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Dynamic analysis of microbeams based on modified strain gradient theory using differential quadrature method

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ABSTRACT

Dynamic analysis of microbeams based on the modified strain gradient elasticity theory (MSGT) is carried out in this study. MSGT theory comprises additional material length scale parameters to effectively capture the size effect. Beams with fixed-fixed, simply supported and fixed-free boundary conditions are analysed. Additionally, frequency analysis for beams based on modified coupled stress and continuum theory is also presented by neglecting one or more length scale parameters. Results obtained for various theories in the present analysis are compared with those available in literature. Differential quadrature method (DQM) is employed to perform the analysis. Two different techniques are presented for implementing different boundary conditions of beam. It is shown that frequencies obtained from the strain gradient theory are higher when compared to the frequencies predicted by modified coupled stress theory and classical theory, when beam thickness becomes comparable to the length scale parameter. Besides, we also show that implementation of DQM is simple, accurate and robust in solving vibrational problems of different nature.

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1. Introduction

Microelectromechanical systems- and nanoelectromechanical systems-based sensors and actuators widely use microbeams/nanobeams as the sensing element. Mechanical properties of beams and plates constitute working principles of these sensors and actuators. Owing to the small size of these beams, size-dependent effects greatly influence dynamic characteristics (Fleck, Muller, Ashby, & Hutchinson, 1994; Lam, Yang, Chong, Wang, & Tong, 2003; Stlken & Evans, 1998). As a result, for improved and accurate performance, it is vital to capture the size effect associated with these low dimensional beams. The continuum theory fails to capture this size effect on microstructures due to absence of any material length scale parameter in the

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governing equation. In order to overcome break down of classical theory, the non-classical continuum theories like nonlocal elasticity theory (Eringen & Edelen, 1972), modified couple stress theory (MCST) (Yang, Chong, Lam, & Tong, 2002), modified strain gradient elasticity theories (MSGT) (Lam et al., 2003), etc., have been proposed which contain one or more material length scale parameters to capture the size effects. MSGT captures effects related with dilation gradient vector, deviatoric stretch gradient tensor and symmetric rotation tensor. MCST is a special case of MSGT when two of the three effects such as dilation gradient vector and deviatoric strech gradient tensor are neglected. Therefore, in this paper, we focus on the modified strain gradient theory for the analysis and compare it with MCST and classical theory.

Differential quadrature method (DQM) proposed by Bellman, Kashef, and Casti, (1972) is an efficient method to solve differential equations using just a few grid points. The pioneering works in the application of DQM have been used to solve structural mechanics problems (Bert, Jang, & Striz, 1988; Bert, Wang, & Striz, 1994; Bert, Xinwei, & Striz, 1993; Wang & Bert, 1993; Wang, Striz, & Bert, 1993, 1994). However, it has also been used to solve fluid- and thermal-related problems (Shu, 1991; Shu & Richard, 1992; Shu & Richards, 1990, 1992; Tornabene, Marzani, Viola, & Elishakoff, 2010). In the past, researchers have used DQM to solve vibrational problems associated with different structural elements such as beams, plates, rods, shells amongst others. Shu and Du (1997) used DQM to find the frequencies of beams and plates with clamped and simply supported boundary conditions. Murmu and Pradhan (2009) used DQM to study vibration response of non-uniform cantilever beams using nonlocal theory. Janghornan (2012) studied the vibration characteristics of tapered nanowires with simply supported as well as clamped boundary conditions using nonlocal elasticity theory. Danesh, Farajpour and Mohammadi (2012) studied the small-scale effect of tapered nanorods using nonlocal elasticity theory. Chang (2012) and Simsek (2012) studied the vibrational characteristics of nonuniform as well as non-homogenous nanorods using nonlocal elasticity theory. Jang, Bert and Striz, (1989) used DQM to carry out the static analysis of beams, columns, membranes and plates with different end conditions. Al Kaisy, Esmaeel and Nassar, (2007) used DQM to study longitudinal vibrations of non-uniform nanorods.

MCST has also been used to study the static and dynamic behaviour of microbeams. Yang et al. presented modified couple stress approach for vibrational analysis of beams (2002). It comprises an additional internal material length scale parameter related with symmetric rotation tensor. Park and Gao (2006) studied characteristics of cantilever beams using MCST theory. Akgoz and Civalek (2013) again used MCST to study dynamic problems of non-uniform cantilever beams. Ma, Gao and Reddy, (2008) used MCTS to study vibrational characteristics of Timoshenko beam. Salamat-Talab, Nateghi and Torabi, (2012) also used MSGT to analyse functionally graded beams with simply supported end conditions. Lam et al. (2003) proposed MSGT which uses an additional equation pertaining to equilibrium of moments of couples in addition to the classical equilibrium equation of moments and forces to govern the behaviour of higher order stresses (Chong, 2002; Lam et al., 2003). Whereas, MCST considers conventional equilibrium of force and moments, Kong, Zhou, Nie and Wang (2009) used MSGT to study the static and dynamic characteristics of cantilever beams. Their analysis incorporated three additional material length scale parameters to capture size effects related with dilation gradient vector, deviatoric stretch gradient tensor and symmetric rotation tensor. However, their analysis was limited to cantilever beams.

In this paper, we present vibrational analysis based on MSGT for beams with different boundary conditions. We solve the governing equation using DQM and also discuss the techniques for implementing different boundary conditions. We show that implementation of DQM is simple, accurate and fast. We first describe different non-classical and classical theories. Subsequently, we briefly discuss DQM and apply it to reduce the governing equation to its discrete form. Finally, we compute the first three frequencies for all types of beams based on different theories and compare the change in frequencies to describe the size-dependent effects.

2. Formulation

In this section, we rewrite the governing equation of motion and the associated boundary conditions of Euler-Bernoulli beams with different end conditions based on MSGT as described by Kong et al. (2009) with the correction proposed by Akgoz and Civalek (2012). Subsequently, we write the governing equations and boundary conditions in their equivalent discretised forms using DQM. The terms comprising spatial derivative are replaced with their equivalent weighted coefficients as discussed earlier. We start with the governing equation in the differential form followed by the boundary conditions.

2.1. *Governing equation*

The governing equation based on modified strain gradient theory for a beam subjected to external load intensity q is derived using the extended Hamilton's principle by taking variation of energies as (Akgoz and Civalek, 2012; Kong et al., 2008, 2009)

$$\delta\left[\int_{t_2}^{t_1}(T-U-W)dt\right]=0, \tag{1}$$

where T is kinetic energy, U is strain energy and W is work done by external forces.

The strain energy, U, of a deformed isotropic solid occupying the volume Ω can be written by considering the behaviour of higher order stresses as (Kong et al., 2009)

$$U=\frac{1}{2}\left(\int_{\Omega}\sigma_{ij}\epsilon_{ij}+p_i\gamma_i+\hat{\tau}_{ijk}\hat{\eta}_{ijk}+m^s_{ij}\chi^s_{ij}\mathrm{d}v\right). \tag{2}$$

Considering the length scales l_0 related with change in volume of material (dilatation gradient, γ_i or $\epsilon_{mm,i}$), l_1 associated with change in shape of the material or body distortion (deviatoric strech gradient tensor $\hat{\eta}_{ijk}$) and l_2 related with strain rate (symmetric rotation gradient tensor, χ^s_{ij}), the stresses can be written in terms of classical material parameters k (bulk modulus) and μ (shear modulus), and nonlocal length scales l_0, l_1 and l_2 as

$$\sigma_{ij}=k\delta_{ij}\epsilon_{mm}+2\mu\epsilon'_{ij},p_i=2\mu l_0^2\gamma_i,\hat{\tau}_{ijk}=2\mu l_1^2\hat{\eta}_{ijk},m^s_{ij}=2\mu l_2^2\chi^2_{ij}, \tag{3}$$

where $\epsilon'_{ij}=\epsilon_{ij}-\frac{1}{3}\epsilon_{mm}\delta_{ij}$ is deviatoric strain, ϵ_{ij} is strain tensor, δ_{ij} is Kronecker delta function.

Using expression of strain tensors as described by Kong et al. (2009) along with the correction mentioned by Akgoz and Civalek (2012), we write the strain energy based on MSGT for a slender beam subjected to bending as (Akgoz and Civalek, 2012; Kong et al., 2009)

$$U=\frac{1}{2}\int_{L}^{0}\left[S\left(w^{(2)}\right)^2+k\left(w^{(3)}\right)^2\right]dx, \tag{4}$$

where w is transverse deflection, $w^{(1)}=\frac{\partial w(x,t)}{\partial x}$, $w^{(2)}=\frac{\partial^2 w(x,t)}{\partial x^2}$, $w^{(3)}=\frac{\partial^3 w(x,t)}{\partial x^3}$, $\ldots$,$w^{(n)}=\frac{\partial^n w(x,t)}{\partial x^n}$, and S and K are defined as

$$S=EI+2\mu A l_0^2+\frac{8}{15}\mu A l_1^2+\mu A l_2^2;\qquad K=I(2\mu l_0^2+\frac{4}{5}\mu l_1^2). \tag{5}$$

The kinetic energy, T, can be written as

$$T=\frac{1}{2}\int_{L}^{0}\rho A(\dot{w})^2dx, \tag{6}$$

where $\dot{w} = \frac{\partial w(x,t)}{\partial t}$ $\ddot{w} = \frac{\partial^2 w(x,t)}{\partial t^2}$, I and A are the moment of inertia and cross-sectional area, respectively.

The corresponding expressions of δU, δT and δW due to external force q, boundary shear force V, boundary classical and non-classical bending moments M and M^h are (Kong et al., 2009)

$$\delta U = \int_L^0 \left[Sw^{(4)} - kw^{(6)}\right] \delta w dx + \left[-Sw^{(3)} + kw^{(5)}\right] \delta w|_0^L, \\ +[Sw^{(2)} - Kw^{(4)}]\delta w^{(1)}|_0^L + Kw^{(3)}\delta w^{(2)}|_0^L \tag{7}$$

$$\delta T = \delta \left[\frac{1}{2}\int_L^0 \rho A(\dot{w})^2 dx\right], \tag{8}$$

$$\delta W = \int_L^0 q(x)\delta w(x,t)dx + [V\delta w]_0^L + [M\delta w^{(1)}]_0^L + [M^h \delta w^{(2)}]_0^L. \tag{9}$$

Using Equations (7), (8), (9) in Equation (1), we obtain the governing equation of beam based on MSGT as

$$Sw^{(4)}(x,t) - kw^{(6)}(x,t) + \rho A\ddot{\omega}(x,t) + q(x) = 0 \tag{10}$$

and the boundary conditions satisfy the equations

$$\left[V(L) - Sw^{(3)}(L) + kw^{(5)}(L)\right]\delta w(L) - \left[V(0) - sw^{(3)}(0) - kw^{(5)}(0)\right]\delta w(0) = 0$$

$$[M(L) + Sw^{(2)}(L) - Kw^{(4)}(L)]\delta w^{(1)}(L) - [M(0) + Sw^{(2)}(0) - Kw^{(4)}(0)]\delta w^{(1)}(0) = 0$$

$$[M^h(L) + Kw^{(3)}(L)]\delta w^{(2)}(L) - [M^h(0) + Kw^{(3)}(0)]\delta w^{(2)}(0) = 0. \tag{11}$$

- It is to be noted that when two material parameters related with dilatation gradients and deviatoric stretch gradients become zero, i.e. $l_0 = l_1 = 0$, then the governing equation reduces to that of MCST as

$$(EI + \mu A l_2^2)w^{(4)}(x,t)\rho A\ddot{w}(x,t) + q(x) = 0, \tag{12}$$

and the boundary condition can be written as

$$\left[V(L) - (El + \mu A l_2^2)w^{(3)}(L)\right]\delta w(L) - \left[V(0) - (El + \mu A l_2^2)w^{(3)}(0)\right]\delta w(0) = 0$$

$$\left[M(L) - (El + \mu A l_2^2)w^{(2)}(L)\right]\delta w^{(1)}(L) - \left[M(0) - (El + \mu A l_2^2)w^{(2)}(0)\right]\delta w^{(1)}(0) = 0. \tag{13}$$

- When all the material parameters $l_0 = l_1 = l_2 = 0$, then the governing equation reduces to that of classical theory

$$(EI)w^{(4)} + \rho A\ddot{w}(x,t) + q(x) = 0 \tag{14}$$

and the boundary condition reduces to

$$\left[V(L) - Elw^{(3)}(L)\right]\delta w(L) - \left[V(0) - Elw^{(3)}(0)\right]\delta w(0) = 0$$

$$\left[M(L) - Elw^{(2)}(L)\right]\delta w^{(1)}(L) - \left[M(0) - Elw^{(2)}(0)\right]\delta w^{(1)}(0) = 0. \tag{15}$$

Equations (10) and (12) show that l_0, l_1 and l_2 capture size effect by including higher order stresses when the thickness/size of the beam is of the order of these length scales. When the beam thickness increases to higher value, the size effects reduce and all the values will be of the same order as that given by the classical beam equation (Equation (14)).

- Frequency analysis of free vibration problem

To perform free vibration analysis, the term corresponding to external force, $q(x)$, can be neglected and the deflection $w(x,t)$ can be approximated as $w(x,t) = w_0(x)e^{\iota\omega t}$. Thus, governing Equation (10) reduces to the form as given by

$$Sw_0^{(4)}(x) - Kw_0^{6)}(x) - \rho A\omega^2 w_0(x) = 0. \tag{16}$$

The above equation contains fourth- and sixth-order differential term of w_0 which is a function of x only. In the next section, we demonstrate DQM in order to solve the above equation for a given boundary conditions. Henceforth, we omit subscript ‘0’ from w_0(x) for simplicity.

2.2. *Length scale parameter*

As discussed in Section 2.1, MSGT consists of three non-classical length scale parameters (l_0, l_1 and l_2) to incorporate the size effect. MCST comprises of a single non-classical length scale parameter (l_2) to capture the size effect. For classical theory, these non-classical length scales become zero.

To compute the length scales under which different theories become significant, Lam et al. (2003) proposed the higher order bending rigidity (b_h) as given by

$$b_h^2 = 6(1-2\nu)l_0^2 + \frac{2}{5}(4-\nu)l_1^2 + 3(1-\nu)l_2^2, \tag{17}$$

where l_0, l_1, l_2 are the length scale parameters and ν is the Poisson's ratio. Additionally, Lam et al. (2003) also experimentally determined the higher-order bending rigidity to be $b_h = 0.24\mu m$ for $\nu = 0.38$. The length scale parameter l corresponding to MCST can be determined by putting $l_0 = l_1 = 0$ and $l_2 = l$ in Equation (17) to obtain the relation

$$l = \sqrt{\frac{b_h^2}{3(1-\nu)}} \tag{18}$$

Based on Equation (18), the value of length scale parameter is found to be $l = 17.6\mu m$ (Dehrouyeh-Semnani, 2015; Park & Gao, 2006) corresponding to MCST.

For determining the length scale parameters corresponding to MSGT, we take $l_0 = l_l = l_2 = l_s$ based on the experimental evidences (Dehrouyeh-Semnani, 2015). Under this assumption, based on Equation (17), we get

$$l_0 = l_1 = l_2 = l_s = \sqrt{\frac{b_h^2}{10.6 - 15.4\nu}}. \tag{19}$$

It gives the length scale parameter as $l_s = 11.01$ μm. Henceforth, we use $l_0 = l_1 = l_2 = l_s = 11.01$ μm for MSGT and $l_2 = l = 17.6$ μm for MCST. For classical theory, $l_0 = l_1 = l_2 = l_s = 0$.

3. Differential quadrature method

DQM is a numerical method which reduces differential equations into algebraic equations. The central idea behind DQM is to approximate the spatial derivative of a function at a point with weighted linear combination of the functional value at all other points in the domain as shown mathematically in Equations (20) and (21)

$$\frac{d^n f(x,t)}{dx^n}\Big|_{x=x_i} \approx \sum_{N}^{j=1} C_{ij}^{(n)} f(x_j,t) \quad i = 1,2\ldots N, \tag{20}$$

or

$$\frac{d^n}{dx^n}\begin{bmatrix} f(x_1,t) \\ f(x_2,t) \\ \vdots \\ f(x_{N-1},t) \\ f(x_N) \end{bmatrix} \approx [C_{ij}^{(n)}] \begin{bmatrix} f(x_1,t) \\ f(x_2,t) \\ \vdots \\ f(x_{N-1,t}) \\ f(x_N,t) \end{bmatrix} i,j = 1,2,3,N, \tag{21}$$

where $f(x_i,t)$ is the functional value at i^{th} sampling point of total N points, $[C_{ij}^{(n)}]$ is the weighing coefficient matrix of the n^{th} order differential equation.

This reduces the differential equation into a set of linear algebraic equations. The number of such algebraic equations depends on the number of sampling points taken. In DQM, the domain is discretised into N sampling points using the Chebyshev-Gauss-Lobatto distribution and is given by

$$X_i = \frac{1}{2}\left[1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right], \quad i = 1, 2, N. \tag{22}$$

The weighing coefficient matrix can now be generated using the sampling distribution as mentioned above using following relations.

$$C_{ij}^{(1)} = \frac{L^{(1)}(x_i)}{(x_i - x_j)L^{(1)}(x_j)}, \quad i, j = 1, 2, N, i \neq j$$

$$C_{ii}^{(1)} = -\sum_{N}^{j=1, j\neq i} C_{ij}^{(1)}, \qquad i, j = 1, 2, N, i = j, \tag{23}$$

where $L^{(1)}$ is the first derivative of Lagrange interpolating polynomials at sampling points given by

$$L^{(1)}(x_i) = \prod_{k=1, k\neq i}^{N} (x_i - x_k), \quad i = 1, 2, , N. \tag{24}$$

The corresponding weighted coefficient matrices for higher order derivatives can be obtained using the relation

$$C_{ij}^{(n)} = n(C_{ij}^{(n-1)}C_{ij}^{(1)} - \frac{C_{ij}^{(n-1)}}{x_i - x_j}), i, j = 1, 2, \ldots, N, i \neq j, n = 2, 3, \ldots, N-1$$

$$C_{ii}^{(n)} = -\sum_{j=1, j\neq i}^{N} C_{ij}^{(n)}, \quad i, j = 1, 2, \ldots N, i = j, n = 2, 3, \ldots, N-1. \tag{25}$$

Using the DQM, Equation (16) is now re-written as

$$S\sum_{j=1}^{N} C_{ij}^{(4)} w_j - K\sum_{j=1}^{N} C_{ij}^{(6)} w_j - \rho A\omega^2 w_i = 0, \quad i = 1, 2N. \tag{26}$$

To solve the above equations, the correct implementation of boundary conditions is vital in order to successfully implement DQM. Different techniques have been proposed to incorporate different boundary conditions. In DQM, we use the δ-technique (Lam et al., 2003) to solve vibrational problems with clamped-free (CF) boundary conditions and the SBCGE technique (Shu & Du, 1997) for clamped-clamped (CC) and simply supported (SS) boundary conditions.

3.1. *Implementation of boundary conditions*

For vibrational analysis of beams, there are two boundary conditions at one end, i.e. $X = 0$ or $i = 1$, and another two at the other end, i.e. $X = 1$ or $i = N$. As discussed earlier, we use the δ technique proposed by Bert et al. (1988) and Jang et al. (1989) for the clamped-free boundary conditions. This technique is simple and eliminates the difficulty of implementing two boundary conditions at one grid point. It applies one condition at each end of the boundary and the other condition at δ distance away from the boundary. In other words, one boundary condition is applied at $X = 0$ and $X = \delta$, and the other condition at $X = 1 - \delta$ and $X = 1$. However, proper selection of δ is vital to ensure correct results (Bert et al., 1988). In δ technique, the boundary conditions are approximated as the conditions are not implemented exactly at the boundary but at grid points adjacent to it. However, this technique works well for clamped-free end conditions but not for others. To do the vibrational analysis for beams with clamped-clamped (CC) and simply-supported (SS) end conditions, we use another approach devised by Shu and Du (1997) referred as SBCGE. The central idea of this technique is to implement all the four boundary conditions at the boundaries and not on adjacent grids. As a result, this technique is more accurate and works for all types of boundary conditions. In this technique, the boundary conditions are directly substituted into the governing equations. In the next section, we demonstrate this technique for CC and SS beams.

3.1.1. *Clamped-free beams*

For cantilever beams with clamped-free end conditions, the boundary conditions given by Equation (11) reduce to

$$w(0) = w^{(1)}(0) = 0, Kw^{(5)}(L) - Sw^{(3)}(L) = 0, Sw^{(2)}(L) - Kw^{(4)}(L) = 0. \tag{27}$$

The equivalent formulation in DQM at the clamped end $i = 1$ or $x = 0$ is

$$w_1 = 0, \sum_{N}^{j=1} C_{1j}^{(1)} w_j = 0, \tag{28}$$

and, at the free end $i = N$ or $X = 1$ is

$$K\sum_{N}^{j=1} C_{Nj}^{(5)} w_j - S\sum_{N}^{j=1} C_{Nj}^{(3)} w_j = 0, \quad S\sum_{N}^{j=1} C_{Nj}^{(2)} w_j - K\sum_{N}^{j=1} C_{Nj}^{(4)} w_j = 0. \tag{29}$$

Using the governing equation at inner domain points and boundary conditions at the end points, we write the equation in matrix form as

$$[K][x] = \omega^2[M][x], \tag{30}$$

where $[K]$ matrix is composed of

i/j	1 2 3	… … … …	… $N-1$ N
1	K_{bb1}	K_{bd1}	K_{bb2}
2	K_{db1}	K_{dd}	K_{db2}
3	K_{bb3}	K_{bd2}	K_{bb4}
.			
.			
.			
.			
.			
$N-1$			
N			

(31)

$$K_{bb} = \begin{bmatrix} K_{bb1} & K_{bb2} \\ K_{bb3} & K_{bb4} \end{bmatrix}, K_{bd} = \begin{bmatrix} K_{bd1} \\ K_{bd2} \end{bmatrix}, K_{db} = \begin{bmatrix} K_{db1}, & K_{db2} \end{bmatrix}, \tag{32}$$

and $[x] = [x_b \quad x_d]^T$, where, subscripts $x_b = [X_1\ X_2\ X_{N-1}\ X_N]$ and $x_d = [X_3\ X_4 X_{n-2}]$ represent boundary grid points and domain grid points, respectively. Similar analogy also holds for $[M]$ matrix. Rewriting Equation (30) in terms of boundary and domain points, we get

$$\begin{bmatrix} K_{bb} & K_{bd} \\ K_{db} & K_{dd} \end{bmatrix} \begin{bmatrix} x_b \\ x_d \end{bmatrix} = \omega^2 \begin{bmatrix} M_{bb} & M_{bd} \\ M_{db} & M_{dd} \end{bmatrix} \begin{bmatrix} x_b \\ x_d \end{bmatrix}. \tag{33}$$

For the boundary conditions, $[M_{bb}] = [M_{bd}] = [M_{db}] = [0]$. Rearranging Equation (33) in the form of an eigenvalue problem, we get final form of the equation as

$$[M_{dd}^{-1}(K_{dd} - K_{db}K_{bb}^{-1}K_{bd})][x_d] = \omega^2[x_d]. \tag{34}$$

Equation (34) is then solved using standard eigenvalue solving algorithm to obtain natural frequencies ω.

3.1.2. *Clamped-clamped and simply supported beams*

For CC and SS beams, the boundary conditions as given by Equation (11) are reduced to

$$w(0) = w^{(n)}(0) = 0, w(L) = w^{(n)}(L) = 0, \tag{35}$$

where, $n = 1$ corresponds to CC and $n = 2$ corresponds to SS beams. The equivalent formulation in DQM at clamped end $i = 1$ or $X = 0$ is

$$w_1 = 0, \sum_{N}^{j=1} C_{1j}^{(n)} w_j = 0, \tag{36}$$

and, at clamped end $i = N$ or $X = 1$ is

$$w_N = 0, \quad \sum_{N}^{j=1} C_{Nj}^{(n)} w_j = 0. \tag{37}$$

However, these four boundary conditions cannot be applied at the two boundaries of the domain in straightaway. Therefore, we apply the Dirichlet conditions at the ends and derivative conditions are discretised using DQ method. The discretised equations are then combined to determine w_2 and w_{N-1} which are substituted back into the governing equations applied to interior (or domain) points. We follow the approach given by Shu & Du (1997) and write the derivative boundary conditions for CC and SS beams in terms of w_2 and w_{N-1} as

$$w_2 = \frac{1}{AXN} \sum_{j=3}^{N-2} AXK_1 w_j$$

$$w_{N-1} = \frac{1}{AXN} \sum_{N-2}^{j=3} AXK_N w_j, \tag{38}$$

where

$$AXK_1 = C_{1,j}^{(n)} C_{N,N-1}^{(n)} - C_{1,N-1}^{(n)} C_{N,j}^{(n)}$$

$$AXK_N = C_{1,2}^{(n)} C_{N,k}^{(n)} - C_{1,j}^{(n)} C_{N,2}^{(n)}$$

$$AXN = C_{N,2}^{(n)} C_{1,N-1}^{(n)} - C_{1,2}^{(n)} C_{N,N-1}^{(n)}. \tag{39}$$

Here, $n = 1$ corresponds to CC case and $n = 2$ corresponds to SS case. Now, we substitute Equation (38) along with the Dirichlet boundary conditions of Equations (36) and (37) in the governing Equation (26) giving

$$\begin{aligned} &S\left(C_{i,2}^{(4)} w_2 + \sum_{j=3}^{N-2} C_{i,j}^{(4)} w_j + C_{i,N-1}^{(4)} w_{N-1}\right) - \\ &K\left(C_{i,2}^{(6)} w_2 + \sum_{j=3}^{N-2} C_{i,j}^{(6)} w_j + C_{i,N-1}^{(6)} w_{n-1}\right) - \rho A \omega^2 w_i = 0. \end{aligned} \tag{40}$$

After substituting Equation (38) into Equation (40), the governing equation is written in the form of Equation (30) applied at $N-4$ grid points. This again reduces to an eigenvalue problem and is solved using available techniques.

4. Results and discussions

The natural frequencies for beams with different end conditions are obtained by solving corresponding equations using standard eigenvalue problem solvers. Beam dimensions and properties used for the analysis in this paper are tabulated in Table 1.

In order to present the robustness and ease of using DQM to solve vibrational problems, we first obtain the fundamental frequencies of beams with different end conditions corresponding to the classical theory by setting $l_0 = l_1 = l_2 = 0$ in our analysis. The nondimensional frequencies Ω obtained using the present method are then compared with results available in literature. Table 2 presents the first three frequencies for different beams. We see that results obtained using the present analysis agree well with those available in literature. It is emphasised here that δ technique is used for beams with clamped-free end conditions while SBCGE method is employed for analysis of other types of beams. It is to be noted that SBCGE is a more accurate method as it ensures that the boundary conditions are incorporated exactly at the boundaries. As a

Table 1. The dimensions and the material properties used for the analysis.

Property	Values
Thickness	20 μm
Width	40 μm
Length	400 μm
E	1.44 GPa
μ	$\frac{0.5E}{(1+v)}$
ρ	1200 kgm^{-3}

Table 2. The first three non-dimensional frequencies for different types of beams. For SS and CC beams, the SBCGE is used, and for clamped-free beams, the δ technique is used to determine the frequencies.

	Ω_1	Ω_2	Ω_3
SS			
Su and Du (1997)	9.8696	39.4784	88.8264
Present	9.8689	39.4657	88.7957
CC			
Su and DU (1997)	22.3733	61.6728	120.9034
Present	22.3733	61.6726	120.9028
Clamped-free			
Singh, Pal, and Pandey, (2015)	3.5160	22.0345	61.6972
Present	3.5199	22.0589	61.7657

result, this method always gives accurate results. The δ technique employs two boundary conditions at a given boundary and other two at δ distance away from the boundary. As a result, it produces accurate results for clamped-free end conditions and is simpler in implementation.

Additionally, we also compare our results pertaining to MCST. The present analysis can be reduced to that of MCST by setting the parameters $l_0 = l_1 = 0$ and $l_2 = l$. For MCST, we compare our results of cantilever beams for $l = 0.5$ and $l = h$ with that of Akgoz and Civalek (2013) who used the RayleighRitz solution method to obtain the natural frequency for different material length scale parameters in Table 3 (Akgoz and Civalek, 2013). Moreover, we also compare the first, second and third mode frequencies obtained by solving classical theory using DQM for cantilever beam with the numerical results given by Kong et al. (2009) as shown in Figure 1(a), (b) and (c), respectively, at $h = 20$ μm and 50 μm. The percentage error is found to be about 10% due to the fact that Kong et al. (2009) computed frequency values for $\rho = 1000$ kg/m^3 whereas we take $\rho = 1200$ kg/m^3 in current analysis based on the data provided by Dehrouyeh-Semnani (2015). Considering the current values of density, the percentage difference reduces to around 2%. Now, to compare the present numerical solution based on MSGT for cantilever beam, we compare our result with numerical solution mentioned by Dehrouyeh-Semnani (2015) for the same dimension and density as shown in Figure 1(a), (b) and (c). The comparison of results indicates that the percentage error is about 2%. Thus, based on the above analysis,

Table 3. Comparison of first three non-dimensional frequencies based on MCST for different material length scale parameters, $l_2 = l$ with the available results in literature.

	l = 0.5 h			$l = h$		
Reference	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
(Akgoz & Civalek, 2013)	5.160	32.338	90.547	8.332	52.215	146.203
Present method (DQM)	5.178	32.447	90.855	8.359	52.392	146.699

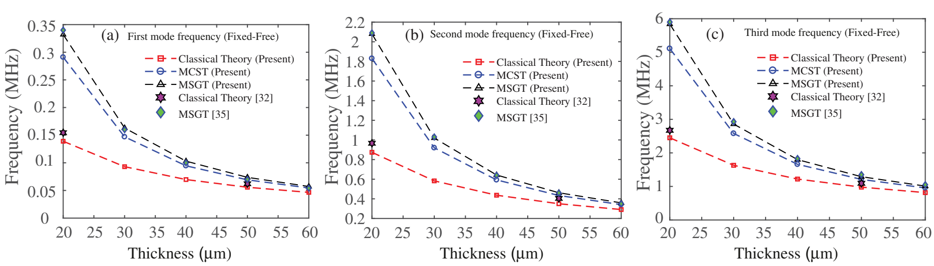

Figure 1. (a) First, (b) second and (d) third mode frequencies of microbeams with fixed-free end conditions based on the continuum theory, MCST ($l = 17.6\mu m$) and MSGT ($l_s = 11.01\mu m$).

we validate the solution approach based on DQM. Now, we use it to analyse the influence of size effect on the evaluation of first, second and third modes frequencies.

Figure 1(a)–(c) shows the variation of frequencies using classical theory, MCST ($l_2 = l = 17.6$ μm) and MSGT ($l_0 = l_1 = l_2 = l = 11.01$ μm) with beam thickness varying from 20 to 60 μm for cantilever beam with $\rho = 1200$ kgm^{-3} and $\nu = 0.38$. The results show that when $h = 20$ μm, $l/h = 0.83$ for MCST and $l/h = 0.55$ for MSGT. The percentage difference in first mode frequencies computed by MCST and MSGT as compared to classical theory are found to be about 109% and 139%, respectively. As the thickness increases to $h = 60$ μm, $l/h = 0.29$ for MCST and $l/h = 0.18$ for MSGT, these differences reduce to 17% and 23%, respectively. As l/h ratio decreases below 0.1, percentage difference reduced even further and became eventually zero as $l/h \rightarrow 0$. Under this condition, frequencies given by classical and non-classical theories (MCST and MSGT) become same. Therefore, the results show that the size effect or nonlocal effect becomes important when beam thickness is of the same order as that of non-classical length scale. Similarly, we observe the same effect in the variation of frequencies corresponding to second and third modes as shown in Figure 1(b), (c).

To observe the size effect in fixed-fixed and simply-supported beams, we also plot the variation of first, second and third mode frequencies for beam thickness 20–60 μm based on classical theory, MCST and MSGT as shown in Figures 2 and 3. Like the case of clamped-free condition, as the beam thickness reduces, the difference in frequencies obtained by all theories eventually becomes zero when the size effect is not significant.

5. Conclusions

In this paper, we have done the dynamic study of microbeams using MSGT. Additionally, we have also presented the behaviour of beams employing MCST. Beams with different end conditions like fixed-fixed, simply supported

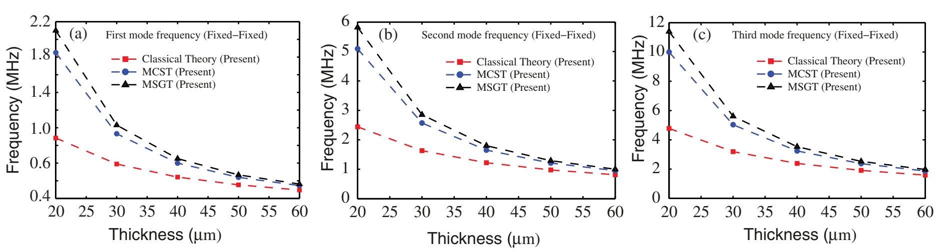

Figure 2. (a) First, (b) second and (d) third mode frequencies of microbeams with fixed-fixed end conditions based on the continuum theory, MCST ($l = 17.6 \mu m$) and MSGT ($l_s = 11.01 \mu m$).

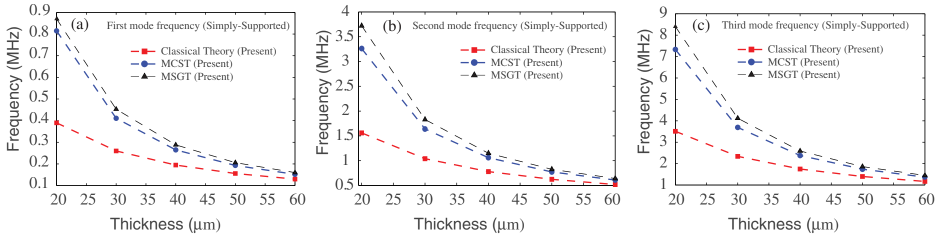

Figure 3. (a) First, (b) second and (d) third mode frequencies of microbeams with simply supported end conditions on both ends based on the continuum theory, MCST ($l = 17.6\mu m$) and MSGT ($l_s = 11.01\mu m$).

and fixed-free boundary conditions are analysed. We have presented the dynamic analysis using DQM which is a fast, accurate and robust method for studying structural problems. It is found that the frequencies for all types of beams predicted by MSGT are higher than that predicted by MCST, which in turn, is higher than that from the classical theory. However, as the thickness of beam increases, difference in frequencies obtained using the three methods continues to diminish. As a result, for application in micro/nanomechanics where size effects are appreciable, our analysis is vital to obtain correct frequencies for beams with different end conditions.

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