Numerical Analysis of a SUSHI Scheme for an Elliptic-parabolic System Modeling Miscible Fluid Flows in Porous Media

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Abstract

In this paper, we demonstrate the convergence of a schema using stabilization and hybrid interfaces applying to partial differential equations describing miscible displacement in porous media. This system is made of two coupled equations: an anisotropic diffusion equation on the pressure and a convection-diffusion-dispersion equation on the concentration of invading fluid. The anisotropic diffusion operators in both equations require special care while discretizing by a finite volume method SUSHI. Later, we present some numerical experiments.

Keywords: Porous media, nonconforming grids, finite volume schemes, SUSHI, convergence analysis.

1 Introduction

The single-phase miscible displacement of one fluid by another in a porous medium is modeling by a system of nonlinear partial differential equations. In the literature, there exists several modelling approaches, in [2–6] the authors are introduced the Peaceman model where the fluids are considered

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incompressible this model is constituted of an elliptic parabolic coupled system. While, if the fluids are compressible, the system becomes parabolic see [7] and [8]. We are interested in the study of The first model, it is constituted of an anisotropic diffusion equation on the pressure and a convection-diffusion-dispersion equation on the concentration of the invading fluid, see [9] for the theoretical analysis of this system of partial differential equations.

The miscible displacement problem studied in this work consists of a nonlinear elliptic-parabolic coupled system, it has been the subject of several studies, C. Chainais-Hillairet, S. Krell and A. Mouton, in [10] and [11], are study the numerical analysis and the convergence of a schema DDFV, on the other hand in [12] Chainais-Hillairet and Droniou were proposed the scheme of mixed finite volume, Hanzhang Hu, Yiping Fu and Jie Zhou study this model by the mixed finite elements in [13]. The authors in [14, 15] studied the Finite Element schemes for both equations. We refer to [27, 28] for the Eulerian-Lagrangian localized adjoint method combined with the mixed finite element methods. See [16] for the convergence analysis for a discontinuous Galerkin finite element. Then, he pressure equation was discretised by finite element method and the concentration equation by method of characteristics in [17–19], let cite other references that study the miscible displacement problems by different methods [24–26].

The unknowns of the problem are p the pressure in the mixture, U its Darcy velocity and c the concentration of the invading fluid. The porous medium is characterized by its porosity $\phi(x)$ and its permeability $A(x)$. We denote by $\mu(c)$ the viscosity of the fluid mixture, $\widehat{c}$ the injected concentration, q^+ and q^- are the injection and the production source terms. The model on a time interval $(0,T)$ and a domain $\Omega \subset \mathbb{R}^2$ is defined by the following system

$$\begin{cases} \operatorname{div}(U) = q^+ - q^- & \text{in } (0,T)\times\Omega, \\ U = -K(x,c)\nabla p & \text{in } (0,T)\times\Omega, \\ \int_\Omega p(.,x)dx = 0 & \text{On } (0,T), \end{cases} \tag{1}$$

$$\phi(x)\partial_t c - \operatorname{div}(D(x,U)\nabla c) + \operatorname{div}(Uc) + q^- c = \widehat{c}q^+ \text{ in } (0,T)\times\Omega. \tag{2}$$

Where

- The domain Ω is an open, bounded, connected subset of $\mathbb{R}^d$ ($d = 2$ or $d = 3$). Which supported tube polygonal ($d = 2$) or polyhedral ($d = 3$), and $\partial\Omega$ stands for its boundary.

- The initial condition is given by

$$c(x,0) = c_0(x), \tag{3}$$

  such that

$$c_0 \in L^\infty(\Omega), \text{ and satisfies } 0 \le c_0 \le 1 \text{ a.e. in } \Omega, \tag{4}$$

- The Neumann boundary conditions are defined by

$$\begin{cases} U.n = 0 & \text{on }]0,T[\times\partial\Omega, \\ D(x,U)\nabla c.n = 0 & \text{on }]0,T[\times\partial\Omega. \end{cases} \tag{5}$$

- The porous medium is characterized by the porosity $\phi(x)$ with

$$\phi \in L^\infty(\Omega) \text{ and there exists } \phi^* > 0 \text{ such that : } \phi_* \le \phi \le \phi_*^{-1} \text{ a.e. in } \Omega. \tag{6}$$

- $K(x,c) = \frac{A(x)}{\mu(c)}$ and D are the diffusion-dispersion tensors including molecular diffusion and mechanical dispersion, verified the following properties:

$$\begin{cases} K : \Omega \times \mathbb{R} \to M_2(\mathbb{R}) \text{ is a Caratheodory matrix-valued function satisfying:} \\ \exists \alpha_K > 0 \text{ s.t. } K(x,s)\xi.\xi \ge \alpha_K|\xi|^2 \text{ for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R}, \text{ and all } \xi \in \mathbb{R}^2, \\ \exists \Lambda_K > 0 \text{ s.t. } |K(x,s)| \le \Lambda_K \text{ for a.e. } x \in \Omega \text{ and all } s \in \mathbb{R}, \end{cases} \tag{7}$$

  and

$$\begin{cases} D : \Omega \times \mathbb{R}^2 \to M_2(\mathbb{R}) \text{ is a Caratheodory matrix-valued function satisfying:} \\ \exists \alpha_D > 0 \text{ s.t. } D(x,V)\xi.\xi \ge \alpha_D(1+|V|)|\xi|^2 \text{ for a.e. } x \in \Omega, \text{ all } (V,\xi) \in \mathbb{R}^2 \times \mathbb{R}^2, \\ \exists \Lambda_D > 0 s.t. |D(x,V)| \le \Lambda_D(1+|V|) \text{ for a.e. } x \in \Omega \text{ and all } V \in \mathbb{R}^2, \end{cases} \tag{8}$$

- D is given by

$$D(x,U) = \phi(x)\left(d_m I + |U|\left(d_l E(U) + d_t(I - E(U))\right)\right), \tag{9}$$

  I is the identity matrix, d_m is the molecular diffusion, d_l and d_t are the longitudinal and transverse dispersion coefficients, and $E(U) = (\frac{U_i U_j}{|U|^2})_{1 \le i,j \le d}$.
- We denote by $\mu(c)$ the viscosity of the fluid mixture as

$$\mu(c) = \mu(0)\left(1 + (M^{\frac{1}{4}} - 1)c\right)^{-4} \text{ on } [0,1], \tag{10}$$

  $M = \frac{\mu(0)}{\mu(1)}$ is the mobility ratio (we extend μ to $\mathbb{R}$ by letting $\mu = \mu(0)$ on $]-\infty,0[$ and $\mu = \mu(1)$ on $]1,\infty[$).

- $\widehat{c}$ is the injected concentration such that

$$\widehat{c} \in L^{\infty}(]0,T[\times\Omega) \text{ satisfies} : 0 \leq \widehat{c} \leq 1 \text{ a.e. in }]0,T[\times\Omega. \tag{11}$$

- q^{+} and q^{-} are the injection and the production source terms

$$\begin{cases} (q^{+},q^{-}) \in L^{\infty}(0,T;L^{2}(\Omega)) \text{ are non negative functions such that (s.t.)}, \\ \int_{\Omega} q^{+}(.,x)dx = \int_{\Omega} q^{-}(.,x)dx \text{ almost everywhere (a.e.) on }]0,T[. \end{cases} \tag{12}$$

Definition 1.1 (Weak solution) *Under the hypotheses* (3–12). *A weak solution of* (1) *and* (2) *is a triple of functions* $(\overline{p},\overline{U},\overline{c})$ *satisfying:* $(\overline{p} \in L^{\infty}(0,T;H^{1}(\Omega)))$, $(\overline{U} \in L^{\infty}(0,T;L^{2}(\Omega)))$ *and* $(\overline{c} \in L^{\infty}(0,T;L^{2}(\Omega)) \cap L^{2}(0,T;H^{1}(\Omega)))$

$$\begin{cases} \displaystyle\int_{\Omega} \overline{p}(t,.)dt = 0 \text{ for all a.e } t \in]0,T[, & \overline{U} = K(x,\overline{c})\nabla\overline{p} \text{ in }]0,T[\times\Omega, \\ a_1(\overline{p},\varphi_1) = \displaystyle\int_0^T\int_{\Omega} q^{+}\varphi_1 + \int_0^T\int_{\Omega} q^{-}\varphi_1 & \forall\varphi_1 \in C^{\infty}([0,T]\times\overline{\Omega}), \\ a_2(\overline{c},\varphi_2) = \displaystyle\int_0^T\int_{\Omega} \widehat{c}q^{+}\varphi_2 & \forall\varphi_2 \in C_c^{\infty}([0,T]\times\overline{\Omega}), \end{cases} \tag{13}$$

with

$$\begin{cases} a_1(\overline{p},\varphi_1) = \displaystyle\int_0^T\int_{\Omega} K(x,c)\nabla\overline{p}\nabla\varphi_1, \\ a_2(\overline{c},\varphi_2) = -\displaystyle\int_0^T\int_{\Omega} \phi(x)\overline{c}\partial_t\varphi_2 + \int_0^T\int_{\Omega} D(x,\overline{U})\nabla\overline{c}\nabla\varphi_2 \\ \qquad\qquad - \displaystyle\int_0^T\int_{\Omega} \overline{c}\overline{U}\nabla\varphi_2 + \int_0^T\int_{\Omega} q^{+}\overline{c}\varphi_2 - \int_{\Omega} \varphi c_0(x)\varphi_2(0,.), \end{cases} \tag{14}$$

In this work, we want to apply one of the finite volume methods dedicated to anisotropic diffusion. We will examine the application of a finite volume scheme using stabilization and hybrid interfaces, which has been proposed by Eymard et al. [20], to the diffusion term in the pressure equation and in the concentration convection-diffusion-dispersion equation of the system describing miscible fluid flows in porous media (Peaceman model). This method is characterized by:

- Using a single mesh that is very general, unstructured and does not take into account the condition of orthogonality (Finite Volume method [29]).
- Avoid to project the gradient on the edges of dual and primal mesh (method DDFV [30]) by the addition of a term of stability which stabilize the gradient obtained by the method of the finite volume 4, so, the number of variables of SUSHI method is less compared to the method (DDFV).

We present and study a numerical scheme and convergence analysis for SUSHI method applying to this model, more precisely. In order to prove the convergence of the SUSHI schemes, we apply a similar strategy as [8] on our numerical approximation instead of the regularization of the Peaceman model. Then, Some numerical tests are also carried out to verify the validity of the numerical scheme proposed.

This article is organized as follows. In Section 2 we present meshes and the associated notations, than, we introduce the different discrete operators (gradient, diffusion and convection operators) and some proprieties. The main result of the paper is detailed in Section 3 as follows Sections 3.1, 3.2 and 3.3 are devoted for the discretization of the system (1–12), in Section 3.4 we present some numerical experiments. Finally we demonstrate the convergence theorem in Section 3.5.

2 The Finite Volume Scheme SUSHI

The SUSHI scheme is based on two schemes, firstly the Hybrid Finite Volume (HVF) scheme introduced in 2007 by R. Eymard et al. [23]. This HVF scheme, adapted to solve the anisotropic diffusion problem, introduces additional unknowns on the edges of the meshes in order to reconstruct the discrete gradient in all directions and thus to correctly treat the anisotropic and heterogeneous problems, and secondly on the cell-centered SUCCES scheme (Finite volume method classic).

The originality of this scheme (HVF) lies in the proof of convergence that only requires weak hypotheses on the mesh. This HVF scheme was then modified and gave birth to the SUSHI schemes in 2009 [20].

There are two variants of the SUSHI schema: a first one where the unknowns on the edges are only introduced where they are needed, for example where there are strong heterogeneities and a second where the unknowns on the edges are introduced on all the edges of the mesh.

In this section, we will present different definitions, notations and conventions of writing that we will use later, on the other hand, we follow the idea of Eymard et al. [20] to build flux using a stabilized discrete gradient. After, we define the discretization of the convection term, then, we give some proprieties and definition of the SUSHI schemes.

2.1 Space and Time Discretization

Definition 2.1 *Let's define some notations of the discretization of* Ω.

- *A discretization of* Ω *is defined by* $\mathcal{D} = (\mathcal{M}, \mathcal{E}, P)$.
- $\mathcal{M}$ *is a family of connected non-empty open subspaces included in* Ω *(set of control volumes* $\mathcal{K}$*).*
- σ *is a non-empty open of* $\mathbb{R}$.
- $\mathcal{E}$ *is the set of the interfaces* σ.
- $\mathcal{E}$ *is decomposed into two subdomains* $\mathcal{E}_{int}$ *and* $\mathcal{E}_{ext}$ *which are respectively the sets of interfaces inside and outside to the mesh* $\mathcal{D}$.
- *For all* $\mathcal{K} \in \mathcal{M}$, $M_\sigma = \{\mathcal{K}, \sigma \in \mathcal{E}_\mathcal{K}\}$.
- x_σ *and* $x_\mathcal{K}$ *are respectively the center and the barycentre of* σ *and* $\mathcal{K}$.
- $m_\mathcal{K}$ *and* m_σ *are respectively the measure of control volume* $\mathcal{K}$ *and of interface* σ.
- $n_{\mathcal{K},\sigma}$ *is the unit vector normal to* σ *outward to* $\mathcal{K}$.
- P *is the set of points of* Ω.
- $C_{\mathcal{K},\sigma}$ *is the cone with vertex* $x_\mathcal{K}$ *and basis* σ.

Definition 2.2 *We consider* $X_\mathcal{D}$, $X_{\mathcal{D},0}$ *and* $X_{\mathcal{D},0,\mathcal{B}}$ *three spaces defined as follow:*

$$X_\mathcal{D} = \{v = ((v_\mathcal{K})_{\mathcal{K}\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}}); v_\mathcal{K} \in \mathbb{R}, v_\sigma \in \mathbb{R}\}, \tag{15}$$

$$X_{\mathcal{D},0} = \{v \in X_\mathcal{D} \text{ such that } \Lambda_\mathcal{K} \nabla^n_{\mathcal{K},\sigma} v.n_{\mathcal{K},\sigma} = 0, \forall\sigma \in \mathcal{E}_{ext}\}, \tag{16}$$

$$X_{\mathcal{D},0,\mathcal{B}} = \left\{ v \in X_{\mathcal{D},0} / \exists \beta^\mathcal{K}_\sigma \in \mathbb{R}\ ;\ v_\sigma = \sum_{\mathcal{K}\in\mathcal{M}} \beta^\mathcal{K}_\sigma v_\mathcal{K} \right\}. \tag{17}$$

With $\mathcal{B}$ *is defined in the next definition.*

Definition 2.3 *Let:*

$$u_\sigma = \sum_{\mathcal{K}\in\mathcal{M}} \beta^\mathcal{K}_\sigma u_\mathcal{K}, \tag{18}$$

where $(\beta^\mathcal{K}_\sigma)_{\mathcal{K}\in\mathcal{M},\sigma\in\mathcal{E}_{int}}$ *is a family of real numbers, with* $\beta^\mathcal{K}_\sigma \neq 0$ *only for some control volumes* $\mathcal{K}$ *close to* σ*, and such that*

$$\sum_{\mathcal{K}\in\mathcal{M}} \beta^\mathcal{K}_\sigma = 1 \text{ and } x_\sigma = \sum_{\mathcal{K}\in\mathcal{M}} \beta^\mathcal{K}_\sigma x_\mathcal{K}. \tag{19}$$

$\mathcal{B}$ *is the set of the eliminated unknowns using* (18), *and* $\mathcal{H} = \mathcal{E}_{int}/\mathcal{B}$.

The projections in the spaces $X_{\mathcal{D}}, X_{\mathcal{D},0}$ and $X_{\mathcal{D},0,\mathcal{B}}$ are defined in the definition next.

Definition 2.4 *$C_0(\overline{\Omega})$ is the set of continues functions which are null in $\partial\Omega$. for all $\psi \in C_0(\overline{\Omega})$, we define*

1. *The projection in $X_{\mathcal{D}}$ by*

$$\begin{aligned} \mathcal{P}_{\mathcal{D}} : C_0(\overline{\mathbb{R}}) &\to X_{\mathcal{D}} \\ \psi &\mapsto \mathcal{P}_{\mathcal{D}}\psi = ((\psi(x_{\mathcal{K}}))_{\mathcal{K}\in\mathcal{M}}, (\psi(x_\sigma))_{\sigma\in\mathcal{E}}). \end{aligned}$$

2. *$\mathcal{P}_{\mathcal{D},\mathcal{B}}\psi = v$ is an element of $X_{\mathcal{D},\mathcal{B}}$ such that $v_{\mathcal{K}} = \psi(x_{\mathcal{K}}), \forall\mathcal{K} \in \mathcal{M}$; $v_\sigma = v_{\mathcal{K}}$, for all $\sigma \in \mathcal{E}_{ext}$; $v_\sigma = \psi(x_\sigma)$, for all $\sigma \in \mathcal{H}$ and $v_\sigma = \sum_{\mathcal{K}\in\mathcal{M}} \beta_\sigma^{\mathcal{K}} \psi(x_{\mathcal{K}})$ for all $\sigma \in \mathcal{B}$.*
3. *$\mathcal{P}_{\mathcal{M}} \in H_{\mathcal{M}}(\Omega)$ ($H_{\mathcal{M}}(\Omega)$ is the set of the piece-wise function in central volume $\mathcal{K} \in \mathcal{M}$) such that $\mathcal{P}_{\mathcal{M}}(\psi(x)) = \psi(x_{\mathcal{K}})$, a.s $\forall x \in \mathcal{K}, \forall\mathcal{K} \in \mathcal{M}$.*

The space $X_{\mathcal{D}}$ is equipped with the semi-norm $|.|_{X_{\mathcal{D}}}$ defined by

$$|v|_{X_{\mathcal{D}}}^2 = \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \frac{m_\sigma}{d_{\mathcal{K}\sigma}} (v_\sigma - v_{\mathcal{K}})^2, \text{ for all } v \in X_{\mathcal{D}}. \tag{20}$$

Note that $|.|_{X_{\mathcal{D}}}$ is a norm on the spaces $X_{\mathcal{D},0}$ and $X_{\mathcal{D},0,\mathcal{B}}$.

Definition 2.5 *The time interval $(0, T)$ $(T > 0)$ is divided to N (N is an integer such that $N > 0$) small intervals have a step δt equals to T/N, where $\delta t = t_{n+1} - t_n$ we introduce this spaces*

$$X_{\mathcal{D},\delta t} = \{(v^n)_{n\in\{0,\dots,N-1\}}, v^n \in X_{\mathcal{D}}\}, \tag{21}$$

$$X_{\mathcal{D},\delta t,0} = \{(v^n)_{n\in\{0,\dots,N-1\}}, v^n \in X_{\mathcal{D},0}\}, \tag{22}$$

$$X_{\mathcal{D},\delta t,\mathcal{B}} = \{(v^n)_{n\in\{0,\dots,N-1\}}, v^n \in X_{\mathcal{D},0,\mathcal{B}}\}. \tag{23}$$

The semi-norm on $X_{\mathcal{D},\delta t}$ is defined by

$$|v|_{X_{\mathcal{D},\delta t}}^2 = \sum_{n=0}^{N-1} \delta t |v^n|_{X_{\mathcal{D}}}^2. \tag{24}$$

2.2 The Discrete Gradient

It is always possible to deduce an expression for $\nabla_{\mathcal{D}} u(x)$ as a linear combination of $(u_\sigma - u_{\mathcal{K}})_{\sigma\in\mathcal{E}_{\mathcal{K}}}$. Let us first define

$$\begin{aligned} \nabla_{\mathcal{K}} : &X_{\mathcal{D}} \to H_{\mathcal{M}}(\Omega)^d \\ &u^{n+1} \mapsto \nabla_{\mathcal{K}} u^{n+1}, \end{aligned}$$

such that

$$u^{n+1} \in X_D, \nabla_{\mathcal{K}} u^{n+1} = \frac{1}{|\mathcal{K}|} \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} |\sigma| [u_\sigma^{n+1} - u_{\mathcal{K}}^{n+1}] n_{\mathcal{K},\sigma}.$$

However, we find that this discrete gradient is zero for any $u_{\mathcal{K}}^{n+1} \in \mathcal{K}$, if u_σ^{n+1} are zero, so it is not coercive. We thus seek a new coherent discrete gradient with the previous and coercive in the $C_{\mathcal{K},\sigma}$ (cone the vertex $x_{\mathcal{K}}$ and basis σ). This corresponds to the previous step gradient to which we add a correction term. We define the discrete gradient as follows:

$$\nabla_{\mathcal{K},\sigma} u^{n+1} = \nabla_{\mathcal{K}} u^{n+1} + \mathcal{R}_{\mathcal{K},\sigma}(u^{n+1}) n_{\mathcal{K},\sigma}, \tag{25}$$

with

$$\mathcal{R}_{\mathcal{K},\sigma}(u^{n+1}) = \frac{\sqrt{d}}{d_{\mathcal{K},\sigma}} (u_\sigma^{n+1} - u_{\mathcal{K}}^{n+1} - \nabla_{\mathcal{K}} u^{n+1} \,.\, [x_\sigma - x_{\mathcal{K}}]).$$

(Recall that d is the space dimension and $d_{\mathcal{K},\sigma}$ is the Euclidean distance between $x_{\mathcal{K}}$ and x_σ) We obtain the following stable discrete gradient

$$\nabla_{\mathcal{K},\sigma} u^{n+1} = \nabla_{\mathcal{K}} u^{n+1} + \mathcal{R}_{\mathcal{K},\sigma} u^{n+1} . n_{\mathcal{K},\sigma}, \tag{26}$$

We may then define $\nabla_{\mathcal{D}}$ as the piece-wise constant function equal to $\nabla_{\mathcal{K},\sigma}$ a.e. in the cone $C_{\mathcal{K},\sigma}$ with vertex $x_{\mathcal{K}}$ and basis σ

$$\nabla_{\mathcal{D}} u^{n+1} = \nabla_{\mathcal{K},\sigma} u^{n+1} \text{ for a.e } x \in C_{\mathcal{K},\sigma}. \tag{27}$$

Then we have

$$\nabla_{\mathcal{K},\sigma} u^{n+1} = \sum_{\sigma' \mathcal{E}_{\mathcal{K}}} Y^{\sigma,\sigma'} (u_{\sigma'}^{n+1} - u_{\mathcal{K}}^{n+1}), \tag{28}$$

with $Y^{\sigma,\sigma'}$ giving by

$$Y^{\sigma,\sigma'} = \begin{cases} \frac{m_\sigma}{m_{\mathcal{K}}} n_{\mathcal{K}\sigma} + \frac{\sqrt{d}}{d_{\mathcal{K},\sigma}} (1 - \frac{m_\sigma}{m_{\mathcal{K}}} n_{\mathcal{K}\sigma} . [x_\sigma - x_{\mathcal{K}}]) n_{\mathcal{K}\sigma} & \text{if } \sigma = \sigma', \\ \frac{m_{\sigma'}}{m_{\mathcal{K}}} n_{\mathcal{K}\sigma'} - \frac{\sqrt{d}}{d_{\mathcal{K},\sigma} m_{\mathcal{K}}} n_{\mathcal{K},\sigma'} . [x_\sigma - x_{\mathcal{K}}] n_{\mathcal{K},\sigma} & \text{otherwise.} \end{cases} \tag{29}$$

2.3 The Discrete Convection Term

To treat the convection term in the concentration equation, we define the following convection operator discrete:

$$\begin{aligned} divc_\sigma : \mathbb{R}^{\mathcal{D}} \times \mathbb{R}^{\mathcal{D}} &\to \mathbb{R}^{\mathcal{D}} \\ (\xi_{\mathcal{D}}, v_{\mathcal{D}}) &\mapsto div(\xi_{\mathcal{D}}, v_{\mathcal{D}}), \end{aligned}$$

with

$$divc_\sigma(\xi_\mathcal{D}, v_\mathcal{D}) = \begin{cases} v_\mathcal{K} n_{\sigma,\mathcal{K}} \xi_\mathcal{K} & \text{if } v_\mathcal{K} n_{\sigma,\mathcal{K}} \geq 0, \\ -v_\mathcal{K} n_{\sigma,\mathcal{K}} \xi_\mathcal{L} & \text{if } v_\mathcal{K} n_{\sigma,\mathcal{K}} < 0. \end{cases} \tag{30}$$

2.4 The Proprieties of the Schemes

Let $\mathcal{D}$ be a discretisation of Ω in the sense of Definition (2.1). The size of the discretisation $\mathcal{D}$ is defined by

$$h_\mathcal{D} = \sup_{\mathcal{K} \in \mathcal{M}} (d(\mathcal{K})), \tag{31}$$

with $d(\mathcal{K})$ is the diameter of $\mathcal{K}$ and the regularity of the mesh is defined by

$$\theta_\mathcal{D} = \max \left(\max_{\sigma_{\mathcal{K},\mathcal{L}} \in \mathcal{E}_{int}} \left(\frac{d_{\mathcal{K},\sigma}}{d_{\mathcal{L},\sigma}} \right), \max_{\mathcal{K} \in \mathcal{M}, \sigma \in \mathcal{E}_\mathcal{K}} \left(\frac{d(\mathcal{K})}{d_{\mathcal{K},\sigma}} \right) \right). \tag{32}$$

For a given set $\mathcal{B} \in \mathcal{E}_{int}$ and for a given family $\beta_\sigma^\mathcal{K}$ satisfying property (18), we introduce a measure of the resulting regularity by

$$\theta_{\mathcal{D},\mathcal{B}} = \max \left(\theta_\mathcal{D}, \max_{\mathcal{K} \in \mathcal{M}, \sigma \in \mathcal{E} \cap \mathcal{B}} \frac{\sum_{\mathcal{L} \in \mathcal{M}} |\beta_\sigma^\mathcal{L}| |x_\mathcal{L} - x_\sigma|^2}{h_\mathcal{D}^2} \right). \tag{33}$$

Definition 2.6 *Let $\mathcal{D}$ be a discretisation of Ω in the sense of Definition* (2.1)*, and let δt be the time step defined in Definition* (2.5)*. For $v \in X_\mathcal{D}$, we define the following related norm*

$$\|\mathcal{P}_\mathcal{M} v\|_{1,2,\mathcal{M}}^2 = \sum_{\mathcal{K} \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}} |\sigma| d_{\mathcal{K},\sigma} \left(\frac{D_\sigma v}{d_\sigma} \right)^2, \tag{34}$$

and for $v \in X_{\mathcal{D},\delta t}$, we define the following related norm

$$\|\mathcal{P}_\mathcal{M} v\|_{1;1,2,\mathcal{M}}^2 = \sum_{n=0}^{N-1} \delta t \|\mathcal{P}_\mathcal{M} v^n\|_{1,2,\mathcal{M}}^2, \tag{35}$$

with $d_\sigma = |d_{\mathcal{K},\sigma} + d_{\mathcal{L},\sigma}|$, $D_\sigma v = |v_\mathcal{K} - v_\mathcal{L}|$ if $\mathcal{M}_\sigma = \{\mathcal{K}, \mathcal{L}\}$, and $d_\sigma = d_{\mathcal{K},\sigma}$, $D_\sigma v = |v_\mathcal{K}|$ if $\mathcal{M}_\sigma = \{\mathcal{K}\}$.

A result stated in [20] gives the relation

$$||\mathcal{P}_{\mathcal{M}}v||^2_{1,2,\mathcal{M}} \leq |v|^2_{X_{\mathcal{D}}}, \forall v \in X_{\mathcal{D},0}. \tag{36}$$

Then, we get

$$||\mathcal{P}_{\mathcal{M}}v||^2_{1;1,2,\mathcal{M}} \leq |v|^2_{X_{\mathcal{D}},\delta t}, \forall v \in X_{\delta t,\mathcal{D},0}. \tag{37}$$

A result stated in [20] gives the relation

$$||\mathcal{P}_{\mathcal{M}}v||^2_{1,2,\mathcal{M}} \leq |v|^2_{X_{\mathcal{D}}}, \forall v \in X_{\mathcal{D},0}. \tag{38}$$

Then, we get

$$||\mathcal{P}_{\mathcal{M}}v||^2_{1;1,2,\mathcal{M}} \leq |v|^2_{X_{\mathcal{D}},\delta t}, \forall v \in X_{\delta t,\mathcal{D},0}. \tag{39}$$

We recall in this section some proprieties of SUSHI scheme. The proof of these proprieties can be found in [21].

Lemma 2.7 *Let $\mathcal{D}$ be a discretisation of Ω in the sense of Definition* (2.1). *Let $\nu > 0$ be such that $\nu \leq \frac{d_{\mathcal{K},\sigma}}{d_{\mathcal{L},\sigma}} \leq \frac{1}{\nu}$ for all $\sigma \in \mathcal{E}$, where $M_\sigma = \{\mathcal{K}, \mathcal{L}\}$. Then there exists C_1 only depending on d, Ω and ν such that*

$$||\mathcal{P}_{\mathcal{M}}v||_{L^2(\Omega)} \leq C_1||\mathcal{P}_{\mathcal{M}}v||_{1,2,\mathcal{M}}, \forall v \in X_{\mathcal{D}}, \tag{40}$$

where $||\mathcal{P}_{\mathcal{M}}v||_{1,2,\mathcal{M}}$ is defined by (34).

Lemma 2.8 *Let $\mathcal{D}$ be a discretisation of Ω in the sense of Definition* (2.1), *and let δt be the time step defined in Definition* (2.5). *Let $\nu > 0$ be such that $\nu \leq \frac{d_{\mathcal{K},\sigma}}{d_{\mathcal{L},\sigma}} \leq \frac{1}{\nu}$ for all $\sigma \in \mathcal{E}$, where $M_\sigma = \{\mathcal{K}, \mathcal{L}\}$. Then there exists C_1' only depending on δt and C_1 such that*

$$||\mathcal{P}_{\mathcal{M}}v||_{L^2(0,T;L^2(\Omega))} \leq C_1'||\mathcal{P}_{\mathcal{M}}v||_{1;1,2,\mathcal{M}}, \forall v \in X_{\mathcal{D},\delta t}, \tag{41}$$

where $||\mathcal{P}_{\mathcal{M}}v||_{1;1,2,\mathcal{M}}$ is defined by (35).

Proof 2.9 *The proof is a result of Lemma* 2.7.

Definition 2.10 *Let $\mathcal{D}$ be a discretisation of Ω in the sense of the Definition* (2.1), *and δt be a time step defined in Definition* (2.5). *We define the $L^2 - norm$ of the discrete gradient by*

$$||\nabla_{\mathcal{D}}v(x)||^2_{L^2(\Omega)^d} = \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \frac{|\sigma| d_{\mathcal{K},\sigma}}{d} |\nabla_{\mathcal{K},\sigma}v|^2, \forall v \in X_{\mathcal{D}},$$

and

$$||\nabla_{\mathcal{D}} w(x,t)||^2_{L^2(0,T;L^2(\Omega)^d)} = \sum_{n=1}^{N} \delta t \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \frac{|\sigma| d_{\mathcal{K},\sigma}}{d} |\nabla_{\mathcal{K},\sigma} w^n|^2, \quad \forall w \in X_{\mathcal{D},\delta t},$$

where $\nabla_{\mathcal{K},\sigma}$ *and* $\nabla_{\mathcal{D}}$ *is defined by* (26) *and* (27)

Lemma 2.11 *Let* $\mathcal{D}$ *be a discretisation of* Ω *in the sense of the Definition* (2.1)*, and* δt *be a time step defined in Definition* (2.5) *and we assume that there exists a positive* θ *such that* $\theta_{\mathcal{D}} \leq \theta$ *for all* $\mathcal{D}$*,*

1. *Then there exist positive constants* C_2 *and* C_3 *only depending on* θ *and* d *such that*

$$C_2|v|^2_{X_{\mathcal{D}}} \leq ||\nabla_{\mathcal{D}} v||^2_{L^2(\Omega)^d} \leq C_3|v|^2_{X_{\mathcal{D}}} \forall v \in X_{\mathcal{D}}. \tag{42}$$

2. *Moreover, we have*

$$C_4|w|^2_{X_{\mathcal{D},\delta t}} \leq ||\nabla_{\mathcal{D}} w||^2_{L^2(0,T;L^2(\Omega)^d} \leq C_5|w|^2_{X_{\mathcal{D},\delta t}} \forall v \in X_{\mathcal{D}\delta t}. \tag{43}$$

Definition 2.12 *Let* $\mathcal{D}$ *be a discretisation of* Ω *in the sense of Definition* (2.1) *and* δt *be a time step defined in Definition* (2.5)*. Let* $u_{\mathcal{D},\delta t} \in X_{\mathcal{D},\delta t}$ *be a solution of the problem. Then,* $\mathcal{P}_{\mathcal{M}} u_{\mathcal{D},\delta t}(x,t)$ *is an approximate solution of the problem.*

Lemma 2.13 (Discrete gradient consistency) *Let* $\mathcal{D}$ *be a discretisation of* Ω *in the sense of Definition 2.1, and* $\theta \geq \theta_{\mathcal{D}}$ *given. Then, for any function* $\psi \in C^2(\overline{\Omega})$*, there exists* C_6 *only depending on* d*,* θ *and* ψ *such that*

$$\|\nabla_{\mathcal{D}} \mathcal{P}_{\mathcal{D}} \psi - \nabla \psi\|_{(L^\infty(\Omega))^d} \leq C_6 h_{\mathcal{D}}, \tag{44}$$

where $\nabla_{\mathcal{D}}$ *is defined by* (25–27)*.*

Lemma 2.14 (A compactness lemma) *Let* Ω *be a bounded open subset of* $\mathbb{R}^d$*,* $T > 0$ *and* $A \subset L^1(0,T;L^1_{Loc}(\Omega))$*. If* A *is relatively compact in* $L^1(0,T;(C^2_c(\Omega))')$ *and for all* ω *relatively compact in* Ω*,*

$$\sup_{u\in A} \|u(.,.+\xi) - u\|_{L^1(]0,T[\times\omega)} \to 0 \text{ as } |\xi| \to 0.$$

Then, A *is relatively compact in* $L^1(0,T;L^1_{Loc}(\Omega))$*.*

Proof 2.15 *The proof of this propriety can be found in [12].*

2.5 Discrete Weak Formulation

In this section we present the discrete weak formulation for the problem (1–2). We consider, for all $n = 0, \ldots, N-1$ and all $\mathcal{K} \in \mathcal{M}$, unknowns $c^{n+1} \in X_{\mathcal{D},\delta t}$, $U^n \in X_{\mathcal{D},\delta t}$ and $p^n \in X_{\mathcal{D},\delta t}$ which stand for approximate values of c, U and p on $[n; n+1]$,

2.5.1 Equation of the pressure

We begin with the discretisation of this equation

$$-\operatorname{div}(K(x,c)\nabla p) = q^+ - q^-. \tag{45}$$

We integer over $\mathcal{K}$ for any $\mathcal{K} \in \mathcal{M}$ and in the interval $]t^n, t^{n+1}[\subset]0, T[$ that which yields

$$\int_{\mathcal{K}}\int_{t^n}^{t^{n+1}} -div(K(x,c)\nabla p) = \int_{\mathcal{K}}\int_{t^n}^{t^{n+1}} (q^+ - q^-),$$

that's gives

$$\delta t \int_{\mathcal{K}} -\operatorname{div}(K(x,c^n)\nabla p^{n+1}) = \delta t \int_{\mathcal{K}} (q^{+,n+1} - q^{-,n+1}),$$

then

$$\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \int_{\sigma} K(x,c^n)\nabla p^{n+1}.n_{\mathcal{K},\sigma} = m_{\mathcal{K}}(q_{\mathcal{K}}^{+,n+1} - q_{\mathcal{K}}^{-,n+1}),$$

finally

$$\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}^1_{\mathcal{K},\sigma}(p^{n+1}) = m_{\mathcal{K}} q_{\mathcal{K}}^{+,n+1} - \mathcal{K} q_{\mathcal{K}}^{-,n+1}. \tag{46}$$

For the border elements, we obtain the equations by discretising the second part of the system (5), then we have

$$K(x, c^n_{\mathcal{K}})\nabla_{\mathcal{K},\sigma} p^{n+1}.n_{\mathcal{K},\sigma} = 0 \ ,\forall \sigma \in \mathcal{E}_{ext}. \tag{47}$$

We use the fact that the flow is continuous at the interface of the two elements, we have then

$$F^1_{\mathcal{K},\sigma}(p^{n+1}) + F^1_{\mathcal{L},\sigma}(p^{n+1}) = 0 \text{ for all } \sigma \in \mathcal{E}_{int} \text{ such that } \mathcal{M}_\sigma = \{\mathcal{K},\mathcal{L}\}. \tag{48}$$

And now we have $card(\mathcal{E}_{int}) + card(\mathcal{E}_{ext}) + card(\mathcal{M})$ unknowns and equations.

Let, multiplying the equation (46) by $v_{\mathcal{K}}^{n+1}$ for all $\mathcal{K} \in \mathcal{M}$ and all $n = 0, ..., N-1$, then sum over $\mathcal{K}$ and over $n = 0, \ldots, N-1$, we get

$$\sum_{n=0}^{N-1} \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}_{\mathcal{K}}^{1}(p^{n+1})v_{\mathcal{K}}^{n+1} = \sum_{n=0}^{N-1} \sum_{\mathcal{K}\in\mathcal{M}} v_{\mathcal{K}}^{n+1} m_{\mathcal{K}}(q_{\mathcal{K}}^{+,n+1} - q_{\mathcal{K}}^{-,n+1}), \tag{49}$$

which gives

$$< p, v >_{F^1} = \sum_{n=0}^{N-1} \sum_{\mathcal{K}\in\mathcal{M}} v_{\mathcal{K}}^{n+1} m_{\mathcal{K}}(q_{\mathcal{K}}^{+,n+1} - q_{\mathcal{K}}^{-,n+1}), \tag{50}$$

with

$$< p, v >_{F^1} = \sum_{n=0}^{N-1} \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}_{\mathcal{K}}^{1}(p^{n+1})[v_{\mathcal{K}}^{n+1} - v_{\sigma}^{n+1}]. \tag{51}$$

We define also

$$[p^{n+1}, v^{n+1}]_{F^1} = \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}_{\mathcal{K}}^{1}(p^{n+1})[v_{\mathcal{K}}^{n+1} - v_{\sigma}^{n+1}]. \tag{52}$$

2.5.2 The constitutive law of the model adopted

For the second equation we have

$$U = -K(x, c)\nabla p, \text{ in }]0, T[\times\Omega. \tag{53}$$

We integer over $\mathcal{D}$ such that $\mathcal{D} \in \mathfrak{D}$ with $\mathfrak{D}$ is the set of diamond Ω, and over the time interval $]t^n, t^{n+1}[\subset]0, T[$, we obtain

$$\int_{t^n}^{t^{n+1}} \int_{\mathcal{D}} U dx dt = \int_{t^n}^{t^{n+1}} \int_{\mathcal{D}} -K(x, c)\nabla p dx dt, \tag{54}$$

after simplifications we obtain the following formula

$$U_{\mathcal{D}}^{n+1} = (-K(x_\sigma, c^n)\nabla p^{n+1})_{\mathcal{D}}.$$

Since for any diamond $\mathcal{D} \in \mathfrak{D}$, we have

$$\mathcal{D} = \begin{cases} \{\mathcal{K}, \sigma\} \cup \{\mathcal{L}, \sigma\} & \text{if } \sigma \in \mathcal{E}_{int}, \\ \{\mathcal{K}, \sigma\} & \text{if } \sigma \in \mathcal{E}_{extt}. \end{cases}$$

Then

$$U_{\mathcal{D}}^{n+1} = \begin{cases} U_{\mathcal{K},\sigma}^{n+1} + U_{\mathcal{L},\sigma}^{n+1}, & \text{if } \sigma \in \mathcal{E}_{int}, \\ U_{\mathcal{K},\sigma}^{n+1}, & \text{if } \sigma \in \mathcal{E}_{ext}, \end{cases}$$

and

$$\nabla^{\mathcal{D}} p^{n+1} = \begin{cases} \nabla_{\mathcal{K},\sigma} p^{n+1} + \nabla_{\mathcal{L},\sigma} p^{n+1}, & \text{if } \sigma \in \mathcal{E}_{int} \\ \nabla_{\mathcal{K},\sigma} p^{n+1}, & \text{if } \sigma \in \mathcal{E}_{ext}. \end{cases}$$

Finally, we get

$$\begin{cases} U_{\mathcal{K},\sigma}^{n+1} + U_{\mathcal{L},\sigma}^{n+1} = -K(x_\sigma, c_{\mathcal{K}}^n) \nabla_{\mathcal{K},\sigma} p^{n+1} - K(x_\sigma, c_{\mathcal{L}}^n) \nabla_{\mathcal{L},\sigma} p^{n+1}, & \text{if } \sigma \in \mathcal{E}_{int}, \\ U_{\mathcal{K},\sigma}^{n+1} = -K(x_\sigma, c_{\mathcal{K}}^n) \nabla_{\mathcal{K},\sigma} p^{n+1}, & \text{if } \sigma \in \mathcal{E}_{ext}, \end{cases} \tag{55}$$

with $\nabla_{\mathcal{K},\sigma} p^{n+1}$, $\nabla_{\mathcal{L},\sigma} p^{n+1}$ are noted in (28).
(55) write in other way by

$$\begin{cases} U_{\mathcal{K},\sigma}^{n+1}.n_{\sigma,\mathcal{K}} + U_{\mathcal{L},\sigma}^{n+1}.n_{\sigma,\mathcal{L}} = \mathcal{F}_{\mathcal{K},\sigma}^1(p^{n+1}) + \mathcal{F}_{\mathcal{L},\sigma}^1(p^{n+1}), & \text{if } \sigma \in \mathcal{E}_{int} \\ U_{\mathcal{K},\sigma}^{n+1}.n_{\sigma,\mathcal{K}} = \mathcal{F}_{\mathcal{K},\sigma}^1(p^{n+1}), & \text{otherwise.} \end{cases} \tag{56}$$

2.5.3 Concentration equation

Now, use discrete the following third equation

$$\phi(x)\partial_t c - div(D(x,U)\nabla c) + div(cU) + q^- c = q^+ \widehat{c}. \tag{57}$$

We integrate over the volume control $\mathcal{K} \in \mathcal{M}$ and over the time interval $]t^n, t^{n+1}[\subset [0,T]$ we obtain

$$\int_{t^n}^{t^{n+1}} \int_{\mathcal{K}} \phi(x)\partial_t c - \int_{t^n}^{t^{n+1}} \int_{\mathcal{K}} div(D(x,U)\nabla c) + \int_{t^n}^{t^{n+1}} \int_{\mathcal{K}} div(cU) + \int_{t^n}^{t^{n+1}} \int_{\mathcal{K}} q^- c = \int_{t^n}^{t^{n+1}} \int_{\mathcal{K}} q^+ \widehat{c}.$$

Then

$$\int_{\mathcal{K}} \phi(x)(c^{n+1} - c^n) - \delta t \int_{\mathcal{K}} div(D(x,U^{n+1})\nabla c^{n+1}) + \delta t \int_{\mathcal{K}} div(c^{n+1}U^{n+1}) + \delta t \int_{\mathcal{K}} q^{-,n+1} c^{n+1} = \delta t \int_{\mathcal{K}} q^{+,n+1} \widehat{c}^{n+1},$$

we get

$$m_{\mathcal{K}}\phi_{\mathcal{K}}(c_{\mathcal{K}}^{n+1}-c_{\mathcal{K}}^{n})+\delta t\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\mathcal{F}_{\mathcal{K},\sigma}^{2}(c^{n+1})+\delta t\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}m_{\sigma}divc_{\sigma}(c_{\mathcal{D}}^{n+1},U_{\mathcal{D}}^{n+1})$$
$$+\delta t m_{\mathcal{K}}q_{\mathcal{K}}^{-,n+1}c_{\mathcal{K}}^{n+1}=\delta t m_{\mathcal{K}}q_{\mathcal{K}}^{+,n+1}\widehat{c}_{\mathcal{K}}^{n+1}. \tag{58}$$

We have $card(\mathcal{M})$ equations and $card(\mathcal{E})+card(\mathcal{M})$ unknowns, for a reasonable system we need to $card(\mathcal{E})$ equations, for that we have

$$D(x_{\mathcal{K}},U_{\mathcal{K}}^{n+1})\nabla_{\mathcal{K},\sigma}c^{n+1}.n_{\mathcal{K},\sigma}=0\ ,\forall\sigma\in\mathcal{E}_{ext}, \tag{59}$$

and since the flux is continuous with the interface of the two elements, then we have

$$F_{\mathcal{K},\sigma}^{2}(c^{n+1})+F_{\mathcal{L},\sigma}^{2}(c^{n+1})=0 \text{ for all } \sigma\in\mathcal{E}_{int} \text{ such that } \mathcal{M}_{\sigma}=\{\mathcal{K},\mathcal{L}\}. \tag{60}$$

We have now $card(\mathcal{E}_{int})+card(\mathcal{E}_{ext})+card(\mathcal{M})$ unknowns and equations.

We multiplying (58) by $w_{\mathcal{K}}^{n+1}$ for all $w_{\mathcal{K}}\in\mathcal{M}$ and all $n=0,\dots,N-1$, then sum over $\mathcal{K}$ and over $n=0,\dots,N-1$, then we get

$$\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}w_{\mathcal{K}}^{n+1}\phi_{\mathcal{K}}(c_{\mathcal{K}}^{n+1}-c_{\mathcal{K}}^{n})+\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\mathcal{F}_{\mathcal{K},\sigma}^{2}(c^{n+1})w_{\mathcal{K}}^{n+1}$$
$$+\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}w_{\mathcal{K}}^{n+1}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}m_{\sigma}divc_{\sigma}(c_{\mathcal{D}}^{n+1},U_{\mathcal{D}}^{n+1})$$
$$+\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}w_{\mathcal{K}}^{n+1}q_{\mathcal{K}}^{-,n+1}c_{\mathcal{K}}^{n+1}=\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}w_{\mathcal{K}}^{n+1}q_{\mathcal{K}}^{+,n+1}\widehat{c}_{\mathcal{K}}^{n+1}.$$

Bearing in mind (60), from above, we get

$$\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}w_{\mathcal{K}}\phi_{\mathcal{K}}(c_{\mathcal{K}}^{n+1})-\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\mathcal{F}_{\mathcal{K},\sigma}^{2}(c^{n+1})[w_{\mathcal{K}}^{n+1}-w_{\sigma}^{n+1}]$$
$$+\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}w_{\mathcal{K}}^{n+1}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}m_{\sigma}divc_{\sigma}(c_{\mathcal{D}}^{n+1},U_{\mathcal{D}}^{n+1})$$
$$+\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}w_{\mathcal{K}}^{n+1}q_{\mathcal{K}}^{-,n+1}c_{\mathcal{K}}^{n+1}$$
$$=\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}w_{\mathcal{K}}^{n+1}[q_{\mathcal{K}}^{+,n+1}\widehat{c}_{\mathcal{K}}^{n+1}+\phi_{\mathcal{K}}(c_{\mathcal{K}}^{n})],$$

thus, we give as a form of bilinear approximation the following formula

$$\begin{cases} < c, w >_{F^2} = \sum_{n=0}^{N-1} \sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1} \phi_{\mathcal{K}}(c_{\mathcal{K}}^{n+1}) + \delta t \sum_{n=0}^{N-1} \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}_{\mathcal{K},\sigma}^2(c^{n+1})[w_{\mathcal{K}}^{n+1} - w_{\sigma}^{n+1}] \\ +\delta t \sum_{n=0}^{N-1} \sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma divc_\sigma(c_{\mathcal{D}}^{n+1}, U_{\mathcal{D}}^{n+1}) + \delta t \sum_{n=0}^{N-1} \sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1} q_{\mathcal{K}}^{-,n+1} c_{\mathcal{K}}^{n+1}. \end{cases} \tag{61}$$

We define also

$$\begin{cases} [c^{n+1}, w^{n+1}]_{F^2} = \sum_{\mathcal{K}\in\mathcal{M}} \frac{w_{\mathcal{K}}^{n+1}\phi_{\mathcal{K}}}{\delta t}(c_{\mathcal{K}}^{n+1}) + \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}_{\mathcal{K},\sigma}^2(c^{n+1})[w_{\mathcal{K}}^{n+1} - w_{\sigma}^{n+1}] \\ + \sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma divc_\sigma(c_{\mathcal{D}}^{n+1}, U_{\mathcal{D}}^{n+1}) + \sum_{\mathcal{K}\in\mathcal{M}} w_{\mathcal{K}}^{n+1} q_{\mathcal{K}}^{-,n+1} c_{\mathcal{K}}^{n+1}. \end{cases} \tag{62}$$

2.6 The Discrete Flux

The discrete flux $\mathcal{F}_{\mathcal{K},\sigma}^1$ and $\mathcal{F}_{\mathcal{K},\sigma}^2$ are expressed in terms of the discrete unknowns. For this purpose we apply the SUSHI scheme proposed in [20]. The idea is based upon the identification of the numerical flux through the mesh dependent bilinear form, using the expression of the discrete gradient

$$\sum_{n=0}^{N-1} \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}_{\mathcal{K},\sigma}^1(p^{n+1})(u_{\mathcal{K}} - u_\sigma) \approx \int_0^T \int_\Omega \nabla_{\mathcal{D}} p^{n+1} K(x,s) \nabla_{\mathcal{D}} u, \forall p^{n+1}, u \in X_{0,\mathcal{D}}, \tag{63}$$

and

$$\sum_{n=0}^{N-1} \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}_{\mathcal{K},\sigma}^2(c^{n+1})(v_{\mathcal{K}} - v_\sigma) \approx \int_0^T \int_\Omega \nabla_{\mathcal{D}} c^{n+1} D(x, U^{n+1}) \nabla_{\mathcal{D}} v, \forall c^{n+1}, v \in X_{0,\mathcal{D}}. \tag{64}$$

The identification of the numerical fluxes using relation (63) and (64) leads to the expression

$$F_{\mathcal{K},\sigma}^1(p^n) = \sum_{\sigma'\in\mathcal{E}_{\mathcal{K}}} K_{\mathcal{K}}^{\sigma,\sigma'}(p_{\mathcal{K}}^{n+1} - p_{\sigma}^{n+1}), \tag{65}$$

$$F^2_{\mathcal{K},\sigma}(c^{n+1}) = \sum_{\sigma'\in\mathcal{E}_\mathcal{K}} D^{\sigma,\sigma'}_\mathcal{K}(c^{n+1}_\mathcal{K} - c^{n+1}_\sigma). \tag{66}$$

Thus

$$\int_\mathcal{K} \nabla_\mathcal{D} p^{n+1} K(x,c^n)\nabla_\mathcal{D} u = \sum_{\sigma\in\mathcal{E}_\mathcal{K}}\sum_{\sigma'\in\mathcal{E}_\mathcal{K}} K^{\sigma,\sigma'}_\mathcal{K}(p^{n+1}_\mathcal{K} - p^{n+1}_{\sigma'})(u_{\sigma'} - u_\mathcal{K}). \tag{67}$$

$$\int_\mathcal{K} \nabla_\mathcal{D} c^{n+1} D(x,U^{n+1})\nabla_\mathcal{D} v = \sum_{\sigma\in\mathcal{E}_\mathcal{K}}\sum_{\sigma'\in\mathcal{E}_\mathcal{K}} D^{\sigma,\sigma'}_\mathcal{K}(c^{n+1}_\mathcal{K} - c^{n+1}_{\sigma'})(v_{\sigma'} - v_\mathcal{K}). \tag{68}$$

With $\sigma, \sigma' \in \mathcal{E}_\mathcal{K}$ and

$$\begin{cases} K^{\sigma,\sigma'}_\mathcal{K} = \displaystyle\sum_{\sigma''\in\mathcal{E}_\mathcal{K}} Y^{\sigma'',\sigma}\Gamma^{\sigma''}_\mathcal{K} Y^{\sigma'',\sigma'} & \text{with } \Gamma^{\sigma''}_\mathcal{K} = \displaystyle\int_{\mathcal{K},C_{\sigma''}} K(x,c^n)dx, \\ D^{\sigma,\sigma'}_\mathcal{K} = \displaystyle\sum_{\sigma''\in\mathcal{E}_\mathcal{K}} Y^{\sigma'',\sigma}\Theta^{\sigma''}_\mathcal{K} Y^{\sigma'',\sigma'} & \text{with } \Theta^{\sigma''}_\mathcal{K} = \displaystyle\int_{\mathcal{K},C_{\sigma''}} D(x,U^{n+1})dx. \end{cases}$$

The local matrices $K^{\sigma,\sigma'}_\mathcal{K}$ and $D^{\sigma,\sigma'}_\mathcal{K}$ are symmetric and positive.

2.7 Final Scheme

Using (26) we have

$$\begin{cases} \nabla_{\mathcal{K},\sigma} p^{n+1} = \nabla_\mathcal{K} p^{n+1} + \mathcal{R}_{\mathcal{K},\sigma} p^{n+1}.n_{\mathcal{K},\sigma}, \\ \nabla_{\mathcal{K},\sigma} c^{n+1} = \nabla_\mathcal{K} c^{n+1} + \mathcal{R}_{\mathcal{K},\sigma} c^{n+1}.n_{\mathcal{K},\sigma}, \end{cases}$$

and

$$divc_\sigma(\xi_\mathcal{D}, v_\mathcal{D}) = \begin{cases} v_\mathcal{K} n_{\sigma,\mathcal{K}}\xi_\mathcal{K} & \text{if } v_\mathcal{K} n_{\sigma,\mathcal{K}} \geq 0, \\ -v_\mathcal{K} n_{\sigma,\mathcal{K}}\xi_\mathcal{L} & \text{if } v_\mathcal{K} n_{\sigma,\mathcal{K}} < 0. \end{cases}$$

The discretisation of the problems (1) and (2) is defined as following:

$$c(x,0) = \frac{1}{m_\mathcal{K}}\int_{\mathcal{K}\in\mathcal{M}} c_0(x)dx. \tag{69}$$

$$\sum_{\mathcal{K}\in\mathcal{M}} m_\mathcal{K} p_\mathcal{K} = 0. \tag{70}$$

$$\begin{cases} D_{\mathcal{K},\sigma}\nabla_\mathcal{D} c.n = 0 \\ U_\mathcal{D}.n = 0. \end{cases} \tag{71}$$

$$
\begin{cases}
\text{Find for all } \mathcal{K} \in \mathcal{M} \text{ and for all instant } n, p^{n+1} \text{ and } c^{n+1} \\
\displaystyle\sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} \sum_{\sigma' \in \mathcal{E}_{\mathcal{K}}} A_{\mathcal{K}}^{\sigma\sigma'}[p_{\sigma}^{n+1} - p_{\mathcal{K}}^{n+1}][v_{\sigma'} - v_{\mathcal{K}}] = m_{\mathcal{K}} v_{\mathcal{K}} (q_{\mathcal{K}}^{+,n+1} - q_{\mathcal{K}}^{-,n+1}), \\
U_{\mathcal{D}}^{n+1} = K(x, c_{\mathcal{K}}^{n})) \nabla_{\mathcal{D}} p^{n+1}, \\
m_{\mathcal{K}} v_{\mathcal{K}} \phi_{\mathcal{K}}(c_{\mathcal{K}}^{n+1}) - \delta t \displaystyle\sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} \sum_{\sigma' \in \mathcal{E}_{\mathcal{K}}} B_{\mathcal{K}}^{\sigma\sigma'}[c_{\sigma}^{n+1} - c_{\mathcal{K}}^{n+1}][v_{\sigma'} - v_{\mathcal{K}}] \\
+\delta t v_{\mathcal{K}} \displaystyle\sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} m_{\sigma} div c_{\sigma}(c_{\mathcal{D}}^{n+1}, U_{\mathcal{D}}^{n+1}) + \delta t m_{\mathcal{K}} v_{\mathcal{K}} q_{\mathcal{K}}^{-,n+1} c_{\mathcal{K}}^{n+1} \\
\quad = \delta t m_{\mathcal{K}} v_{\mathcal{K}} [q_{\mathcal{K}}^{+,n+1} \widehat{c}_{\mathcal{K}}^{n+1} + \phi_{\mathcal{K}} c_{\mathcal{K}}^{n}].
\end{cases}
\tag{72}
$$

with

$$
\begin{cases}
A_{\mathcal{K}}^{\sigma,\sigma'} = \displaystyle\sum_{\sigma'' \in \mathcal{E}_{\mathcal{K}}} Y^{\sigma'',\sigma}.D_{\mathcal{K},\sigma''}^{1} Y^{\sigma'',\sigma'}, \\
B_{\mathcal{K}}^{\sigma\sigma'} = \displaystyle\sum_{\sigma'' \in \mathcal{E}_{\mathcal{K}}} Y^{\sigma'',\sigma}.D_{\mathcal{K},\sigma''}^{2} Y^{\sigma'',\sigma'},
\end{cases}
\tag{73}
$$

and

$$
\begin{cases}
Y^{\sigma\sigma'} \text{ given by (29)}, \\
D_{\mathcal{K},\sigma''}^{1} = \displaystyle\int_{\mathcal{C}_{\mathcal{K},\sigma''}} K(x, c_{\mathcal{K}}^{n}) dx, \\
D_{\mathcal{K},\sigma''}^{2} = \displaystyle\int_{\mathcal{C}_{\mathcal{K},\sigma''}} D(x, U^{n+1}) dx.
\end{cases}
\tag{74}
$$

$\mathcal{C}_{\mathcal{K},\sigma''}$ is the cone with vertex $x_{\mathcal{K}}$ and basis σ''.

2.8 A Priori Estimates

Lemma 2.16 *Let Ω be an open bounded connected polygonal domain of $\mathbb{R}^2$ and let $\mathcal{D}$ be a SUSHI mesh of Ω in the sense of Definition* (2.1). *Assume* (4), (6–7) *and* (12) *hold and that the Scheme* (72) *has a solution $(p_{\mathcal{D},\delta t}, U_{\mathcal{D},\delta t}, c_{\mathcal{D},\delta t})$. Then, there exists $C' > 0$ depending only on Ω, α, C_1, C_2, C_5 and $\Lambda_{\mathcal{K}}$, such that we have for all $n \in [0, ..., N-1]$:*

$$
\begin{aligned}
||\mathcal{P}_{\mathcal{M}} p_{\mathcal{D},\delta t}||_{1;1,2,\mathcal{M}}^{2} + ||\nabla p_{\mathcal{D},\delta t}||_{L^2(0,T;L^2(\Omega))}^{2} + ||U_{\mathcal{D},\delta t}||_{L^2(0,T;L^2(\Omega))}^{2} \\
\leq C' ||q^{+} - q^{-}||_{L^{\infty}(0,T;L^2(\Omega))}.
\end{aligned}
\tag{75}
$$

Proof 2.17 *For the proof see [1].*

Lemma 2.18 *Let Ω be an open bounded connected polygonal domain of $\mathbb{R}^2$ and let $\mathcal{D}$ be a SUSHI mesh of Ω in the sense of the Definition* (2.1). *Assume* (4), (6–8), (11) *and* (12) *hold and that the Scheme* (72) *has a solution* $(p_{\mathcal{D},\delta t}, U_{\mathcal{D},\delta t}, c_{\mathcal{D},\delta t})$. *Then, there exists $C" > 0$ depending only on Ω, $\alpha_{\mathcal{D}}$, ϕ_*, c_0, C_2, C_6 and q^+ such that we have*

$$\frac{\phi_*}{2}||c_{\mathcal{D}}^N||^2_{L^2(\Omega)} + \alpha_D|||U_{\mathcal{K}}^n|^{1/2}\nabla_{\mathcal{D}}c_{\mathcal{D},\delta t}||^2_{L^2(0,T;L^2(\Omega))} \\ +(1+\alpha_D)\|\nabla c_{\mathcal{D},\delta t}\|_{L^2(0,T;L^2(\Omega))}) \leq C". \qquad (76)$$

Proof 2.19 *For the proof see [1].*

2.9 Existence and Uniqueness of $(c_{\mathcal{D}}^n; U_{\mathcal{D}}^n; p_{\mathcal{D}}^n)$

Lemma 2.20 *Let $\mathcal{D}$ be a SUSHI mesh of Ω (Ω is an open bounded connected polygonal domain of $\mathbb{R}^2$). Let $T > 0$ and δt be a time step such that $N = \frac{T}{\delta t}$ is an integer. Assume* $(3-12)$ *hold. Then, the Scheme* $(72-74)$ *admits a unique solution* $(c_{\mathcal{D}}^n; U_{\mathcal{D}}^n; p_{\mathcal{D}}^n)_{1\leq n\leq N}$.

Proof 2.21 *To demonstrate this lemma we adapt the demonstration of Theorem 3.4 in [11].*

3 The Main Results

3.1 Numerical Results

In this section, we apply the schema using stabilisation and hybrid interface (SUSHI), firstly to the diffusion equation in the test number 1, then to the convection-diffusion-reaction equation in second test, finally to a three examples of two-dimensional miscible displacement problems of one incompressible fluid by another in porous media (test 3, 4 and 5) to examine its performance.

3.1.1 Test 1 (Convergence of the pressure equation)

As a first example we want to demonstrate the convergence of the pressure equation with Dirichlet boundary, we take the exact solution $p_1(x,y) = sin(\pi x)^2 sin(\pi y)^2$ in a first case and in a second case $p_2(x,y) = x^2y^2$ $(x-1)^2$ $(y-1)^2$ the permeability $K(x,y)$ is given by

$$K_1(x,y) = 80\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},$$

Table 1 Convergence results of the SUSHI on the pressure p, with $p_{ext} = p_1$ and $K = K_1$

Level	$\|p - p_{ext}\|_{L^2(\Omega)}$	$\|p - p_{ext}\|_{L^1(\Omega)}$	$\|p - p_{ext}\|_{L^\infty(\Omega)}$
1	0.0149	0.0045	0.1920
2	0.0033	0.0010	0.0848
3	8.0426e-04	2.4747e-04	0.0414
4	2.0011e-04	6.1305e-05	0.0206

Table 2 Convergence results of the SUSHI on the pressure p, with $p_{ext} = p_2$ and $K = K_2$

Level	$\|p - p_{ext}\|_{L^2(\Omega)}$	$\|p - p_{ext}\|_{L^1(\Omega)}$	$\|p - p_{ext}\|_{L^\infty(\Omega)}$
1	0.0159	0.0017	0.3076
2	0.0050	4.7231e-04	0.1941
3	0.0015	1.2424e-04	0.1124
4	4.2783e-04	3.1108e-05	0.0623

or by

$$K_2(x,y) = \frac{1}{x^2+y^2}\begin{bmatrix} 10^{-3}x^2 + y^2 & (10^{-3}-1)xy \\ (10^{-3}-1)xy & 10^{-3}y^2 + x^2 \end{bmatrix},$$

in both cases $\Omega = (0,1)^2$. Then we get the convergence Tables 1 and 2 in norm L^2, L^1 and L^∞ defined by the formulas (77).

3.1.2 Test 2 (Convergence of the parabolic-elliptic equations)

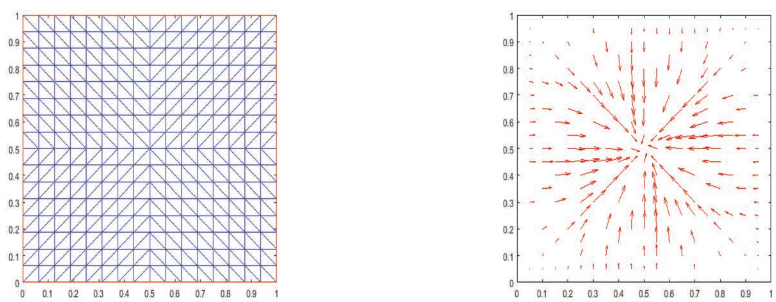
Now, we consider the following parabolic-elliptic system:

$$\begin{aligned}
-div(\frac{1}{\mu(c)}\nabla p) &= f_1 \text{ in } \Omega \times [0,T], \\
U &= \frac{1}{\mu(c)}\nabla p \text{ in } \Omega \times [0,T], \\
\int_\Omega p(x)dx &= 0, \\
\phi\frac{\partial c}{\partial t} - div(D(u)\nabla c) + div(uc) &= f_2 \text{ in } \Omega \times [0,T], \\
U = 0 \text{ and } D(u)\nabla c &= 0 \text{ on } \partial\Omega \times [0,T], \\
c(0,.) &= 0 \text{ in } \Omega.
\end{aligned}$$

Under the following assumptions : $\Omega = [0, 1]^2$, $t \in [0, T]$ $T > 0$. Let f_1, f_2 are chosen such that the exacts solutions are

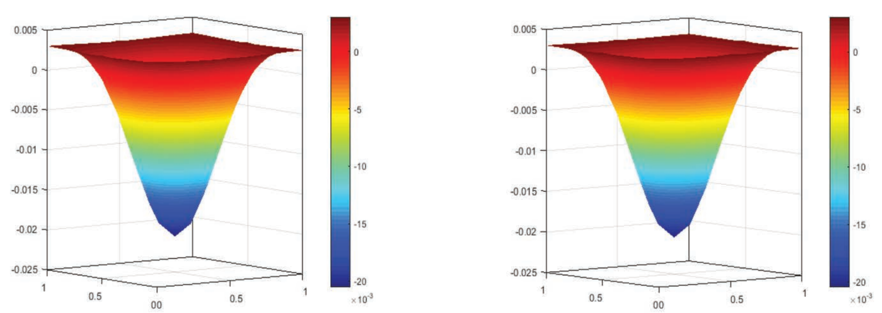
$$\begin{cases} c(x,t) = sin^2(\pi x) sin^2(\pi y) t, \\ p(x,t) = -0.5c^2 - 2c + \frac{9}{128}t^2 + 0.25t, \\ U(x,t) = \pi t [sin(2\pi x) sin^2(\pi y), sin(2\pi y) sin^2(\pi x)], \end{cases}$$

where $D(u) = |U| + 0.02$, $d_m = 0.02$, $d_l = d_t = 1$, $\mu(c) = c + 2$, $\phi = 1$, and the time steps is $dt = 10^{-4}$. We obtained the following results (1(b)–3(b)) and the tables of convergence (3–5):



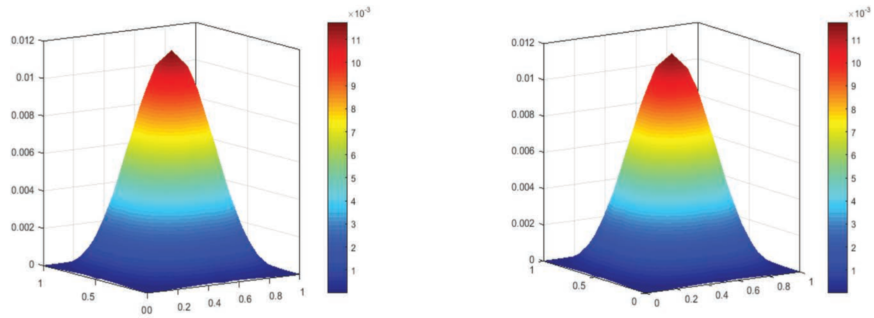
(a) Triangular meshes with a refinement level $i = 3$ (b) Darcy velocity U

Figure 1 Pressure gradient at $t = 0.0120$ and the mesh used with $h = 0.1188$.



(a) Pressure exact solution p (b) Pressure numerical solution p

Figure 2 The exact and numerical solutions of the pressure equation at $t = 0.0120$.



(a) Concentration exact solution (b) Concentration numerical solution

Figure 3 The exact and numerical solutions of the concentration equation at $t = 0.0120$.

The surface plots for the pressure, the gradient of the pressure and the concentration of the invading fluid at $t = 10^{-3}$ with the step of the mesh is $h = 0.1188$ are presented respectively in Figures 1(b)–3(b). The Tables 3, 4 and 5 are present respectively the norms $L^1(\Omega)$, $L^2(\Omega)$ and $L^\infty(\Omega)$ between the exact solution and the numerical solution of the pressure and the concentration, these norms are defined by the following formulas:

$$\begin{cases} errl1 = \|u_\mathcal{T} - u(x)\|_{L^1(\Omega)} = \displaystyle\sum_{\mathcal{K}\in\mathcal{M}} m_\mathcal{K}|u_\mathcal{K} - u(x_\mathcal{K})|, \\ errl2 = \|u_\mathcal{T} - u(x)\|_{L^2(\Omega)} = \left(\dfrac{\displaystyle\sum_{\mathcal{K}\in\mathcal{M}} m_\mathcal{K}|u_\mathcal{K} - u(x_\mathcal{K})|^2}{\displaystyle\sum_{\mathcal{K}\in\mathcal{M}} m_\mathcal{K}|u(x_\mathcal{K})|^2}\right)^{1/2}, \\ errlinf = \|u_\mathcal{T} - u(x)\|_{L^\infty(\Omega)} = \displaystyle\max_{\mathcal{K}\in\mathcal{M}} |u_\mathcal{K} - u(x_\mathcal{K})|. \end{cases} \quad (77)$$

with $u_\mathcal{T}$ is the numerical solution and $u(x)$ is the exact solution.

Table 3 $L^1 - norm$ Convergence results of the SUSHI method on the pressure p and concentration c at $t = 10^{-3}$

Level	N.U	Step	$\|p - p_{ext}\|_{L^1(\Omega)}$	Order	$\|c - c_{ext}\|_{L^1(\Omega)}$	Order
1	56	0.4751	4.5716e-05	–	0.007	–
2	212	0.2375	1.2788e-05	1.1275	0.0018	1.3184
3	824	0.1188	3.2903e-06	1.1205	4.5244e-04	1.2572
4	3248	0.0594	9.2565e-07	1.1005	5.0308e-05	1.2709
5	12896	0.0297	4.0595e-07	1.0593	6.4287e-06	1.0569

Table 4 $L^2 - norm$ Convergence results of the SUSHI method on the pressure p and concentration c at $t = 10^{-3}$

Level	N.U	Step	$\|p - p_{ext}\|_{L^2(\Omega)}$	Order	$\|c - c_{ext}\|_{L^2(\Omega)}$	Order
1	56	0.4751	5.1180e-05	–	0.0086	–
2	212	0.2375	1.5027e-05	1.1240	0.0022	1.3353
3	824	0.1188	3.9538e-06	1.1202	5.6307e-04	1.2708
4	3248	0.0594	1.1598e-06	1.0986	5.8721e-05	1.2931
5	12896	0.0297	5.7188e-07	1.0517	1.0516e-06	1.0163

Table 5 $L^\infty - norm$ Convergence results of the SUSHI method on the pressure p and concentration c at $t = 10^{-3}$

Level	N.U	Step	$\|p - p_{ext}\|_{L^\infty(\Omega)}$	Order	$\|c - c_{ext}\|_{L^\infty(\Omega)}$	Order
1	56	0.4751	0.0014	–	0.0602	–
2	212	0.2375	8.0704e-04	1.0838	0.0329	1.2150
3	824	0.1188	5.2442e-04	1.7468	0.0157	1.2167
4	3248	0.0594	8.2744e-05	0.5705	0.0083	1.2533
5	12896	0.0297	7.9262e-05	1.0039	9.8790e-04	1.2314

And to calculate the order of convergence, let $or1$, $or2$ and $orinf$ are the orders of convergence, they are defined by for $i \geq 2$:

$$\begin{cases} or1 = \frac{ln(errl1_i)}{ln(errl1_{i-1})}, \\ or2 = \frac{ln(errl2_i)}{ln(errl2_{i-1})}, \\ orinf = \frac{ln(errlinf_i)}{ln(errlinf_{i-1})}. \end{cases}$$

3.1.3 Test 3 (Peaceman model with continuous permeability)

In this numerical test, the spatial domain is $\Omega = (0, 1000) \times (0, 1000) ft^2$, and the time period is $[0, 3600]$ days. The injection and the production well

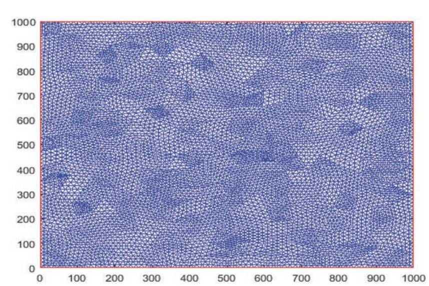


Figure 4 Triangular mesh used with a refinement level $i = 5$.

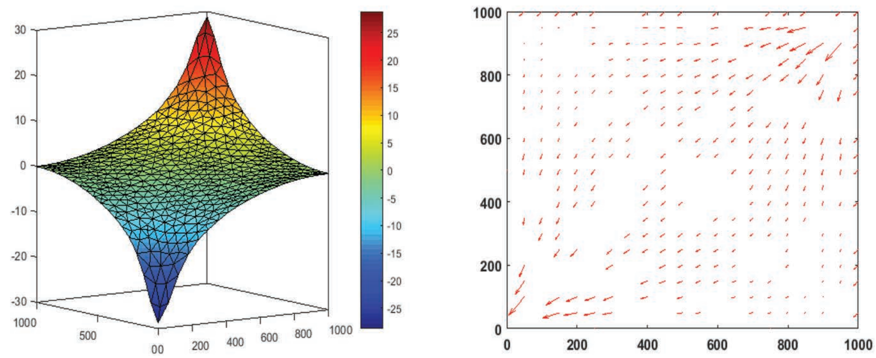


Figure 5 The pressure (left) and the gradient of the pressure (right) at $t = 3600$ days.

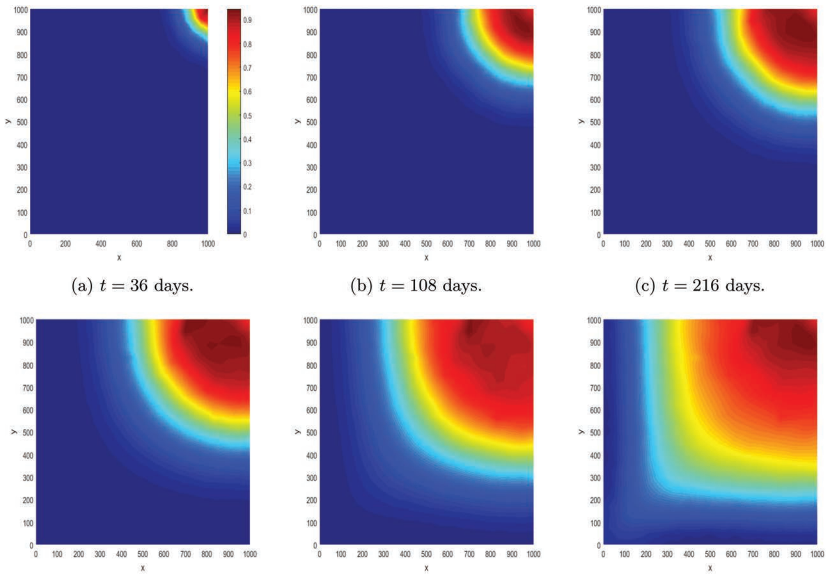
are respectively located at the upper-right corner $(1000, 1000)$ and the lower-left corner $(0, 0)$ with an injection rate $q^+ = 30ft^2/day$ and a production rate $q^- = 30ft^2/day$. The viscosity of the oil is $\mu(0) = 1.0cp$, the injection concentration is $\widehat{c} = 1.0$. The initial concentration is $c_0(x) = 0$ and the porosity of the medium is specified as $\phi(x) = 0.1$. We consider that the porous medium is homogeneous and isotropic and the permeability tensor is given by $K = 80I$ with I is the identity matrix. Let $M = 1$, and $\mu(c) = 1.0cp$. We assume that $\phi d_m = 1.0ft^2/day$, $\phi d_l = 5.0ft$ and $\phi d_t = 0.5ft$. on an unstructured mesh (4), we present the pressure and the speed (5) and the concentration at the different values of t in the Figures 6(a)–6(f).

3.1.4 Test 4 (Peaceman model with discontinuous permeability)

Later, let $c_0(x) = 0$ and the porosity of the medium is specified as $\phi(x) = 0.1$. We consider that the porous medium is homogeneous and isotropic and the permeability tensor is given by $K = (801_{y<500}+201_{y>500})I$. Let $M = 1$, and $\mu(c) = 1.0cp$. We assume that $\phi d_m = 1.0ft^2/day$, $\phi d_l = 5.0ft$ and $\phi d_t = 0.5ft$. In a structured mesh (7a) we calculate the norm V defined in Figure 7(b) in each step time and we remark in the Table 6 that $\sum_{\mathcal{K}\in\mathcal{M}} m_{\mathcal{K}} p_{\mathcal{K}}$ close to zeros.

3.1.5 Test 5 (Peaceman model with an adverse mobility ratio)

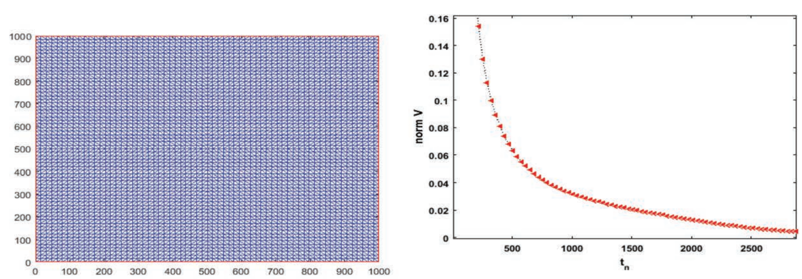
In this test case, we are interested in the case where the relation between the equation of the pressure and that of the concentration is strong with a discontinuity of the permeability $K(x, c)$, i.e $\mu(c) = (1 + (M^{1/4} - 1)c)^{-4}$ with $M = 41$ and if $(x, y) \in [200, 400] \times [200, 400] \cup [600, 800] \times [200, 400] \cup [200, 400] \times [600, 800] \cup [600, 800] \times [600, 800]$, $K(x, y) = 80$



(a) $t = 36$ days. (b) $t = 108$ days. (c) $t = 216$ days.

(d) $t = 1$ years. (e) $t = 3$ years. (f) $t = 10$ years.

Figure 6 Surfaces plot of concentration, at different value of t, with $\delta t = 36$ days and the mesh of the domain is made of 928 triangles of maximal edge length $50ft$.



(a) Triangular meshes with a refinement level $i = 5$. (b) The norm $V = \frac{\|c^{t_{k+1}} - c^{t_k}\|_{L^2(\Omega)}}{\|c^{t_{k+1}}\|_{L^2(\Omega)}}$.

Figure 7 Norm V of concentration (7b) and structured triangular meshes (7a).

Table 6 The value of the integral of the pressure $\int_\Omega p(x)dx$ at $t = 3600$, with $K = (801_{y<500} + 201_{y>500})I$

Refinement level	$\sum_{\mathcal{K}\in\mathcal{M}} m_{\mathcal{K}} p_{\mathcal{K}}$
1	4.330e-01
2	1.083e-01
3	2.76e-02
4	7e-03
5	1.8e-03

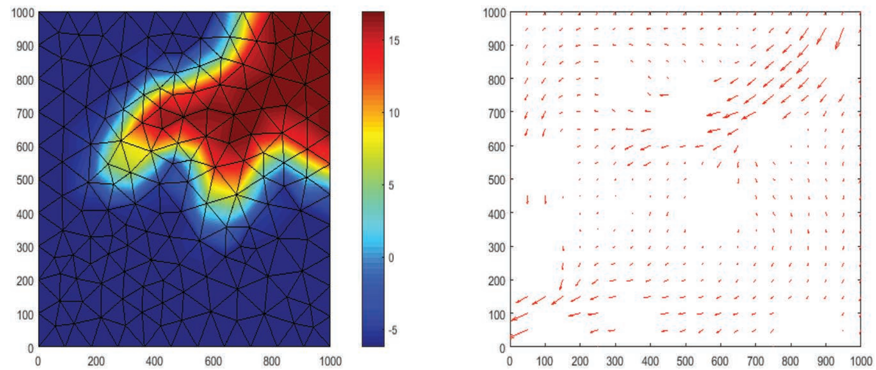


Figure 8 The pressure (left) and the gradient of the pressure (right) at $t = 3600days \approx 10years$.

else $K(x, y) = 20$. Let $c_0(x) = 0$ and the porosity of the medium is specified as $\phi(x) = 0.1$, and we assume that $\phi d_m = 0ft^2/day$, $\phi d_l = 5.0ft$ and $\phi d_t = 0.5ft$.

The Figure 8 represents the pressure and the pressure gradient at $t = 10$ years, and the Figure 9 represents the surfaces plot of concentration, at $t = 36$ days, $t \simeq 1$ years, $t \simeq 3$ years and $t \simeq 10$ years, with $\delta t = 36$ days.

3.2 Analysis Convergence

Theorem 3.1 *Let $\mathcal{D}_m$ be a family of discretisation in the sense of Definition* (2.1)*, for any $\mathcal{D} \in \mathcal{D}_m$, let $\mathcal{B} \in \mathcal{E}_{int}$ and $(\beta_\sigma^{\mathcal{K}})$ satisfying by* (18)*. Assume that there exists $\theta > 0$ such that $\theta_{\mathcal{D},\mathcal{B}} < \theta$ for all $\mathcal{D} \in \mathcal{T}$, where $\theta_{\mathcal{D},\mathcal{B}}$ is defined by* (33)*. Let $(\delta t_m)_{m\leq 1}$ be a sequence of time steps such that $T/\delta t_m$ is an integer and $\delta t_m \to 0$ as $m \to +\infty$.*

Then, the Scheme (72) *defines a sequence of approximation solution $(p_m = p_{\mathcal{D}_m,\delta t_m}, U_m = U_{\mathcal{D}_m,\delta t_m}, c_m = c_{\mathcal{D}_m,\delta t_m}) \in X_{\mathcal{T}_m,\delta t_m} \times X_{\mathcal{D}_m,\delta t_m} \times X_{\mathcal{T}_m,\delta t_m}$, there exists $\overline{p} \in L^\infty(0, T; H^1(\Omega)), \overline{U} \in L^\infty(0, T; L^2(\Omega)), \overline{c} \in L^\infty(0, T; L^2(\Omega))$, and, up to a subsequence, we have the following convergence results when $m \to \infty$*

$$
\begin{aligned}
&p_m \to \overline{p} \text{ weakly} - * \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ and strongly in} \\
&\qquad L^p(0, T; L^q(\Omega)), \forall p < \infty, q < 2; \\
&\nabla^{h_m} p_m \to \nabla\overline{p} \text{ weakly} - * \text{ in } (L^\infty(0, T; L^2(\Omega)))^2 \text{ and strongly in} \\
&\qquad (L^2((0, T) \times \Omega))^2;
\end{aligned}
$$

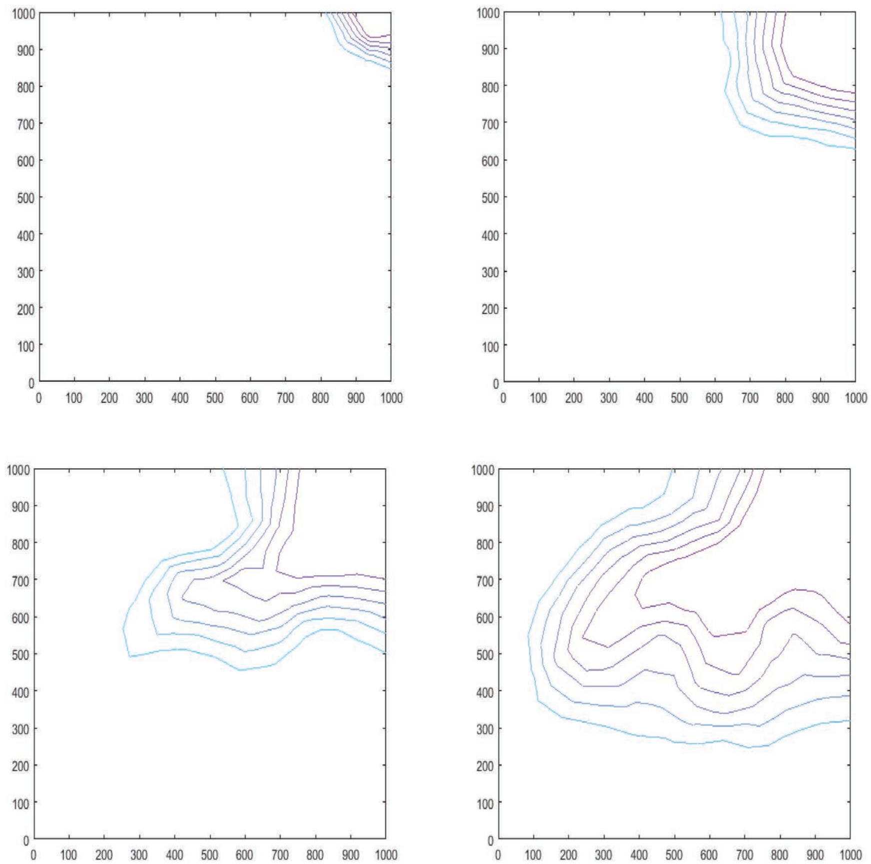


Figure 9 Surfaces plot of concentration, at $t = 36$ days, $t \simeq 1$ years, $t \simeq 3$ years and $t \simeq 10$ years, with $\delta t = 36$ days.

$$U_m \to \overline{U} \text{ weakly} - * \text{ in } (L^\infty(0,T;L^2(\Omega)))^2 \text{ and strongly in } (L^2((0,T)\times\Omega))^2;$$
$$c_m \to \overline{c} \text{ weakly} - * \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ and strongly in } L^p(0,T;L^q(\Omega)), \forall p<\infty, q<2;$$
$$\nabla^{h_m} c_m \to \nabla\overline{c} \text{ weakly in } L^2((0,T)\times\Omega))^2.$$

Moreover, $(\overline{p}, \overline{U}, \overline{c})$ *is a weak solution to* $(1{-}2)$.

3.2.1 Convergence of the pressure

Lemma 3.2 *Let F be a family of discretisation in the sense of Definition 2.1. Let δt_m be a sequence of times steps such that $T/\delta t_m$ is an integer and $\delta t_m \to 0$ as $m \to \infty$. We consider a sequence of functions $(v_m)_m$ with $v_m = v_{\mathcal{D}_m,\delta t_m} \in X_{\mathcal{D}_m,\delta t}$ when $m \to \infty$ such that*

$$\begin{aligned}
&v_m \to v \textit{ weakly in } L^2((0,T)\times\Omega),\\
&\quad(\textit{respectively weakly-* in } L^\infty(0,T;L^2(\Omega)))\\
&\nabla_{\mathcal{D}_m} v_m \to \chi \textit{ weakly in } (L^2((0,T)\times\Omega))^2,\\
&\quad(\textit{respectively weakly-* in } L^\infty(0,T;L^2(\Omega))).
\end{aligned}$$

Then, we have

$$\nabla v = \chi \textit{ and } v \in L^2(0,T;H^1(\Omega))(\textit{respectively in } L^\infty(0,T;H^1(\Omega)))$$

Proof 3.3 *An adaptation of the proof of Lemma 4.2 in [20], leads to prove that $\nabla v = \chi$ in the distribution sense on $]0,T[\times\Omega$, and therefore $v \in L^2(0,T;H^1(\Omega))$ or $v \in L^\infty((0,T)\times\Omega)$.*

Lemma 3.4 *Under the assumptions of Theorem (3.1), there exists $\overline{p} \in L^\infty(0,T;H^1(\Omega))$ and $\overline{U} \in (L^\infty(0,T;L^2(\Omega)))^2$, such that the sequences $(p_m)_m, (U_m)_m$ defined by the Scheme (72) have the following convergence result when $m \to \infty$*

$$\begin{aligned}
&p_m \to \overline{p} \textit{ weakly} - * \textit{ in } L^\infty(0,T;L^2(\Omega))\\
&\quad\textit{and strongly in } L^p(0,T;L^q(\Omega)), \forall p < \infty, q < 2;\\
&\nabla^{h_m} p_m \to \nabla\overline{p} \textit{ weakly} - * \textit{ in } (L^\infty(0,T;L^2(\Omega)))^2\\
&\quad\textit{and strongly in } (L^2((0,T)\times\Omega))^2;\\
&U_m \to \overline{U} \textit{ weakly} - * \textit{ in } (L^\infty(0,T;L^2(\Omega)))^2\\
&\quad\textit{and strongly in } (L^2((0,T)\times\Omega))^2;
\end{aligned}$$

and $(\overline{p},\overline{U})$ is a weak solution to (1).

Proof 3.5 ***step 1:*** *Using Lemma 2.16 (a priori estimate), we get when $m \to \infty$*

$$\begin{aligned}
p_m &\to \overline{p} \textit{ weakly-* in } L^\infty(0,T;L^2(\Omega));\\
\nabla_{h_m} p_m &\to v \textit{ weakly-* in } (L^\infty(0,T;L^2(\Omega)))^2;\\
U_m &\to \tilde{U} \textit{ weakly-* in } (L^\infty(0,T;L^2(\Omega)))^2;
\end{aligned}$$

Lemma 3.2 implies that

$$\overline{p} \in L^{\infty}(0, T; H^1(\Omega)), \text{ with } \nabla\overline{p} = v.$$

by (1)*, we have* $\int_\Omega p_m(t,.)dx = 0$ *for all* $t \in]0,T[$*, it gives that* $\int_\Omega \overline{p}(t,.)dx = 0$ *for all* $t \in]0,T[$*. we define the function* $A_{\mathcal{D}} : \Omega \times \mathbb{R} \to M_d(\mathbb{R})$ *by* $A_{\mathcal{D}}(x,s) = A_{\mathcal{K}}(s)$ *with* $s \in \mathbb{R}$ *and* $x \in \mathcal{K}$*. Let* $\breve{c} :]0,T[\times \to \mathbb{R}$ *such that*

$$\begin{cases} \breve{c} = c_{\mathcal{K}}^n, & \text{on } [n\delta t, (n+1)\delta t[\times\mathcal{K} \text{ with } n = 0, \dots, N; \\ \breve{c} = c_{\mathcal{K}}^0, & \text{on } [0, \delta t[\times\mathcal{K}; \\ \breve{c} = c(. - \delta t, .), & \text{on } [\delta t, T[\times\Omega. \end{cases}$$

We have $\breve{c} \to \overline{c}$ *in* $L^1(0,T;L^1(\Omega))$ *as* $\delta t \to 0$ *and* $h_d \to 0$*. for all* $Z \in L^2(]0,T[\times\Omega)^d$, $\int_0^T \int_\Omega Z.U = \int_0^T \int_\Omega -A_{\mathcal{D}}(.,\breve{c})^T Z.v$ *with* $U = -A_{\mathcal{D}}(.,\breve{c}).v$*. Using Lemma 7.6 in [12], we get* $\int_0^T \int_\Omega Z.U \to \int_0^T \int_\Omega -A_{\mathcal{D}}(.,c)^T Z.\nabla\overline{p}$*. That's give* $\overline{U} = \tilde{U}$*.*

Let $\Psi \in C_c^2(]0,T[\times\Omega)$ *such that*

$$\mathcal{P}_{\mathcal{D},\mathcal{B}}\Psi^n = \begin{cases} \Psi^n(x_{\mathcal{K}}), & \text{for all } \mathcal{K} \in \mathcal{M} \\ \Psi^n(x_\sigma), & \text{for all } \sigma \in \mathcal{H} \\ \sum_{\mathcal{K}\in\mathcal{M}} \beta_\sigma^{\mathcal{K}} \Psi^n(x_{\mathcal{K}}), & \text{for all } \sigma \in \mathcal{B} \\ \Psi^n(x_{\mathcal{K}}), & \text{for all } \sigma \in \mathcal{E}_{ext}. \end{cases}$$

Let us take $v = \mathcal{P}_{\mathcal{D},\mathcal{B}}\Psi$ *in* (50)*, we get*

$$\int_0^T \int_\Omega A(x,c)\nabla_{\mathcal{D}_m} p_m.\nabla_{\mathcal{D}_m}\mathcal{P}_{\mathcal{D},\mathcal{B}}\Psi dxdt = \int_0^T \int_\Omega \mathcal{P}_{\mathcal{D},\mathcal{B}}\Psi(q^+ - q^-)dxdt. \tag{78}$$

Since $\nabla_{\mathcal{D}_m} p_m \to \nabla\overline{p}$ *weakly in* $L^\infty(0,T;L^2(\Omega))^2$ *and the consistency of the discret gradient, we have*

$$\mathcal{P}_{\mathcal{D},\mathcal{B}}\Psi \to \nabla\Psi$$

Hence

$$\lim_{h_{\mathcal{D}_m}\to 0} \int_0^T \int_\Omega A(x,c)\nabla_{\mathcal{D}_m} p_m.\nabla_{\mathcal{D}_m}\mathcal{P}_{\mathcal{D},\mathcal{B}}\Psi dxdt = \int_0^T \int_\Omega A(x,c)\nabla\overline{p}.\nabla\Psi dxdt.$$

Therefore, p is the unique solution of (13)*, and we get that the whole family* $(p_{\mathcal{D}})_{\mathcal{D}\in\mathcal{F}}$ *converges to* p *as* $h_{\mathcal{D}} \to 0$.

step 2: *Let* $\psi \in C_c^\infty(]0,T[\times\Omega)$ *be begin. Thanks to the Cauchy-Schwartz inequality, we have*

$$\int_0^T \int_\Omega |\nabla_{\mathcal{D}} p_{\mathcal{D}}(x) - \nabla p(x)|^2 dx \leq 3(T_5^{\mathcal{D}} + T_6^{\mathcal{D}} + T_7), \tag{79}$$

with

$$T_5^{\mathcal{D}} = \int_0^T \int_\Omega |\nabla_{\mathcal{D}} p_{\mathcal{D}}(x) - \nabla_{\mathcal{D}} \mathcal{P}_{\mathcal{D}}\psi(x)|^2 dxdt, \tag{80}$$

$$T_6^{\mathcal{D}} = \int_0^T \int_\Omega |\nabla_{\mathcal{D}} \mathcal{P}_{\mathcal{D}}\psi(x) - \nabla\psi(x)|^2 dxdt, \tag{81}$$

and

$$T_7 = \int_0^T \int_\Omega |\nabla\psi(x) - \nabla p(x)|^2 dxdt. \tag{82}$$

Thanks to lemma of consistence in [20], we have $\lim_{h_{\mathcal{D}}\to 0} T_6^{\mathcal{D}} = 0$. *Thanks to Lemma 2.7 there exists* C_7 *such that*

$$||\nabla_{\mathcal{D}} w||^2_{L(0,T;L^2(\omega)^d)} \leq C_5 |w|^2_{\mathcal{D},\delta t} \leq C_7[w,w]_{F^1}, \forall w \in X_{\mathcal{D}}. \tag{83}$$

With $C_7 = \frac{C_5}{\alpha_K}$. *Taking* $w = p_m - \mathcal{P}_{\mathcal{D}}\psi$, *we have*

$$T_5^{\mathcal{D}} \leq C_7[p_m - \mathcal{P}_{\mathcal{D}}\psi, p_m - \mathcal{P}_{\mathcal{D}}\psi]_{F^1}. \tag{84}$$

By step 1, we get

$$\lim_{h_{\mathcal{D}}\to 0}[p_m - \mathcal{P}_{\mathcal{D}}\psi, p_m - \mathcal{P}_{\mathcal{D}}\psi]_{F^1} = \int_0^T \int_\Omega \nabla(p-\psi).K(x,c)\nabla(p-\psi)(x)dxdt. \tag{85}$$

Thats gives

$$\lim_{h_{\mathcal{D}}\to 0}[p_m - \mathcal{P}_{\mathcal{D}}\psi, p_m - \mathcal{P}_{\mathcal{D}}\psi]_{F^1} \leq \bar{\lambda} \int_0^T \int_\Omega |\nabla p(x) - \nabla\psi(x)|^2 dxdt.$$

which yields

$$\lim_{h_{\mathcal{D}}\to 0} \sup T_5^{\mathcal{D}} \leq C_7 \bar{\lambda} \int_0^T \int_\Omega |\nabla p(x) - \nabla\psi(x)|^2 dx. \tag{86}$$

Then, there exists C_8 independent of $\mathcal{D}$, such that

$$\int_0^T \int_\Omega |\nabla_\mathcal{D} p_\mathcal{D}(x) - \nabla p(x)|^2 dx \leq C_7 \int_0^T \int_\Omega |\nabla \psi(x) - \nabla p(x)|^2 dx + 3T_7^\mathcal{D}.$$
$$\leq C_8 \int_0^T \int_\Omega |\nabla \psi(x) - \nabla p(x)|^2 dx.$$

We may choose ψ such that $\int_0^T \int_\Omega |\nabla \psi(x) - \nabla p(x)| dx \leq \epsilon$, $\epsilon > 0$ and $h_\mathcal{D}$ small enough, then

$$\lim_{h_\mathcal{D} \to 0} \int_0^T \int_\Omega |\nabla_\mathcal{D} p_\mathcal{D}(x) - \nabla p(x)|^2 dx dt = 0. \tag{87}$$

Hence, we have the strongly convergent of $\nabla_\mathcal{D} p_m \to \nabla \overline{p}$. That's implies: $U_m \to \overline{U}$.
Then we have the proof.

3.2.2 Convergence of the concentration

Lemma 3.6 (compactness concentration) *Under the assumption of Theorem* (3.1)*, c is relatively compact in $L^1(0,T; L^1_{Loc}(\Omega))$.*

Proof 3.7 ***step 1:*** *Let $\tilde{c} : [0,T] \times \Omega \to \mathbb{R}$ continuous function in time and affine on each time interval. Hence, for all $n = 0, \ldots, N-1$ and all $t \in [n\delta t, (n+1)\delta t]$,*

$$\tilde{c}(t,.) = \frac{t - n\delta t}{\delta t} c_\mathcal{K}^{n+1} + \frac{(n+1)\delta t - t}{\delta t} c_\mathcal{K}^n \text{ on } \mathcal{K}.$$

Let $w \in C_c^2(\Omega)$ and $n \in \{0, \ldots, N-1\}$. Multiplying (58) *by $w_\mathcal{K}$, we get*

$$\begin{aligned}
&\int_\Omega \varphi_\mathcal{D}(x) \partial_t \tilde{c}(t,x) w(x) dx \\
&\quad = < \varphi_\mathcal{K} \frac{c_\mathcal{K}^{n+1} - c_\mathcal{K}^n}{\delta t}, w_\mathcal{K} > \\
&\quad = < div_\mathcal{K}(D \nabla_\mathcal{D} c_\mathcal{K}^{n+1}), w_\mathcal{K} > - < div c_\sigma(U_\mathcal{D}^{n+1}, c_\mathcal{K}^{n+1}), w_\mathcal{K} > - \\
&\quad < q_\mathcal{K}^{-,n+1} c_\mathcal{K}^{n+1}, w_\mathcal{K} > + < q_\mathcal{K}^{+,n+1} \hat{c}_\mathcal{K}^{n+1}, w_\mathcal{K} >.
\end{aligned}$$

Let

$$T_1 =< div_{\mathcal{K}}(D\nabla_{\mathcal{D}} c_{\mathcal{K}}^{n+1}), w_{\mathcal{K}} >, \tag{88}$$
$$= - < D_{\mathcal{D}}(U_{\mathcal{D}}^{n+1})\nabla_{\mathcal{D}} c_{\mathcal{K}}^{n+1}, \nabla_{\mathcal{D}} w_{\mathcal{K}} >, \tag{89}$$
$$= - \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_{\mathcal{K}} D_{\mathcal{K}}(U_{\mathcal{K}}^{n+1})\nabla_{\mathcal{K},\sigma} c_{\mathcal{K}}^{n+1}.\nabla_{\mathcal{K},\sigma} w_{\mathcal{K}}. \tag{90}$$

Then, the hypothesis (9) on D implies

$$|T_1| \leq \Lambda_{\mathcal{D}} |w|_X \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_{\mathcal{K}}(1 + |U_{\mathcal{K}}^{n+1}|)|\nabla_{\mathcal{K},\sigma} c_{\mathcal{K}}^{n+1}|. \tag{91}$$

The second term

$$\begin{aligned}
T_2 &= - < div c_{\sigma}(U_{\mathcal{D}}^{n+1}, c_{\mathcal{K}}^{n+1}), w_{\mathcal{K}} >,\\
&= - \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_{\sigma} \left((U_{\mathcal{D}}^{n+1}.n_{\mathcal{K},\sigma})^+ c_{\mathcal{K}}^{n+1} - (U_{\mathcal{D}}^{n+1}.n_{\mathcal{K},\sigma})^- c_{\mathcal{L}}^{n+1}\right) w_{\mathcal{K}},\\
&= - \sum_{\sigma=\mathcal{K}/\mathcal{L}\in\mathcal{E}_{int}} m_{\sigma} \left(U_{\mathcal{D}}^{n+1}.n_{\mathcal{K},\sigma}\right)^+ c_{\mathcal{K}}^{n+1}(w_{\mathcal{K}} - w_{\mathcal{L}})\\
&\quad - \sum_{\sigma=\mathcal{K}/\mathcal{L}\in\mathcal{E}_{int}} m_{\sigma} \left(U_{\mathcal{D}}^{n+1}.n_{\mathcal{K},\sigma}\right)^- c_{\mathcal{K}}^{n+1}(w_{\mathcal{K}} - w_{\mathcal{L}}),\\
&= - \sum_{\sigma=\mathcal{K}/\mathcal{L}\in\mathcal{E}_{int}} m_{\sigma} \left(U_{\mathcal{D}}^{n+1}.n_{\mathcal{K},\sigma}\right) c_{\mathcal{K}}^{n+1}(w_{\mathcal{K}} - w_{\mathcal{L}})\\
&\quad - \sum_{\sigma=\mathcal{K}/\mathcal{L}\in\mathcal{E}_{int}} m_{\sigma} \left(U_{\mathcal{D}}^{n+1}.n_{\mathcal{K},\sigma}\right)^- (c_{\mathcal{K}}^{n+1} - c_{\mathcal{L}}^{n+1})(w_{\mathcal{K}} - w_{\mathcal{L}}).
\end{aligned}$$

That's gives

$$|T_2| \leq |w|_X \sum_{\sigma=\mathcal{K}/\mathcal{L}\in\mathcal{E}_{int}} d_{\mathcal{K},\sigma}|U_{\mathcal{D}}^{n+1}||c_{\mathcal{K}}^{n+1}| + |w|_X \sum_{\sigma=\mathcal{K}/\mathcal{L}\in\mathcal{E}_{int}} d_{\mathcal{K},\sigma}|U_{\mathcal{D}}^{n+1}||\nabla_{\sigma,\mathcal{K}} c_{\mathcal{K}}^{n+1}|. \tag{92}$$

We focus now on the last two terms $T_3 = - < q_{\mathcal{K}}^{-,n+1} c_{\mathcal{K}}^{n+1}, w_{\mathcal{K}} >$ and $T_4 =< q_{\mathcal{K}}^{+,n+1} \hat{c}_{\mathcal{K}}^{n+1}, w_{\mathcal{K}} >$. They verify

$$|T_3| \leq |w|_X \|q^{-,n+1}\|_{L^2(\Omega)} \|c_{\mathcal{D}}^{n+1}\|_{L^2(\Omega)} \tag{93}$$

$$|T_4| \leq |w|_X \|q^{+,n+1}\|_{L^2(\Omega)} \|\widehat{c}^{n+1}\|_{L^2(\Omega)} \tag{94}$$

Finally we obtain for all $w \in X_{\mathcal{D}}$

$$\begin{aligned}
&< \varphi_{\mathcal{K}} \frac{c_{\mathcal{K}}^{n+1} - c_{\mathcal{K}}^{n}}{\delta t}, w_{\mathcal{K}} > \\
&\quad \leq |w|_X (\Lambda_{\mathcal{D}} \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_{\mathcal{K}} (1 + |U_{\mathcal{K}}^{n+1}|) |\nabla_{\mathcal{K},\sigma} c_{\mathcal{K}}^{n+1}| \\
&\qquad + \sum_{\sigma=\mathcal{K}/\mathcal{L}\in\mathcal{E}_{int}} d_{\mathcal{K},\sigma} |U_{\mathcal{D}}^{n+1}| |c_{\mathcal{K}}^{n+1}| \\
&\qquad + \sum_{\sigma=\mathcal{K}/\mathcal{L}\in\mathcal{E}_{int}} d_{\mathcal{K},\sigma} |U_{\mathcal{D}}^{n+1}| |\nabla_{\sigma,\mathcal{K}} c_{\mathcal{K}}^{n+1}| \\
&\qquad + \|q^{+,n+1}\|_{L^2(\Omega)} \|\widehat{c}_{\mathcal{D}}^{n+1}\|_{L^2(\Omega)} + \|q^{-,n+1}\|_{L^2(\Omega)} \|c_{\mathcal{D}}^{n+1}\|_{L^2(\Omega)}).
\end{aligned}$$

Multiplying by δt and summing over n we get

$$\begin{aligned}
&\sum_{n=0}^{N-1} \delta t \|\varphi \partial \tilde{c}_{\mathcal{D}}^{n+1}\|_{(C_c^2(\Omega))'} \\
&\quad \leq \Lambda_D \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_{\mathcal{K}} (1 + |U_{\mathcal{K}}^{n+1}|) |\nabla_{\mathcal{K},\sigma} c_{\mathcal{K}}^{n+1}| \\
&\qquad + \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\mathcal{K}/\mathcal{L}\in\mathcal{E}_{int}} d_{\mathcal{K},\sigma} |U_{\mathcal{D}}^{n+1}| |c_{\mathcal{K}}^{n+1}| \\
&\qquad + \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\mathcal{K}/\mathcal{L}\in\mathcal{E}_{int}} d_{\mathcal{K},\sigma} |U_{\mathcal{D}}^{n+1}| |\nabla_{\sigma,\mathcal{K}} c_{\mathcal{K}}^{n+1}| \\
&\qquad + \sum_{n=0}^{N-1} \delta t \|q^{-,n+1}\|_{L^2(\Omega)} \|c_{\mathcal{D}}^{n+1}\|_{L^2(\Omega)} \\
&\qquad + \sum_{n=0}^{N-1} \delta t \|q^{+,n+1}\|_{L^2(\Omega)} \|\widehat{c}_{\mathcal{D}}^{n+1}\|_{L^2(\Omega)}.
\end{aligned}$$

Using the Cauchy-Schwartz inequality and Lemma 4.3, then we have $\partial_t(\varphi_{\mathcal{D}}\tilde{c})$ is bounded in $L^2(0,T;(C_c^2(\Omega))')$. $\tilde{c}$ and $\varphi_{\mathcal{D}}$ are respectively bounded in $L^\infty(0,T;L^2(\Omega))$ and $L^2(\Omega)$, that's implies $\varphi_{\mathcal{D}}\tilde{c}$ is bounded in $L^\infty(0,T;L^2(\Omega))$. $\varphi_{\mathcal{D}}\tilde{c}$ is also bounded in $H^1(0,T;(C_c^2(\Omega))')$ ($L^2(\Omega)$ is *continuously embedded in $(C_c^2(\Omega))'$ and $L^\infty(\Omega)$ is continuously embedded in $L^2(\Omega)$). In the fact $L^2(\Omega)$ is compactly embedded in $(C_c^2(\Omega))'$ and using Aubin's compactness theorem proved by Gallouet and Latché [22] we deduce that $\varphi_{\mathcal{D}}\tilde{c}$ is relatively compact in $C([0,T];(C_c^2(\Omega))')$.*

Step 2: *$\varphi_{\mathcal{D}}c(t,.) = \varphi_{\mathcal{D}}\tilde{c}(n\delta t,.)$ on Ω. $H^1(0,T;(C_c^2(\Omega))')$ is continuously embedded in $C^{1/2}([0,T];(C_c^2(\Omega))')$.*

Hence, $\varphi_{\mathcal{D}}\tilde{c}$ is also bounded in $C^{1/2}([0,T];(C_c^2(\Omega))')$ and there exists C not depending on δt or $\mathcal{D}$ such that for all $n = 0,\ldots,N-1$ and all $t \in [n\delta t,(n+1)\delta t]$,

$$\begin{aligned}\|\varphi_{\mathcal{D}}c(t,.) - \varphi_{\mathcal{D}}\tilde{c}(t,.)\|_{(C_c^2(\Omega))'} &= \|\varphi_{\mathcal{D}}\tilde{c}((n+1)\delta t,.) - \varphi_{\mathcal{D}}\tilde{c}(t,.)\|_{(C_c^2(\Omega))'} \\ &\le C_9\sqrt{\delta t}.\end{aligned}$$

Implies that as $\delta t \to 0$, $\varphi_{\mathcal{D}}c - \varphi\tilde{c} \to 0$ in $L^\infty(0,T;(C_c^2(\Omega))')$. $\varphi_{\mathcal{D}}\tilde{c}$ is relatively compact in $L^1(0,T;(C_c^2(\Omega))')$.
The Lemma 5.4 in [20] gives, for all $\xi \in \mathbb{R}$

$$\|c(t,.+\xi) - c(t,.)\|_{L^1(\mathbb{R}^d)} \le |\xi|\sqrt{d}\|c\|_{1,1,\mathcal{M}}.$$

Integrating on $t \in [n\delta t,(n+1)\delta t[$ and summing on $n = 0,\ldots,N-1$, this implies

$$\|c(t,.+\xi) - c(t,.)\|_{L^1(]0,T[\times\mathbb{R}^d)} \le |\xi|\sqrt{d}\sum_{n=0}^{N-1}\|c\|_{1,1,\mathcal{M}}. \tag{95}$$

Thanks to the estimates of Lemma 4.3 we have

$$\|c(t,.+\xi) - c(t,.)\|_{L^1(\mathbb{R}^d)} \to 0 \text{ as } \xi \to 0 \text{ independently of } \delta t \text{ and } \mathcal{D}. \tag{96}$$

Since $\varphi_{\mathcal{D}}$ and c are respectively bounded in $L^\infty(\Omega)$ and $L^\infty(0,T;L^2(\Omega))$, and using the estimates of Lemma 4.3 we have

$$\begin{aligned}&\|(\varphi_{\mathcal{D}}c)(.,.+\xi) - (\varphi_{\mathcal{D}}c)\|_{L^1(]0,T[\times\mathbb{R}^d)} \\ &\quad= \|\varphi_{\mathcal{D}}(.+\xi)(c(.,.+\xi) - c) + (\varphi_{\mathcal{D}}(.+\xi) - \varphi_{\mathcal{D}})c\|_{L^1(]0,T[\times\mathbb{R}^d)} \\ &\quad\le C\|c(.,.+\xi) - c\|_{L^1(]0,T[\times\mathbb{R}^d)} + C\|\varphi_{\mathcal{D}}(.+\xi) - \varphi_{\mathcal{D}}\|_{L^1(]0,T[\times\mathbb{R}^d)}.\end{aligned}$$

$\varphi_{\mathcal{D}} \to \varphi$ in $L^2(\Omega)$ as $h_{\mathcal{D}} \to 0$, the propriety $\|\varphi_{\mathcal{D}}(.+\xi) - \varphi_{\mathcal{D}}\|_{L^2(\omega)} \to 0$ as $\xi \to 0$ and (95) implies that $\|\varphi_{\mathcal{D}}c(.,.+\xi) - \varphi_{\mathcal{D}}c\|_{L^1(0,T;L^1(\omega))} \to 0$ as

$\xi \to 0$ *independently of* $\mathcal{D}$ *and* δt*. Using Lemma 2.14 and the fact that* $\varphi_{\mathcal{D}} c$ *is relatively compact in* $L^1(0,T;(C_c^2(\Omega))')$*, we get* $\varphi_{\mathcal{D}} c$ *is relatively compact in* $L^1(0,T;L^1_{Loc}(\Omega))$*. Let* $f \in L^1(0,T;L^1_{Loc}(\Omega))$ *such that* $\varphi_{\mathcal{D}} c \to f$ *in* $L^1(0,T;L^1_{Loc}(\Omega))$ *as* $\delta t \to 0$ *and* $h_{\mathcal{D}} \to 0$*. Hence, the hypotheses* (6) *and the dominated convergence theorem then shows that* $c = \frac{1}{\varphi_{\mathcal{D}}}\varphi_{\mathcal{D}} c \to \frac{1}{\varphi} f$ *in* $L^1(0,T;L^1_{Loc}(\Omega))$ *(we also using the fact that* $\varphi_{\mathcal{D}} \to \varphi$ *in* $L^2(\Omega)$*), which concludes the proof.*

Lemma 3.8 *Under the assumptions of Theorem* (3.1)*, the function* $\overline{c}$*.* $\overline{U}$ *introduced in* (3.4) *satisfy* (13)*.*

Proof 3.9 *Let* $\psi \in C^\infty([0,T] \times \overline{\Omega})$*, multiplying* (58) *by* $\psi^n(x_{\mathcal{K}})$ *and sum on* $\mathcal{K} \in \mathcal{M}$ *and on* $n = 0, \dots, N-1$*, we get*

$$T_6 + T_7 + T_8 + T_9 = T_{10}.$$

With

$$T_6 = \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{K}\in\mathcal{M}} m_{\mathcal{K}} \varphi_{\mathcal{K}} \frac{c_{\mathcal{K}}^{n+1} - c_{\mathcal{K}}^{n}}{\delta t} \psi^{n+1}(x_{\mathcal{K}}),$$

$$T_7 = -\sum_{n=0}^{N-1} \delta t \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}^2_{\mathcal{K},\sigma}(c^{n+1}) \psi^{n+1}(x_{\mathcal{K}}),$$

$$T_8 = \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma div c_\sigma(c_{\mathcal{D}}^{n+1}, U_{\mathcal{D}}^{n+1}) \psi^{n+1}(x_{\mathcal{K}}),$$

$$T_9 = \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{K}\in\mathcal{M}} q_{\mathcal{K}}^{-,n+1} c_{\mathcal{K}}^{n+1} \psi^{n+1}(x_{\mathcal{K}}),$$

$$T_{10} = \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{K}\in\mathcal{M}} m_{\mathcal{K}} q_{\mathcal{K}}^{+,n+1} \widehat{c}_{\mathcal{K}}^{n+1} \psi^{n+1}(x_{\mathcal{K}}).$$

Limite of $\mathbf{T_6}$***:***

$$T_6 = \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{K}\in\mathcal{M}} m_{\mathcal{K}} \varphi_{\mathcal{K}} \frac{c_{\mathcal{K}}^{n+1} - c_{\mathcal{K}}^{n}}{\delta t} \psi^{n+1}(x_{\mathcal{K}}),$$

$$= \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{K} \in \mathcal{M}} m_{\mathcal{K}} \varphi_{\mathcal{K}} c_{\mathcal{K}}^{n+1} \frac{\psi_{\mathcal{K}}^{n+1} - \psi_{\mathcal{K}}^{n}}{\delta t} - \sum_{\mathcal{K} \in \mathcal{M}} m_{\mathcal{K}} \varphi_{\mathcal{K}} c_{\mathcal{K}}^{0} \psi^{1}(x_{\mathcal{K}}),$$
$$= \int_0^T \int_\Omega \varphi c \xi_{\mathcal{K},\mathcal{D}} - \int_\Omega \varphi_{\mathcal{D}} c_0 \mathcal{P}_{\mathcal{M}}(\psi^1).$$

Where $\varphi_{\mathcal{D}} = \varphi_{\mathcal{K}}$ on $\mathcal{K}$, $\xi_{\mathcal{K},\mathcal{D}} = \frac{\psi^{n+1}(x_{\mathcal{K}}) - \psi^{n}(x_{\mathcal{K}})}{\delta t}$ on $[(n\delta t, (n+1)\delta t] \times \mathcal{K}$ and $\mathcal{P}_{\mathcal{M}}(\psi^1) = \psi_{\mathcal{K}}^1$ on $\mathcal{K}$. ψ is regular, then $\xi_{\mathcal{K},\mathcal{D}} \to -\partial_t \psi$ uniformly on $[0,T] \times \Omega$ and $\mathcal{P}_{\mathcal{M}}(\psi^1) \to \psi(0,.)$ uniformly on Ω; $\varphi_{\mathcal{D}} \to \varphi$ strongly in $L^2(\Omega)$. The weak- convergence of c in $L^\infty(0,T;L^2(\Omega))$ then implies*

$$T_6 \to \int_0^T \int_\Omega \varphi \overline{c} \partial_t \psi - \int_\Omega \varphi c_0 \psi(0,.). \tag{97}$$

Limit of T_7:

$$T_7 = -\sum_{n=0}^{N-1} \delta t \sum_{\mathcal{K} \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} \mathcal{F}_{\mathcal{K},\sigma}^2(c^{n+1}) \psi_{\mathcal{K}}^{n+1},$$
$$= \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{K} \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} \mathcal{F}_{\mathcal{K},\sigma}^2(c^{n+1})(\psi_{\mathcal{K}}^{n+1} - \psi_{\sigma}^{n+1}),$$
$$= \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{K} \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} \sum_{\sigma' \in \mathcal{E}_{\mathcal{K}}} D_{\mathcal{K}}^{\sigma,\sigma'}(c_{\mathcal{K}}^{n+1} - c_{\sigma'}^{n+1})(\psi_{\mathcal{K}}^{n+1} - \psi_{\sigma'}^{n+1}),$$
$$= \int_0^T \int_\Omega \nabla_{\mathcal{D}} c^{n+1} D(x, U^n) \nabla_{\mathcal{D}} \psi^{n+1}.$$

Since $U \to \overline{U}$ strongly in $L^2(]0,T[\times\Omega)^d$, we have $D(.,U) \to D(.,\overline{U})$ strongly in $L^2(]0,T[\times\Omega)^{d\times d}$ (extract a subsequence of U which converges a.e. and use Vitali's theorem). $\nabla_{\mathcal{D}} \psi \to \nabla \psi$ uniformly on $]0,T[\times\Omega$, we deduce that $D(.,U)^T \nabla_{\mathcal{D}} \psi^{n+1} \to D(.,\overline{U})^T \nabla \psi$ in $L^2(]0,T[\times\Omega)^d$ and the weak convergence of $\nabla_{\mathcal{D}} c \to \nabla \overline{c}$, we get

$$T_7 \to \int_0^T \int_\Omega D(.,U) \nabla \overline{c} . \nabla \psi. \tag{98}$$

Limit of T_8:
Applying the technic Using in proof of Lemma (3.6)*, we get*

$$\begin{aligned}T_8 &= \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma div c_\sigma(c_{\mathcal{D}}^{n+1}, U_{\mathcal{D}}^{n+1})\psi_{\mathcal{K}}^{n+1},\\ &= \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\mathcal{K}/\mathcal{L}\in\mathcal{E}_{int}} m_\sigma \left(U_{\mathcal{D}}^{n+1}.n_{\mathcal{K},\sigma}\right) c_{\mathcal{K}}^{n+1}(\psi_{\mathcal{K}}^{n+1} - \psi_{\mathcal{L}}^{n+1}) +\\ &\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\mathcal{K}/\mathcal{L}\in\mathcal{E}_{int}} m_\sigma \left(U_{\mathcal{D}}^{n+1}.n_{\mathcal{K},\sigma}\right)^- (c_{\mathcal{K}}^{n+1} - c_{\mathcal{L}}^{n+1})(\psi_{\mathcal{K}}^{n+1} - \psi_{\mathcal{L}}^{n+1}).\end{aligned}$$

That's gives

$$\begin{aligned}&\left| T_8 - \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\mathcal{K}/\mathcal{L}\in\mathcal{E}_{int}} m_\sigma \left(U_{\mathcal{D}}^{n+1}.n_{\mathcal{K},\sigma}\right) c_{\mathcal{K}}^{n+1}(\psi_{\mathcal{K}}^{n+1} - \psi_{\mathcal{L}}^{n+1}) \right|\\ &\quad = \left| \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\mathcal{K}/\mathcal{L}\in\mathcal{E}_{int}} m_\sigma \left(U_{\mathcal{D}}^{n+1}.n_{\mathcal{K},\sigma}\right)^- (c_{\mathcal{K}}^{n+1} - c_{\mathcal{L}}^{n+1})(\psi_{\mathcal{K}}^{n+1} - \psi_{\mathcal{L}}^{n+1}) \right|,\\ &\quad \le \|\psi\|_{X_{\mathcal{D},\delta t}} \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}} d_{\mathcal{K},\sigma} |U_{\mathcal{D}}^{n+1}||\nabla_{\sigma,\mathcal{K}} c_{\mathcal{K}}^{n+1}|.\end{aligned}$$

and

$$\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\mathcal{K}/\mathcal{L}\in\mathcal{E}_{int}} m_\sigma \left(U_{\mathcal{D}}^{n+1}.n_{\mathcal{K},\sigma}\right) c_{\mathcal{K}}^{n+1}(\psi_{\mathcal{K}}^{n+1} - \psi_{\mathcal{L}}^{n+1}) \to \int_0^T \int_\Omega cU.\nabla\psi. \tag{99}$$

Using Lemma 2.18 and (99)*, we have*

$$T_8 \to \int_0^T \int_\Omega cU.\nabla\psi. \tag{100}$$

Limit of T_9 and T_{10}:
We have

$$T_9 = \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{K}\in\mathcal{M}} q_{\mathcal{K}}^{-,n+1} c_{\mathcal{K}}^{n+1} \psi^{n+1}(x_{\mathcal{K}}) = \int_0^T \int_\Omega q^- c\psi_{\mathcal{D}} \to \int_0^T \int_\Omega q^- \overline{c}\psi. \tag{101}$$

It is also easy to pass to the limit in

$$T_{10} = \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{K}\in\mathcal{M}} m_{\mathcal{K}} q_{\mathcal{K}}^{+,n+1} \widehat{c}_{\mathcal{K}}^{n+1} \psi^{n+1}(x_{\mathcal{K}}) = \int_0^T \int_\Omega q^+ \widehat{c}_{\mathcal{D}} \psi_{\mathcal{D}}.$$

The function $\widehat{c}_{\mathcal{D},\mathcal{K}}$ *equal to* $\widehat{c}_{\mathcal{K}}^{n}$ *on* $[n\delta t,(n+1)\delta t[\times\Omega$ *converges to* $\widehat{c}$ *in* $L^2(]0,T[\times\Omega)$*, then*

$$T_{10} \to \int_0^T \int_\Omega q^+ \widehat{c}\psi. \tag{102}$$

Using (97)*,* (98)*,* (100)*,* (101)*, and* (102) *in* $T_6 + T_7 + T_8 + T_9 + T_{10}$ *we deduce that* $\overline{c}$ *is a weak solution to* (13) *with the function* $\overline{U}$ *limit of* U*.*

Proof 3.10 *The proof of Theorem 3.1 is a result of Lemma 3.4 and Lemma 3.8.*

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