Numerical comparison of several a posteriori error estimators for 2D stress analysis

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ABSTRACT. *The reliabilities of several a posteriori error estimators, including those of Gago, Zienkiewicz-Zhu, and more recently those proposed by Beckers and Zhong are compared through a set of examples in plane elasticity. The examples range from those having analytic solutions to those having progressively stronger singularities. The examples generally use either 4 (or 8) node quadrilaterals for initial comparisons. The results of these examples indicate that, in certain cases, some of the error estimators are unreliable and do not appear to be asymptotically exact. Further studies are suggested to investigate the general validity of the initial conclusions.*

RÉSUMÉ. *Les fiabilités de quelques estimateurs d'erreur, y compris ceux de Gago, Zienkiewicz-Zhu, et plus récemment de ceux proposés par Beckers et Zhong, sont comparées à travers un ensemble d'exemples de l'élasticité plane. Ces exemples s'étendent de ceux qui ont une solution analytique à ceux qui présentent des singularités de plus en plus fortes. Les éléments utilisés sont généralement des quadrangles à 4 ou 8 nœuds pour des comparaisons de départ. Les résultats montrent que, dans certains cas, quelques-uns des estimateurs d'erreur ne sont pas fiables et semblent ne pas être asymptotiquement exacts. De nouvelles études sont suggérées afin d'examiner la validité générale de ces conclusions préliminaires.*

KEY WORDS : *discretization error, a posteriori error estimators, effectivity index, uniformity index.*

MOTS-CLÉS : *erreur de discrétisation, estimateurs d'erreur a posteriori, indice d'effectivité, indice d'uniformité.*

1. Introduction

The notion of *a posteriori error estimation* for the finite element method in solving linear self-adjoint elliptic problems was first introduced by Ladevèze [LAD 77] and Babuska [BAB 78]. In [LAD 77] Ladevèze introduced the notion of "error in the constitutive relations" for 1D and 2D heat conduction problems. Today, *a posteriori error estimators* can be classified into two main categories :

a) estimators related to *equilibrium defaults* (equilibrium residuals, inter-element traction jumps, surface traction defaults) of finite element solutions. For these error estimators, the boundary conditions of the problem are needed in order to estimate the discretization error;

b) estimators based on *post-processing techniques* of the finite element solution (approximation of the exact stresses, displacements, derivatives of the displacements, etc...). The boundary conditions of the problem are not needed for these error estimators so that they are easier to implement and to connect with any finite element code.

The methods used in the error estimation may be considered at global, local or regional levels. A *global method* demands the solution of a system of equations at the global level. A *local method* requires only supplementary calculations at the local level, involving only patches of nodes, elements or interfaces. A *regional method* is between the global and local ones.

The error estimators in the category a) may be explicit or implicit. In an *explicit error estimator*, the estimated error is explicitly expressed as a function of the equilibrium defaults. In an *implicit error estimator*, the estimated error is obtained by solving numerically the *residual equations* in which the equilibrium defaults are used as boundary conditions.

In [BAB 78] a first explicit error estimator related to the equilibrium residuals has been reported by Babuska and Rheinboldt for 1D problems with linear elements. Gago extended this error estimator to 2D elements and added also the inter-element traction jumps to the formulas [GAG 82, KEL 83]. An interpretation of the Gago estimator has recently been made by Zhong and Beckers showing that the Gago estimator is heuristic [ZHO 90a]. New explicit estimators were then proposed in which the surface traction defaults have also been taken into account and a better technique for estimating the exact inter-element tractions has been adopted leading to more reliable results [BEC 90, ZHO 91b].

The implicit error estimators differ one from another in mainly two aspects : firstly, the method used to approximate the boundary conditions of the residual equations; and secondly, the numerical method used to solve the residual equations. Kelly tried to obtain a self-equilibration of the equilibrium defaults at the element level by using a global search method and used an equilibrium model to solve the residual equations [KEL 84]. This lead to an upper bound to the global exact error in the energy norm. Ohtsubo and co-workers used a simplified local method to estimate the self-equilibration, and they used a displacement model of higher degree to solve the residual equations [OHT 90]. Oden and co-workers have tested other types of boundary conditions to solve the residual equations in each element or in each patch of elements [ODE 86]. Ladevèze and co-workers proposed to construct a statically admissible stress field from the finite element solution and from the boundary conditions so as to obtain an upper bound to the global exact error [LAD 83, ROU 89]. A graphical interpretation of the Ladevèze estimator can be found in [MAU 90].

In the category b), Kikuchi and co-workers proposed some interpolation-type error estimators based on the estimation of higher order derivatives of the displacement field by using some post-processing techniques [KIK 86]. Zienkiewicz and Zhu proposed to construct a continuous stress field from the finite element stress field by using different methods, including the global L_2-projection method (or the original Zienkiewicz-Zhu estimator), the "lumped mass" method, the method of simple nodal averages, and more recently a method based on the superconvergent patch recovery technique (or the new Zienkiewicz-Zhu estimator) [ZIE 87, ZIE 92]. Numerical examples in [SHE 89, ZHO 91a, STR 92, ZIE 92] showed that the original Zienkiewicz-Zhu estimator is not reliable for elements of even degree. The new Zienkiewicz-Zhu estimator seems to give very reliable results [ZIE 92]. Zhong and Beckers proposed to estimate nodal stresses by a method called "averaging + extrapolation" using some optimal extrapolation points inside each element [ZHO 90b, ZHO 90c].

Some of the above error estimators have been mathematically analyzed [AIN 89, BAB 92]. It has been shown that some of them can be asymptotically exact while others may not be so. In general, the reliability of the error estimators depends strongly on the mesh size and geometry but relatively weakly on the smoothness of the exact solution.

In this paper several of the above-mentioned error estimators in both categories a) and b) will be examined through a set of examples having different orders of singularities. The meshes will generally be composed of 4-node or 8-node quadrilaterals with different geometries and different levels of refinement. Some reliability measures will be defined so that these error estimators can be compared on the same basis. Numerical experiences on the Ladevèze estimator will be shown in another paper.

2. A posteriori error estimators

2.1. Definition of the structural problem

Consider a 2D bounded domain Ω with its boundary $\Gamma = \Gamma_u \cup \Gamma_t$. In Ω is prescribed a body force field $\bar{f} = \{\bar{f}_x , \bar{f}_y\}^T$. On Γ_u the displacements $\bar{u} = \{\bar{u} , \bar{v}\}^T$ are known a priori. On Γ_t the surface tractions $\bar{t} = \{\bar{t}_x , \bar{t}_y\}^T$ are imposed.

Denote by $\boldsymbol{u} = \{u(x,y)\ ,\ v(x,y)\}^T$, $\boldsymbol{\epsilon} = \{\epsilon_x\ ,\ \epsilon_y\ ,\ \gamma_{xy}\}^T$ and $\boldsymbol{\sigma} = \{\sigma_x\ ,\ \sigma_y\ ,\ \tau_{xy}\}^T$ the exact displacement, strain and stress fields respectively. Then the exact solution satisfies the following equations :

— the equilibrium equations :

$$\begin{cases} div\,\boldsymbol{\sigma} + \bar{\boldsymbol{f}} = 0 & in\ \Omega \\ \boldsymbol{t} = \bar{\boldsymbol{t}} & on\ \Gamma_t \end{cases} \tag{1}$$

— the constitutive equations :

$$\boldsymbol{\sigma} = \boldsymbol{H}\boldsymbol{\epsilon} \tag{2a}$$

where $\boldsymbol{H}$ is the Hooke's elastic matrix. For plane stress problems with isotropic materials, we have

$$\boldsymbol{H} = \begin{pmatrix} H_{11} & H_{12} & 0 \\ H_{12} & H_{11} & 0 \\ 0 & 0 & H_{33} \end{pmatrix} = \frac{E}{(1-\nu^2)} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu^2)}{2} \end{pmatrix} \tag{2b}$$

where E is Young's modulus, and ν is Poisson's ratio.

— the compatibility equations :

$$\begin{cases} \boldsymbol{\epsilon} = \boldsymbol{L}\boldsymbol{u} & in\ \Omega \\ \boldsymbol{u} = \bar{\boldsymbol{u}} & on\ \Gamma_u \end{cases} \tag{3a}$$

where

$$\boldsymbol{L} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \tag{3b}$$

The exact strain energy of the structure is :

$$0.5\|u\|_E^2 = 0.5\int_\Omega \sigma^T H^{-1}\sigma d\Omega \tag{4}$$

2.2. Finite element displacement model

Decompose the domain by a set of finite elements Ω_i , $i = 1 ,\cdots, N_e$. The mesh contains N_1 corner nodes and N_2 interface nodes. Let Γ_{ij} be the interface common to elements i and j. Let $N_n = N_1 + N_2$ be the total number of nodes. Denote the quantities related to the finite element solution by a subscript h which represents the mesh size. For the displacement model, the finite element displacement field is expressed as :

$$u_h = \sum_{j=1}^{N_n} N_j q_j \tag{5}$$

where N_j is the shape function at node j, and q_j is the displacement of node j. The finite element displacement field satisfies the compatibility equations (3a)-(3b) and the constitutive equations (2a)-(2b), but generally it does not satisfy the equilibrium equations (1). This means that the finite element solution contains generally equilibrium defaults in three forms defined as follows :

— equilibrium residuals r in each element :

$$r = \{r_x , r_y\}^T \doteq div\sigma_h + \bar{f} \quad in\ \Omega_i , i = 1 ,\cdots, N_e \tag{6}$$

— inter-element traction jumps J on each interface Γ_{ij} common to elements i and j:

$$J = \{J_x , J_y\}^T \doteq t_h^{(i)} + t_h^{(j)} \tag{7}$$

— surface traction defaults G on Γ_t :

$$G = \{G_x , G_y\}^T \doteq \bar{t} - t_h \tag{8}$$

where the symbol " $\doteq$ " means "equal by definition".

It has been shown that these equilibrium defaults are globally in equilibrium [KEL 83]. But they do not directly provide an error measure. Since the finite element solution is the best numerical solution in the energy sense, i.e., the finite element solution u_h satisfies :

$$\|e_h\| \doteq \|u-u_h\|_E = \underset{v_h}{Minimum} \; \|u-v_h\|_E \tag{9}$$

for all finite element displacements v_h which are kinematically admissible, the energy norm $\|\cdot\|_E$ is then the natural error norm for the finite element displacement model.

2.3. *Error estimators for 2D plane stress problems*

Denote by ε_i the estimated error in the energy norm for element i, ε the total estimated error in the energy norm :

$$\varepsilon \equiv \left(\sum_{i=1}^{N_e} \varepsilon_i^2\right)^{\frac{1}{2}} \approx \left(\sum_{i=1}^{N_e} \|e_h\|_{E,i}^2\right)^{\frac{1}{2}} \equiv \|e_h\|_E \tag{10}$$

In the following, several error estimators will be defined.

2.3.1. The Gago estimator

By extending the Babuska's method for 1D problems, Gago has proposed the following explicit error estimator [GAG 82, KEL 83] :

$$\varepsilon_i^2 = \frac{h_i^2}{24a}\left(\|r_x - \bar{r}_x\|_{0,i}^2 + \|r_y - \bar{r}_y\|_{0,i}^2\right) + \frac{h_i T}{24a} \sum_{mn} \int_{\Gamma_{mn}} \left(J_x^2 + J_y^2\right) d\Gamma \tag{11a}$$

for 4-node elements, and

$$\varepsilon_i^2 = \frac{h_i^2}{96a}\left(\|r_x\|_{0,i}^2 + \|r_y\|_{0,i}^2\right) + \frac{h_i T}{96a} \sum_{mn} \int_{\Gamma_{mn}} \left(J_x^2 + J_y^2\right) d\Gamma \tag{11b}$$

for 8-node elements, respectively. In (11a) and (11b), h_i is the size of element i (in the numerical examples, it has been replaced by the square root of the element area), T is the thickness of the element, $\bar{r}_x$ and $\bar{r}_y$ denote the mean values of the residuals in the directions x and y respectively, $\|\cdot\|_{0,i}$ represents the

L_2-norm for element i, a is a constant depending on the material properties : $a = E/[(1+\nu)(1-2\nu)]$ for plane stress problems. For convenience, expression (11) is called the ***Jr-estimator*** and will be denoted by ***Jr-***. The derivation of this error estimator, as shown in [GAG 82, KEL 83], does not seem to be rigorous.

2.3.2. The $\tilde{G}$-estimator

Based on the nodal superconvergence assumption of the finite element displacement field, an "exact" relation can be found between the energy norm of the error and the integral of the surface traction defaults for each rectangle. This leads to the following error estimator (see [ZHO 91a] for the complete derivation) :

$$\varepsilon_i^2 = AT \sum_{mn} \int_{\Gamma_{mn}} \frac{K_n \tilde{G}_n^2 + K_t \tilde{G}_t^2}{L_{mn}} \, d\Gamma \qquad (12)$$

where A is the area of element i, and L_{mn} is the length of a boundary segment Γ_{mn} of the element. $\tilde{G}_n$ and $\tilde{G}_t$ are approximations of the exact normal and tangential traction defaults respectively, which are defined on the element boundaries. K_n and K_t are two constants related to the coefficients of the Hooke's elastic matrix : $K_n = H_{11}/(6H_{11}^2 + 2H_{12}^2)$, $K_t = 1/(8H_{33})$ for 4-node quadrilaterals, and $K_n = H_{11}/(8H_{11}^2 + 2H_{12}^2)$, $K_t = 1/(10H_{33})$ for 8-node quadrilaterals. For 3-node and 6-node triangles these constants should be modified. Expression (12) is called the ***$\tilde{G}$-estimator*** and will be denoted by ***$\tilde{G}$-***.

2.3.3. The *r*-estimator for 8-node elements

By using the same superconvergence assumption, another explicit error estimator related only to the equilibrium residuals can be deduced for 8-node quadrilaterals [ZHO 91b] :

$$\varepsilon_i^2 = \frac{A}{60H_{11}} \left(\| r_x \|_{0,i}^2 + \| r_y \|_{0,i}^2 \right) \qquad (13)$$

This error estimator is called the ***r-estimator*** and will be denoted by ***r-***. Note that such a relation does not exist for 4-node quadrilaterals.

2.3.4. The global L_2-projection method

Among all the error estimators in category b), the most popular is that proposed by Zienkiewicz and Zhu [ZIE 87]. The idea is to construct a continuous stress field (denoted by $\tilde{\sigma}$) from the finite element one, expressed as :

$$\tilde{\sigma} = \sum_{j=1}^{N_n} N_j s_j \tag{14}$$

where s_j denotes the value of a unique defined stress field at node j, and then to estimate the exact energy of the element error by the following expression :

$$\varepsilon_i^2 = \int_{\Omega_i} (\tilde{\sigma} - \sigma_h)^T H^{-1} (\tilde{\sigma} - \sigma_h)\, d\Omega \tag{15}$$

which is called the *$\tilde{\sigma}$-estimator.*

There may exist various ways to determine the nodal values of the continuous stress field. This leads to different $\tilde{\sigma}$-estimators. Usually, the *projection method* [ZIE 87] can be used, which consists in solving the following linear systems of equations

$$A s^k = b^k, \quad k = 1,\cdots,3 \tag{16a}$$

where three systems of equations exist in (16a), one for each stress component k. The coefficients of A are defined by :

$$A_{ij} = \int_{\Omega} N_i N_j \, d\Omega \tag{16b}$$

and are thus similar to those of a consistent mass matrix of the structure with unit mass density. s^k is the vector of the k^{th} component of nodal stresses, s_j^k is the k^{th} component of stress at node j. The coefficients of b^k are defined by :

$$b_j^k = \int_{\Omega} N_j \sigma_h^k \, d\Omega \tag{16c}$$

The $\tilde{\sigma}$-estimator deduced from the projection method will be denoted by $\tilde{\sigma}(L_2)$-. Note that this method for obtaining a continuous stress field was initially studied by Oden and Brauchli [ODE 71].

2.3.5. The "lumped mass" method

Due to the fact that the solution of the equations in (16a) is time-consuming, simpler methods are preferable in practice to evaluate s_j. Many possibilities have been suggested [ZIE 87]. One of them is to replace the matrix A with the diagonal matrix $\bar{A}$, where $\bar{A}_{ij} = 0$ when $i \neq j$, and $\bar{A}_{ii} = \Sigma_{j=1}^{N_n} A_{ij}$. then

$$s_i^k = \frac{b_i^k}{\bar{A}_{ii}} \ , \quad i = 1 ,\cdots, N_n \tag{17}$$

This is called the *"lumped mass" method*, and the deduced error estimator will be denoted by $\tilde{\sigma}(L_m)$-. It has proved rather effective for interior nodes. However, it is not satisfactory for nodes on the boundaries of the structure.

2.3.6. The method of "averaging + extrapolation"

Another "local" method for evaluating s_j has been proposed in [ZHO 90b, ZHO 91b] for linear and bi-linear elements. This method consists in determining the value s_j^k at an interior node j by a weighted mean value of the finite element stresses :

$$s_j^k = \sum_{e=1}^{M_j} w_e (\sigma_h^k)_e \tag{18a}$$

where M_j is the total number of elements connected to node j, $(\sigma_h^k)_e$ is the value of the k^{th} component of the finite element stress field at the barycentre of element e. The weight w_e is chosen as :

$$w_e = \frac{\alpha_e / L_e}{\sum_{e=1}^{M_j} \alpha_e / L_e} \ , \quad e = 1 ,\cdots, M_j \tag{18b}$$

where α_e is the angle included at the node j, and L_e is the distance between the element barycentre and the node. For nodes on the boundaries of the structure, a method of linear extrapolation is used instead. This last local method is called the *method of "averaging + extrapolation"*, and the deduced error estimator will be denoted by $\tilde{\sigma}(\alpha_e/L_e)$-. For elements of degree two, nodal and interface stresses can be determined as weighted mean values using optimal stress extrapolation points within each element [ZHO 90c].

A simpler local method determines s_j as the simple mean value of σ_h evaluated at node j for each element connected to node j. This is not considered further in this paper, but recent numerical studies have been reported [ROB 92, ROB 93].

2.4. Reliability measures for the error estimators

In order to evaluate the reliability of an error estimator, two indices can be introduced. The first one is called the *global effectivity index* [GAG 82, ZIE 87, BEC 92, ...], defined as :

$$\theta \doteq \frac{\varepsilon}{\|e_h\|_E} \tag{19}$$

If θ is near to one, then the error estimator is said to be globally effective, or globally reliable. Since this index can not reveal the reliability of the error estimator at the element level, a second index has to be introduced :

$$SD = \left(\frac{1}{N_e} \sum_{i=1}^{N_e} (\theta_i - \bar{\theta})^2 \right)^{1/2} \tag{20}$$

which is called the *uniformity index* [ZHO 90a, BEC 92]. A similar index has been proposed in [ODE 89]. In (20) θ_i is the elemental effectivity index, and $\bar{\theta}$ is the mean value of the elemental effectivity indices. If SD is near to zero, then the error estimator is said to be uniform. A reliable error estimator should be at the same time effective and uniform. If $\theta \to 1$ and $SD \to 0$ when the mesh size $h \to 0$, then the error estimator is said to be *asymptotically exact.*

Of course, other reliability indices can be used, such as a quality index, or a robustness index [BAB 91]. However, their evaluation is complicated and will not be considered here.

2.5. Richardson's extrapolation and dual analysis

If the exact solution of a problem is known, then all the reliability indices defined above can be easily evaluated. However, most of the problems encountered in practice do not have analytical solutions. It is then more important to evaluate the reliability of the error estimators for problems whose exact solution is not available. In this case the uniformity index can not be evaluated, but the global effectivity index can still be precisely calculated if the strain energy of the structure can be precisely obtained.

When the convergence of the energy norm of the error is monotonic and asymptotic, then there exists an asymptotic relation between the global energy of the error and the total number of degrees of freedom (*DOF*) :

$$\|e_h\|_E^2 = C\left(\frac{1}{DOF}\right)^{2r_c} \tag{21}$$

where r_c is called the *asymptotic convergence rate* of the global energy norm of the error. It is also a measure of the order of stress singularity of the structure. C is a constant independent of the mesh size. The exact energy norm $\|u\|_E$ of the structure can be estimated by a procedure called *Richardson's extrapolation* [RIC 10]. Three analyses are needed to determine the two constants C, r_c and the energy norm. Denote such an estimate by $\|u_R\|_E$. Then if the boundary displacement conditions of a structure are homogeneous and consistent loads are applied, the Pythagoras' theorem of the discretization error can be applied

$$\|e_h\|_E^2 = \|u\|_E^2 - \|u_h\|_E^2 \tag{22}$$

so that the exact error can be approximated by

$$\|e_h\|_E^2 \approx \|u_R\|_E^2 - \|u_h\|_E^2 \tag{23}$$

In order that the approximate global effectivity indices be sufficiently precise, a sequence of fine meshes should be used in the procedure of Richardson's extrapolation. This procedure may be applied to a sequence of displacement models only.

Another method to obtain a precise estimation of the exact strain energy of the structure is to perform *dual analyses* [FRA 65] in which the same problem is solved by using both displacement models and equilibrium models. Then if the displacement boundary conditions are homogeneous and consistent loads are applied, a displacement model gives a lower bound to the exact strain energy while an equilibrium model gives an upper bound. Richardson's extrapolation may then be applied to both sequences of displacement and equilibrium models so as to obtain two estimates which bound the exact strain energy. For equilibrium models, the signs in (22) and (23) should be altered.

In the following numerical examples, the exact precision on the strain energy is defined as :

$$\eta^2 \doteq \frac{\|e_h\|_E^2}{\|u\|_E^2} \tag{24}$$

while the estimated precision by the procedure of Richardson's extrapolation will be defined as :

$$\eta_R^2 \doteq \frac{\|u_R\|_E^2 - \|u_h\|_E^2}{\|u_R\|_E^2} \tag{25}$$

3. Numerical examples

In the following three numerical examples, plane stress conditions are prescribed to the structures. For simplicity, all the physical units will be omitted. The geometry of the structures and the meshes are fully described so that they can be easily reproduced by the reader. This helps to verify the reported numerical results.

3.1. Example of bending and shearing of a rectangular beam

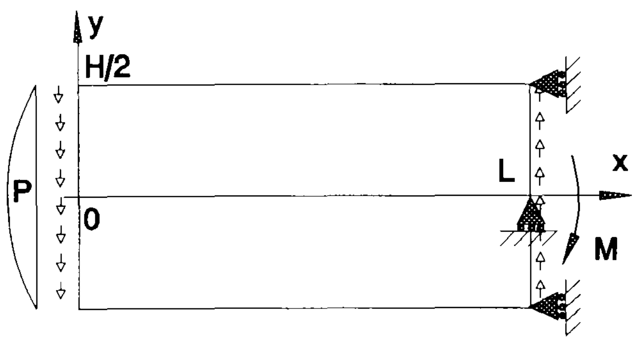

Figure 1. *Rectangular beam — Boundary conditions*

This numerical example is taken from [ZHO 91b]. It is a rectangular cantilever beam under bending and parabolic shear, with only rigid body modes constrained (Figure 1). The structure is isostatic. The boundary conditions are such that the exact displacement field can be expressed as :

$$
\begin{cases}
u(x,y) = -\dfrac{Py}{EI}\left[\dfrac{1}{2}(L^2-x^2) + \dfrac{2+\nu}{6}\left(y^2-\dfrac{1}{4}H^2\right)\right] \\[2ex]
v(x,y) = -\dfrac{P}{EI}\left[\dfrac{1}{3}L^3 - \dfrac{1}{2}L^2x + \dfrac{1}{6}x^3 + \dfrac{4+5\nu}{24}H^2(L-x) + \dfrac{\nu}{2}xy^2\right]
\end{cases}
$$

where $I = TH^3/12$. The exact stress field is :

$$
\begin{cases}
\sigma_x(x,y) = \dfrac{P}{I}xy \\[2ex]
\sigma_y(x,y) = 0 \\[2ex]
\tau_{xy}(x,y) = \dfrac{P}{I}\left(\dfrac{1}{8}H^2 - \dfrac{1}{2}y^2\right)
\end{cases}
$$

The parameters of the structure are : $E = 3 \cdot 10^7$, $\nu = 0.3$, $T = 1$, $L = 8$, $H = 4$, $M = 2000$, $P = 250$. The exact strain energy is $0.5 \cdot \|u\|_E^2 = 0.039833333$.

Two 8 x 4 initial meshes are designed, one uniform and another non-uniform (Figure 2). The parameters of the initial non-uniform mesh are :

$$
y_i = \frac{H}{2}\sqrt{\frac{i}{2}}, \quad x_i = L\sqrt{\frac{i}{8}}
$$

Tables 1 to 4 assemble all the results of error estimation for a sequence of meshes uniformly refined from the initial ones (numbered 1).

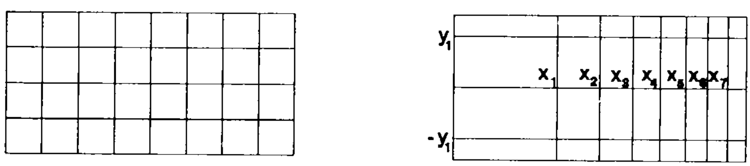

Figure 2. *Rectangular beam — Initial meshes*

Mesh		1	2	3	4
DOF		90	306	1122	4290
η^2		0.034	0.0088	0.0022	0.00055
$\tilde{G}$-	θ	0.90	0.90	0.90	0.90
	SD	0.06	0.04	0.02	0.01
Jr-	θ	0.54	0.58	0.61	0.62
	SD	0.08	0.05	0.04	0.03
$\tilde{\sigma}(L_2)$-	θ	0.90	0.95	0.98	0.99
	SD	0.20	0.10	0.07	0.05
$\tilde{\sigma}(L_m)$-	θ	1.11	1.06	1.03	1.02
	SD	0.25	0.14	0.09	0.05
$\tilde{\sigma}(\alpha_e/L_e)$-	θ	0.97	0.99	1.00	1.00
	SD	0.23	0.11	0.06	0.03

Table 1. *Rectangular beam — Convergence of the indices* θ *and SD of the error estimators, uniform initial mesh, 4-node elements*

Mesh		1	2	3	4
DOF		242	866	3266	12674
η^2		$0.98 \cdot 10^{-4}$	$0.62 \cdot 10^{-5}$	$0.39 \cdot 10^{-6}$	$0.24 \cdot 10^{-7}$
$\tilde{G}$-	θ	0.93	0.92	0.91	0.91
	SD	0.02	0.02	0.01	0.01
r-	θ	0.81	0.83	0.80	0.83
	SD	0.02	0.02	0.02	0.01
Jr-	θ	0.50	0.49	0.48	0.47
	SD	0.03	0.03	0.03	0.01
$\tilde{\sigma}(L_2)$-	θ	0.06	0.06	0.06	0.03
	SD	0.03	0.03	0.03	0.02
$\tilde{\sigma}(L_m)$-	θ	11.8	16.8	21.4	33.8
	SD	7.14	13.1	20.5	31.8
$\tilde{\sigma}(\alpha_e/L_e)$-	θ	1.34	1.33	1.33	1.32
	SD	0.03	0.02	0.02	0.01

Table 2. *Rectangular beam — Convergence of the indices* θ *and SD of the error estimators, uniform initial mesh, 8-node elements*

Mesh		1	2	3	4
DOF		90	306	1122	4290
η^2		0.038	0.010	0.0026	0.00067
$\tilde{G}$-	θ	0.83	0.87	0.89	0.91
	SD	0.17	0.16	0.12	0.09
Jr-	θ	0.53	0.57	0.59	0.61
	SD	0.09	0.09	0.07	0.05
$\tilde{\sigma}(L_2)$-	θ	0.85	0.93	0.96	0.98
	SD	0.26	0.18	0.13	0.09
$\tilde{\sigma}(L_m)$-	θ	1.18	1.08	1.04	1.02
	SD	0.51	0.26	0.22	0.17
$\tilde{\sigma}(\alpha_e/L_e)$-	θ	1.04	1.00	1.00	1.00
	SD	0.07	0.05	0.03	0.02

Table 3. *Rectangular beam — Convergence of the indices* θ *and SD of the error estimators, non-uniform initial mesh, 4-node elements*

Mesh		1	2	3	4
DOF		242	866	3266	12674
η^2		$0.57\cdot10^{-3}$	$0.37\cdot10^{-4}$	$0.24\cdot10^{-5}$	$0.48\cdot10^{-6}$
$\tilde{G}$-	θ	0.92	0.88	0.86	0.85
	SD	0.16	0.16	0.14	0.13
r-	θ	0.69	0.69	0.71	0.72
	SD	0.16	0.16	0.14	0.13
Jr-	θ	0.45	0.42	0.43	0.44
	SD	0.20	0.16	0.13	0.12
$\tilde{\sigma}(L_2)$-	θ	0.34	0.26	0.22	0.14
	SD	0.97	0.82	0.60	0.50
$\tilde{\sigma}(L_m)$-	θ	6.37	9.07	12.4	17.9
	SD	21.2	35.4	58.0	93.2
$\tilde{\sigma}(\alpha_e/L_e)$-	θ	1.20	1.19	1.20	1.21
	SD	0.14	0.12	0.09	0.09

Table 4. *Rectangular beam — Convergence of the indices* θ *and SD of the error estimators, non-uniform initial mesh, 8-node elements*

3.2. Example of L-shaped domain with applied tractions

This example is taken from [ZIE 87, SHE 89] where a stress singularity occurs at the re-entrant corner. A uniform initial mesh and a "geometrical mesh" [SZA 86] have been designed (Figure 3). A geometrical mesh is the one which is progressively graded toward a singular point. The geometrical mesh used in the present example has two layers, with a geometrical progression factor 0.15, and is composed of both triangles and quadrilaterals.

The parameters of the structure are : $E = 1.0$, $\nu = 0.3$, $T = 1$. To estimate the exact strain energy of the structure, a series of meshes uniformly refined from the uniform initial one have been created. Both displacement elements of degree two and equilibrium elements of degree one have been used. The results are shown in Table 5. Note that, by analogy, the Fraeijs de Veubeke displacement plate bending elements [FRA 68] can be used in place of the real equilibrium elements [BEC 73]. In fact, all the results related to the equilibrium models have been obtained here by using this analogy.

By applying the procedure of Richardson's extrapolation for the displacement model, a convergence rate $r_c = 0.306$ has been found (the correct asymptotical convergence rate is 0.272 [SZA 86]), and the extrapolated strain energy is :

$$0.5 \cdot \| u_R \|_E^2 = 15565.833$$

Similarly, for the equilibrium model, $r_c = 0.292$ and

$$0.5 \cdot \| u_R \|_E^2 = 15567.086$$

The mean value of these two extrapolated strain energies is 15566.460. The global effectivity indices shown in Tables 6, 7 and 8 are based on this mean value. Note that this value is practically the same as the one shown in [ZIE 92], equal to 15566.200, which has been obtained by an h-p code. In [SHE 89] a third value, 15566.660, has been used. All these values are sufficiently precise so that the results shown in the following tables are correct.

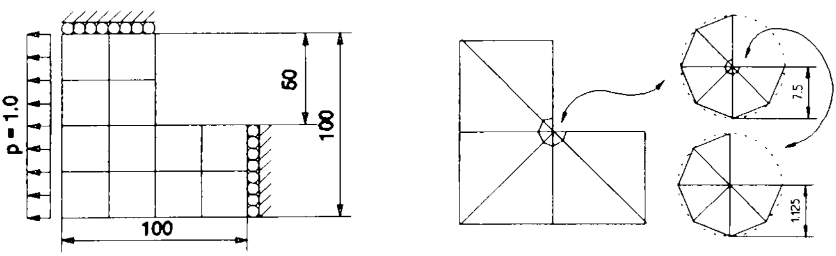

Figure 3. *L-shaped domain - Uniform initial mesh and geometrical mesh*

Mesh	Finite element model			
	8-node displacement elements		equilibrium quadrilaterals with piece-wise linear stress field	
	DOF	Strain energy	DOF	Strain energy
1	96	15299.175	63	15765.426
2	336	15460.452	245	15647.029
3	1248	15520.529	969	15601.959
4	4800	15545.970	3857	15582.676
5	18816	15557.205	15393	15574.045

Table 5. *L-shaped domain — Convergence of the strain energies of the displacement and equilibrium models*

Mesh		1	2	3	4
DOF		36	120	432	1632
η_R^2		0.11	0.039	0.014	0.0048
θ	$\tilde{G}$-	0.75	0.80	0.83	0.85
	Jr-	0.37	0.44	0.46	0.46
	$\tilde{\sigma}(L_2)$-	0.71	0.80	0.82	0.82
	$\tilde{\sigma}(L_m)$-	0.95	0.96	0.95	0.94
	$\tilde{\sigma}(\alpha_e/L_e)$-	0.84	0.92	0.93	0.94

Table 6. *L-shaped domain — Convergence of the indices* **θ** *of the error estimators, uniform initial mesh, 4-node elements*

Mesh		1	2	3	4
DOF		96	336	1248	4800
η_R^2		0.017	0.0068	0.0029	0.0013
θ	$\tilde{G}$-	0.86	0.88	0.90	0.91
	r-	0.56	0.55	0.55	0.55
	Jr-	0.53	0.54	0.55	0.55
	$\tilde{\sigma}(L_2)$-	0.62	0.61	0.61	0.61
	$\tilde{\sigma}(L_m)$-	2.14	1.70	1.45	1.35
	$\tilde{\sigma}(\alpha_e/L_e)$-	1.01	0.97	0.94	0.93

Table 7. *L-shaped domain — Convergence of the indices* **θ** *of the error estimators, uniform initial mesh, 8-node elements*

Degree		1	2
DOF		38	116
η_R^2		0.24	0.018
θ	$\tilde{G}$-	0.88	1.31
	$\tilde{\sigma}(L_2)$-	0.46	0.65
	$\tilde{\sigma}(L_m)$-	0.63	*
	* $\tilde{\sigma}(\alpha_e/L_e)$-	1.37	2.23

* extremely high

Table 8. *L-shaped domain — The indices* θ *of the error estimators for the geometrical mesh*

3.3. Example of crack problem in linear elasticity

Figure 4. *Crack problem — Boundary conditions and uniform initial mesh*

Consider now a crack problem in linear elasticity (Figure 4). The parameters of the structure are : $E = 1.0$, $\nu = 0.3$, $T = 1.0$. Due to the symmetry, only a half of the structure will be analyzed with an uniform initial mesh. Note that although the finite element solution is not changed by taking account of the symmetry, the reliabilities of all the error estimators, except the r-estimator, are a little changed.

From Table 9, one can obtain the convergence rates $r_c = 0.252$ and 0.244 for the displacement and equilibrium models respectively. Note that the asymptotical convergence rate for the displacement model is 0.250 [SZA 86]. By the procedure of Richardson's extrapolation, one can obtain $0.5\cdot\|u_R\|_E^2 = 8085.8890$ and $0.5\cdot\|u_R\|_E^2 = 8085.6330$, so that one can take their average $0.5\cdot\|u_R\|_E^2 = 8085.7610$ as a good approximation of the exact strain energy. The effectivity indices shown in Tables 10 and 11 are based on this value.

Mesh	Finite element model			
	8-node displacement elements		equilibrium quadrilaterals with piece-wise linear stress field	
	DOF	Strain energy	DOF	Strain energy
1	125	7106.5239	53	8783.0531
2	441	7570.8356	265	8416.0116
3	1649	7821.7079	1169	8246.9442
4	6369	7952.2561 *	4883	8165.7145
5	25025	8018.8749	20033	8125.8524

Table 9. *Crack problem — Convergence of the strain energies of the displacement and equilibrium models*

Mesh		1	2	3	4
DOF		47	157	569	2161
η_R^2		0.22	0.13	0.069	0.036
θ	$\bar{G}$-	0.67	0.70	0.73	0.74
	Jr-	0.28	0.31	0.33	0.34
	$\bar{\sigma}(L_2)$-	0.62	0.67	0.70	0.71
	$\bar{\sigma}(L_m)$-	0.77	0.81	0.83	0.84
	$\bar{\sigma}(\alpha_e/L_e)$-	0.80	0.84	0.86	0.88

Table 10. *Crack problem — Convergence of the indices* θ *of the error estimators, uniform initial mesh, 4-node elements*

Mesh		1	2	3	4
DOF		125	441	1649	6369
η_R^2		0.12	0.064	0.033	0.017
θ	$\bar{G}$-	0.78	0.80	0.81	0.83
	r-	0.49	0.49	0.50	0.51
	Jr-	0.46	0.47	0.48	0.49
	$\bar{\sigma}(L_2)$-	0.61	0.63	0.64	0.65
	$\bar{\sigma}(L_m)$-	1.18	1.18	1.19	1.20
	$\bar{\sigma}(\alpha_e/L_e)$-	0.93	0.94	0.95	0.97

Table 11. *Crack problem — Convergence of the indices* θ *of the error estimators, uniform initial mesh, 8-node elements*

4. Discussions on the numerical results

*1) The Gago estimator **Jr**-*

This error estimator provides generally significant underestimation of the exact error and may not be asymptotically exact even for smooth solutions and for uniform meshes [ZHO 90a].

2) The $\tilde{G}$-estimator

The $\tilde{G}$-estimator is generally quite reliable at both the global and local levels. The results obtained from the geometrical mesh for the L-shaped domain indicate that the $\tilde{G}$-estimator is also the most robust one against element distortions (ref. Figure 3 and Table 8). This suggests that, for general meshes, error estimators related to the equilibrium defaults might be more reliable than those based on stress smoothing.

3) The r-estimator for 8-node elements

As with the Gago estimator, the r-estimator for 8-node elements provides generally an underestimation of the exact error although it is quite uniform. Perhaps the L_2-norm of the residuals is not suitable for indicating the discretization error.

4) The global L_2-projection method $\tilde{\sigma}(L_2)$-

The original Zienkiewicz-Zhu estimator, as described in (15) - (16c), is in fact a least squares matching of the finite element stress field into a continuous stress field represented by (15), i.e., the equations (16a)-(16c) are equivalent to

$$\underset{s}{minimize} \int_{\Omega} (\tilde{\sigma} - \sigma_h)^T (\tilde{\sigma} - \sigma_h) d\Omega \tag{26}$$

This means that, among all the possible $\tilde{\sigma}$-estimators, the $\tilde{\sigma}(L_2)$-estimator gives the lowest estimation of the exact error measured in the least squares sense.

It has been shown that if the exact solution is smooth and when uniform meshes are used, then the stress field derived from the finite element solution by using elements of even degree (p = 2, 4, ...) is already quasi-continuous [ZHO 90C, ZHO 91b], so that the minimization procedure (26) leads to a continuous stress field which is very near to the one obtained by the finite element method. This means that the global effectivity indices may tend to zero rather than one. For non-uniform meshes or for more singular solutions, the estimator seems to be globally more reliable but is unfortunately less uniform.

For elements of odd degree ($p = 1, 3, ...$), the error estimator is quite reliable for smooth solutions and for uniform meshes. In this case various constant corrective factors proposed by Zienkiewicz and Zhu can be used to improve the global reliability of the error estimator [ZIE 87]. However, the use of such factors can improve its global effectivity only for relatively coarse meshes. They can not improve its uniformity.

5) The "lumped-mass" method $\tilde{\sigma}(L_m)$-

For elements of degree one, the $\tilde{\sigma}$-estimator based on the "lumped mass" method is globally comparable to the one based on the global L_2-projection method, but it is very poor from the standpoint of uniformity.

For elements of degree two, this error estimator is completely defective for smooth solutions. It can not be used for meshes containing 6-node triangles (ref. Table 8) since the lumped masses may be zero for these elements. Moreover, the difference between these two $\tilde{\sigma}$-estimators is much greater than in the case of elements of degree one. For more singular solutions, it seems to be globally more reliable for 8-node elements, but this is at the expense of higher uniformity indices.

Note that more sophisticated mass lumping techniques produce non-zero nodal masses for 6-node triangles. The derived error estimators are then very similar to the one based on simple nodal averaging of the stresses [SHE 89].

6) The method of "averaging + extrapolation" $\tilde{\sigma}(\alpha_e/L_e)$-

For elements of degree one, the $\tilde{\sigma}(\alpha_e/L_e)$-estimator is the most effective and the most uniform one among all the error estimators examined in this paper. It seems to be asymptotically exact for smooth solutions.

For elements of degree two, it is less reliable for smoother solutions but seems to be more reliable for more singular solutions when undistorted meshes are used.

This error estimator seems to be very similar to the new Zienkiewicz-Zhu error estimator [ZIE 92], especially for uniform meshes.

5. Concluding remarks

1) The numerical tests as shown in this paper are necessary but of course not sufficient to justify the validity of the error estimators. Further studies should be undertaken so as to determine the conditions under which these error estimators are asymptotically exact, in particular the $\tilde{\sigma}(\alpha_e/L_e)$-estimator and the $\tilde{G}$-estimator which appear to be generally the most reliable.

2) An error estimator should be sufficiently robust against element distortions, i.e., the element distortions that are practically acceptable should not cause a strong loss of the reliability of the error estimators. Further studies should be undertaken, in particular to determine the sensitivities of the $\tilde{\sigma}(\alpha_e/L_e)$- and $\tilde{G}$-estimators to element distortions. In addition, it would be desirable to overestimate rather than underestimate the exact error in the case where the quality of the finite element solutions is less reliable.

3) It is impossible to develop an error estimator which is reliable for all types of problems, for all solutions having different order of singularities, and for all meshes of different geometries. In fact, the asymptotical exactness of an error estimator is not a crucial criterion for practical engineering analysis, since a numerical solution is always allowed to have a limited level of precision. It is then only required that an error estimator be sufficiently reliable for the numerical solutions which are within a certain range of engineering precision.

4) The error estimators in both categories a) and b) vary considerably in their ability to quantify error, and this ability is generally problem and model dependent. From the results reported in this paper it is considered that the explicit error estimators $\boldsymbol{Jr}$- and $\boldsymbol{r}$- in category a) should be rejected. On the other hand, implicit error estimators [LAD 83, MAU 90, STR 92, ...], might give more reliable results when correctly modelling the boundary conditions for the solution of the residual equations. In category b), the $\tilde{\sigma}$-estimators based on the projection of σ_h, or the simplified alternatives, are not always reliable, and their use should be discouraged. However, those based on extrapolation of stress from optimal or superconvergent points [ZHO 91b, ZIE 92] appear to give better results, and further studies of them should be pursued.

5) For coarse meshes, especially for meshes which are only a single element deep in some direction, the use of the error estimators of category a) is recommended.

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