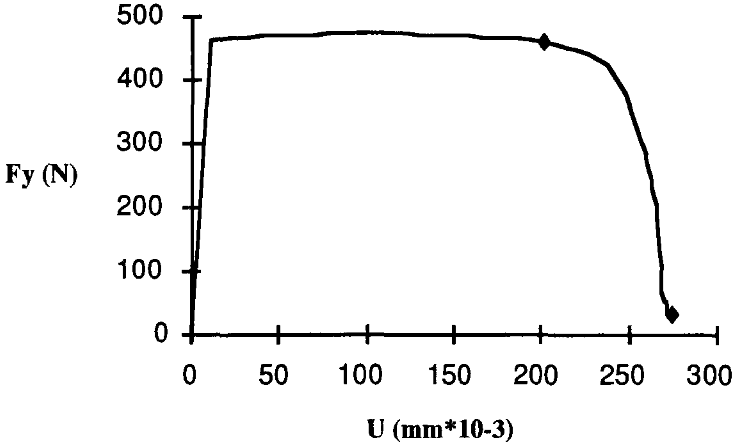
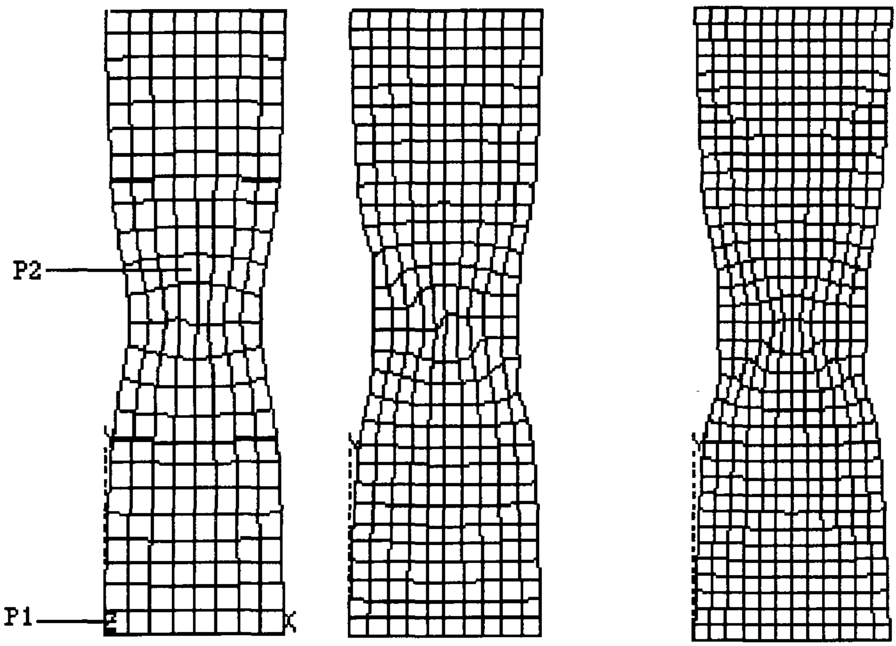
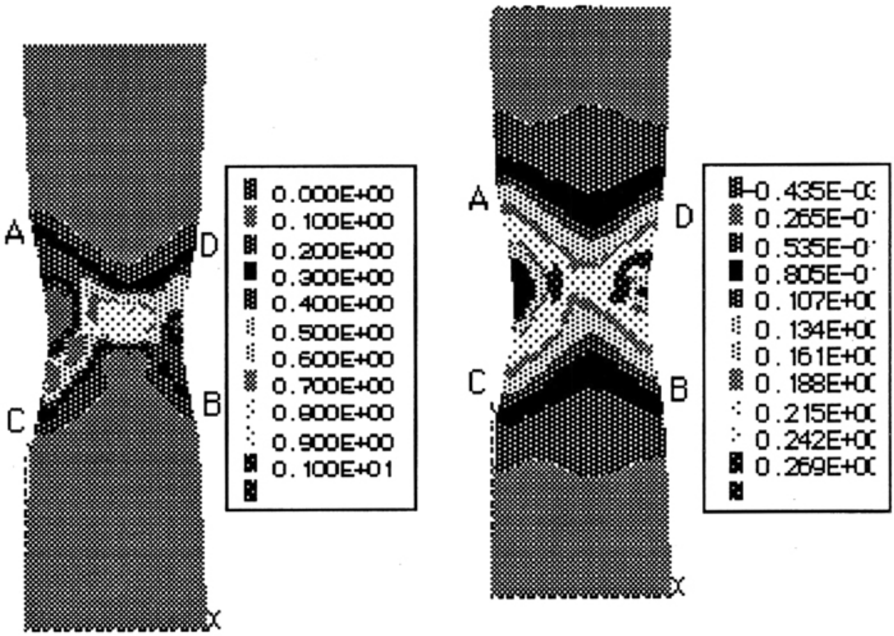


Figure 11. *Global force-displacement curve for "homogeneous" boundary conditions*



(a) Mesh n° 1 *(b) Mesh n° 2* *(c) Mesh n° 3*

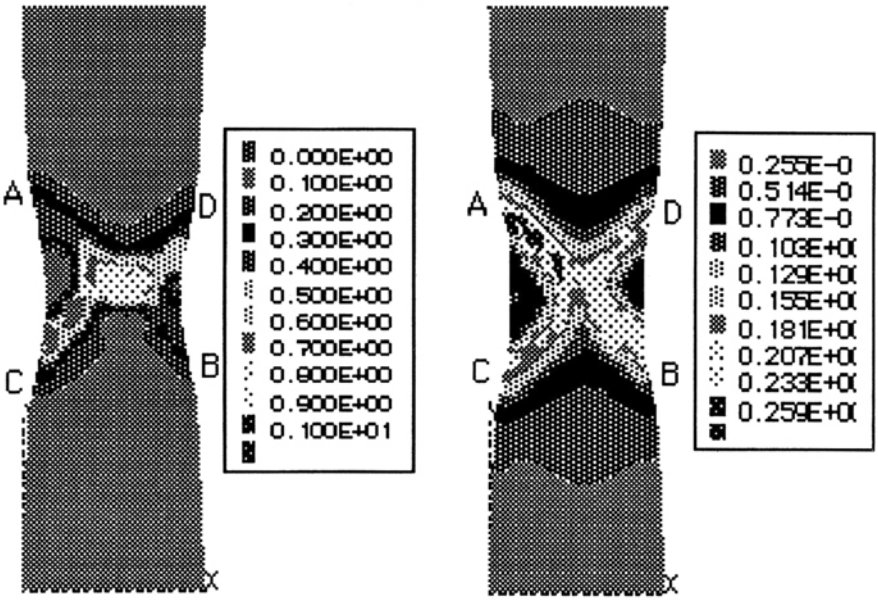
Figure 12. *Last deformed configurations for the three meshes*



(a) Damage distribution

(b) Accumulated plastic strain distribution

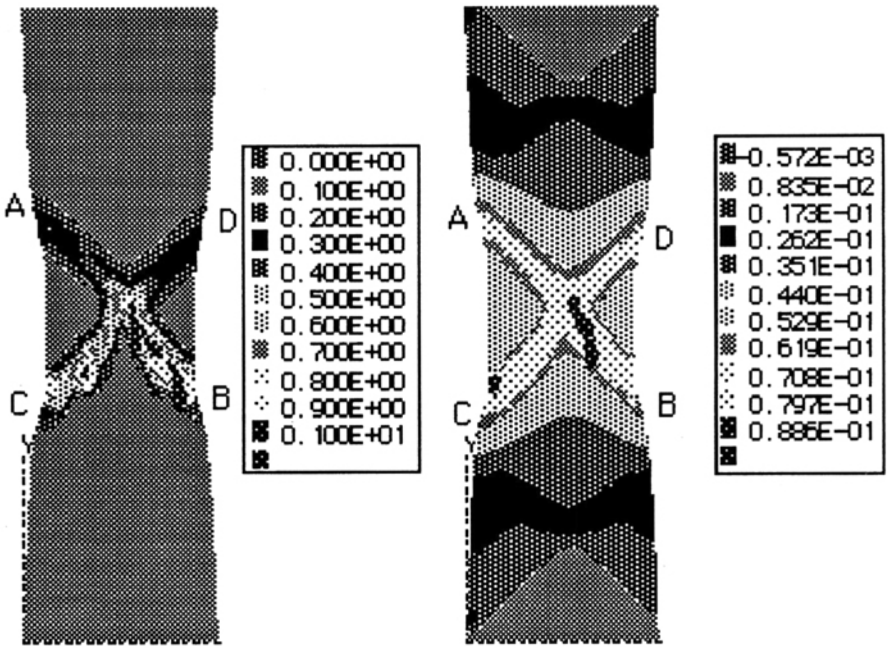
Figure 13. *Final damage and accumulated strain distributions for the mesh n° 1 (U=0.180mm)*



(a) Damage distribution

(b) Accumulated plastic strain distribution

Figure 14. *Final damage and accumulated strain distributions for the mesh n° 2 (U=0.185mm)*



(a) Damage distribution *(b) Accumulated plastic strain distribution*

Figure 15. *Final damage and accumulated strain distributions for the mesh n°3 (U=0.153mm)*

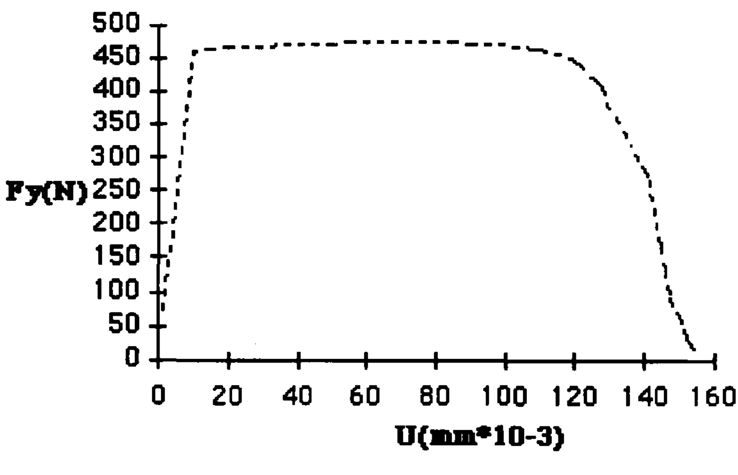


Figure 16. *Global force-displacement curve (mesh 3)*

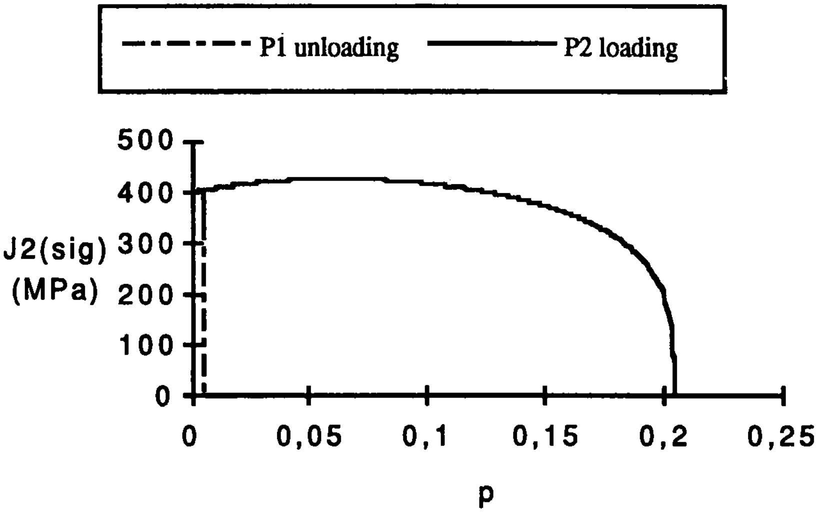


Figure 17. *Local stress- accumulated plastic strain curves for the points P1 and P2*

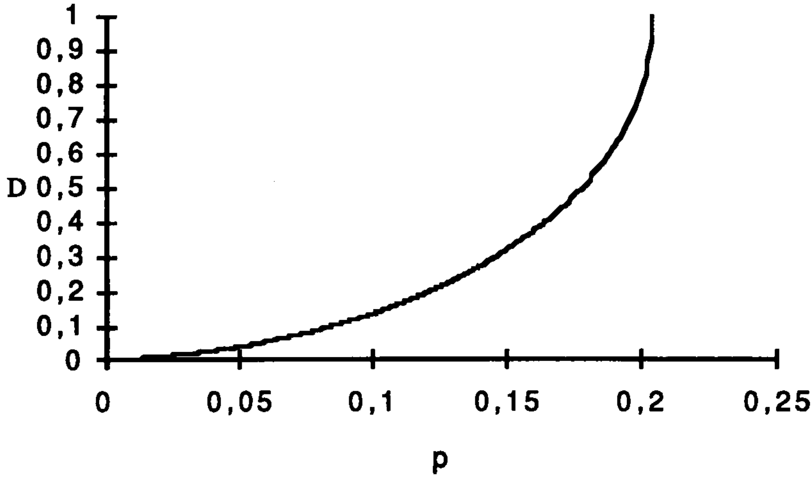


Figure 18. *Damage evolution curves for the point P2*

Acknowledgment

The research described in this paper was sponsored by the DRET, Groupe 8. This support and the interest of Mr. Grellier are gratefully acknowledged. Also the authors are gratefully acknowledged to Prof. G. Touzot (now Director of the INSA Rouen), Drs. S. Aunay (now at FRAMASOFT-CSI), M. El Mouatassim (now at RENAULT), P. Breitkopf and J.M. Beranguier from UTC (Service informatique) for the help when using the software SIC.

References

[AUN 90-1] AUNAY S., "Architecture de logiciel de modélisation et traitement distribué", thèse de doctorat, université de Compiègne, 1990.

[AUN 90-2] AUNAY S., TOUZOT G., "Une nouvelle architecture de logiciel", *StruCoMe 90, Actes des conférences.* Dataid, 1990.

[BAL 88] BALL J.M. (Editor), *Material instabilities in continuum mechanics and related mathematical problems*, Oxford Science Publication, 1988.

[BAZ 88] BAZANT J.P. and PIGAUDIER-CABOT G., "Nonlocal continuum damage, localization instability and convergence", *J. Applied Mechanics*, vol. 55, p. 287, 1988.

[BEN 85] BENALLAL A., BILLARDON R. and LEMAITRE J., "Continuum damage theory and local approach of fracture", Euromech 138, Stuttgart, RFA, 1985.

[BEN 89] BENALLAL A., BILLARDON R. and GEYMONAT G., "Conditions de bifurcation à l'intérieur et aux frontières pour une classe de matériaux non standard", CRAS, Paris, t. 308, série II, p. 893-898, 1989.

[BEN 91] BEN HATIRA F., FORSTER Ch. and SAANOUNI K., "Modélisation mécanique et numérique des structures anélastiques endommageables", Rapport DRET, n° 90-115, UTC, 1991.

[BEN 92] BEN HATIRA F., "Modélisation de l'élastoplasticité endommageable en transformations finies", thèse de doctorat de l'UTC, Compiègne 1992.

[BIL 87] BILLARDON R., "Mesh sensitivity of fully coupled strain and damage finite element analyses", Int. Seminar "High temperature fracture mechanisms and mechanics", Dourdan, France, 1987.

[BRA 85] BRAUDEL H.J., ABOUAF H. AND CHENOT J.L., "An implicit and incrementally objective formulation for the solution of elastoplastic problems at finite strain by the FEM", Rapport GRECO n° 152, 1985.

[BUI 81] BUI H.D. and EHRLACHER A., "Propagation of damage in elastic and plastic solids", ICF 5, Cannes, France, 1981, edited by D. FRANCOIS.

[CHA 78] CHABOCHE J.L., "Description thermodynamique et phénoménologique de la viscoplasticité cyclique avec endommagement", thèse de doctorat d'Etat,université Paris VI, 1978.

[CHA 84] CHAMBON R. and DESRUES J., "Quelques remarques sur le problème de la localisation en bande de cisaillement", *Mech. Rest. Comm.*, 11, p. 145-153, 1984.

[CHO 87] CHOW C.L., WAG J. "An anisotropic theory of continuum damage mechanics for ductile fracture", *Eng. Fract. Mech.*, vol. 27, n° 5, p. 547-558, 1987.

[COR 79] CORDEBOIS J.P., SIDOROFF F., "Damage induced elastic anisotropy", Colloque Euromech 115, Villard de Lans, France, 1979.

[COR 82] CORDEBOIS J.P., LADEVEZE P., "Necking criterion applied in sheet metal forming", in *Plastic behavior of anisotropic solids*, edited by J.P. BOEHLER, Editions du CNRS, Paris, p. 449, 1982.

[DAF 79] DAFALIAS Y.F., "Anisotropic hardening of initially orthotropic materials", ZAMM, 59, p. 437-446, 1979.

[DAF 87] DAFALIAS Y.F., "Issues on the constitutive formulation at large elastoplastic deformations : Part I : kinematics", *Acta Mechanica*, 69, p. 119, 1987.

[DEB 88] DEBORDES O., EL MOUATASSIM M., TOUZOT G., "Local numerical integration of large strain elastoplastic constitutive laws", on *Constitutive laws for Engineering Materials*, vol. II, p. 1373, Edited by DESAI and *al.*,1988.

[DEB 87] DEBORST R., "Computation of post-bifurcation and post failure behavior of strain-softening solids", *Computers structures*, vol. 25, n° 2, p. 211, 1987.

[DOG 89] DOGUI A., "Plasticité anisotrope en grande déformation", thèse de doctorat d'Etat, université Lyon I, 1989.

[DRA 85] DRAGON A., "Plasticity and ductile fracture damage", *Eng. Fract. Mech.*, 21, p. 875-885, 1985.

[ELM 89] ELMOUATASSIM M., "Modélisation en grandes déformations des solides massifs par E.F.", thèse de doctorat de l'université de Compiègne, 1989.

[GEL 85] GELIN J.L., "Modèles numériques et expérimentaux en grandes déformations plastiques et endommagement", thèse d'Etat, Paris VI, 1985.

[HEA 92] HEALY B.E., DODDS R.H., "A large strain plasticity model for implicit finite element analysis", *Computational Mechanics*, 9, p. 95, 1992.

[HIL 62] HILL R., "Acceleration waves in solids", *J. Mech. Phys. Solids*, vol. 10, p. 1-16, 1962.

[HUG 80] HUGHES T.J.R., WINGET J., " Finite rotation effects in numerical integration of rate constitutive equations arising in large deformation analysis", Int. *J. for Numerical Methods in Engineering*, vol. 15, p. 1862-1867,. 1980.

[JU 90] JU J.W., "Energy based coupled elastoplastic damage models at finite strains", *J. Eng. Mech.*, 115, 11, p. 2507, 1990.

[KAC 58] KACHANOV L.M., "Time of the rupture process under creep conditions", *IZV Akad. Nauk.*, SSR, 8, p. 26, 1958.

[KLE 86] KLEIBER M., "On plastic localization and failure in plane strain and round void containing tensile burs", *Int. J. of Plasticity*, vol. 2, p. 205-221, 1986.

[LAD 80] LADEVEZE P., "Sur la théorie de la plasticité en grandes déformations", *Internal Report*, ENS-LMT, n° 9, 1980.

[LEC 74] LECKIE F.A., HAYHURST D.R., "Creep rupture of structures", *Phil. Trans. Roy. Soc. of London*, A 240, p. 233, 1974.

[LEE 69] LEE E.H., "Elastic plastic deformation at finite strains", *J. Appl. Mech.*, vol. 36, p. 1-6, 1969.

[LEH 91] LEHMANN Th., "Thermodynamical foundations of large inelastic deformations of solid bodies including damage", *Int. J. of Plasticity*, 7, p. 79, 1991.

[LEM 85] LEMAITRE J., CHABOCHE J.L., *Mécanique des matériaux solides*, Dunod, Paris, 1985.

[LEM 86] LEMAITRE J., "Local approach of fracture", *Eng. Fract. Mech.*, vol. 25, n° 5/6, 1986.

[LEM 87] LEMAITRE J., "Formulation unifiée des lois d'évolution d'endommagement", CRAS, Paris, t. 305, Série II, p. 1125-1130, 1987.

[LOR 83] LORET B., "On the effects of plastic rotation in the finite deformation of anisotropie elastoplastic materials", *Mechanics of Materials*, 2, p. 287, 1983.

[MAN 73] MANDEL J., "Equations constitutives et directeurs dans les milieux plastiques et viscoplastiques", *Int. J. Solids Structures*, vol. 9, p. 725-740, 1973.

[MAZ 89] MAZARS J., BAZANT Z.P. (Editors), *Cracking and damage, strain localization and size effect*, Elsevier Applied Science, 1989.

[NGU 84] NGUYEN Q.S., "Bifurcation et stabilité des systèmes irréversibles obéissant au principe de dissipation maximale", *J. de mécanique*, n° 3, p. 41, 1984.

[NGU 87] NGUYEN Q.S., "Bifurcation and post-bifurcation analysis in plasticity and brittle fracture", *J. Mech. Phys. Solids*, vol. 35, n° 3, 1987.

[ORT 87] ORTIZ M., LEROY Y., NEEDLEMAN A., "A finite element method for localized failure analysis", *Comp. Meth. Appl. Mech. Eng.*, 61, p. 189-214, 1987.

[PIG 87] PIGAUDIER-CABOT G., BAZANT Z.P., "Nonlocal damage theory", *Journal of Engineering Mechanics*, ASCE, vol. 113, p. 1512, 1987.

[PRE 84] PREVOST J.H., "Localization of deformation in elastic plastic solids", *Int. J. Num. Anal. Methods in Geomechanics*, vol. 8, p. 187, 1984.

[REE 83] REED K.W., ATLURI S.N., "Analysis of large quasistatic deformations of inelastic bodies by a new hybrid-stress finite element algorithm", *Computer Methods in Appl. Mech. Eng.*, 39, p. 245, 1983.

[RIC 76] RICE J.R., "The localization of plastic deformation", *14th Int. Cong. of Theoritical and Applied Mechanics*, vol. I, Delft, August 1976.

[ROU 80] ROUSSELIER G., "Finite deformation constitutive relations including ductile fracture damage", *in Three dimensional constitutive relations and ductile fracture*, edited by S. Nemat-Nasser, N.H. Publishing, 1980.

[RUD 75] RUDNICKI J.W., RICE J.R., "Conditions for the localization of deformation in pressure sensitive dilatant materials", *J. Mech. Phys. Solids*, vol. 23, p. 371, 1975.

[SAA 85] SAANOUNI K., CHABOCHE J.L., "Numerical aspects of a local approach for predicting creep crack growth", *Actes du 3e colloque Tendances actuelles en calcul des structures*, Bastia, Editions Pluralis, p. 369, 1985.

[SAA 87] SAANOUNI K., LESNE P.M., "Nonlocal damage model for creep crack growth prediction", Int. Seminar "High temperature fracture mechanisms and mechanics", Dourdan, France, 1987.

[SAA 88] SAANOUNI K., "Sur l'analyse de la fissuration des milieux élasto-visco-plastiques par la théorie de l'endommagement continu", thèse d'Etat, université de Compiègne, 1988.

[SAA 89] SAANOUNI K., CHABOCHE J.L., LESNE P.M., "On the creep crack-growth prediction by a non local damage formulation", *Eur. J. Mech. A/Solids*, 8, n° 6, p. 437-459, 1989.

[SAA 93] SAANOUNI K., BEN HATIRA F., FORSTER Ch., " On the anelastic flow with damage", accepted for publication in the *Int. J. of Damage Mechanics*, 1992.

[SAN 83] SANDLER I.S., WRIGHT J.P., "Strain softening", Proceedings of the workshop on the theoretical foundation for large scala computations of nonlinear material behavior, Evanston, Illinois, 1983.

[SCH 86] SCHREYER H.L., CHEN Z., "One dimensional softening with localization", *J. of Applied Mechanics*, vol. 53, p. 791, 1986.

[SID 73] SIDOROFF F., "The geometrical concept of intermediate configuration and elastic-plastic finite strain", *Archive of Mechanics*, 25, 2, p. 299, 1973.

[SIM 87] SIMO J.C., "A three dimensional finite deformation viscoelastic model accounting for damage effects", *Compt. Meth. Appl. Mech. Eng.*, 60, p. 153-173, 1987.

[STO 82] STOLZ C., "Contribution à l'étude des grandes déformations en élastoplasticité", Thèse, ENCP, Paris, 1982.

[TVE 81] TVERGAARD V., NEEDLEMAN A., LO K.K.,"Flow localization in the plane strain tensile test", *J. Mech. Phys. Solids*, vol. 29, n° 2, 1981.

[TVE 82] TVERGAARD V., "Ductile fracture by cavity nucleation between large voids", *J. Mech. Phys. Solids*, 30, p. 265-286, 1982.

[TVE 86] TVERGAARD V., "Analysis of creek crack growth by grain boundary cavitation", *Int. J. of Fracture*, 31, p. 183, 1986.

[VOY 92] VOYIADJIS G.S. and KATTAN P.I. ,"A plasticity-damage theory for large deformation of solids-I theoretical formulation", *Int. J. Engng. Sci.*, vol. 30, n° 9, p. 1089-1108, 1992.