Energy release rate and J integral for cracks propagating in nonhomogeneous media
Part I – Problem formulation

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ABSTRACT. *In this paper the energy release rate (or the j integral) is calculated for cracks propagating in media with spatially varying material characteristics. Without any assumption on the direction of such a variation with respect to the crack axis at it was made in a similar approach [ATK 75], an analytical expression for the energy release rate under general loading conditions is derived and put in a form suitable for numerical analysis of an arbitrary 3D crack configuration. For cracks in nonhomogeneous media, it is shown that the equivalent J integral should contain, in the 2D deformation field, an additional surface integral (volume integral in the 3D case) to keep its welknown independence property when computed with different integration paths.*

RÉSUMÉ. *L'objet de ce papier est de donner les expressions du taux de restitution de l'énergie (ou de l'intégrale de Rice J) lorsqu'une fissure se propage dans un milieu dont les propriétés élastiques dépendent de l'espace. Pour les fissures en milieu non homogène on montre qu'il est nécessaire d'ajouter à la formulation habituelle une intégrale de surface, calculée à l'intérieur du contour d'intégration, pour garder la propriété d'indépendance du contour du taux de restitution de l'énergie.*

KEY WORDS : *inactive mechanics, energy release rate, J integral, nonhomogenous median.*

MOTS-CLÉS : *mécanique de la rupture, taux de restitution de l'énergie, matériaux non homogènes.*

1. Introduction

The access of techniques for calculating the energy release rate (or the J integral) to characterize a material's static crack extension under general loading condition is an old branch of fracture mechanics. In the literature, there exists an abundant contribution to this subject. As an example, mathematical surveys of this topic has been conducted by Ohtsuka [OHT 81], evaluations of the energy release rate by the virtual crack extension technique by Parks [PAR 74] (further exploitations were due to De Lorenzi [DEL 74]), J integral estimation with stiffness gradient method by Derbalian et al. [DER 88].

In the case of nonhomogeneous materials, a number of investigators have given a resurgence interest in the elastic interface problem, for which the characteristic oscillating stress singularity was early determined by Williams [WIL 59], and solutions for specific problems given by Cherepanov [CHER 62]. Review articles in this topic include those by Atkinson [ATK 75] assuming that the variation of elastic module is perpendicular to crack's growth direction, and more recently by Delale et al. [DEL 88] where the interfacial region of two bonded dissimilar homogeneous elastic half-planes was modeled by a very thin layer of nonhomogeneous material. Built on basic stress equilibrium equations, analytical solutions for the elastic stress intensity factors were derived for simple-shaped structures under simple loading conditions. But for arbitrary structures under general loading conditions, more sophisticated and more general calculation techniques are required. In particular, the versatile finite element method should be applied to 2D and 3D problems for the energy release rate of crack tip at different positions.

The work presented in this paper is motivated by the paper by Destuynder et al. [DES 81]. In their work, a Lagrangian method, called Θ method, was introduced to examine the variation of an arbitrary integral-defined quantity when its integrating domain undergoes a small perturbation. It has been shown in [DES 83] that such a method is particularly well amenable to fracture mechanics where in 2D deformation problems the J integral is calculated by a surface integration rather than by a line integration. The Θ method has over the last few years incorporated into the finite element system CASTEM 2000, developed at Mechanical and Technological Department of France Atomic Energy Commission (CEA/DMT). Its efficiency and advantages have been revealed by an number of applications to both elastic and elasto-plastic problems under mechanical or thermal loadings, in 2D as well as 3D homogeneous medias (See, for example, [SUO 92, BRO 91]). Recently, it was decided to apply the Θ method to cracks propagating in nonhomogeneous media, especially to study the situation where the crak tip is extremely close to, even located at, the interfacial region of two bonded dissimilar homogeneous bodies. The investigations were started by a review of the Θ method in nonhomogeneous material case without any assumption about the variation direction of material's characteristics with respect to the crack axis as made in [ATK 75], and without any simplifying assumption on the applied loads as considered in [DES 81]. After mathematical formulations, a set of numerical validations were performed for various crack configurations in both linear and non-linear elasticity with spatially varying elastic constitutive tensor and thermal expansion coefficient. In the present paper, only the mathematical formulation of the problem is presented. Example computations already performed will be given in an other forthcoming paper.

The main results obtained in this paper are :

1) for cracks propagating in nonhomogeneous materials, an analytical expression for the energy release rate under general loading conditions is derived and put in a form suitable for numerical analysis of arbitrary 3D crack configurations ;

2) it is shown that the equivalent J integral should contain, in 2D deformation field, an additional surface integral (volume integral in 3D case) to keep its well-known independence property when computed with different integration paths.

2. Problem formulation

In this paper we shall extend the Θ method introduced in [DES 81] for calculating the energy variation in nonhomogeneous material case. To examine the variation of the functional [a10] (See appendix), we now imagine an arbitrary infinitesimal perturbation, η, in the body's geometry, but with no change in its boundary $\partial\Omega$ (hence no change in the external surface tractions f). To be clear, assume that after this domain perturbation all physical quantities are designated with a superscript η. Therefore, Ω^η represents the changed body and M^η is the position vector of a material point in the body Ω^η, where material's thermal expansion coefficient and elastic constitutive tensor are designated by $\alpha(M^\eta)$ and $R(M^\eta)$, respectively. F^η and T^η are the body forces and temperature change in the body Ω^η, (σ^η, U^η) the corresponding stress and displacement fields defined by a system of equations like [a4] - [a5] :

$$\sigma^\eta = R(M^\eta)\,[\xi^\eta(U^\eta) - \alpha(M^\eta)\,T^\eta\,Id] \qquad \sigma^\eta \in \Sigma \qquad [1]$$

$$\int_{\Omega^\eta} Tr\left(\sigma^\eta\ \mathrm{grad}\,V\right) = \int_{\Gamma_f} f\,V + \int_{\Omega^\eta} F^\eta\ V \qquad \forall V \in \Psi \qquad [2]$$

where $\varepsilon^\eta(U^\eta) = \frac{1}{2}$ (grad U^η + grad $U^{\eta T}$) is the strain tensor with "grad" representing the gradient with respect to the coordinates of point M^η. Using (σ^η, U^η), the total potential energy of the body Ω^η, noted W^η, is calculated by :

$$W^\eta = -\frac{1}{2}\int_{\Omega^\eta} \mathrm{Tr}\left(\sigma^\eta \ \mathrm{grad}\, U^\eta\right) - \frac{1}{2}\int_{\Omega^\eta} \mathrm{Tr}\left(\sigma^\eta\right) \alpha\left(M^\eta\right) T^\eta \qquad [3]$$

We assume at present the existence of a *coordinate mapping function* $\mathcal{H}$ that maps the perturbed body Ω^η into the original body Ω in such a way that the outer boundary of body is Ω^η is mapped into the outer boundary of body Ω. This mapping is assumed one-to-one, so that a special point in each body corresponds to one and only one in the other body. Since the perturbation between the two bodies is sufficiently small, we can make the restriction on the mapping function $\mathcal{H}$ from Ω to Ω^η that a point M is mapped into M^η according to :

$$\mathcal{H} \rightarrow M^\eta = M + \eta\, \Theta(M) \qquad [4]$$

where $\Theta(M)$ is a displacement vector of class $W^{1,\infty}(\Omega)$. Using the mapping function [4] we can write :

$$\alpha(M^\eta) = \alpha(M + \eta\, \Theta) \approx \alpha(M) + \eta\, \nabla\alpha(M)\, \Theta \text{ and} \qquad [5]$$

$$R(M^\eta) = R(M + \eta\, \Theta) \approx R(M) + \eta\, \nabla R(M)\, \Theta \qquad [6]$$

By differentiating [4] with respect to the point M we get :

$$\nabla M^\eta = Id + \eta\, \nabla\Theta \qquad [7]$$

This leads to the Jacobian determinant for the transformation $\Omega \rightarrow \Omega^\eta$

$$\det(Jac) = \det(\nabla M^\eta) \sim 1 + \eta\, \mathrm{div}\Theta \qquad [8]$$

By multiplying [7] on both sides with [grad M (Id - η $\nabla\Theta$)] we obtain its inverse expression : grad M ~ Id - $\eta\nabla\Theta$. It follows that for arbitrary vectors (.) :

$$\mathrm{grad}\,(.) = \frac{\partial(.)}{\partial M^\eta} = \nabla(.)\ \mathrm{grad}\ M \approx \nabla(.)\ (Id - \eta\nabla\Theta) \qquad [9]$$

With the mapping $\mathcal{H}$ we can translate all physical quantities like the stress, strain and displacement from the perturbed configuration Ω^η into the original unperturbed configuration Ω. Let F^t and T^t be the images of F^η and T^η in the configuration Ω, and (σ^t, U^t) the images of fields (σ^η, U^η) in the same configuration, then using [5], [6], [8] and [9] we get from [1] and [2] :

$$\sigma^t = (R + \eta\, \nabla R\, \Theta)\,[\varepsilon(U^t) - \eta\, \varepsilon(U^t,\Theta) - \alpha\, T^t\, Id - \eta\, \nabla\alpha\, \Theta\, T^t\, Id]$$
$$\sigma^t \in \Psi \qquad [10]$$

$$\int_\Omega Tr\left[\sigma^t\, \nabla V\, (Id - \eta \nabla\Theta)\right](1 + \eta\, div\Theta) = \int_{\Gamma_f} f\, V + \int_\Omega F^t\, V\, (1 + \eta\, div\Theta)$$
$$\forall\, V \in \Psi \qquad [11]$$

where the strain tensors $\varepsilon(U^t)$ and $\varepsilon(U^t, \Theta)$ are defined by [a7] and [a8], respectively. Similarly, from [3] we get the image of the total potential energy W^η, noted W^t, in the configuration, giving :

$$W^t = -\frac{1}{2}\int_\Omega Tr\left[\sigma^t\, \nabla U^t\, (Id - \eta\nabla\Theta)\right]\left(1 + \eta\, div\Theta\right)$$
$$-\frac{1}{2}\int_\Omega Tr\left(\sigma^t\right)(\alpha + \eta\nabla\alpha\Theta)\, T^t\, (1 + \eta\, div\Theta) \qquad [12]$$

Destuynder et al. [DES 81] demonstrated that if η is sufficiently small, there exist :

$$F^t(M) \approx F + \eta\, \nabla F\, \Theta \text{ and } T^t(M) \approx T + \eta\, \nabla T\, \Theta \qquad [13]$$

$$\sigma^t(M) \approx \sigma + \eta\, \sigma^1 \text{ and } U^t(M) \approx U + \eta\, U^1 \qquad [14]$$

where (σ, U) are solutions of problem [a4] - [a5]. Substituting [13] and [14] into [10] - [11] and using the fact that each equality must be satisfied for all admissible quantity η, we get for $\forall\, V \in \Psi$:

$$\sigma^1 = R(M)\,[\varepsilon(U^1) - \varepsilon(U, \Theta) - \alpha\, \nabla T\, \Theta\, Id - \nabla\alpha\, \Theta\, T\, Id]$$
$$+ \nabla R(M)\, \Theta\, [\varepsilon(U) - \alpha\, T\, Id]$$
$$\sigma^1 \in \Sigma \qquad [15]$$

$$\int_{\Omega} \mathrm{Tr}\left(\sigma^{1}\, \nabla V\right) = \int_{\Omega} \mathrm{Tr}\,(\sigma \nabla V \nabla \Theta) - \int_{\Omega} \mathrm{Tr}\,(\sigma \nabla V)\, \mathrm{div}\Theta$$

$$+\int_{\Omega} F\, V\, \mathrm{div}\Theta + \int_{\Omega} \nabla F \Theta V \qquad [16]$$

Similarly, by substituting [13] and [14] into [12] we get the variation of the total potential energy W corresponding to a small perturbation η int the body's geometry :

$$G = -\frac{\partial W}{\partial \eta} = -\lim_{\eta \to 0} \frac{W^{t} - W}{\eta} = \frac{1}{2} \int_{\Omega} \mathrm{Tr}\left[\sigma\left(\nabla U^{1} - \nabla U\, \nabla \Theta\right)\right]$$

$$+\frac{1}{2} \int_{\Omega} \mathrm{Tr}\left(\sigma^{1}\, \nabla U\right) + \frac{1}{2} \int_{\Omega} \mathrm{Tr}(\sigma\, \nabla U)\, \mathrm{div}\Theta + \frac{1}{2} \int_{\Omega} \mathrm{Tr}\,(\sigma)\, \alpha\, \nabla T\, \Theta$$

$$+\frac{1}{2} \int_{\Omega} \mathrm{Tr}\,(\sigma) \nabla \alpha \Theta T + \frac{1}{2} \int_{\Omega} \mathrm{Tr}\left(\sigma^{1}\right) \alpha\, T$$

$$+\frac{1}{2} \int_{\Omega} \mathrm{Tr}\,(\sigma)\, \alpha\, T\, \mathrm{div}\Theta \qquad [17]$$

On the other hand, combining expressions [a4] and [15] and making use of [a6] we have the following equality :

$$\sigma\left[\varepsilon\left(U^{1}\right) - \varepsilon\,(U,\Theta)\right] = \left[\sigma^{1} - \nabla R\, \Theta\, (\varepsilon\,(U) - \alpha\, T\, \mathrm{Id})\right]$$

$$(\varepsilon\,(U) - \alpha\, T\, \mathrm{Id}) + \sigma\, [\alpha\, \nabla T\, \Theta\, \mathrm{Id} + \nabla \alpha\, \Theta\, T\, \mathrm{Id}]$$

Therefore :

$$\int_{\Omega} \mathrm{Tr}\left[\sigma\left(\nabla U^{1} - \nabla U\, \nabla \Theta\right)\right] =$$

$$\int_{\Omega} \mathrm{Tr}\left[\left(\sigma^{1} - \nabla R\, \Theta\, (\varepsilon\,(U) - \alpha\, T\, \mathrm{Id})\right) (\varepsilon\,(U) - \alpha\, T\, \mathrm{Id})\right]$$

$$+\int_{\Omega} \mathrm{Tr}\,(\sigma)\, (\alpha\, \nabla T\, \Theta + \nabla \alpha\, \Theta\, T)$$

This leads to write the energy variation [17] in an other form :

$$G = -\frac{\partial W}{\partial \eta} = \int_{\Omega} \mathrm{Tr}\left(\sigma^{1}\, \nabla U\right) - \frac{1}{2} \int_{\Omega} \mathrm{Tr}\left[\nabla R\, \Theta\, (\varepsilon(U) - \alpha\, T\, \mathrm{Id})\right.$$

$$\left(\varepsilon(U) - \alpha\, T\, Id\right)\Big] + \frac{1}{2}\int_{\Omega} Tr\left(\sigma\, \nabla U\right) div\Theta + \int_{\Omega} Tr\left(\sigma\right)\left(\alpha\, \nabla T\, \Theta + \nabla\alpha\, \Theta\, T\right)$$
$$+\frac{1}{2}\int_{\Omega} Tr\,(\sigma)\,\alpha\, T\, div\Theta \qquad [18]$$

In equilibrium state by putting in the equation [16] V = U [18] becomes :

$$G = G_{hom} + G_{add} \qquad [19]$$

$$\text{with } G_{hom} = \int_{\Omega} Tr\,(\sigma\, \nabla U\, \nabla\Theta) - \frac{1}{2}\int_{\Omega} Tr\left[\sigma\,(\nabla U - \alpha\, T\, Id)\right] div\Theta$$
$$+\int_{\Omega} Tr\,(\sigma)\,\alpha\, \nabla T\, \Theta + \int_{\Omega} F\, U\, div\Theta + \int_{\Omega} \nabla F\, \Theta\, U \qquad [20]$$

$$\text{and } G_{add} = \int_{\Omega} Tr\,(\sigma)\, \nabla\alpha\, \Theta\, T$$
$$-\frac{1}{2}\int_{\Omega} Tr\left[\nabla R\, \Theta(\nabla U - \alpha\, T\, Id)(\nabla U - \alpha\, T\, Id)\right] \qquad [21]$$

where G_{hom} corresponds to an energy variation when the material is homogeneous and G_{add} to an additional term when the material is nonhomogeneous. In the absence of body forces and thermal loadings (F = 0 and T = 0), and in the case of homogeneous materials ($\nabla\alpha = 0$ and $\nabla R = 0$), we have :

$$G = \int_{\Omega} Tr\,(\sigma\, \nabla U\, \nabla\Theta) - \frac{1}{2}\int_{\Omega} Tr\,(\sigma\, \nabla U)$$

This is the formula given in the paper by Destuynder et al. [DES 81] (a similar expression has also been derived by De Lorenzi [DEL 82]). However, the present formulations are more general and cover a wider mechanical problems.

3. Situations in fracture mechanics

The formula [19] - [21] present a general 3D analytical expression for the energy variation, available for geometry perturbation of all kinds. It may be well applied to fracture mechanics for calculating the energy release rate. In such a case, the variation in body's geometry is specified as to be a small increase in the crack length. For clarity, we limit here our analysis to a 2D linear elastic body containing a straight edge crack of lenght ℓ shown in figure 1. In the expressions [20] and [21], the displacement vector Θ is a mapping function which maps the body containing the crack into a body with a slightly increased crack length. Until now we have not been concerned with the form of this mapping. Theoretically, there are infinitely many forms of the displacement vector Θ which are all available for the formulas [20] and [21]. However, of specific meaning for fracture mechanics is the scheme where Θ varies only over a small band region as adopted by Destuynder et al. [DES 83], and briefly described below.

For simplicity, one supposes that the crack flat surfaces are parallel to the X-axis as shown in figure 2. Note however that this scheme is also available for 3D crack configurations and thin shell structures as well with a crack growing off at an angle to the X-axis. To map the body in figure 1 into a body with a increased crack length, the body is divided by two contours Γ_1 and Γ_2 into three parts, A, B and C. Let the vector $\Theta \equiv (0, 0)$ in region C, $\Theta \equiv (1, 0)$ in region A, and let Θ vary from (1, 0) to (0, 0) in the band B. Hence the vector Θ represents the direction in which the crack length will increas.

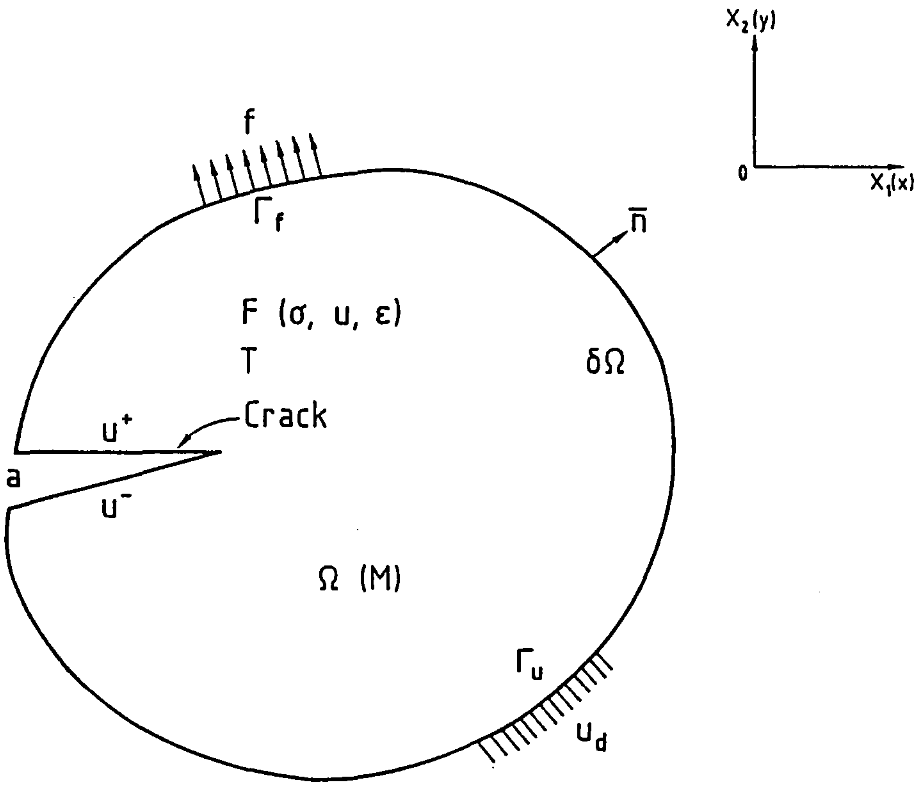

Figure 1. *Typical modeling for mapping function Θ*

Since the vector Θ in fracture mechanics is taken to be constant in almost all parts of the body Ω, varying only in a small band region, expression [20] is simplified by using the divergence formula. That gives :

$$\int_\Omega F\ U\ \mathrm{div}\Theta + \int_\Omega \nabla F\ \Theta\ U = -\int_\Omega F\ \nabla U\ \Theta + \int_{\partial\Omega} F\ U\ \Theta\ n$$

where the last term is zero since $\Theta \equiv (0, 0)$ on the boundary $\partial\Omega$ of the body Ω. This leads to obtain the following expression for the energy release rate :

$$G = \int_\Omega \mathrm{Tr}\ (\sigma\ \nabla U\ \nabla\Theta) - \frac{1}{2}\int_\Omega \mathrm{Tr}\left[\sigma\ (\nabla U - \alpha\ T\ \mathrm{Id})\right] \mathrm{div}\Theta$$
$$+\int_\Omega \mathrm{Tr}\ (\sigma)\ \alpha\ \nabla T\ \Theta - \int_\Omega F\ \nabla U\ \Theta + \int_\Omega \mathrm{Tr}\ (\sigma)\ \nabla\alpha\ \Theta\ T$$
$$-\frac{1}{2}\int_\Omega \mathrm{Tr}\left[\nabla R\ \Theta\ (\nabla U - \alpha\ T\ \mathrm{Id})\ (\nabla U - \alpha\ T\ \mathrm{Id})\right] \qquad [22]$$

Consequently, with the mapping function Θ described above, the first and second volume integrals in [22] are, for plane strain/stress problems, computed only over the band B, because div $\Theta \equiv \nabla\Theta \equiv 0$ in the regions (A $\cup$ C). However, for axisymmetric conditions these two integrations should be performed over both regions A and B. In fact, in a cylindrical coordinate system r, θ, z where the z axis in the axis of symmetry and the r axis is the radius axis, the gradient and divergence of the vector Θ with respect to the coordinates of a point M are :

$$\nabla\Theta = \begin{bmatrix} \frac{\partial\Theta_r}{\partial r} & \frac{\partial\Theta_r}{\partial z} & 0 \\ \frac{\partial\Theta_z}{\partial r} & \frac{\partial\Theta_z}{\partial z} & 0 \\ 0 & 0 & \frac{\Theta_r}{r} \end{bmatrix} \text{ and } \mathrm{div}\Theta = \frac{\partial\Theta_r}{\partial r} + \frac{\partial\Theta_z}{\partial z} + \frac{\Theta_r}{r}$$

which are not zero in the region A due to the term $\frac{\Theta_r}{r}$. In all cases, the 3th - 6th terms in [22] have always to be integrated over both regions A and B. As a special case the region A can be shrunk into one point - the crack tip. In this case all volume integrals in [22] has the same integration region - a small volume B around the crack tip.

4. J integral for cracks in nonhomogeneous media

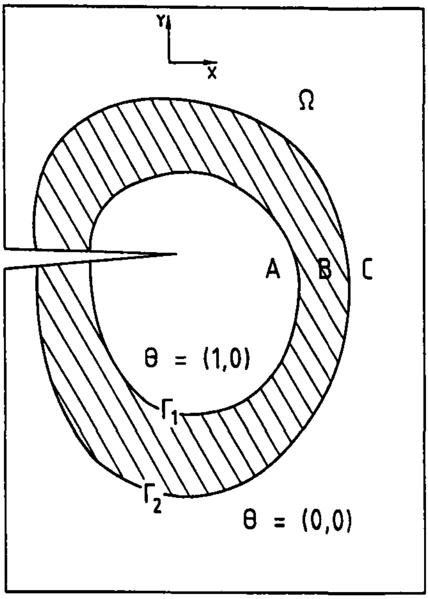

Figure 2. *J integral in 2D deformation field. s is an arbitrary curve surrounding crack typ, n an unit outward vector perpendicular to s, v the surface enclosed by s and d the complementary domain of v in the body* Ω

Originally, the line independent J integral [RIC 68] was introduced for an approximate analysis of stress concentration by cracks, and subsequently used by a number of investigators as a parameter to characterize material's static crack extension under elasto-plastic condition. For cracks propagating in a 2D elastic homogeneous media with body and termal loadings, the well-known expression of the J integral in terms of stress σ and displacement U, is :

$$J_{hom} = \frac{1}{2}\int_s \mathrm{Tr}\left[\sigma\left(\nabla U - \alpha\, T\, \mathrm{Id}\right)\right]n_1 - \int_s n\,\sigma\,\frac{\partial U}{\partial X} + \int_v \mathrm{Tr}(\sigma)\,\alpha\,\frac{\partial T}{\partial X} - \int_v F\,\frac{\partial U}{\partial X} \qquad [23]$$

where s is a curve surrounding the crack tip, v the surface enclosed by s and n an unit outward vector perpendicular to s. The curve s starts from the lower flat crack surface and continues along the path s to the upper flat surface as depicted in figure 2.

In this section we will show that in nonhomogeneous material case the calculation of the energy release rate by [22] amounts to the calculation of a line integral of the J type. For this purpose, let d be the complementary domain of v in the body Ω (i.e. $\Omega = d \cup v$) so that integrations over whole body Ω is the sum of integrations over d and v, namely :

$$\begin{aligned} G = & \int_v Tr\,(\sigma\, \nabla U\, \nabla\Theta) - \frac{1}{2}\int_v Tr\left[\sigma\,(\nabla U - \alpha\, T\, Id)\right] div\Theta \\ & + \int_v Tr\,(\sigma)\, \alpha\, \nabla T\, \Theta - \int_v F\, \nabla U\, \Theta + \int_v Tr\,(\sigma)\, \nabla\alpha\, \Theta\, T \\ & - \frac{1}{2}\int_v Tr\left[\nabla R\, \Theta\, (\nabla U - \alpha\, T\, Id)\,(\nabla U - \alpha\, T\, Id)\right] \\ & + \int_d Tr\,(\sigma\, \nabla U\, \nabla\Theta) - \frac{1}{2}\int_d Tr\left[\sigma\,(\nabla U - \alpha\, T\, Id)\right] div\Theta \\ & + \int_d Tr\,(\sigma)\, \alpha\, \nabla T\, \Theta - \int_d F\, \nabla U\, \Theta + \int_d Tr\,(\sigma)\, \nabla\alpha\, \Theta\, T \\ & - \frac{1}{2}\int_d Tr\left[\nabla R\, \Theta\, (\nabla U - \alpha\, T\, Id)\,(\nabla U - \alpha\, T\, Id)\right] \end{aligned} \qquad [24]$$

Assume that the curve s (or volume v) is completely located in the region where Θ is a constant unit vector (i.e. in the region A in figure 1). Then the first and second integrations in [24] are zero because $\nabla\Theta = div\ \Theta = 0$ for all points in v. Further, using the divergence theorem it follows :

$$\int_d Tr\,(\sigma\, \nabla U\, \nabla\Theta) = \int_{\partial d} N\, \sigma\, \nabla U\, \Theta - \int_d div\,(\sigma\, \nabla U)\, \Theta \text{ and}$$

$$\int_d Tr\left[\sigma\,(\nabla U - \alpha\, T\, Id)\right] div\Theta = \int_{\partial d} Tr\left[\sigma\,(\nabla U - \alpha\, T\, Id)\right] N\, \Theta - \int_d \nabla\left\{Tr\left[\sigma\,(\nabla U - \alpha\, T\, Id)\right]\right\} \Theta$$

where ∂d is the boundary of the band d (figure 2), including the boundary $\partial\Omega$ of the body Ω, the curve s as well as two segments on the flat crack surfaces, N an unit vector perpendicular to ∂d (figure 2). By substituting the two equalities above into [24], the energy release rate is transformed into :

$$\begin{aligned} G = & \int_{\partial d} N\, \sigma\, \nabla U\, \Theta - \frac{1}{2}\int_{\partial d} Tr\left[\sigma\,(\nabla U - \alpha\, T\, Id)\right] N\, \Theta \\ & + \int_v Tr\,(\sigma)\, \alpha\, \nabla T\, \Theta - \int_v F\, \nabla U\, \Theta + \int_v Tr\,(\sigma)\, \nabla\alpha\, \Theta\, T \\ & - \frac{1}{2}\int_v Tr\left[\nabla R\, \Theta\, (\nabla U - \alpha\, T\, Id)\,(\nabla U - \alpha\, T\, Id)\right] + \int_d \xi\, \Theta \end{aligned} \qquad [25]$$

with :

$$\xi = \frac{1}{2}\nabla\left\{\mathrm{Tr}\left[\sigma\,(\nabla U - \alpha\, T\, Id)\right]\right\} - \mathrm{div}(\sigma\,\nabla U) - F\,\nabla U + \mathrm{Tr}\,(\sigma)\,\alpha\,\nabla T$$
$$+\mathrm{Tr}\,(\sigma)\,\nabla\alpha\, T - \frac{1}{2}\mathrm{Tr}\left[\nabla R\,(\nabla U - \alpha\, T\, Id)\,(\nabla U - \alpha\, T\, Id)\right] \quad [26]$$

We demonstrate now that [26] is zero for all points in the body Ω. In fact, under the summation convention for repeated indices, we can write :

$$\mathrm{Tr}\left[\sigma\,(\nabla U - \alpha\, T\, Id)\right] = \sigma_{ij}\left(U_{i,j} - \alpha\, T\, Id\right)$$

Therefore, its K^{th} component of gradient is expressed as :

$$\frac{1}{2}\nabla\left\{\mathrm{Tr}[\sigma\,(\nabla U - \alpha\, T\, Id)]\right\} = \frac{1}{2}\Big[\sigma_{ij,k}\, U_{j,i} + \sigma_{ij}\, U_{j,ki}$$
$$-\sigma_{ij,k}\,\alpha\, T\, Id - \sigma_{ij}\,\alpha\, T_{,k}\, Id - \sigma_{ij}\,\alpha_{,k}\, T\, Id\Big] \quad [27]$$

Similarly, the K^{th} component of all other terms in [26] are :

$$-\mathrm{div}(\sigma\,\nabla U) - F\,\nabla U = -\sigma_{ij,i}\, U_{j,k} - \sigma_{ij}\, U_{j,ki} - F_j\, U_{j,k} \quad [28]$$

$$\mathrm{Tr}(\sigma)\,\alpha\,\nabla T\, Id + \mathrm{Tr}(\sigma)\,\nabla\alpha\, T\, Id = \sigma_{ij}\,\alpha\, T_{,k}\, Id + \sigma_{ij}\,\alpha_{,k}\, T\, Id \quad [29]$$

and :

$$-\frac{1}{2}\mathrm{Tr}\left[\nabla R\,(\nabla U - \alpha\, T\, Id)\,(\nabla U - \alpha\, T\, Id)\right] =$$
$$-\frac{1}{2}R_{ijmn,k}\left[\left(U_{m,n} - \alpha\, T\, Id\right)\left(U_{j,i} + \alpha\, T\, Id\right)\right] \quad [30]$$

By substituting [27] - [30] into [26] and using the stress equilibrium equation $\sigma_{ij,i} + F_j = 0$, it results :

$$\xi = \frac{R_{ijmn}}{2}\left[\sigma_{ij,k}\, U_{j,i} - \sigma_{ij}\, U_{j,ki} - \sigma_{ij,k}\,\alpha\, T\, Id + \sigma_{ij}\left(\alpha\, T_{,k} + \alpha_{,k}\, T\right) Id\right]$$
$$-\frac{R_{ijmn,k}}{2}\left[\left(U_{m,n} - \alpha\, T\, Id\right)\left(U_{j,i} + \alpha\, T\, Id\right)\right] \quad [31]$$

However for linear elastic materials, there exists the following stress-strain relationship :

$$\sigma_{ij} = R_{ijmn} \left(U_{m,n} - \alpha \, T \, Id\right) \qquad [32]$$

Hence the K^{th} component of gradient of the stress σ_{ij} is :

$$\sigma_{ij,k} = R_{ijmn} \left(U_{m,nk} - \alpha \, T_{,k} \, Id - \alpha_{,k} \, T \, Id\right)$$
$$+R_{ijmn,k} \left(U_{m,n} - \alpha \, T \, Id\right) \qquad [33]$$

Using [32] and [33], the expression [31] is transformed into :

$$\xi = \frac{R_{ijmn}}{2} \left[\left(U_{m,nk} \, U_{j,i} - U_{m,n} \, U_{j,ki}\right) + \left(\alpha \, T_{,k} + \alpha_{,k} \, T\right)\right.$$
$$\left.\left(U_{m,n} - U_{j,i}\right) Id + \alpha \, T \, Id \left(U_{j,ki} - U_{m,nk}\right)\right] \qquad [34]$$

Since $R_{ijmn} = R_{nmji}$ (symmetry property of constitutive tensor R), the value of expression [34] is obviously zero because on the right hand side of this expression index n is replaced by i and m by j. For this reason, the last volume integration in [25] cancells. Further, assuming that N Θ = 0 on the two crak faces and knowing that the vector Θ is zero on the body's boundary $\partial\Omega$, the line integrals [25] over ∂d are reduced to a line integrals over s. Note that on the curve s, N is an unit inward vector perpendicular to s. Hence by putting n = -N expression [25] becomes :

$$G = \frac{1}{2} \int_s Tr \left[\sigma \left(\nabla U - \alpha \, T \, Id\right)\right] n \, \Theta - \int_s n \, \sigma \, \nabla U \, \Theta$$
$$+\int_v Tr\,(\sigma)\, \alpha \, \nabla T \, \Theta - \int_v F \, \nabla U \, \Theta + \int_v Tr\,(\sigma) \, \nabla\alpha \, \Theta \, T$$
$$-\frac{1}{2} \int_v Tr \left[\nabla R \, \Theta \left(\nabla U - \alpha \, T \, Id\right) \left(\nabla U - \alpha \, T \, Id\right)\right] \qquad [35]$$

As it was assumed above, the curve s is located in the region where Θ is a constant unit vector. In 2D deformation problems, if the crack plane is parallel to the X axis as illustrated in figure 2, we take Θ = (1, 0) for all points in v. In this case, the integral [35] is written as :

$$G = J = J_{hom} + J_{add} \qquad [36]$$

with J_{hom} being defined by [23] and :

$$J_{add} = \int_v Tr(\sigma)\frac{\partial \alpha}{\partial X}T - \frac{1}{2}\int_v Tr\left[\frac{\partial R}{\partial X}(\nabla U - \alpha T\, Id)(\nabla U - \alpha T\, Id)\right] \quad [37]$$

In [36] J_{hom} corresponds to the J integral when the material is homogeneous (formula [23]) and J_{add} an additional surface integration (volume integration in 3D case) when the material is nonhomogeneous.

The demonstration of the independency of J [36] with respect to its integration paths s is classical. For this purpose, we consider any closed curve s* enclosing an area v*. An application of Green and Gauss' formulas enables us to transform the line integral [36] into a surface integral over the area v*, giving :

$$J(s^*) = \int_{v^*} \frac{1}{2}\frac{\partial}{\partial X} Tr\left[\sigma\,(\nabla U - \alpha\, T\, Id)\right] - \int_{v^*} \sigma\, \frac{\partial(\nabla U)}{\partial X}$$
$$-\int_{v^*}\left(F_j + \frac{\partial \sigma_{ij}}{\partial X_j}\right)\frac{\partial U_j}{\partial X} + \int_{v^*}\sigma_{ii}\left(\alpha\,\frac{\partial T}{\partial X} + \frac{\partial \alpha}{\partial X}\,T\right)$$
$$-\frac{1}{2}\int_{v^*} Tr\left[\frac{\partial R}{\partial X}\,(\nabla U - \alpha\, T\, Id)\,(\nabla U - \alpha\, T\, Id)\right] \quad [38]$$

But :

$$\frac{1}{2}\frac{\partial}{\partial X} Tr\left[\sigma\,(\nabla U - \alpha\, T\, Id)\right] = \frac{1}{2}\frac{\partial}{\partial X} Tr\left[R\,(\nabla U - \alpha\, T\, Id)\right.$$
$$\left.(\nabla U - \alpha\, T\, Id)\right] = \frac{1}{2}\, Tr\left[\frac{\partial R}{\partial X}\,(\nabla U - \alpha\, T\, Id)\,(\nabla U - \alpha\, T\, Id)\right]$$
$$+\sigma\left[\frac{\partial(\nabla U)}{\partial X} - \alpha\,\frac{\partial T}{\partial X}\, Id - \frac{\partial \alpha}{\partial X}\, T\, Id\right]$$

By substituting this equality into [38] and using the stress equilibrium equation $F_i + \sigma_{ij,j} = 0$, it follows that J is zero for any closed curve. Knowing that on the portions of the path along the flat crack surfaces, we have $\sigma\bar{n} = 0$ (crack faces are assumed to be free of tractions and $n_i\, dC = 0$ (in the case where crack faces are parallel to the X-axis), it results that J along a contesclockwise curve and J along a clockwise curve sum to zero. Hence, in nonhomogeneous material case, J has also the same value for all open paths surrounding the crack tip as in homogeneous material case.

Five comments shall be made about some expressions obtained previously :

1) In the special case where the variation of material's elastic constitutive tensor and thermal expansion coefficient are parallel to the crack plane (hence $\partial R/\partial X = 0$ and $\partial \alpha/\partial X = 0$), J_{add} (or G_{add}) is zero (Cf. [37] or [21]). In such a case, the J integral (or the energy release rate) can be calculated for cracks exactly as in a homogeneous material.

2) Formula [19] is valid for general isotropic as well as anisotropic materials in 2D as well as 3D crack configurations. For simplicity, it is deduced assuming linear elasticity. Using the same calculation technique, a similar formula can easily be derived for nonhomogeneous materials following the deformation theory of plasticity.

3) Because [36] is obtained by setting $\Theta = (1, 0)$, this line integral by itself is only valid for the specialized case where the crack plane is lined up with a co-ordinate axis and only if this crack continue to extend in its original plane. Formula [35] is, however, more general being valid even in the case where the crack grows off at an angle to crack's original plane.

4) If a numerical analysis is, for instance, carried out using the finite element method, the volume integral formulation for the energy release rate, [19] - [21], should be preferred over the line integral formulation [35] or [36]. The former is a natural extension of the volume integrals already carried out in a finite element analysis, while the latter give rise to additional encumbrances in defining the adequate contour over which to integrate and in performing the actual path integration. This point is particularly true when higher order finite elements are used in 3D crack configurations where J integral technique requires the calculation of surface integral around a particular point on the crack profile. However, this is general difficult to achieve in the finite element model.

5) In order to calculate the energy release rate (or the J integral) in the case of two bonded dissimilar homogeneous bodies, Delale et al. [DEL 88] modeled the interfacial region by a very thin layer of nonhomogeneous material over which material's characteristics like α and R were supposed to vary continually. The Θ method previously described may be applied to this problem as well, for which the derivatives $\nabla\alpha$ and ∇R in [21] (or $\partial \alpha/\partial X$ and $\partial R/\partial X$ in [37]) are either zero or non-zero depending on whether a material point is located in this thin band. This means that under the assumption of Delate et al., the volume integrations [21] or [37] for the additional terms G_{add} or J_{add} are performed only over a thin layer of nonhomogeneous material. Clearly, this allows us to obtain only an approximate estimation of the energy release rate (or the J integral). It is hoped that the Θ method can not only resolve problems of this kind in such a manner, but provides a more sophisticated tool for calculating the energy release rate. In fact, since we have the analytical expression for G_{add} or J_{add}, using the mathematical theory of distribution we can accurately compute the value of G_{add} or J_{add} when the width of the thin nonhomogeneous material band tends to zero. Along this line, the real energy release rate (or the J integral) for the problem of two bonded dissimilar

homogeneous bodies could be derived. We would like to give a detailed discussion of this problem in the second past of the paper.

5. Summary

For cracks propagating in a nonhomogeneous medium, an analytical expression for the energy release rate has been derived from an continuum mechanics viewpoint for arbitrary 3D crack configurations under general loading conditions. Similar investigations were performed in [DES 81] where the material was assumed to be homogeneous and the thermal and body forces were excluded, and more early in the paper by Atkinson [ATK 75] where the variation of elastic modulus was assumed to be perpendicular to the crack's growth direction, therefore valid only for specific fracture problems. The formulations presented in the paper have more general validity.

The resulting expression for the energy release rate can effectively be used in finite element applications since it can take full advantage of the numerical integration already available in a finite element program. Applications of the method recently performed concern a cracked compact specimem with 1) varying material characteristics over whole body, and 2) two bonded dissimilar homogeneous bodies. Numerical results for these applications and comparison with independent methods will be given in an other forthcoming paper.

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Appendix

We consider the situation where an elastic body Ω with a boundary $\partial\Omega$ in $\mathbb{R}^3$ occupies, in its non-deformed state, a bounded open domain with local Lipschitzian property. In loaded state, let (σ, U) represent the stress and displacement solutions and ∇U ($= \partial U_i/\partial X_j = U_{i,j}$) the derivative of displacement components U_i with respect to the coordinates X_i of a point M in a Cartesian coordinate system. The trace manipulation of an endomorphism C (corresponding to a matrix C_{ij}) is written as :

$$Tr(C) = \sum C_{ii} = C_{ii}$$

where, for brevity, the summation convention for repeated indices is used. Similarly, we write :

$$Tr(\sigma \nabla U) = \sum \sigma_{ij} U_{j,i} = \sigma_{ij} U_{j,i} \qquad [a1]$$

For the reason of simplicity, we shall write all integrals in the form :

$$\int_\Omega Tr(C)\, d\Omega = \int_\Omega Tr(C)$$

Let Γ_f and Γ_u be two portions of the body's boundary (with $\Gamma_f \cap \Gamma_u = \varnothing$) where a surface traction f and a specified displacement condition U_d are prescribed. Let the body Ω be also subjected to a body force F of class $L^2(\Omega)$ and a system of thermal loads with T representing temperature change between the current studied state and that of reference. Given an open subset $\mathcal{O}$ in $\mathbb{R}^3$, let Σ and Ψ represent the statically admissible stress space and kinematically admissible displacement space, respectively :

$$\Sigma(\mathcal{O}) = \left\{\sigma = \left(\sigma_{ij}\right) \in \left(L^2(\Omega)^9\right), \sigma_{ij} = \sigma_{ij} \text{ in } \Omega\right\} \qquad [a2]$$

$$\Psi(\mathcal{O}) = \left\{V = \left(V_i\right) \in \left(H^1(\Omega)^3\right), V = U_d \text{ on } \Gamma_u\right\} \qquad [a3]$$

Then in virtue of Hellinger - Reissner variational principle [ODE 76 and WASH 68], (σ, U) are sure and unique solutions in Σ x Ψ of linear Hooke's law and virtual work equation :

$$\sigma = R(M)\, [\varepsilon(U) - \alpha(M)\, T\, Id] \quad \sigma \in \Sigma \qquad [a4]$$

$$\int_{\Omega} \mathrm{Tr}\,(\sigma \nabla V) = \int_{\Gamma_f} f\ V + \int_{\Omega} F\ V \quad \forall V \in \Psi \qquad [a5]$$

In the expressions above, α is material's thermal expansion coefficient, Id a 3 x 3 unit tensor (2 x 2 in 2D case), R material's elastic constitutive tensor belonging to $C^{\infty}(\Omega)$ and satisfying the property of symmetry as follows :

$$R_{ijkl} = R_{jikl} = R_{lkji} = R_{ijlk} \qquad [a6]$$

In nonhomogeneous material case, α and R are all functions of point M. Assuming a small transformation, the strain tensor $\varepsilon(U)$ is linearized as :

$$\varepsilon(U) = \frac{1}{2}\left(\nabla U + \nabla U^{T}\right) \qquad [a7]$$

where T represents the transpose of matrix. In this paper we define a strain tensor $\varepsilon(U, V)$ associated with two independent displacements, U and V, as :

$$\varepsilon(U, V) = \frac{1}{2}\left(\nabla U\ \nabla V + \nabla V^{T}\ \nabla U^{T}\right) \qquad [a8]$$

Equation [a5] is written in the functional form. Using divergence and Green's formula, one can deduce the local equilibrium equations of stress in quasi static problems and corresponding boundary conditions :

$$\mathrm{div}\ \sigma + F = 0 \text{ in } \Omega \text{ and } \sigma\ n = f \text{ on } \Gamma_f \qquad [a9]$$

where $n = (n_1, n_2, n_3)$ denotes an unit outward vector perpendicular to the body's boundary $\partial\Omega$.

With the loads mentioned above, the total strain energy W in equilibrium state is :

$$W = -\frac{1}{2}\int_{\Omega} \mathrm{Tr}\,(\sigma\ \nabla U) - \frac{1}{2}\int_{\Omega} \mathrm{Tr}\,(\sigma)\ \alpha\ T \qquad [a10]$$

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