A Numerical Stability Study on Truss Structures

Yuan-yao Qian* — Jean-Louis Batoz

Division MNM, Laboratoire LG2MS, URA-CNRS 150
BP 649, Université de technologie de Compiègne
60206 Compiègne, France

** To whom correspondence should be addressed. And present address :*
Département de Génie civil, université Laval
Québec, Canada G1K 7P4

ABSTRACT. *This paper, which presents a numerical study on non-linear stability problems of truss structures, concentrates on three main parts. First, the derivative of the tangential stiffness matrix is used to give the stability analysis with direct calculation of the critical points and the branch-switching function. And then a quadratically convergent path-following algorithm is obtained. Compared with the classical arc-length methods, it is more efficient in fast convergence and saving CPU time. Finally it studies four geometrically non-linear problems in details.*

RÉSUMÉ. *Cet article, qui présente une étude numérique des problèmes de stabilité non linéaire des structures en treillis, traite de trois parties principales. Tout d'abord, la dérivée de la matrice de rigidité tangentielle est utilisée pour effectuer l'analyse de stabilité avec le calcul direct des points critiques et la fonction "branch-switching". Ensuite un algorithme quadratiquement convergent est présenté pour suivre la courbe charge-déplacements de façon continue. Comparativement aux méthodes de longueur d'arc classiques, celle-ci converge plus rapidement et utilise moins de temps de calcul. Finalement, quatre problèmes géométriquement non linéaires seront étudiés en détail.*

KEY WORDS : *bifurcation, critical point search, derivative of stiffness, higher order prediction, path-follow algorithm, stability, truss structures.*
MOTS-CLÉS : *algorithme de continuation, bifurcation, dérivée de rigidité, prédiction d'ordre élevé, recherche des points critiques, stabilité, structures en treillis.*

Revue européenne des éléments finis. Volume 1 - n° 4/1992, pages 461 à 479

1. Introduction

The stability problems have been long researched in engineering mechanics. With the growing interests in non-linear behavior of the engineering structures, the stability study becomes more important.

One of the main topic, which has been widely investigated by many authors[e. g. AL89, KQ91, MU77, SP85, TH73], is the stability conditions with the classification of critical stability points. By the existing criteria, one can theoretically say that it is possible to determine all the stable and unstable equilibrium states of non-linear structures. In fact, many research activities have been reported on this topic[e. g. AB78, BR80, EC83, RI72, RI79]. However, numerically it is difficult to approach an arbitrary critical state by the well-known arc-length method. This numerical difficulty is another main topic in stability study.

To trace the non-linear response by finite element methods exist there three basic algorithms. They are the methods with classical load increment control, with incremental displacement control[BATO79] and with the incremental arc-length control[e. g. CR81, FR84, RI79, WEM71]. The first two methods fail when more than one load level exists for a given displacement or, in other words, when there exists a turn back of displacement response during loading procedure. Since the algorithm of arc-length methods can pass every turning point very easily, it receives a considerable attention and becomes the most commonly used algorithm in finite element analysis of non-linear structures in recent years, see e. g. Bergan [BE78], Crisfield [CR81, CR86], Riks [RI79] and Wempner [WEM71] et al. The new review on solution procedure can be found in [CR91].

Unfortunately, not all the non-linear response paths follow the primary post critical branches. So that, a branch-switching function which is difficult to be supplied by the arc-length method is required at every bifucation point. It is noted that most of the currently employed methods used for the stability analysis of nonlinear structures rely on the inspection of determinant of tangential stiffness matrix[e. g. AB78, BR80, EC83, RA81, RI84, WA88]. And the critical point is calculated by a series of reduced incremental steps. However, in some mathematical literature, a constraint equation which characterizes the presence of critical stability point is appended to the solution process of non-linear equations[MO80, SE79, WER84] in order to obtain the limit or bifurcation point with needed precision. With almost the same idea, Wriggers et al [WR88, WR90] have developed an extended system in finite element formulation for the purpose of direct calculation of limit or bifurcation points.

The paper reports a study on the non-linear stability analysis of truss structures. It starts by extracting some basic equations and ideas from earlier work by many other authors. A special attention is paid to the determination of critical states and the post critical behavor of unstable structures. The commonly used stability theories, see e. g. [AL89, KQ91, MU77, TH73], are adopted through the constraints to the solution processes of critical point searches. Reformulation of some of the previous expressions[WR88, WR90] is developed and simplified for truss structures. It shows such a stability analysis of non-linear equations that the pre- or post-critical state is reached step by step through the path-following, and the critical state is approched in one step from the starting state which is either pre- or post-critical. Moreover, during the whole procedure of corrective iteration for critical point calculation the approximation of critical eigenvector remains unit which results in a reduction of iteration number. Numerical examples presented in this paper demonstrate the efficiency of this newly contributed unitary technique.

Since the derivative of tangential stiffness matrix is obtained and used in the stability investigation, it permits us to form a quadratically convergent path-following algorithm which includes the 2nd order approximation of the non-lineaer solution. On this method, there have already existed some studies, see e. g. [ER91, WA91]. The paper combines this advantage of high order prediction with the algorithm for non-linear stability analysis. And it suggests and compares two kinds of incremental arc-length which have different kinds of performance in practical computation for non-linear analysis.

The paper also carefully gives a numerical study on some non-linear stability problems of 2D and 3D truss structures. It presents a whole response of studied structures for either pre- or post-critical behavior. The branch-switching function is used in every approched critical state to switch on the considered secondary branch during the path-following procedure. The final results on the response curves are in some interesting or maybe fantastic shapes.

2. Finite Element Formulation and Critical Point Search

2.1. *Finite element formulation*

Following any typical finite element formulation[BATH82, ZI88], the Green-Lagrange strain vector could be expressed as

$$\varepsilon = [\, \mathbf{B}_l + \frac{1}{2}\mathbf{B}_{nl}(\mathbf{v}) \,] \cdot \mathbf{v} \tag{1}$$

with

$$\delta\varepsilon = [\, \mathbf{B}_l + \mathbf{B}_{nl}(\mathbf{v}) \,] \cdot \delta\mathbf{v} \;. \tag{2}$$

Herein $\mathbf{B}_l \cdot \mathbf{v}$ and $\mathbf{B}_{nl}(\mathbf{v}) \cdot \mathbf{v}$ are the linear and non-linear parts of the strain vector respectively. And the second Piola-Kirchhoff stress vector is given by its incremental form

$$\delta S = \mathbf{D} \cdot \delta\varepsilon = \mathbf{D} \cdot [\, \mathbf{B}_l + \mathbf{B}_{nl}(\mathbf{v}) \,] \cdot \delta\mathbf{v} \;. \tag{3}$$

Then, one can arrive at the principle of virtual work

$$\int_{\Omega^0} S^T \cdot \delta\varepsilon \cdot d\Omega^0 = \delta W \qquad [4]$$

with δW as the virtual work undertaken by the external forces. This obtained principle is in Total-Lagrange description. The derivative of equation [4] shows the basic incremental equation for non-linear finite element analysis

$$K_T \cdot \delta v = \delta Q \, , \qquad [5]$$

where

$$K_T = \int_{\Omega^0} \left\{ [B_l + B_{nl}(v)]^T \cdot D \cdot [B_l + B_{nl}(v)] \right\} \cdot d\Omega^0 + \int_{\Omega^0} G^T \cdot S \cdot G \cdot d\Omega^0 \qquad [6]$$

is the tangential stiffness matrix and Q is the vector of external loads. In equation [6], the matrix G is defined as a connection matrix.

2.2. *Stability conditions and the classification of critical points*

For static problems, the most commonly used stability criterion is to inspect the tangential stiffness matrix K_T whether it has some negative eigenvalues or not[KQ91, MU77]. The criterion states that the structure is

stable, if and only if all the eigenvalues of its tangential stiffness matrix are positive;

critical stable, if and only if its tangential stiffness matrix doesn't have any negative eigenvalue and it has at least one zero eigenvalue;

unstable, if its tangential stiffness matrix has some negative eigenvalues.

According to this criterion, the critical stability points are characterized by the following equation

$$\det(K_T) = 0 \qquad [7]$$

or

$$K_T \cdot \Phi = 0 \, . \qquad [8]$$

Herein Φ is noted as the critical eigenvector of K_T. These two equations are used to constrain the solution to approach the characterized critical state late.

There exist two different kinds of the critical points. One is the limit points, and the other is the bifurcation points. The classification of critical points is not the main topic of this paper. Some previous researches have been dealt with this definition, see e. g. [SP85, WR90]. Here, we use the standard classification as *the critical point is a limit (turning) point, if*

$$\Phi^T \cdot Q \neq 0 \quad ; \tag{9}$$

the critical point is a bifurcation point, if

$$\Phi^T \cdot Q = 0 \quad . \tag{10}$$

2.3. *Algorithm for critical point search*

Generally, the whole response curve of considered non-linear structure under static load can be path-followed step by step if it doesn't bifurcate. However, it is very difficult to obtain the critical response for the stability consideration. This is just because of the normally used path-following algorithms. On the other hand, if the response curves of all the secondary branches of post-bifurcation are needed, the bifurcation point must be calculated first. This also shows why it is necessary to develop an efficient algorithm, which is compatible with the previous path-following algorithms, for the critical point search.

In fact, the principle of virtual work [4] holds at any equilibrium state. To describe the critical state, the constraint [8] is appended to [4]. It gives [KQ91, WA88, WR88, WR90]

$$\int_{\Omega^0} S^T \cdot \delta\varepsilon \cdot d\Omega^0 = \delta W \quad , \tag{11a}$$

$$K_T(v) \cdot \Phi = 0 \, . \tag{11b}$$

To achieve such a critical state which specified by displacement field v and load vector Q, we start from an equilibrium state achieved, and an iteration procedure is introduced below. This achieved equilibrium state is specified by v_0 and Q_0 respectively. And it can be either pre- or post-critical. We chose the initial approximation of Φ to be Φ_0.

Let us respectively denote v_i, Q_i and Φ_i to be the approximations of v, Q and Φ after ith corrective iteration. It means

$$v_\infty = v, \tag{12a}$$

$$Q_\infty = Q, \tag{12b}$$

$$\Phi_\infty = \Phi \tag{12c}$$

if the corrective iteration is converged. Then the linearization of eqs. [11a and b] gives [BATH82, WR88, WR90]

$$\mathbf{K}_T(\mathbf{v}_{i-1}) \cdot \Delta \mathbf{v}_i - \Delta\lambda_i \cdot \mathbf{q} = -\mathbf{R}(\mathbf{v}_{i-1}) \ , \qquad [13a]$$

$$[\ \mathbf{K}_T(\mathbf{v}_{i-1}) \cdot \mathbf{\Phi}_{i-1}\ ]_{,\mathbf{v}} \cdot \Delta \mathbf{v}_i + \mathbf{K}_T(\mathbf{v}_{i-1}) \cdot \Delta \mathbf{\Phi}_i = -\mathbf{K}_T(\mathbf{v}_{i-1}) \cdot \mathbf{\Phi}_{i-1} \ , \qquad [13b]$$

in which **R** is the residual and λ is the load factor with

$$\mathbf{Q}_i = \lambda_i \cdot \mathbf{q} \ , \qquad [14a]$$

$$\mathbf{Q} = \lambda \cdot \mathbf{q} \ . \qquad [14b]$$

And the improved approximations after ith iteration are found to be

$$\mathbf{v}_i = \mathbf{v}_{i-1} + \Delta \mathbf{v}_i \ ; \qquad [15]$$

$$\mathbf{\Phi}_i = \mathbf{\Phi}_{i-1} + \Delta \mathbf{\Phi}_i \ ; \qquad [16]$$

$$\lambda_i = \lambda_{i-1} + \Delta\lambda_i \ . \qquad [17]$$

Finally, the critical point which specified by v and $\mathbf{Q}$ can be calculated with desired precision by one iteration process.

To go into more detail, the iterative equations [13a and b] have such a solution as follows[BATO79, WR88, WR90].

$$\Delta \mathbf{v}_i = \Delta\lambda_i \cdot \Delta \mathbf{v}_i^1 + \Delta \mathbf{v}_i^2 \ , \qquad [18a]$$

$$\Delta \mathbf{\Phi}_i = \Delta\lambda_i \cdot \Delta \mathbf{\Phi}_i^1 + \Delta \mathbf{\Phi}_i^2 \ , \qquad [18b]$$

with

$$\mathbf{K}_T(\mathbf{v}_{i-1}) \cdot \Delta \mathbf{v}_i^1 = \mathbf{q} \ , \qquad [19a]$$

$$\mathbf{K}_T(\mathbf{v}_{i-1}) \cdot \Delta \mathbf{v}_i^2 = -\mathbf{R}(\mathbf{v}_{i-1}) \ , \qquad [19b]$$

and

$$\mathbf{K}_T(\mathbf{v}_{i-1}) \cdot \Delta \mathbf{\Phi}_i^1 = -[\ \mathbf{K}_T(\mathbf{v}_{i-1}) \cdot \mathbf{\Phi}_{i-1}\ ]_{,\mathbf{v}} \cdot \Delta \mathbf{v}_i^1 \ , \qquad [20a]$$

$$\mathbf{K}_T(\mathbf{v}_{i-1}) \cdot \Delta \mathbf{\Phi}_i^2 = -[\ \mathbf{K}_T(\mathbf{v}_{i-1}) \cdot \mathbf{\Phi}_{i-1}\ ]_{,\mathbf{v}} \cdot \Delta \mathbf{v}_i^2 - \mathbf{K}_T(\mathbf{v}_{i-1}) \cdot \mathbf{\Phi}_{i-1} \ . \qquad [20b]$$

The right superscripts 1 and 2 denote the first and second parts of the marked vectors.

In this paper, the approximation of critical vector $\mathbf{\Phi}$ remains unit during the iteration procedure. It requires that

$$\Phi_{i-1}{}^{T}\cdot\Phi_{i-1} = 1\,, \quad \text{and} \quad \Phi_{i-1}{}^{T}\cdot\Delta\Phi_i = 0\ . \tag{21}$$

The adoption of equation [21] has two advantages. One is the determination of incremental factor $\Delta\lambda_i$ which gives

$$\Delta\lambda_i = \frac{-\,\Phi_{i-1}^{T}\cdot\Delta\Phi_i^{\,2}}{\Phi_{i-1}^{T}\cdot\Delta\Phi_i^{\,1}}\ . \tag{22}$$

Another advantage is the acceleration of iterative convergence. With this technique, the corrective iteration of critical point search is more fast convergent. In the section of numerical study, a comparison with previous work [WR88] has been presented.

It is necessary to note that the starting eigenvector Φ_0 plays a very important role in the studied procedure of iteration. The choice of this vector influences the result in two aspects. An incorrect choice could fail in convergence. And an appropriate choice could result in both fast convergence and obtaining automatically a critical eigenvector which is required by the branch-switching function. Normally, the least dominant eigenvector of tangential stiffness matrix at starting equilibrium state is the best choice of Φ_0. Moreover, the different eigenvector at starting state results in different critical mode if the critical state is a bifurcation one of higher degree. This property helps us to follow any needed secondary branches.

2.4. *Branch-switching function*

Only if the critical state is reached, the branch-switching function brings a possibility to follow any secondary branches of post-bifurcation response. This function is performed through a perturbation of deformation at critical state (see e. g. [EC83]):

$$v_{pert.} = v + \varepsilon_p \cdot \Phi\ . \tag{23}$$

Herein ε_p is a small perturbation factor. It is obvious that if the critical point is multi-bifurcated the different critical mode introduces different secondary branche.

3. Iteration with Higher Order Prediction

In last section, we have shown that the derivative of tangential stiffness matrix is used in critical point search. Here, the derivative is used to formulate an iterative algorithm with second order prediction. For truss structures, the derivative can be easily calculated and almost no more computational efforts are needed, see e. g. [ER91, WA91, WR88]. However, the higher order prediction shows a reduction of total CPU time and the number of corrective iterations because it introduces a second order approximation in first iteration loop.

Since the iterative equation [13a] comes from the linearization of virtual work principle, it is valid not only for the critical point search but for path-following procedure. Then we have the following.

3.1. *Iteration of 1st order approximation*

The typical arc-length methods with 1st order approximation are formulated by a combination of iterative equation [13a] and a constraint equation. There exist many different kinds of constraints on arc-length increment. For example[BATO79, CR81, RI79, RI84]

$$\Delta v^T \cdot \Delta v = \Delta s \cdot \Delta s \ , \tag{24a}$$

or

$$\Delta v^T \cdot \Delta v + \Delta\lambda \cdot \Delta\lambda = \Delta s \cdot \Delta s \tag{24b}$$

where Δs is the increment of the 'arc-length' and

$$\Delta v = \Delta v_1 + \cdots + \Delta v_i \ , \tag{24c}$$

$$\Delta\lambda = \Delta\lambda_1 + \cdots + \Delta\lambda_i \ . \tag{24d}$$

These two suggested arc-lengths have different kinds of performance in non-linear analysis. The experience in numerical study shows that the first arc-length which is determined only by the displacement parameters is more appropriate for stability analysis where snap-through is considered. This is because it is difficult to make a compatibility between load factor and displacement parameters. For example, look at the 4th problem of the late numerical study. If the arc-length is defined by [24b], it would be dominated by load factor $\Delta\lambda$ in pre-critical analysis because of the small deformation. Oppositely, the load factor makes almost no contribution to this arc-length for post bifurcation study. In this case, the choice of the value of the arc-length increment Δs becomes difficulty.

If one constraint is adopted, e. g. [24a], it should be reformulated in an iterative form

$$\Delta v_1^T \cdot \Delta v_1 = \Delta s \cdot \Delta s \ , \tag{25a}$$

and

$$\Delta v^T \cdot \Delta v_i = 0 \qquad \text{(for } i > 1) \ . \tag{25b}$$

Under such arc-length constraint, the iterative equation [13a] results

$$\Delta \mathbf{v}_1 = \Delta\lambda_1 \cdot \Delta \mathbf{v}_1{}^1 \quad , \tag{26a}$$

$$\Delta \mathbf{v}_i = \Delta\lambda_i \cdot \Delta \mathbf{v}_i{}^1 + \Delta \mathbf{v}_i{}^2 \qquad (\text{ for } i > 1) \ , \tag{26b}$$

where $\Delta \mathbf{v}_i{}^1$ and $\Delta \mathbf{v}_i{}^2$ are defined by [19a] and [19b] respectively. And

$$\Delta\lambda_1 = \pm \frac{\Delta s}{\sqrt{\Delta \mathbf{v}_1{}^{1\ T} \cdot \Delta \mathbf{v}_1{}^1}} \quad , \tag{27a}$$

$$\Delta\lambda_i = - \frac{\Delta \mathbf{v}^T \cdot \Delta \mathbf{v}_i{}^2}{\Delta \mathbf{v}^T \cdot \Delta \mathbf{v}_i{}^1} \qquad (\text{ for } i > 1) \ . \tag{27b}$$

This is one widely used arc-length algorithm for path-following procedures.

3.2. *Iteration of 2nd order approximation*

The second order approximation of corrective iteration is only considered in first loop of iteration. That means there exists no residual force **R** for this corrective iteration. Let us denote $\Delta^2\lambda$ and $\Delta^2\mathbf{v}$ as the second order approximation of $\Delta\lambda_1$ and $\Delta \mathbf{v}_1$ respectively. So the real increments of load factor λ and the deformation vector v after first corrective iteration are[ER91, WA91, WR88]

$$\Delta \mathbf{v} = \Delta \mathbf{v}_1 + \frac{1}{2} \Delta^2 \mathbf{v} \quad , \tag{28a}$$

$$\Delta\lambda = \Delta\lambda_1 + \frac{1}{2} \Delta^2\lambda \quad . \tag{28b}$$

Back to the linearization of the principle of virtual work [11a], we can obtain the second order derivative of this principle. It gives the control equation for second order approximation as

$$\mathbf{K}_T(\mathbf{v}_0) \cdot \Delta^2 \mathbf{v} - \Delta^2\lambda \cdot \mathbf{q} \ = \ - [\ \mathbf{K}_T(\mathbf{v}_0) \cdot \Delta \mathbf{v}_1\ ]_{,\mathbf{v}} \cdot \Delta \mathbf{v}_1 \quad . \tag{29a}$$

And the derivative of constraint [25a] on arc-length increment shows that

$$\Delta \mathbf{v}_1{}^T \cdot \Delta^2 \mathbf{v} = 0 \quad . \tag{29b}$$

To solve both equations [29a] and [29b], we have the second order approximation of first corrective iteration. Let us denote $\Delta v_1{}^3$ as the solution of

$$K_T(v_0) \cdot \Delta v_1{}^3 = - [\, K_T(v_0) \cdot \Delta v_1 \,]_{,v} \cdot \Delta v_1 \quad . \tag{30}$$

Then

$$\Delta^2 v = \Delta^2\lambda \cdot \Delta v_1{}^1 + \Delta v_1{}^3 \ . \tag{31a}$$

Here $\Delta v_1{}^1$ has the same definition as the solution of equation [19a] and

$$\Delta^2\lambda = - \frac{\Delta v_1{}^{1T} \cdot \Delta v_1{}^3}{\Delta v_1{}^{1T} \cdot \Delta v_1{}^1} \ . \tag{31b}$$

It is noted that except the derivative of stiffness matrix there exists no more computation in second order approximation. And the derivative has been already formulated in critical point calculation. This is why it is worthy to append the second order prediction to path-following algorithms [ER91, WA91 et al]. The following numerical study shows that the high order prediction causes a considerable reduction(about 25%) of total CPU time. It is coincident with what has been reported in Ref. [WA91] on the reduction of CPU time.

4. Numerical Study

In this section, 4 non-linear truss structures are studied. A comparison of some problems with previous studies is made.

For all these 4 problems, we adopt both the path-following algorithms with and without quadratical predictor. It is found that a reduction of about 25% for the numbers of corrective iterations is possible. This reduction is the same as what Wagner found before in his study on different structures[WA91]. Since for elastic truss structures, it is easy to obtain the analytical expression of the derivative of tangential stiffness matrix, the reduction for iteration numbers results in a saving on CPU time.

The units are not indicated specially, because all the units adopted are International Standard Units.

4.1. *Problem 1 Guyed roof system*

This system has been studied by Bogner in his report to WPAFB 68[BO68]. Our new contribution to the study of this system is the calculation of the limit points which is not reported in [BO68].

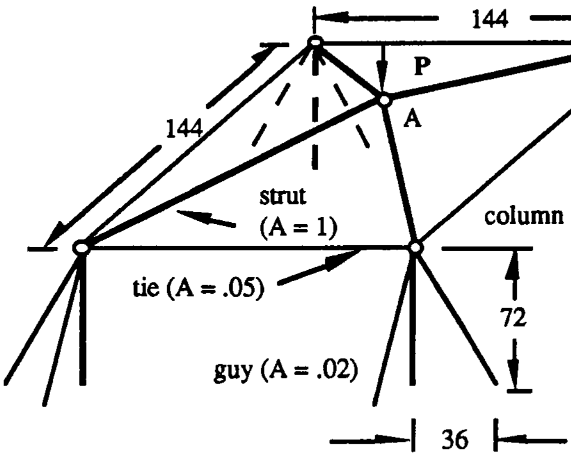

Fig. 1 *Guyed roof system*

The structure is shown in Figure 1. It is initially prestressed by the tension stresses in guy elements. The resulting forces in the members in this initial state are

F(struts) = 0.0 , F(columns) = -2434.9 ,
F(ties) = 608.8 , F(guys) = 1362.5 .

And the loading force is $P = 10000.00 \cdot \lambda$.

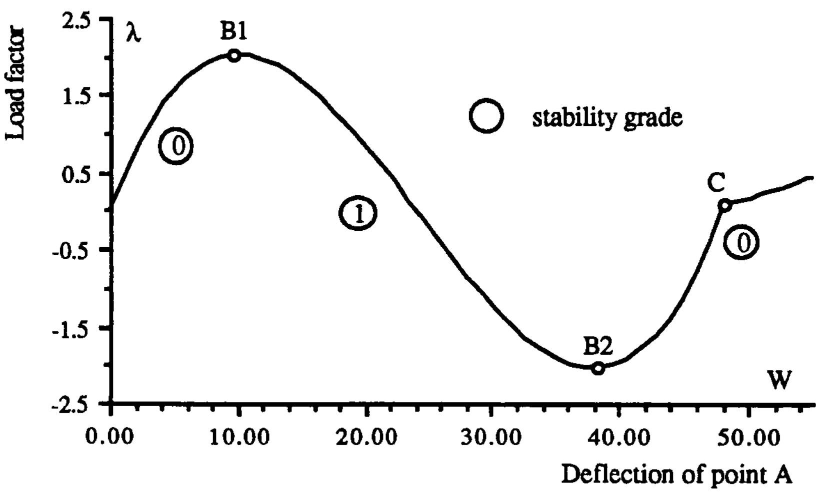

Fig. 2 *Response curve of guyed roof system*

The response curve for the apex (point A) of the structure is shown in Fig. 2. It includes the snap-through instability phenomena coupled with typical tension structure behavior. Both points B1 ($w = 9.692$ and $\lambda = 2.022$) and B2 ($w = 38.35$ and $\lambda = -2.034$) are critical limit points. The curve between B1 and B2 corresponds to unstable equilibrium path. The sharp bend of the curve at point C only means the rapid change in stiffness which takes place as the columns rotate from one side of the vertical to another. Point C is not a critical stability point. The response curve near C corresponds a stable equilibrium path. This phenomenon of sharp bend will be found in another example of Problem 3.

We define such a kind of stability grade as the number of negative eigenvalues of tangential stiffness matrix. In Fig. 2, this stability grade is also indicated for every portion of the curve. It is clear that only such portions of the curve which have zero stability grade correspond to stable configuration.

4.2. *Problem 2 2D truss structure in 'toggle' form*

This is a very simple structure shown by Fig. 3, but its response is characterized by most of the stability phenomena. Both the critical limit points and the critical bifurcation points are found during the load-displacement history. The structure is constructed by 10 truss elements. The load force is $P = 1.0 \cdot \lambda$.

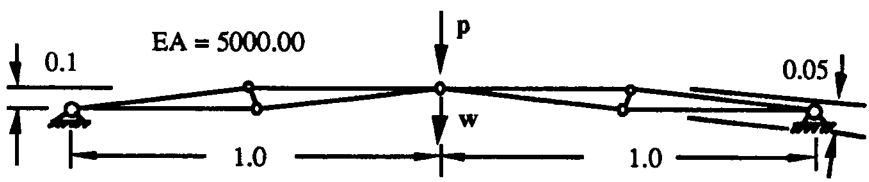

Fig. 3 *2D truss structure in toggle form*

The response curves for both primary and bifurcated secondary route are presented in Fig. 4. The first and the last critical points along the primary route are bifurcation points. And the other two critical points are limit points. The stability grades are also indicated for every portion of the response curves to show the stability or instability.

Both bifurcation points exhibit two modes of bifurcation, but they follow a same post-critically secondary branch. In this problem, two different starting vectors of Φ are used to obtain two bifurcation modes. These two modes correspond to the buckling modes of left and right half of the structure.

For the computation of all the critical points in this problem, the starting vector Φ is chosen to be the eigenvector of tangential stiffness matrix at the starting equilibrium state. Moreover, only the eigenvector which corresponds to the smallest positive eigenvalue could be adopted as the starting vector of Φ for the computation of either limit or bifurcation point with first buckling mode. For the computation of second bifurcation mode, another eigenvector which corresponds to the smallest eigenvalue of the remaining ones is used.

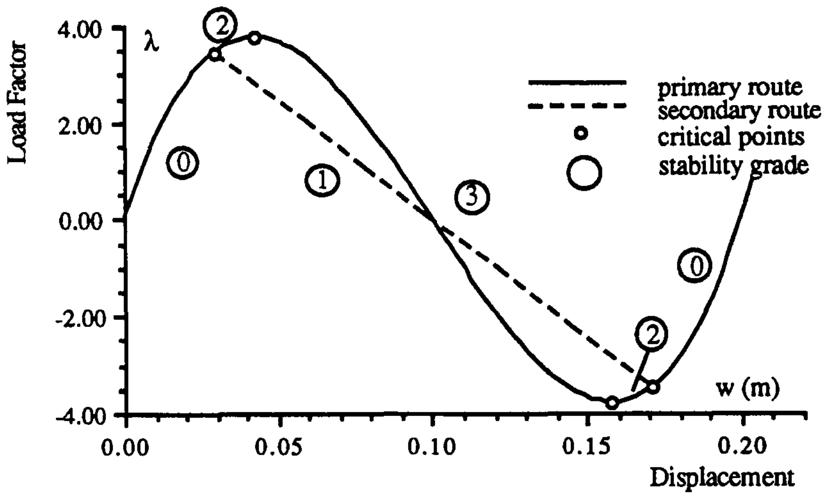

Fig. 4 *Response curve of 2D structure*

During the procedure of critical point searches, the approximation vector of critical mode is remained to be unit. Table 1 shows a comparison between present study and Ref. [WR88] in computation of critical stability points. It is noted that the iteration number is considerably reduced in present study. One major reason can be established for this reduction of iteration numbers is that the application of the mentioned unitary technique.

Table 1 *A comparison in critical stability point computation*

critical point	λ	w	iteration number in present study	iteration number in [29]
1	3.474	0.02938	2	5
2	3.777	0.04227	2	4
3	-3.777	0.1577	2	8
4	-3.474	0.1706	3	21

4.3. *Problem 3 3D truss structure in star-shape*

The structure considered is a 3D dome structure in star-shape. It is widely used to describe the non-linear phenomenon and to explain the path-following algorithms. In those published papers, see e.g. [WR88], the computation is stoped after obtaining the primary route of the response curves.

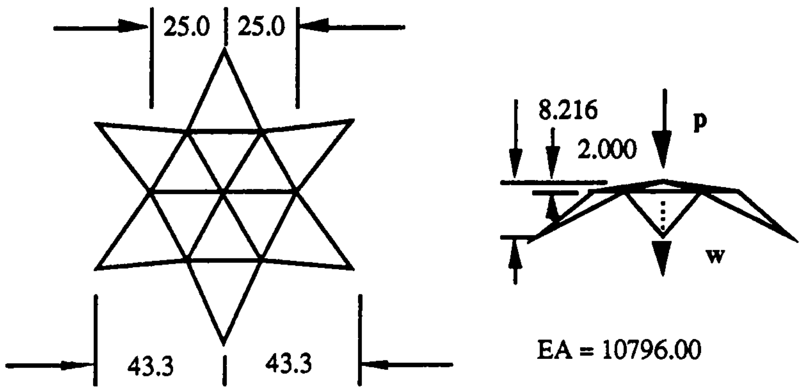

Fig. 5 *Dome structure*

The geometry of the structure is given by Fig. 5. The structure is discretized by 24 truss elements. And the external force is $P = 10.0 \cdot \lambda$.

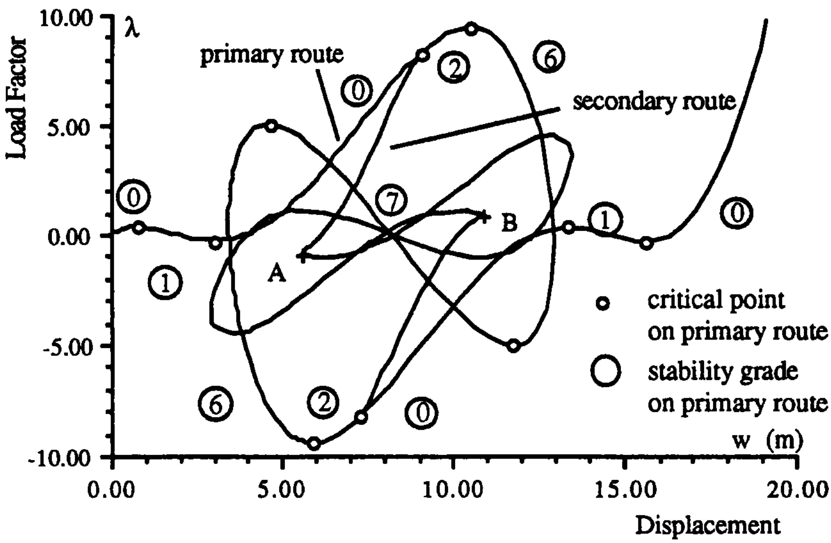

Fig. 6 *Response curve 1 of dome structure*

The primary route of the response curves is fully obtained through the path-following algorithm and is indicated in both Fig. 6 and 7. All the critical stability points along this route are found. And the calculation is fast convergent for every critical point search. Table 2 presents the critical points in an alphabetic order. The stability grade is also given in Fig. 6 for every portion of this primary route.

The first two critical points are limit points which correspond to the local snap-through of the upper dome. And the third one is a bifurcation point which associates with two buckling modes. The secondary routes initiated by these two modes are carefully followed and depicted in Fig. 6 and 7 separately. The structure exhibits a very complicated and maybe fantastic post bifurcation behavior.

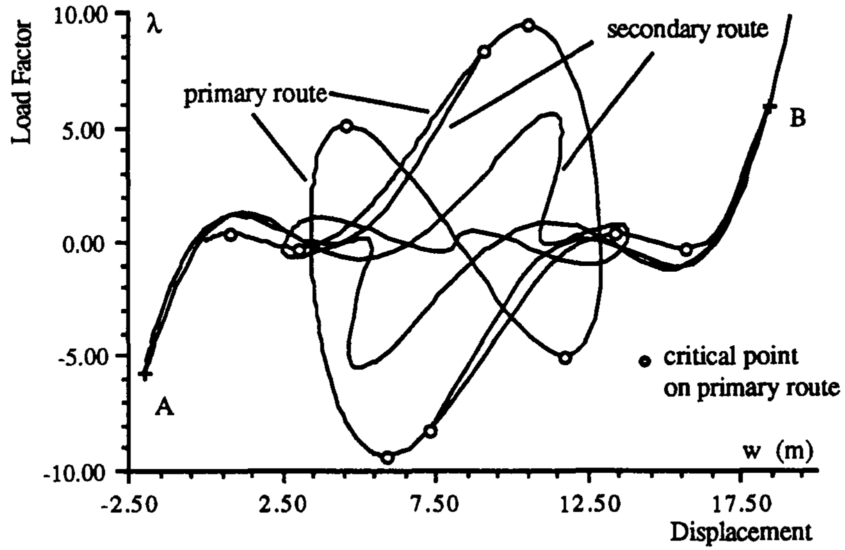

Fig. 7 *Response curve 2 of dome structure*

To go into more detail about the global snap-through of such a structure with a highly geometric non-linearity, it is found that each buckling mode of the third critical point is symmetric to one of the two symmetry axes of the dome. There also exist two different secondary routes because of the different behaviors of these two symmetry axes. Just like what we have done for last problem, the two modes are obtained through the different choices of two starting vectors of Φ. And then the branch-switching function is used to initiate an arbitrary post-bifurcation branch.

Table 2 *Critical points for loaded dome structure*

No. of critical point	1	2	3	4	5
λ	0.3407	-0.2980	8.264	9.409	-5.028
w	0.7686	3.028	9.097	10.51	11.79
No. of critical point	**6**	**7**	**8**	**9**	**10**
λ	5.028	-9.409	-8.264	0.2980	-0,3407
w	4.645	5.919	7.335	13.40	15.66

In both Fig. 6 and 7, points A and B are not the critical stability points. The sharp bends at A and B are due to the rotation of several truss elements from upper side of the horizontal to the other.

4.4. *Problem 4 Three-dimensional mast* [WR88]

The structure shown by Fig. 8 is a 3D mast. It is subjected to a point load of P = 5000.0 · λ. Totally 79 truss element are used for the analysis.

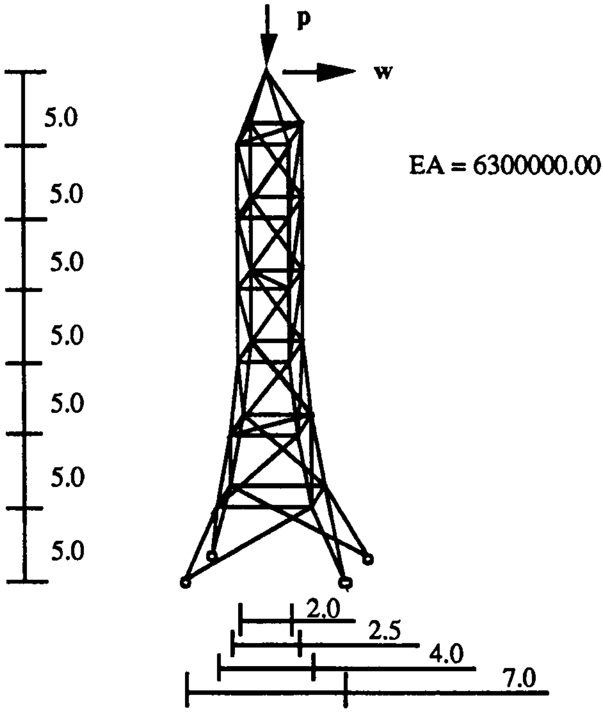

Fig. 8 *3D mast structure*

The response curves of primary route and two secondary routes are depicted in Fig. 9. And the stability grades for the primary route are also given. The critical point, say w = 0.0 and $\lambda = 13.84$, is a double bifurcation point. With the suggested algorithm, we can only obtain the bifurcation point and one buckling mode. The other mode is obtained by solving the eigenvalue problem $\mathbf{K_T} \cdot \mathbf{\Phi} = \mathbf{0}$ at $\lambda = 13.84$. This is the only problem we have met that it is impossible to obtain all the bifurcation modes through the different choices of starting vector $\mathbf{\Phi}$.

The study of this problem shows that the incremental arc-length suggested by [24a] is more appropriate for non-linear stability analysis. If the arc-length increment of [24b] is adopted, the solution will meet a poor convergence or divergence. Since the primary route for this problem is a linear path of zero deflection w, the incremental load factor dominates the arc-length for pre-critical analysis. And it is clear for this problem that the displacement increment makes a major contribution to the arc-length for post-critical analysis of secondary branches. It is difficult to find a good compatibility between pre and post buckling analysis, if [24b] is used.

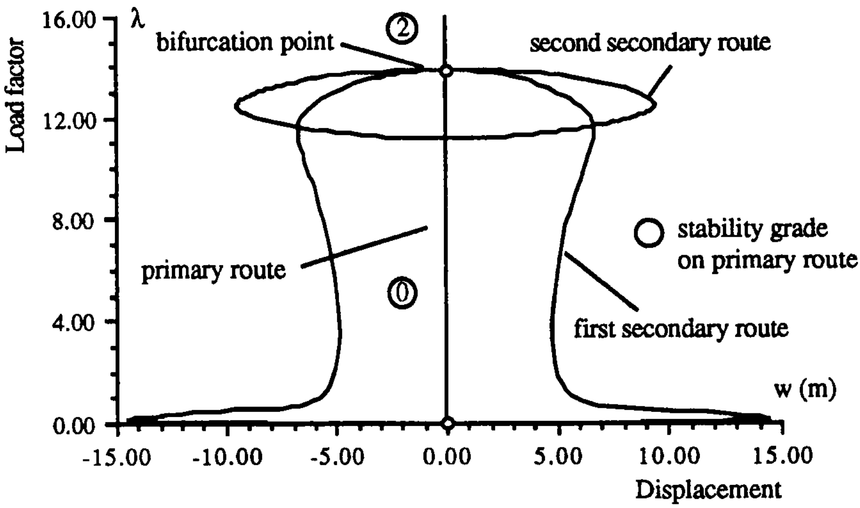

Fig. 9 *Response curve of 3D mast*

5. Conclusion

A numerical stability study on truss structures is presented in this paper. The algorithm described allows global path-following of non-linear stability problems. It combines both the advantages of critical point search and quadratic convergence.

Numerical analysis shows that the method used to calculate the critical point is very efficient for truss structures. It results in fast convergence that the approximation of critical eigenvector remains unit during the whole corrective iteration procedure. Together with the branch-switching function, the stability analysis here permits us to follow not only the pre-critical response curves of non-linear structures but also any arbitrary secondary branches of post-critical response curves with snap-through.

Another worthy application of the derivative of tangential stiffness matrix which has been already formulated in critical stability point calculation is to develop a so-called quadratically convergent path-following algorithm. Since the derivative can be directly formulated for truss structures, almost no more computational efforts are needed. However, the mentioned application brings a reduction of total CPU time for path-following.

The numerically studied structures are some common truss structures. They behave elastically. Although for some structures, the displacement is small, they have the property of highly geometric non-linearity. The global primary response curves of studied structures are carefully followed. And the secondary branches of bifurcated problems are fully calculated. For such simple truss structures, the results show some complicated and maybe fantastic responses. It is very interesting for either theoretical investigations or practical applications.

The presented study is for truss structures, but the methods used could be expended to other problems and have a generlized validity.

References

[AB78] Abbot, J. P., An efficient algorithm for the determination of certain bifurcation points, *J. Comp. Appl. Math.*, 4, pp. 19-27, 1978.

[AL89] Allman, D. J., On the general theory of the stability of equilibrium of discrete conservative systems, *Aero. J.*, 93, pp. 29-35, 1989.

[BATH82] Bathe, K. -J., *Finite Element Procedures in Engineering Analysis*, Pentice-Hall Inc., Englewood Cliffs, New Jersey 07632, 1982.

[BATO79] Batoz, J. L. and Dhatt, G., Incremental displacement algorithms for non-linear problems, *Int. J. Num. Meth. Eng.*, 14, pp. 1262-1267, 1979.

[BE78] Bergan, P. G., Horrigmoe, G., Krakeland, B. and Soreide, T. H., Solution techniques for non-linear finite element problems, *Int. J. Num. Meth. Eng.*, 12, pp. 1677-1696, 1978.

[BO68] Bogner, F. K., Analysis of tension structures, *Proc. 2nd Conf. Matrix Methods in Struct. Mech.*, Air Force Inst. of Tech., Wright Patterson A. F. Base, Dayton, Ohio, USA, 1968.

[BR80] Brendel, B. and Ramm, R., Linear and nonlinear stability analysis of cylindrical shells, *Comp. & Struct.*, 12, pp. 549-558, 1980.

[CR81] Crisfield, M. A., A fast incremental iterative solution procedure that handles snap-through, *Comp. & Struct.*, 13, pp. 55-62, 1981.

[CR86] Crisfield, M. A., New solution procedure for linear and non-linear finite element analysis, in *The Mathematics of Finite Elements and Applications V* (Ed. J. Whitman), Academic Press, London, 1986.

[CR91] Crisfield, M. A. and Shi, J., A review of solution procedures and path-following techniques in relation to the non-linear finite element analysis of structures, in *Nonlinear Computational Mechanics* (Ed. P. Wriggers and W. Wagner), Springer-Verlag, Berlin-Heidelberg-New York, pp. 47-68, 1991.

[EC83] Eckstein, U., Nichtlineare Stabilitätsberechnung elastischer Schalen-tragwerke, *TWM Nr. 83-3*, Institut für Konstruktiven Ingenieurbau, Ruhr-Universität Bochum, Germany, 1983.

[ER91] Eriksson, A., Derivatives of tangential stiffness matrices for equilibrium path descriptions, *Int. J. Num. Meth. Eng.*, 32, pp. 1093-1113, 1991.

[FR84] Fried, I., Orthogonal trajectory accession to the nonlinear equilibrium curve, *Comp. Meth. Appl. Mec. Eng.*, 47, pp. 283-297, 1984.

[KQ91] Krätzig, W. B. and Qian, Y. -y., On stability conditions for nonlinear static and dynamic buckling responses of arbitrary structures, in *Nonlinear Computational Mechanics* (Ed. P. Wriggers and W. Wagner), Springer-Verlag, Berlin-Heidelberg-New York, pp. 85-97, 1991.

[MO80] Moore, G. and Spence, A., The calculation of turning points of nonlinear equations, *SIAM J. Num. Anal.*, 17, pp. 567-576, 1980.

[MU77] Müller, P. C., Stabilität und Matrizen, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[RA81] Ramm, E., Strategies for tracing the nonlinear response near limit points, in *Nonlinear Finite Element Analysis in Structural Mechanics* (Ed. W. Wunderlich, E. Stein, K. -J. Bathe), Springer-Verlag, Berlin-Heidelberg-New York, 1981.

[RI72] Riks, E., The application of Newton's method to the problem of elastic stability, *J. Appl. Mech.*, **39**, pp. 1060-1066, 1972.

[RI79] Riks, E., An incremental approach to the solution of snapping and buckling problems, *Int. J. Solid & Struct.*, **15**, pp. 529-551, 1979.

[RI84] Riks, E., Some computational aspects of the stability analysis of nonlinear structures, *Comp. Meth. Appl. Mech. Eng.*, **47**, pp. 219-259, 1984.

[SE79] Seydel, R., Numerical computation of branch points in nonlinear equations, *Num. Math.*, **33**, pp. 339-352, 1979.

[SP85] Spence, A. and Jepson, A. D., Folds in the solution of two parameter systems and their calculation. Part I, *SIAM J. Numer. Anal.*, **22**, pp. 347-368, 1985.

[TH73] Thompson, J. M. T. and Hunt, G. W. A General Theory of Elastic Stability, Wiley, London, 1973.

[WA88] Wagner, W., and P. Wriggers, A simple method for the calculation of postcritical branches, *Eng. Comput.*, **5**, pp. 103-109, 1988.

[WA91] Wagner, W., A path-following algorithm with quadratic predictor, *Comp. & Struct.*, **39**, pp. 339-348, 1991.

[WEM71] Wempner, G. A., Discrete approximations related to nonlinear theories of solids, *Int. J. Solids Struct.*, **7**, pp. 1581-1599, 1971.

[WER84] Werner, B. and Spence, A., The computation of symmetry-breaking bifurcation points, *SIAM J. Num. Anal.*, **21**, pp. 388-399, 1984.

[WR88] Wriggers, P., Wagner, W. and Miehe, C., A quadratically convergent procedure for the calculation of stability points in finite element analysis, *Comp. Meth. Appl. Mech. Engng.*, **70**, pp. 329-347, 1988.

[WR90] Wriggers, P. and Simo, J. C., A general procedure for the direct computation of turning and bifurcation points, *Int. J. Num. Meth. Eng.*, **30**, pp. 155-176, 1990.

[ZI88] Zienkiewicz, O. C. and Taylor, R. L., *The Finite Element Method*, McGraw-Hill, London, 1988.