Shape optimization of shell structures

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ABSTRACT. *Shells are known to be optimal in many ways, provided certain basic shell oriented design rules are followed. The shape, thickness and material distribution play a dominant role. Minimum material, a specific frequency response, maximum load carrying capacity, a pure membrane stress state are typical design objectives. In the present contribution the form finding and thickness variation are embedded in the concept of structural optimization which combines design modelling, structural and sensitivity analyses and mathematical optimization schemes to a general design tool. The structural response may be based on linear elastic, eigenvalue and geometrically nonlinear analyses. In particular, the imperfection sensitivity with respect to buckling is discussed. A few selected examples demonstrate the versatility of optimization schemes in shell design, among these are the tuning of a bell and the form finding of a classical reinforced concrete dome shell.*

RÉSUMÉ. *Les coques sont reconnues comme étant des structures optimales sous différents aspects dans la mesure où certaines règles de base de conception sont respectées. La forme, l'épaisseur et la répartition du matériau constituant la coque jouent un rôle prépondérant. Les critères d'optimisation typiques sont la minimisation du volume de matériau utilisé, une réponse fréquentielle spécifique, une capacité portante maximum et un comportement en membrane pure. Cet article concerne l'optimisation de forme et d'épaisseur en utilisant un outil général de conception qui combine l'analyse des structures, le calcul des sensibilités et les algorithmes d'optimisation. L'étude de la structure peut être basée sur des analyses linéaires élastiques, non linéaires géométriques ou pour les vibrations libres. En particulier, nous discutons des effets des imperfections sur la stabilité. Quelques exemples démontrent le caractère général des schémas d'optimisation pour la conception des coques. Parmi ces exemples, nous présentons l'étude d'une cloche et d'un dôme classique en béton armé.*

KEYWORDS : *shells, shape optimization, form finding, sensitivity analysis.*
MOTS-CLÉS : *coques, optimisation de forme, analyse de sensibilité, analyses par éléments finis, paramétrisation.*

1. INTRODUCTION

Shells belong to the most common and most efficient structural elements in nature and technology. They are used whenever high resistance, large spans and minimum material are required or a shel(l)ter and containment function is needed. In this respect they also may be termed optimal structures showing excellent structural performance and in many cases also architectural beauty. However, as symptomatic of optimized systems shells can be extremely sensitive with respect to their mechanical behavior as well as their aesthetics [20]. Any design should consider this sensitivity to become finally successful.

Besides the thickness distribution the overall shape of the shell is directly related to this aspect. It is well known that an extremely thin-walled shell heavily relies on a load carrying principle based on a membrane stress state avoiding bending as far as possible. Furthermore, the stress state has to reflect the characteristics of the chosen material: a fabric membrane needs enough prestress avoiding wrinkling under compression, a reinforced concrete shell ought to be mainly in compression. This ideal situation can of course rarely be achieved, if a so-called geometrical, mostly analytically defined shape is adopted. In this case extra structural elements like additional reinforcement, prestress, stiffeners, edge beams etc. ("prostheses") are needed to put the shell into the desired position. In contrast natural or structural shapes try to avoid most of these extra stiffening components. They are obtained by an inverse approach in which the objective of a desired structural response is prescribed and the initial design, e.g. the shape and thickness distribution, is looked for. The interrelationship between shape and structural response has been intensively discussed by the authors in [21], [22]. It has been mentioned that one of the described methods, namely structural optimization, seems to be the most general and versatile technique as design tool. This statement is based on the fact that each design follows essentially an optimization process. Specifically with respect to the form-finding of a shell structure the objective can be stated as follows [22]:

Find the shape and thickness distribution of a shell, so that

- the boundary conditions and all possible load cases are considered,
- material properties are taken into account (e.g. no tension for masonry or concrete),
- stresses and displacements are limited to certain values,
- an almost uniform membrane stress state results,
- buckling, excessive creep and negative environmental effects are avoided,
- a reasonable life time is guaranteed

and hopefully

- manufacturing and service costs are justified and the design is aesthetically pleasing.

These requirements interact with each other and are in some cases even contradictory so that a compromise has to be made. For example, a free form shell may lead to expensive formwork and an efficient concrete shell might look rather bulky.

Structural optimization is currently understood as a synthesis of several individual disciplines like structural and sensitivity analyses, computer aided geometrical design (CAGD), mathematical optimization, interactive graphics etc. Apparently, it is a computational method, consequently only those requirements described above may be part of the process which can be cast into a mathematical formulation. Unfortunately, also the term 'optimization' is misleading since it suggests that there is only the one optimal solution. Firstly, only parts of the complete task can be included up to now so that always a model problem is investigated. Secondly, even for this restricted model a local optimum is reached. In other words, creativity of the design is still kept as part of the game; fortunately, the process is – in this rather general perspective – extremely parameter sensitive and allows a lot of design freedom. Structural optimization is nothing else then an additional design aid. Its applications can be classified into:

- homogenization problems (uniform stress state etc.),
- optimal use of material (trimming, maximum load carrying capacity),
- optimal structural response (tuning, e.g. frequencies).

In this paper we concentrate on shape and thickness optimization. The tuning of axisymmetric shells to certain frequencies and buckling loads by shape modification is described in [16]. Minimum weight and cost design of rotational shells is addressed in [23]. Shape optimization schemes are used in [15] to eliminate bending and minimize membrane stresses of arbitrary shells with constant thickness. Shape optimization of prismatic curved shells and axisymmetric shells is investigated in [9], [10] where also further references are given. The literature for optimization of nonlinear shell structures is rather limited, e.g. [25]. In this context a different shape sensitivity with respect to the influence of small geometrical imperfections on the buckling and failure load of optimized structures has to be mentioned.

The present paper summarizes a general scheme for shape and thickness optimization of free–form shells developed in the authors' research group [3], [7], [11], [12], [19]. This includes the geometrical parameterization, sensitivity analysis, the application of certain mathematical programming schemes and – most important – a general concept to incorporate all kinds of objectives, constraints and design variables. The formulation is currently extended to geometrically nonlinear structures including buckling and its related imperfection sensitivity [24].

2. OPTIMIZATION MODEL

2.1 General Procedure for Shape Optimization of Shells

Hanging fabric models or their numerical simulation are excellent techniques in the form–finding process for a membrane oriented shell design [17]. They are simple and ideally suited in the initial phase since they always give a rough picture of a potential shape. However, their application has certain limits.

Usually, the material of the membrane used in the experiment or analysis is not related to that of the real shell. Wrinkling of the hanging fabric in general cannot be avoided and it is not clear, how different load cases can be incorporated. Furthermore, experimental data have to be processed anyway for a subsequent structural analysis.

Therefore, a more general approach to shell design is advocated here following the principles of structural optimization. They ideally reflect the individual design stages every engineer usually goes through, namely

1. choose a reasonable initial shape,
2. evaluate the structural response for several load cases,
3. check stresses, displacements, buckling load and other safety requirements and serviceability conditions, costs,
4. compare the quality of the design with any chosen optimality criteria,

5. if necessary, evaluate a new trend applying sensitivity analyses and propose an improved design,
6. repeat the process until all criteria and constraints are satisfied.

To allow some flexibility and generality the process has to be structured in optimization, i.e. geometry definition, mechanical behavior, design objectives and constraints and mathematical optimization module are strictly separated (Fig. 1).

This procedure is absolutely different to the hanging model or other principles where the shape generating rule itself (i.e. mechanical response to given loads) is already the criterion for optimality. Nevertheless, the same results can be achieved if equivalent objectives, material data, boundary and load conditions are chosen. But beyond this the methods of structural optimization can handle problems with many load cases, arbitrary design objectives and constraints not necessarily related only to mechanical behavior, loads like body forces and support conditions which change with every modification of shape.

2.2 Definition of Optimization Problem

The rather abstract mathematical statement of a non-linear optimization problem

minimize the objective $f(\mathbf{x})$

subject to

$$
\begin{aligned}
&\text{equality constraints:} && g_j(\mathbf{x}) = 0 \quad ; \; j = 1,\ldots,m_e \\
&\text{inequality constraints:} && g_j(\mathbf{x}) \leq 0 \quad ; \; j = m_e + 1,\ldots,m \\
&\text{bounds for optimization variables:} && \underline{\mathbf{x}}_L \leq \mathbf{x} \leq \overline{\mathbf{x}}_u
\end{aligned} \tag{1}
$$

has to be redefined in mechanical terms (Fig. 2).

For shells coordinates and thicknesses of certain selected design or structural nodes are selected as optimization variables. In order to allow a smooth and efficient solution, the number of design variables should be kept as small as possible but still allowing enough freedom for a general shape. Besides the most common objective "weight" there are other functions of natural significance like strain energy minimization which is equivalent to maximizing the stiff-ness. This means that the bending strains in a shell are mini-

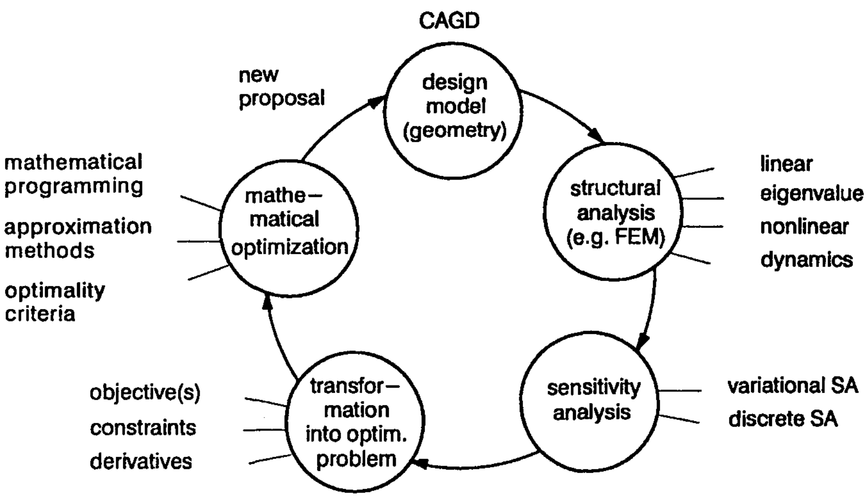


Figure 1 – Design = optimization loop

mized so that a membrane stress state is achieved. Stress levelling with a prescribed target of stress σ_{avg} can be applied to generate a shell mainly in compression. Tuning to a certain response, for example a single frequency or a desired spectrum or maximizing the failure load are classical objectives in engineering. In all cases we have to keep in mind that optimized structures may result in extreme parameter sensitivity.

Inequality constraints are taken into account to check the safety and reliability requirements which have to be satisfied. Typical constraints of this type are stress and displacement limits. If the stiffness or the critical load factor are to be maximized a prescribed structural mass is introduced via an equality constraint. This constraint prohibits an accumulation of mass which would otherwise produce unrealistic massive structures.

For multi–objective optimization the problem has to be generalized (Fig. 3) allowing only a compromise. Here, either several weighted objectives are combined to one compromise or one dominant function $f_1(\mathbf{x})$ is chosen as leading objective whereas the other functions are introduced as constraints. Alternatively, Pareto optimal solutions are located on the so–called functional – efficient curve A – B in the criteria space with C as the min – max solution. Also different load cases can be han-

- Design variables $\mathbf{x}$
  - coordinates of selected nodes $\quad \mathbf{r}$
  - thickness of selected nodes $\quad \mathbf{t}$
  - other cross sectional parameters $\quad \mathbf{A}$
- Objectives $f(\mathbf{x})$
  - weight or volume $\quad f_w = \int \varrho \, dv$
  - strain energy $\quad f_E = \frac{1}{2}\int \sigma \, \varepsilon \, dv$
  - stress leveling $\quad f_s = \int (\sigma - \sigma_{avg})^2 \, da$
  - tuning function $\quad f_\lambda = \Sigma \frac{(\lambda_i - \lambda_{io})^2}{\lambda_{io}^2}$
  - fundamental frequency $\quad \omega$
  - critical load factor $\quad -\lambda$
- Constraints
  - weight or volume $\quad g_w = \frac{W}{W_{all}} - 1 = 0$
  - displacements $\quad g_u = \frac{u}{u_{all}} - 1 \le 0$
  - stresses $\quad g_s = \frac{\sigma}{\sigma_{all}} - 1 \le 0$
  - frequencies $\quad g_\omega = \frac{\omega}{\omega_{all}} - 1 \le 0$
  - load factor $\quad g_\lambda = 1 - \frac{\lambda}{\lambda_{all}} \le 0$

***Figure 2** – Typical variables and functions in optimal shape design of shells*

- general objective $\quad f(\mathbf{x}) = \sum_{i=1}^{k} w_i \, f_i(\mathbf{x})$ $\quad w_i$ ... weighting factors
- leading objective $\quad f(\mathbf{x}) = f_1(\mathbf{x})$ $\quad k$ ... number of individual objective functions

  $f_2 \; .... \; f_k$ as constraints
- Pareto optimization

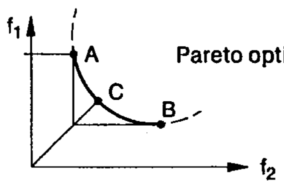


***Figure 3** – Multi–objective optimization*

dled in a similar way if for example the values for one objective of several load cases are added to one general function.

3. GEOMETRIC REPRESENTATION [3]

3.1 Design Model

Shape finding means optimization of geometry. Characteristic optimization variables are therefore geometric parameters defining the structural shape. The number of variables can be reduced dramatically without loss of generality if CAGD concepts are used. By these methods shapes of free formed shells can be described by the coordinates of a few so−called "design nodes" which can be chosen as variables. Additionally, the continuous thickness variation can be optimized where thicknesses at design nodes are taken as discrete variables.

The general methods of CAGD are the basis of modern pre−processors to design structural geometries in two and three dimensions. Shapes are approximated piecewise by "design patches". Within each design patch the resulting shape r_a is parameterized in terms of shape functions H_i, patch parameters u, v, w and design nodes r_{di} which describe the location of the patch in space:

$$r_a(u,v,w) = \sum_{i=1}^{n} H_i(u,v,w)\ r_{di} \tag{2}$$

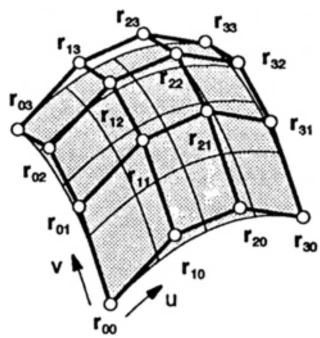


Figure 4 – Bézier patch

There are many different shape functions available, e.g. Lagrangian interpolation, Coons transfinite interpolation, Bézier and B−spline approximations. In shape design of free form shells one dimensional cubic Bézier and B−splines and two dimensional bi−cubic Bézier patches (Fig. 4) appear to be superior to others.

Continuity conditions between adjacent patches of composite surfaces can be formulated in superimposed "continuity patches". They are generated automatically and preserved during manual user interactions and shape optimization. Fig. 5 shows different types of continuity patches depending on whether they are connecting two or four design patches or are defined at an isolated corner. In either case four nodes are independent and control the locations of the remaining

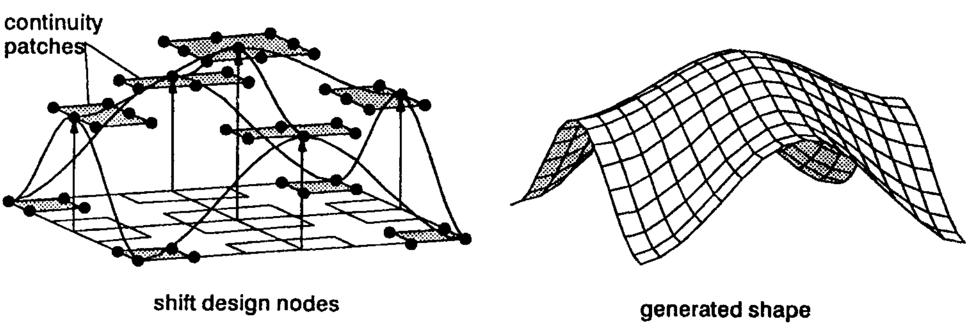


Figure 5 – Interactive surface modification, continuity patches connecting four Bézier patches

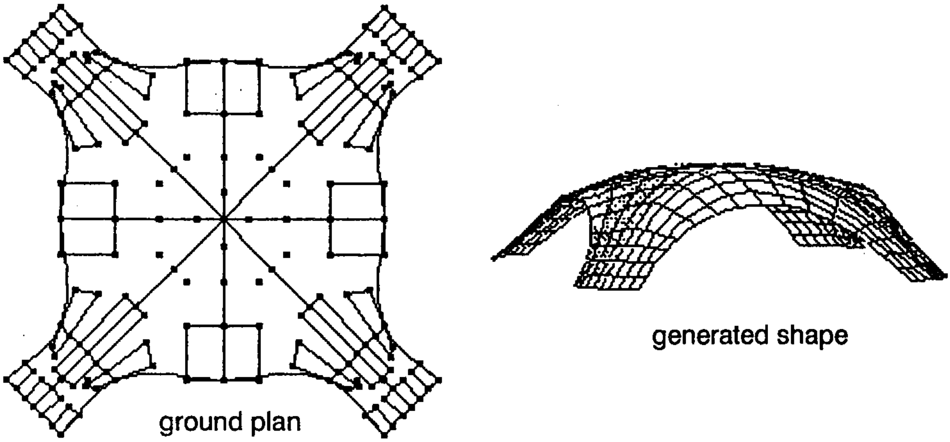


***Figure 6** – Free formed shell (16 Bézier elements)*

nodes leading to a reduction of geometrical degrees of freedom which is very welcome in structural optimization to stabilize the procedure.

The idea of continuity patches is very helpful in interactive design of free formed shells because they can serve as initial shapes for subsequent optimization runs or as valuable interactive pre–processor tools for input preparation of complex shapes. Fig. 6 shows the plan of a free form shell described by a total of 16 Bézier patches and the generated shape modeled by 8–noded isoparametric shell elements.

3.2 Linking

The concept of linking is à necessary technique to introduce certain geometrical constraints. One application has already been mentioned above in the context of continuity requirements. Another one is the interaction of the analysis model (e.g. finite element model) and the design model (geometrical macro element model). A commonly used rule which links variables $\mathbf{r}_a$ of the analysis model via the design model with variables $\mathbf{x}$ of the optimization model is defined as:

$$\mathbf{r}_a = \mathbf{r}_a^0 + \mathbf{L}_{ax}\mathbf{x} + \mathbf{H}_{ad}(\mathbf{r}_d) \qquad (3)$$

with:

$$\mathbf{r}_d = \mathbf{r}_d^0 + \mathbf{L}_{dx}\mathbf{x} \qquad (4)$$

In these relations $\mathbf{r}_a^0$ and $\mathbf{r}_d^0$ denote coordinates of analysis and design models, respectively, which remain constant during the optimization process. Linking matrices $\mathbf{L}_{ax}$ and $\mathbf{L}_{dx}$ describe linear relations between optimization variables $\mathbf{x}$ and variable coordinates of analysis and design model, respectively. $\mathbf{H}_{ad}$ denote nonlinear relations between design and analysis model and are identified as shape functions. In addition, specific linking rules can be introduced to prescribe move directions of nodes, linear combinations of nodal variables, symmetry conditions, projection rules etc. [3].

The concept of linking can drastically reduce the number of optimization variables. It also allows the user to guide the solution interactively into a reasonable design.

4. ANALYSIS

4.1 Structural Analysis

Once the design model is defined a finite element mesh can easily be interpolated within each design element (eqn. (2)). This concept cannot totally avoid mesh distortions during shape optimization. However, by choosing reasonable patches and prescribed move directions the effect can at least be diminished. In most cases the structural response for shape optimization is based on linear elastic analyses or eigenvalue formulations for buckling or vibration. The equilibrium condition of the discretized system $\mathbf{G} = \mathbf{0}$ leads to the usual stiffness expression. Its homogeneous counterpart defines an eigenvalue formulation for the load parameter λ. The homogeneous equation of motion yields the free vibration problem:

equilibrium	$\mathbf{G}(\mathbf{u},\lambda) = \mathbf{0}$	(5 a)
⇒	$\mathbf{K}\,\mathbf{u} = \mathbf{R}$	(5 b)
stability	$\mathbf{K}_T\,\Phi = \mathbf{0}$	(6 a)
⇒	$(\mathbf{K} - \lambda\,\mathbf{K}_g)\,\Phi = \mathbf{0}$	(6 b)
free vibration	$\mathbf{K}\,\mathbf{u} + \mathbf{M}\,\ddot{\mathbf{u}} = \mathbf{0}$	(7 a)
⇒	$(\mathbf{K} - \omega^2\,\mathbf{M})\,\Phi = \mathbf{0}$	(7 b)

$\mathbf{K}$, $\mathbf{K}_g$, $\mathbf{M}$ and $\mathbf{R}$ are the elastic stiffness matrix, geometric stiffness matrix, mass matrix and external load vector, respectively. $\mathbf{K}_T$ is the sum of $\mathbf{K}$ and $\mathbf{K}_g$. $\mathbf{u}$ denotes the vector of the nodal displacement parameters.

Shells are known to be extremely sensitive with respect to small deviations of their ideal shape. The buckling load of the real, imperfect shell may be drastically lower than that of the perfect structure. This is in particular true for optimized structures. It brings up the question, how nonlinear effects and imperfection sensitivity can be included in the optimization procedure. This problem is discussed in detail in [24] for maximizing the critical load of geometrically nonlinear structures and is briefly outlined in the following.

The discretized nonlinear equilibrium equations $\mathbf{G}(\mathbf{u},\lambda) = \mathbf{0}$ are usually linearized, leading to the tangent stiffness matrix $\mathbf{K}_T$, and iteratively solved by a Newton type of iteration, in general combined with a path–following scheme, e.g. the arc – length –method [24]. The maximum load, which is either a bifurcation or a limit point, is controlled by the stability criterion (6 a). Usually the criterion is utilized during the loading process to check whether or not the structure is in a stable equilibrium position or a bifurcation into another equilibrium path occurs.

In order to pinpoint a critical load exactly a so–called extended system can be utilized [30].

$$\left\{\begin{array}{c} \mathbf{G}(\mathbf{u},\lambda) \\ \mathbf{K}_T(\mathbf{u},\lambda)\,\Phi \\ \ell(\Phi) \end{array}\right\} = \mathbf{0} \qquad (8\,a)$$

linearization

$$\begin{bmatrix} \mathbf{K}_T & \mathbf{0} & -\mathbf{R} \\ (\mathbf{K}_T\,\Phi)_{,u} & \mathbf{K}_T & (\mathbf{K}_T\,\Phi)_{,\lambda} \\ \mathbf{0}^T & \ell_{,\Phi} & 0 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u} \\ \Delta\Phi \\ \Delta\lambda \end{bmatrix} = - \begin{bmatrix} \mathbf{G} \\ \mathbf{K}_T\,\Phi \\ \ell \end{bmatrix} \qquad (8\,b)$$

Here displacements $\mathbf{u}$, buckling mode Φ and the critical load parameter λ are the unknowns. $\ell(\Phi)$ is an additional constraint on the length of the buckling mode Φ. Again the linearized equation (8 b) is the basis of a Newton type iteration. $()_,$ denotes the corresponding partial derivatives. A solution of the unsymmetrical set of equations (8 b) is bypassed by a partitioning method. This, in turn, requires the factorization of the tangent stiffness matrix $\mathbf{K}_T$ which is singular at critical points. In [24], [30] different schemes are proposed to augment (8 b) and circumvent this difficulty.

Although the extended system can be started already at the undeformed configuration, the direct solution for the critical point may not converge. Thus a few steps are computed by the path following algorithm coming close to the critical point before the extended system is turned on.

Once the critical point of the perfect structure including the buckling mode is determined, the original structure is perturbed by a fraction of this mode and the resulting imperfect structure is investigated again by the extended system. Its critical load – usually a limit point – is the basis for the maximization of the load factor by the optimization procedure (Fig. 7). A new design is obtained. The critical point of the new perfect structure is computed starting with the extended system at the critical stage of the old perfect structure. This gives the new buckling mode used as imperfection. And again the imperfect structure is investigated in the same manner by the extended system. The procedure is repeated until convergence is reached, i.e. no further increase of the critical load can be observed. For details see [24].

4.2 Sensitivity Analysis [11]

The sensitivity analysis supplies gradient informations on objective and constraints with respect to optimization variables. In general, any function t (objective or constraint) depends on optimization variables **x** and state variables **u**, e.g. for displacements. Thus,

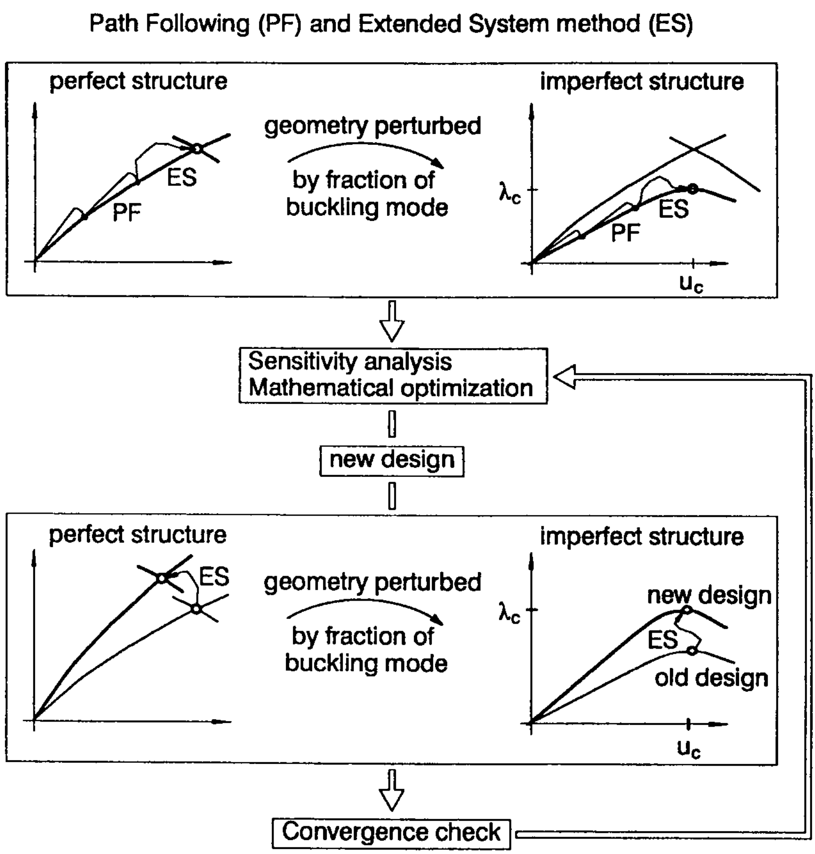


Figure 7 – Optimization including imperfection sensitivity

the total derivative of t with respect to **x** is given as:

$$\frac{dt}{dx} = \frac{\partial t}{\partial u}\;\frac{du}{dx} + \frac{\partial t}{\partial x} \tag{9}$$

where the determination of the response sensitivity **du/dx** is part of the job. It can be carried out by several different techniques. They can be divided into variational or discrete methods, depending on whether the gradients are obtained before or after discretization [1], [8]. Nevertheless, the same results will be obtained if variation, discretization and derivation are done consistently [11]. Within the present approach all three variants of discrete sensitivity analysis (DSA) are adopted: numerical, semi–analytical and analytical.

The analytical derivation of the state variables is given as:

$$\frac{du}{dx} = K^{-1}\left(\frac{dR}{dx} - \frac{dK}{dx}\,u\right) \tag{10}$$

where **R** is the load vector and **K** is the system stiffness matrix. The major concern is the calculation of the pseudo load vector:

$$\overline{R} = \frac{dR}{dx} - \frac{dK}{dx}\,u \tag{11}$$

which results in analytical derivations of **R** and **K**.

Since the element stiffness matrix is decomposed into several individual matrices (e.g. kinematic operator, constitutive matrix, determinant of Jacobian) the analytical derivation has to be carried out for all individual matrix elements.

However, the analytical version of DSA appears to be very reliable for all kinds of applications at the cost of a higher programming effort. Additionally, it is even more efficient than the numerical forward difference scheme if only parts of the structure are affected by shape variations. It definitely depends on the optimization problem which method is to be preferred. Therefore, all of them should be provided.

For the optimization of the critical load (section 4.1) the total derivative of the critical load factor $d\lambda_c/dx$ has to be determined which in turn depends on the derivative du_c/dx of the related displacement field. For details see [24].

5. MATHEMATICAL OPTIMIZATION

The optimization problem eq. (1) is in most cases definitely nonlinear since all functions (objective f(**x**) and constraints g(**x**)) are nonlinear functions of the optimization variables **x**. A local solution is characterized as a stationary point of the corresponding Lagrangian function:

$$L(x,u) = f(x) + v^T g(x) \tag{12}$$

v are the Lagrange multipliers or dual variables. The necessary condition for the stationarity of L or the corresponding constrained minimum of f(**x**) is defined by the Kuhn–Tucker conditions. They give a set of nonlinear equations to determine the optimal solution **x***, **v***:

$$\frac{\partial L^*}{\partial x} = \frac{\partial f^*}{\partial x} + \left(\frac{\partial g^*}{\partial x}\right)^T v^* = 0$$

$$\frac{\partial L^*}{\partial v_j} = g_j^* = 0 \qquad ;\; j = 1, \dots m_e \tag{13}$$

$$\left.\begin{aligned}\frac{\partial L^*}{\partial v_j} = g_j^*\, v_j^* &= 0\\ v_j^* &\ge 0\end{aligned}\right\} \;;\; j = m_e + 1, \dots m$$

where f*, **g*** and L* are the function values at the optimal solution. Without loss of practical relevance for the presented range of applications the problem functions are stated to be continuous in gradients and curvature.

The methods of non-linear programming can be divided into (i) primal methods (e.g. method of feasible directions), (ii) penalty and barrier methods (e.g. sequential unconstrained minimization technique), (iii) dual methods and (iv) Lagrange methods [14]. They are distinguished by the type and number of independent variables they use.

Lagrange methods can be stated to be the most sophisticated numerical optimization techniques and they are applicable for all kinds of constrained problems. They are designed to solve the Kuhn-Tucker conditions (13) directly and are operating in the full space of primal and dual variables. Iterative solution of (13) by subsequent linearization leads to a natural extension of the classical Newton-Raphson procedure which became known e.g. as SQP-method (sequential quadratic programming [26]). Since in the context of structural optimization the evaluation of second derivatives with respect to optimization variables is far too expensive quasi-Newton variants are used. In the k-th iteration step the corresponding quadratic subproblem states as:

minimize:

$$\frac{1}{2}\, \mathbf{d}^{kT}\, \mathbf{B}^k\, \mathbf{d}^k + \left(\frac{\partial f(\mathbf{x}^k)}{\partial \mathbf{x}}\right)^T \mathbf{d}^k$$

subject to:

$$\frac{\partial g_j(\mathbf{x}^k)}{\partial \mathbf{x}}\, \mathbf{d}^k + g_j(\mathbf{x}^k) = 0 \quad ; \; j = 1, \ldots m_e$$

$$\frac{\partial g_j(\mathbf{x}^k)}{\partial \mathbf{x}}\, \mathbf{d}^k + g_j(\mathbf{x}^k) \leq 0 \quad ; \; j = m_e + 1, \ldots m$$

$$\text{with:} \quad \begin{array}{l} d_i^k \leq \overline{x}_i - x_i^k \\ -\, d_i^k \leq x_i^k - \underline{x}_i \end{array} \quad ; \; d_i^k = x_i^{k+1} - x_i^k$$

where $\mathbf{B}^k$ is the current approximation of the second derivatives of the Lagrangian with respect to $\mathbf{x}$.

SQP-methods have been used rather infrequently within structural optimization so far but with the increasing complexity of problems like shape optimization these methods get more and more accepted [27]. The authors can report very good experiences with all kinds of structural optimization problems.

The performance of the iterative design loop can be improved in certain applications if in every iteration step objective function and constraints are replaced by proper approximations. Usually, approximations are derived from first order linearizations with respect to problem oriented, generalized variables. The idea came up first for sizing of statically determinate trusses where cross sectional areas as the design variables enter the stress constraints in the denominator. Consequently, if areas are substituted by their reciprocal value the correct solution emerges. The idea has been generalized and applied also in shape optimization. Depending on the kind of approximation these techniques became known as hybrid approximation, convex linearization or method of moving asymptotes (MMA), for a review see [2], [3]. The special advantages of approximation methods are convex and separable sub-problems which can be solved efficiently by specialized solution schemes, e.g. dual optimizers. In [6] an extended version of the method of the moving asymptotes (EMMA) is proposed which essentially demonstrates that the method can be embedded in the SQP-formulation as special case.

6. EXAMPLES

6.1 Bi-parabolic Roof Shell [5]

This example is used to demonstrate the effects of different objective functions and the variety of shapes which can be generated by only two variables. The structural situation is shown in Fig. 8. A shell of rectangular plan (b = 6 m, l = 12 m) and uniform constant thickness (t = 0.05 m) is supported by diaphragms at the smaller edges. The shape is generated by four Bézier patches. The design nodes are linked (i) to preserve double symmetry and (ii) to describe a bi–parabolic surface which can be controlled by two vertical coordinates as indicated. In the initial design both coordinates are set to $x_{1/2}$ = 3 m describing a cylindrical shell. The structure is loaded by a uniform vertical load p = 5 kN/m^2 (snow). Support conditions are fixed hinges. Due to symmetry of loads and structure only one quarter of the shell has to be analysed. This was done by 72 eight – noded isoparametric shell elements which are 2x2 reduced integrated.

In a first optimization run strain energy was chosen as objective function without stress constraint assuming the structure is sufficiently reinforced to resist also high tension forces. The resulting shape (Fig. 9 a) is an anticlastic surface (HP), very similar to a minimal surface which acts almost like a membrane in tension and compression. Since the structural thickness is fixed, the result is alternatively restricted by an upper bound (6 m) on variable x_1.

To get a more suitable design for concrete the objective "stress leveling" was used to reduce tension stresses in the lower fibres of the structure which are caused by interactions of normal forces and bending moments. A target stress of σ_a = – 100 kN/m^2 was prescribed.

The optimal structure (Fig. 9 b) is a synclastic shape (EP) where the area of tension in lower fibres is reduced to a minimum. Tension cannot be avoided totally because of the simple

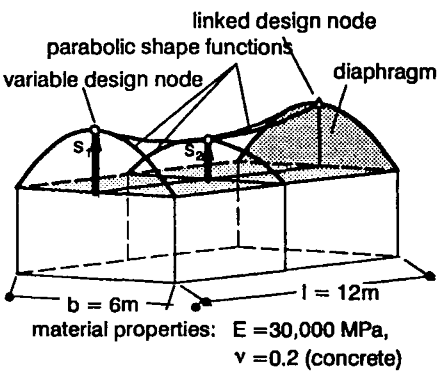


***Figure 8** – Parabolic roof shell: problem statement*

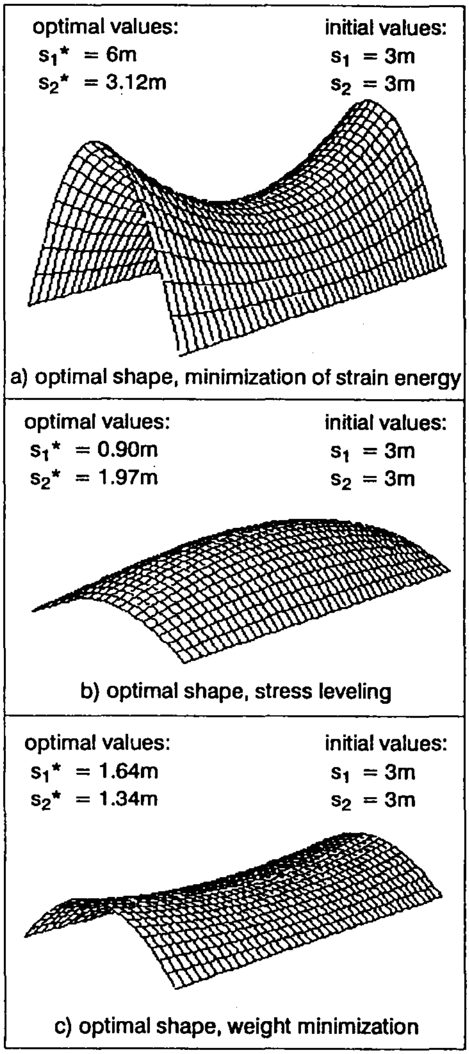


***Figure 9** – Parabolic roof shell: initial shape and optimization results*

shape function and the rectangular plan of the structure. It is remarkable that the diaphragms – although possible – do not disappear. If they vanish, the resulting shape has a horizontal tangent plane at the corner leading to negative curvature and increased bending.

If "weight" is used as objective function any shape between the "minimal surface" and a plate can be determined which is forced by additional constraints on stresses and displacements. Fig. 9 c shows one result obtained with constraints on v. Mises effective stresses which are not allowed to exceed an arbitrarily chosen value of $\sigma_m = 400$ kN/m^2.

6.2 Tuning of a Bell [4], [12]

The major design aspect for a bell is to preserve high tonal quality. For this reason the tuning of the basic frequencies denoting the partial tones of the bell is introduced as objective of the optimization problem without any other constraints. The frequency requirements of a minor– and a major–third bell are very much influenced by the number n of goal frequencies λ_{i0} and the individual weighting factors w_i used in the objective f_λ.

$$f_\lambda = \sum_{i=1}^{n} \frac{(\lambda_i - \lambda_{i0})^2}{\lambda_{i0}^2} w_i$$

The harmony of the lowest five partial tones leads to the following frequency requirements [Hz]:

	major–third bell	minor–third bell
hum	512	512
fundamental	1024	1024
third	<u>1290</u>	<u>1218</u>
fifth	1534	1534
octave	2048	2048

This tuning problem is a multi–criterion optimization problem. As optimization strategy a SQP–method is used. Since all frequency requirements are met exactly at the optimal solution, the individual weighting factors w_i are only important for the convergence of the algorithm.

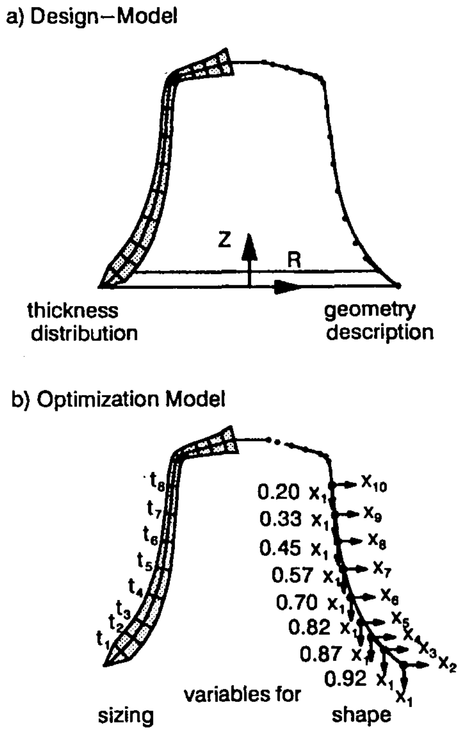


Figure 10 – Definition of the optimization model of the bell

The variables for the optimization model are selected with special care to obtain a well posed optimization problem. As shown in Fig. 10 eight sizing and ten shape values in the design model are used as relevant optimization variables. Some restrictions for design coordinates and nodal thickness values are introduced to obtain a useable optimization result. The number of variables can be further reduced by using linking schemes for sizing and shape variables, like the vertical shape coordinates in Fig. 10 which altogether are linked to the variable x_1.

From the above mentioned 18 relevant optimization variables 15 are used as independent variables to improve the minor–third bell. Seven of them are sizing variables. The resulting shape (Fig. 11) shows only slight modifications of the initial shape.

To obtain the major–third bell, the height of the bell is introduced as additional variable. The shape of this bell has an increased height and a moderate bump at half the height of the bell, also described by Schoofs [28].

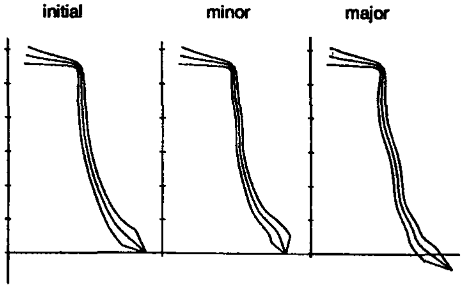
In both optimization procedures the objective function becomes zero within a given tolerance bound. This means that the frequency requirements are fulfilled exactly (see final frequencies in the table of Fig. 11). The iteration history of the objective function and the lower three eigen–frequencies are shown in Fig. 11 for both, the minor– and major–third bell. The difference is pointed out by the third eigen–frequency which is increased for the major–third bell.

6.3 Kresge–Auditorium at MIT [22]

6.3.1 Structure, Objective

Saarinen's famous Kresge auditorium (1955) at the MIT (Fig. 12) belongs to the class of shells with a geometrical form. The three point supported dome is a 1/8 segment of a sphere with a radius of 34.29 m (Fig. 13), a side length of about 48.5 m and an elevation at the vertex of 14.5 m and at the crown of the arches of 8.23 m. The shell has a thickness of 8.9 cm which increases to 12.7 cm at the edge beam and to 49.5 cm at the supports. The edge beam is 25.4 cm wide and varies from 50.8 cm depth at the crown to 91.4 cm at the support. A steel casting is added which is pinned to the abutment. The dead load is 3.98 kN/m², a uniform live load, representing roughly also the wind, is 1.44 kN/m².

to demonstrate that rigid geometrical shapes do not lead to an appropriate shell–like behavior. This is not only verified by linear elastic analyses, it can be confirmed by geometrically and materially nonlinear analyses,



name	initial	final minor	final major
hum	512.9	512.2	512.2
fundamental	1004.0	1023.8	1024.0
third	1239.0	1218.2	1289.6
fifth	1534.2	1534.4	1533.9
octave	2064.8	2049.0	2046.5

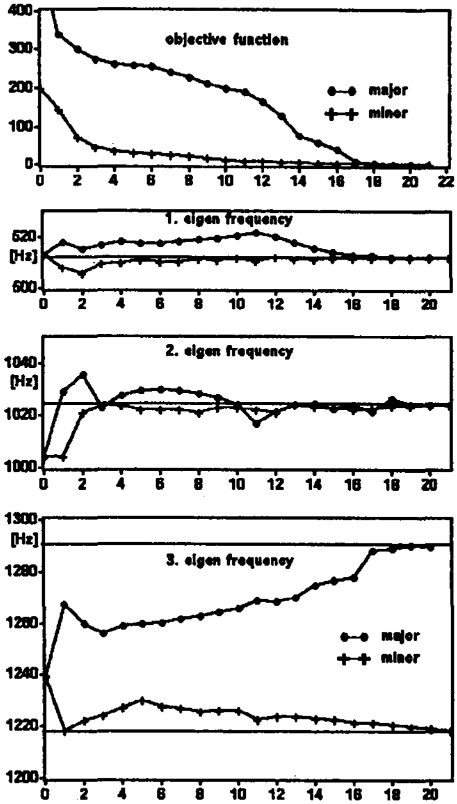


Figure 11 – Optimization results and iteration history of minor– and major–third bell

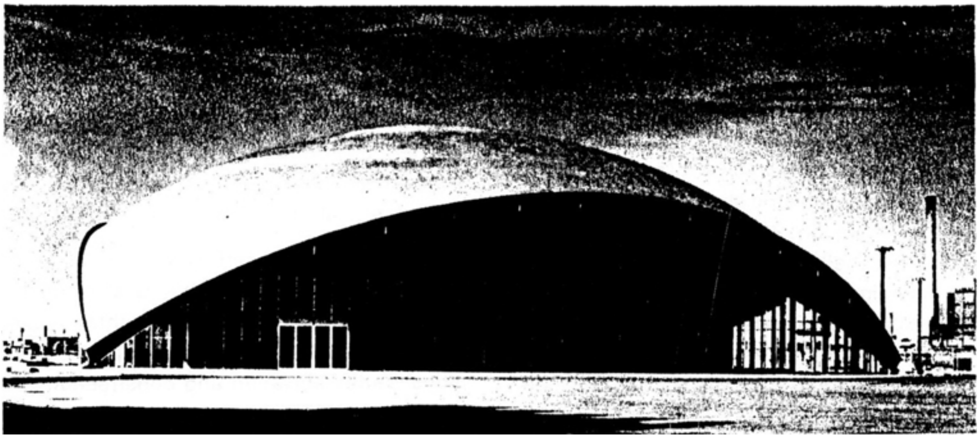


Figure 12 – Kresge auditorium at MIT, Cambridge, MA

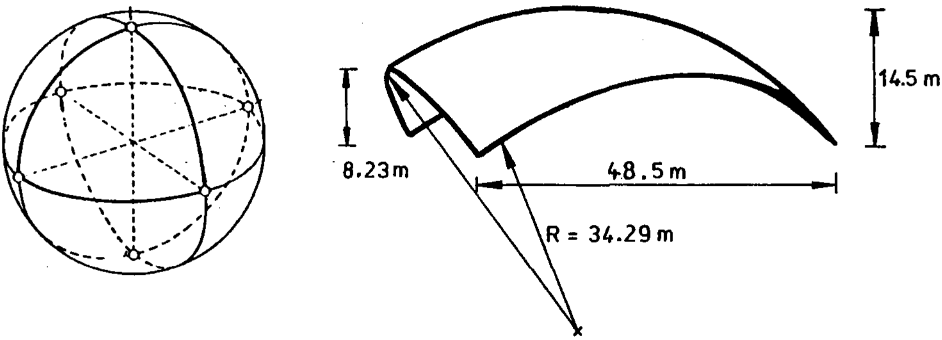


Figure 13 – 1/8 – segment of sphere

- to answer the question how a natural shape of a dome of this size and kind should look like.

In optimization the total strain energy is used as objective thus minimizing the bending in the shell. In the first instance it looks like that this design objective may cause a conflict since a membrane oriented stress state mainly in compression turns out to be more sensitive with respect to buckling and geometrical imperfections (see results for geometrically nonlinear analysis). The objective has to be judged in the context of the chosen material, i.e. reinforced concrete with a low tension capacity.

6.3.2 Scope of Study

The initial analyses are based on linear elastic material properties with a Young's modulus of $3 \cdot 10^4$ N/mm^2 and Poisson's ratio of 0.2. This holds also for the shape optimization. Afterwards the original and the optimized shells are investigated by geometrically nonlinear analyses. Finally, the nonlinear behavior of reinforced concrete is added. Only dead load is applied based on a concrete weight 25 kN/m^3. This load which leads to 2.23 kN/m^2 for a thickness of 8.9 cm is augmented in the entire shell area by a uniform load of 1.75 kN/m^2 for the extra coverage. No live load is considered. In order to avoid high stress concentrations the shell corner is slightly cut off at the supports (at a distance of 1.20 m from the ideal corner point).

The finite element analyses are based on 8–node isoparametric shell elements with reduced integration. 42 elements are used for 1/6 of the shell. Initial stability and geometrically nonlinear analyses of the perfect and imperfect structure are performed to understand the buckling sensitivity of the shell. In this case the entire structure is modelled to allow for unsymmetrical failure modes. The imperfections are based on the fundamental eigenmode scaled to a fraction of the thickness.

The material nonlinear analyses are based on an incremental inelastic, orthotropic material model with a smeared crack approach for the concrete [13], [18]. The layer–wise model utilizes a 2D failure curve, a nonlinear stress – strain curve in compression including softening and tension stiffening. The chosen material properties are: initial modulus of elasticity $E_o = 3 \cdot 10^4$ N/mm², uniaxial strength in compression $f_c =$ 30 N/mm² and tension $f_t =$ 2.2 N/mm², strain at maximum compression $\varepsilon_c = 0.004$, maximum strain of tension stiffening $\varepsilon_{ut} = 0.0025$. Each steel layer has multi–linear 1D properties. Double layered reinforcement with 2.57 cm²/m in both directions is used. An elastic, linear hardening model with $E = 2.1 \cdot 10^5$ N/mm² and $E_h = 0.2 \cdot 10^5$ N/mm² is applied. The yield limit f_y is 240 N/mm².

6.3.3 Shape Optimization

- Analysis of original dome

For simplicity, the thickness is varied only in the direction of the supports with a maximum value of 0.35 m. The edge beams have a width of 0.25 m, their thickness varies from 0.51 to 0.91 m. In the analysis it is simulated by a thicker shell element. The displacements at the vertex and crown of the arches are 1.7 cm and 5.9 cm, respectively. Considerable bending occurs not only at the support but also beside the edge beams, leading to tension stresses of up to 17 N/mm². Of course if no edge beams are added unacceptable values for the displacements (maximum .29 cm) and stresses (> 25 N/mm²) result.

- Shell with "optimized" shape

The above described techniques for shape optimization were applied to the three–point supported shell having the same plan dimensions, vertex elevation and construction volume (amount of concrete). For simplicity only two design elements (bicubic Beziér elements) are introduced, defined only by five geometric variables. In addition, extra thickness parameters are chosen allowing an "optimal" thickness variation. Two studies were performed: In the first one an edge beam with a varying thickness can develop, in the other one the shell is supposed to have a free edge. In this case the elements of the boundary are prevented from geometrically bending downwards indirectly developing into an edge beam. As side constraints the minimum and maximum thickness is chosen as 8.9 cm (thickness of MIT–shell) and 100 cm, respectively. No other constraints are used.

Figure 14 c shows the shape of the optimized shell with a free edge. It can be seen that it differs substantially from the original sphere. it is more parabolic at the edge and has a more pronounced curvature perpendicular to the boundary. The main difference is the large increase of the elevation of the crown which comes along with a negative Gaussean curvature. The thickness varies from 8.9 cm to 42.8 cm. The displacements at the center and at the crown of the arch are reduced to 0.2 cm and 0.3 cm, respectively. The stresses are clearly smaller (max. v. Mises stress 6.4 N/mm²), bending almost vanishes. Although there is tension it is almost negligible (about 1/10 compared to the previous solution).

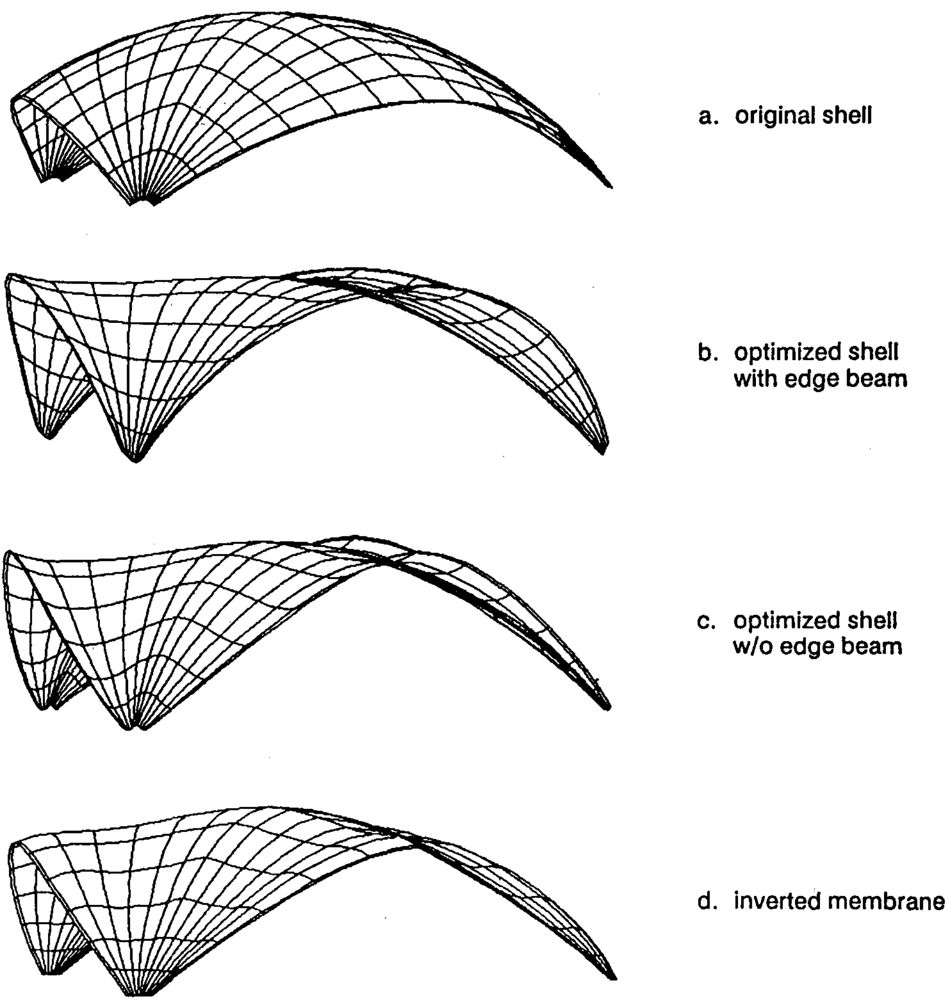


Figure 14 – Shapes of shells

The shell exhibiting an edge beam has a slightly different shape (Fig. 14 b). The thickness of the edge beams varies from 8.9 cm at the crown to 100 cm at the corners. The shell itself is also thinner in this region. The wavy character above the support disappears because now the shell carries the loads more via the edge beams to the foundation. Although the displacements are further reduced the overall stress state is similar. The results of this investigation could be improved if geometrical as well as structural models are refined.

- Shell as inverted membrane

A uniform dead load is applied to an extremely thin membrane with very low bending stiffness having the same plan dimensions. The deformation of the membrane with free edges is monitored by a geometrically nonlinear analysis until the center deflection reaches

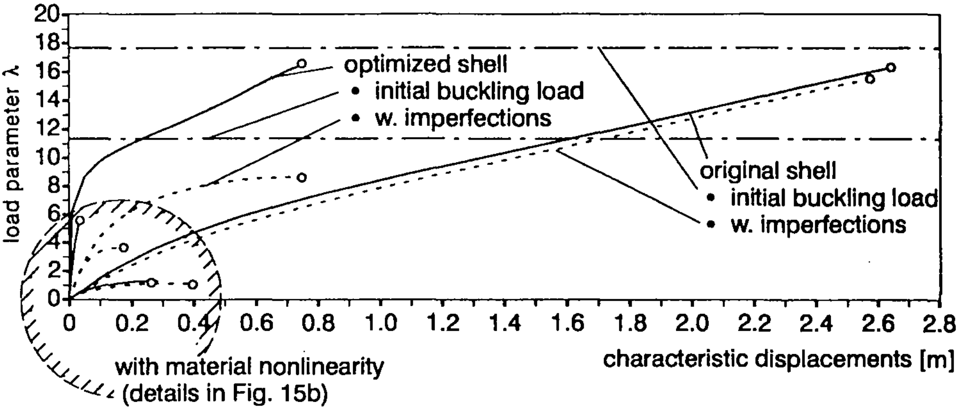


***Figure 15a** – Load – deflection diagram: initial buckling and geometrically nonlinear analyses*

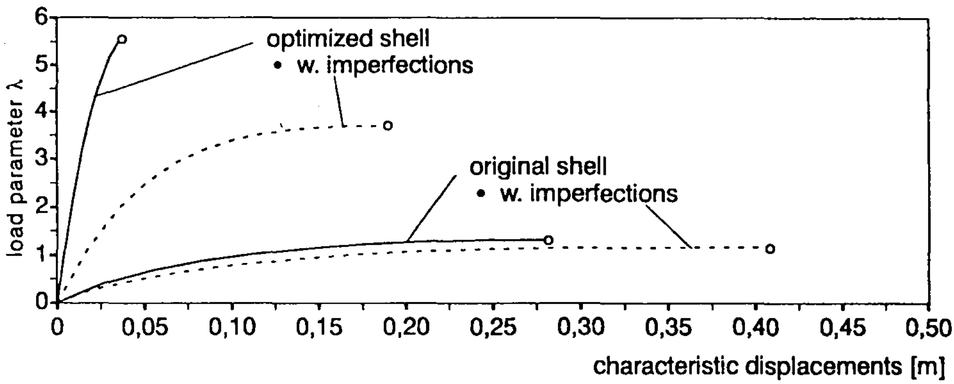


***Figure 15b** – Load – deflection diagram: geometrically and materially nonlinear analyses*

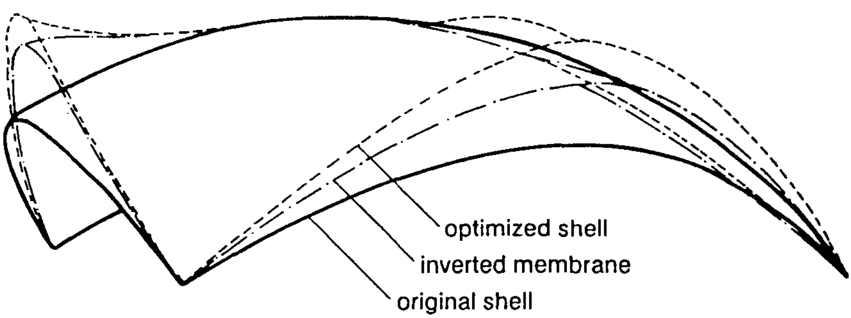


***Figure 16** – Comparison of shapes*

the value of the total height of the original sphere. This shape is inverted (Fig. 14 d). A thickness variation similar to the original structure is assumed and a linear elastic analysis is performed.

The final shape differs from the optimized shape primarily near the free edge. Opposite to this it exhibits a positive Gaussian curvature (which was artificially prevented in the optimization model). Furthermore, slight membrane wrinkles can be visualized at the support. But the structural analysis leads to a similar positive response compared to the optimized shell: small displacements, low stresses, little tension, less bending.

The three shapes are compared in Fig. 16. v. Mises and principal stresses are plotted in [22].

6.3.4 Nonlinear Analyses

The original shell with edge beam and the optimized shell with free edges are further investigated first by a geometrically nonlinear but elastic analysis (Fig. 15 a). Afterwards the materially nonlinear formulation is added (Fig. 15 b) simulating the collapse of both structures.

The initial buckling analysis of the original shell leads to a load multiplier of $\lambda = 17.6$ with a symmetrical failure mode which is 29 percent of the Zoelly load for a perfect spherical shell under pressure The load multiplier λ is defined with respect to the variable weight of the concrete and the uniform load of the coverage. A large deflection analysis indicates distinct nonlinearities but leads almost to the same buckling load ($\lambda = 16.2$), see the load deflection diagram Fig. 15 a. The same holds for the imperfect structure with a maximum imperfection amplitude of 1.5 · thickness ($\lambda = 15.3$). This means that the "amputated" shell itself anticipates the enormous imperfection sensitivity usually present in complete spherical shells and spherical caps.

This is different for the optimized shell. Because of the free edge the buckling load is slightly lower in this case ($\lambda = 11.3$ in an initial buckling analysis, $\lambda = 16.4$ in a large deflection analysis, Fig. 15 a). As expected the optimized structure is more sensitive with respect to initial imperfections. For a maximum imperfection amplitude of 1.5 · thickness the nonlinear elastic buckling load drops to $\lambda = 8.6$.

The real collapse behavior is very much influenced by material failure and therefore yields to lower failure loads compared to elastic analyses. The structural response already depicted in the left lower corner of diagram 15 a is zoomed in Fig. 15 b. Now the anticipated result is obtained. Although the optimized shell with a free edge is still imperfection sensitive to a certain extent its safety margin is still sufficient. The original shell with edge beams but without mullions results in an extremely poor behavior with a maximum load multiplier λ slightly above 1.0.

It needs to be mentioned that the above described study is still of academic nature and is intended only to demonstrate some characteristic features of shells. For a real design much more investigations (environmental effects, long time behavior, non–uniform loads etc.) have to be undertaken to verify the feasibility of the design.

6.3.5 Conclusion for MIT–Shell

The comparison of all four shapes (Fig. 16) gives a good insight what a "natural" form means. In the sphere the stress flow is directed towards the cut off free edge and does not find a stiffening element unless a heavy edge beam or a support is added. Contrary to this the other forms develop an arch with a sufficient stiffness by increasing the curvature perpendicular to the free edge. This causes a considerable reduction in displacements and stresses. Whether this curvature is positive (inverted membrane solution) or negative (enforced by constraints in the optimized solution) does not play an important role. It is interesting to note that the three-point supported shell, investigated in [18] in which three kinks between the corners and the center were admitted, carries the loads mainly in the direction of these lines.

7. CONCLUSIONS

The methods of structural optimization have been presented as general computational tools to find the shape of shells subjected to many different combinations of objectives and constraints. This includes of course the form finding of membranes. The corresponding objectives (e.g. prestress) and constraints (e.g. a desired design space) can be formulated as an optimization problem as well, for example applying the least-square principle or the minimum surface solution.

In general, shape optimization is based on linear elastic structural response including at best linear buckling or eigen-frequency analyses. In this paper the formulation is extended also to geometrically nonlinearities including instability phenomena. A key point of this approach is that it allows to include the imperfection sensitivity with respect to buckling. Further studies are currently under way to verify the method for shape optimization of nonlinear shell structures. The effects of material non-linearities as well as time dependent influences have not been considered but are challenging tasks for further studies.

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