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Vibration analysis of nonlinear systems modelled by a mass attached to a stretched elastic wire

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ABSTRACT

Herein, an accurate approximate analytical solution for oscillation of nonlinear systems modelled by a mass attached to a stretched elastic wire is derived using the homotopy analysis method and homotopy Pade technique. Approximate analytical expressions for the frequency and displacement with respect to time are obtained. Comparison between the presented results and numerical solutions shows that the first-order approximation of homotopy Pade technique leads to accurate results.

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Nonlinear vibration; analytical solution; elastic wire; homotopy analysis method; homotopy Pade technique

1. Introduction

The oscillation of a mass attached to a stretched elastic wire represents a strong nonlinear system, which can successfully model engineering structures such as cable-stayed bridges, isolation systems and truss-structures (Cao, Wiercigroch, Pavlovskaia, Grebogi, & Thompson, 2006; Mickens, 1996a, 1996b; Sun, Wu, & Lim, 2007). Despite the great potential of a mass attached to a stretched elastic wire in modelling various systems in physics and engineering, it is difficult and challenging to solve the governing nonlinear equation analytically. As a result, researchers have been focused to find approximate analytical solutions and several techniques have been developed to solve these kinds of nonlinear equations, such as perturbation techniques (Nayfeh, 1985; Mickens, 1996a, 1996b), harmonic balance-based methods (Beléndez, Hernández, Márquez, Beléndez, & Neipp, 2006; Gottlieb, 2006; Lim, Wu, & Sun, 2006) and the variational iteration method (Fesanghary, Pirbodaghi, Asghari, & Sojoudi, 2009). Due to the nature of a nonlinear phenomenon, these techniques can only be used within limited ranges of the physical parameters and can only be applied to specific equations (Amore & Fernández, 2005).

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Moreover, these analytical methods have their own set of limitations for solving nonlinear ordinary differential equations. For example, perturbation methods are typically restricted to weak nonlinear problems. To address this issue, Homotopy analysis method (HAM) which has successfully solved strong nonlinear equations (Liao, 1992; Hoseini et al., 2009; Pirbodaghi, Ahmadian et al., 2009) was developed.

In dimensionless form, a mass attached to the centre of a stretched elastic wire has the equation of motion as (Mickens, 1996a, 1996b):

$$\frac{d^2u(t)}{dt^2} + u(t) - \frac{\lambda u(t)}{\sqrt{1+u(t)^2}} = 0 \tag{1}$$

$$u(0) = a, \quad \frac{du}{dt}(0) = 0 \tag{2}$$

Equation (1) is an example of a conservative nonlinear oscillatory system (oscillating between $-a$ and a). Its solution and frequency are dependent on the amplitude a and the constant parameter $\lambda(0<\lambda<1)$. Under the transformation $\tau = \omega t$, Equation (1) can be rewritten as follows:

$$\left(\omega^2\frac{d^2u}{d\tau^2} + u\right)^2\left(1+u^2\right) - \lambda^2u^2 = 0, \tag{3}$$

$$u(0) = a, \quad \frac{du(0)}{d\tau} = 0 \tag{4}$$

where ω is the natural frequency. First, approximate analytical solutions for frequency and displacement are obtained using HAM. Then, the homotopy Pade technique is applied to accelerate the convergence rate of the solutions.

The principles of the HAM and its applicability for various kinds of nonlinear engineering problems are given in (Liao, 1992; Hoseini et al., 2009). To illustrate the basic ideas of the HAM, consider a nonlinear differential equation as:

$$N[u(t)] = 0, \tag{5}$$

where N is a general nonlinear differential operator, and $u(t)$ is an unknown function of the parameter t. Under the transformation $\tau = \omega t$, the Equation (5) can be rewritten as:

$$N[u(\tau), w] = 0 \tag{6}$$

Then, the homotopy function is constructed as follows:

$$\bar{H}(\phi, q, \hbar, H(\tau)) = (1-q)L[\phi(\tau,q) - u_0(\tau)] - q\hbar H(\tau)N[\phi(\tau,q), \omega(q)] \tag{7}$$

where ϕ, $\hbar$ and $H(\tau)$ are a function of τ and q, the nonzero auxiliary parameter and the nonzero auxiliary function, respectively. The parameter L denotes an auxiliary linear operator. As q increases from zero to one, $\phi(\tau, q)$ varies from the initial approximation to the exact solution. Setting $H(\phi, q, \hbar, H(\tau)) = 0$, the zero-order deformation is constructed as:

$$(1 - q)L[\phi(\tau, q) - u_0(\tau)] = q\hbar H(\tau)N\,[\phi(\tau, q), \omega(q)] \tag{8}$$

with the following initial conditions:

$$\phi(0, q) = a, \quad \frac{\mathrm{d}\phi(0, q)}{\mathrm{d}\tau} = 0 \tag{9}$$

The functions $\phi(\tau, q)$ and $\omega(q)$ can be expanded as power series of the q using Taylor's theorem as:

$$\phi(\tau, q) = \phi(\tau, 0) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^n \phi(\tau, q)}{\partial q^n}\bigg|_{q=0} q^n = u^0(\tau) + \sum_{n=1}^{\infty} u_n(\tau) q^n \tag{10}$$

$$\omega(q) = \omega_0 + \sum_{n=1}^{\infty} \frac{1\partial^n}{n!} \frac{\omega(q)}{\partial q^n}\bigg|_{q=0} q^n = \omega_0 + \sum_{n=1}^{\infty} \omega_n q^n \tag{11}$$

where $u_n(\tau)$ and ω_n are called the nth-order deformation derivatives. Differentiating zero-order deformation equation with respect to q and then setting $q = 0$, yields the first-order deformation equation ($n = 1$) which gives the first-order approximation of the $\omega(\tau)$ as follows:

$$L[u_1(\tau)] = \hbar H(\tau)N[u_0(\tau), \omega_0] \tag{12}$$

with the following initial conditions:

$$u_1(0) = 0, \quad \dot{u}_1(0) = 0 \tag{13}$$

The higher order approximations of the solution can be obtained by calculating the nth-order ($n > 1$) deformation equation. The nth-order deformation equation can be calculated by differentiating Equations (12) and (13) n times with respect to q as follows (Liao, 1992; Pirbodaghi, Ahmadian et al. 2009; Pirbodaghi, Fesanghary, & Ahmadian, 2011):

$$L[u_n(\tau) - u_{n-1}(\tau)] = \hbar H(\tau)R_n(\vec{u}_{n-1}, \vec{\omega}_{n-1}) \tag{14}$$

where $\vec{u}_{n-1}$, $\vec{\omega}_{n-1}$ and $R_n(\vec{u}_{n-1}, \vec{\omega}_{n-1})$ are defined as follows:

$$\vec{u}_{n-1} = \{u_0, u_1, u_2, \ldots, u_{n-1}\} \tag{15}$$

$$\vec{\omega}_{n-1} = \{\omega_0, \omega_1, \omega_2, \ldots, \omega_{n-1}\} \tag{16}$$

$$R_n(\vec{u}_{n-1}, \vec{\omega}_{n-1}) = \frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1} N[\phi(\tau, q), \omega(q)]}{\mathrm{d}q^{n-1}}\bigg|_{q=0} \tag{17}$$

$$u_n(0) = \dot{u}_n(0) = 0 \tag{18}$$

2. Application of the HAM

Free oscillation of a system without damping is a periodic motion and a harmonic function is the simplest type of periodic motion. Such a relationship can be expressed by the following base functions (Pirbodaghi & Hoseini, 2010; Tse, Morse, & Hinkle, 1978):

$$\{\cos(m\tau), \quad m = 1, 2, 3, \dots\} \tag{19}$$

In order to satisfy the initial conditions, the initial guess of $u(\tau)$ for zero-order deformation equation is chosen as follows:

$$u_0(\tau) = a\cos\tau \tag{20}$$

To construct the homotopy function, the auxiliary linear operator for vibration of a conservative system is selected as (Liao, 1992; Hoseini et al., 2009):

$$L[u(\tau;q)] = \omega_0^2\left(\frac{\partial^2 u(\tau;q)}{\partial\tau^2} + u(\tau;q)\right) \tag{21}$$

The auxiliary linear operator L is chosen in such a way that all solutions of the corresponding high-order deformation equations exist and can be expressed by the general form of the base function. From Equation (3), the nonlinear operator is written as:

$$N[u(\tau;q), \omega] = \left(\omega^2 \frac{\mathrm{d}^2 u(\tau;q)}{\mathrm{d}\tau^2} + u(\tau;q)\right)^2 \left(1 + u(\tau;q)\right) - \lambda^2 u(\tau;q)^2, \tag{22}$$

The solution must comply with the general form of the base functions. Therefore, the auxiliary function, $H(\tau)$, should be chosen as follows:

$$H(\tau) = \cos(\tau) \tag{23}$$

Due to odd nonlinearity of considered conservative system, it is found that $R'_m = H(\tau)R_m$ can be expressed by:

$$R'_m = \sum_{n=0} \mathrm{d}_n(\omega_{m-1})\cos((2n+1)\tau), \tag{24}$$

According to the property of the linear operator, if the term $\cos(\tau)$ exist in R'_m, the secular term $\tau\sin(\tau)$ will appear in the final solution. Therefore, the coefficient of $\cos(\tau)$ in R'_m must be equal to zero:

$$d_0(\omega_{m-1}) = 0 \tag{25}$$

Solving Equation (25), ω_{m-1} is obtained. For the first-order approximation of HAM, R'_1 as follows:

$$\begin{aligned} R_1 = &\left(\tfrac{3}{4}a^2\omega_0^4 + \tfrac{5}{8}a^4\omega_0^4 - \tfrac{3}{2}a^2\omega_0^2 - \tfrac{5}{4}a^4\omega_0^2 - \tfrac{3}{4}\lambda^2a^2 + \tfrac{3}{4}a^2 + \tfrac{5}{8}a^4\right)\cos(\tau) \\ &+\left(\tfrac{5}{16}a^4\omega_0^4 - \tfrac{5}{8}a^4\omega_0^2 - \tfrac{1}{2}a^2\omega_0^2 + \tfrac{1}{4}a^2\omega_0^4 - \tfrac{1}{4}\lambda^2a^2 + \tfrac{1}{4}a^2 + \tfrac{5}{16}a^4\right)\cos(3\tau) \\ &+\left(\tfrac{1}{16}a^4\omega_0^4 - \tfrac{1}{8}a^4\omega_0^2 + \tfrac{1}{16}a^4\right)\cos(5\tau) \end{aligned} \tag{26}$$

Thus, ω_0 can be written as:

$$\omega_0 = \sqrt{1 - \frac{\sqrt{30\lambda^2a^2 + 36\lambda^2}}{5a^2 + 6}} \tag{27}$$

Solving Equation (12) for $m = 1$, u_1 is obtained as:

$$u_1 = \hbar\lambda^2 a^4 \frac{-6\cos(\tau) + 5\cos(3\tau) + \cos(5\tau)}{64\sqrt{30\lambda^2 a^2 + 36\lambda^2 - 320a^2 - 384}} \tag{28}$$

Consequently, from the coefficient of $\cos(\tau)$ in R'_2, ω_1 is obtained as follows:

$$\omega_1 = -\frac{\sqrt{3\hbar}\,\lambda^2a^3((40 + 56a^2)\sqrt{6\lambda^2(6 + 5a^2)} - 299\lambda^2a^2 - 240\lambda^2)}{192\omega_0((6 + 5a^2)\sqrt{6\lambda^2(6 + 5a^2)} - 25a^4 - 60a^2 - 36)\sqrt{\lambda^2(6 + 5a^2)}} \tag{29}$$

The higher order approximations for frequency and deflection can be similarly derived. Solving Equation (12) for $m = 2$ yields the following result for u_2:

$$\begin{aligned} u_2 = &\left(\frac{6\lambda^2a^4\hbar}{L_1} + \frac{L_3}{L_2}\right)\cos(\tau) + \left(-\frac{5\lambda^2a^4\hbar}{L_1} + \frac{L_4}{L_2}\right)\cos(3\tau) \\ &+ \left(-\frac{\lambda^2a^4\hbar}{L_1} + \frac{L_5}{L_2}\right)\cos(5\tau) + \frac{L_6}{L_2}\cos(7\tau) + \frac{L_7}{L_2}\cos(3\tau) \end{aligned} \tag{30}$$

From the coefficient of $\cos(\tau)$ in R'_3, ω_2 is obtained as follows:

$$\omega_2 = -\frac{L_8}{L_9} \tag{31}$$

where $L1$, $L2$, …, $L9$ are given in Appendix 1. The higher order approximations for ω and $u(\tau)$ (ω_{n-1} and $u_m(\tau)$, $m > 2$) can be similarly derived. The [1, 1] homotopy Pade approximation of ω and $u(\tau)$ can be written in the following form (Pirbodaghi, Hoseini et al., 2009):

$$\omega_{[1,1]\text{pade}} = \frac{\omega_1\omega_0 + \omega_1^2 - \omega_2\omega_0}{\omega_1 - \omega_2} \tag{32}$$

$$u_{[1,1]\text{pade}} = \frac{u_1 u_0 + u_1^2 - u_2 u_0}{u_1 - u_2} \tag{33}$$

Also $[i, j]$ homotopy Pade approximation is determined by the first $(i + j + 1)$ th-order approximations of HAM solution.

3. Results

Table 1 lists the obtained frequencies of the system using our approximate analytical technique and compares it with numerical solutions using Runge–Kutta method for different values of λ and initial conditions.

It can be observed from Table 1 that there is an excellent agreement between the results obtained from the homotopy Pade technique and results of Runge–Kutta method. The maximum relative error between [1, 1] homotopy Pade results and numerical results is .3% which is very impressive. Figure 1 illustrates the effect of auxiliary parameter $\hbar$ on the frequency for different-order approximation of HAM solutions. It is obvious that with the increase of the order of approximation, the frequency becomes independent of $\hbar$ and remains fixed.

It can be noted that for $\hbar = -.68017$, the second-order approximation gives the frequency with the highest accuracy. This indicates that the auxiliary parameter plays an important role in the HAM. Figures 2 and 3 show the displacement of the system for $\lambda = .5$, $a = 1$ and $\lambda = .95$, $a = 1$, respectively.

Figure 4 shows the phase-space curves for amplitudes for several λ values. The phase plot shows the behaviour of the oscillator when λ varies. It is periodic with a centre at (0, 0). This situation also occurs in the unforced, undamped cubic Duffing oscillators.

Table 1. Comparison of frequency corresponding to various parameters of system.

a	λ	ω_0	[1, 1] homotopy Pade	Runge–Kutta
.1	.1	.94890	.94889	.94888
.1	.95	.23224	.23146	.23137
1	.1	.96236	.96121	.96110
5	.1	.98924	.98830	.98781
5	.5	.94498	.93998	.93732
5	.95	.89257	.88228	.87656
10	.1	.99454	.99403	.99371
10	.5	.97239	.96978	.96810
10	.95	.94686	.94170	.93833

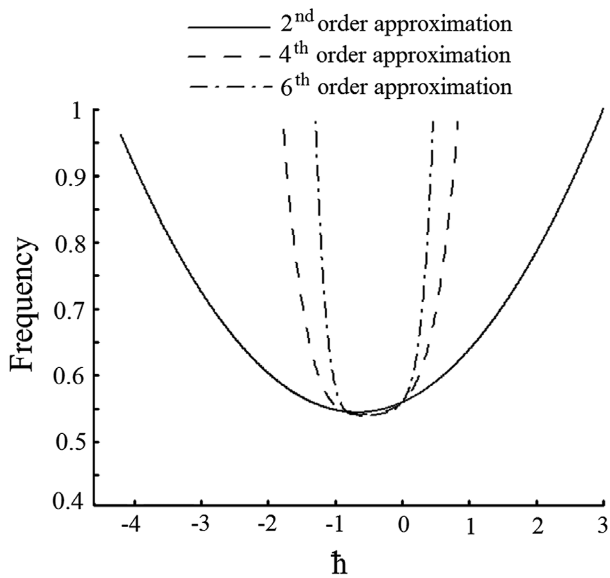

Figure 1. The effect of auxiliary parameter $\hbar$ on the frequency ($\lambda = .95$, and $a = 1$).

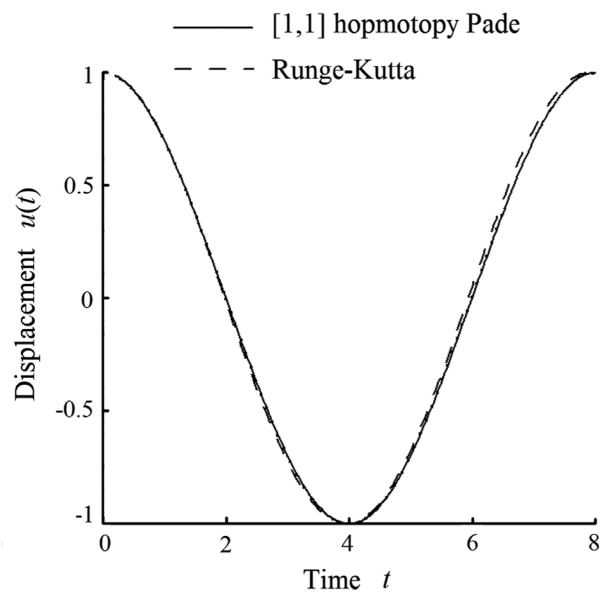

Figure 2. Displacement, $u(t)$, vs. time, t, for $\lambda = .5$ and $a = 1$.

It is evident that the [1, 1] homotopy Pade provides excellent approximations to exact periodic solution for this case study. As shown in this study, the HAM solution is quickly convergent and its components can be simply calculated. Also, compared to other analytical methods, it can be observed that the results of HAM require less computational effort and only a first-order approximation leads to highly accurate solutions.

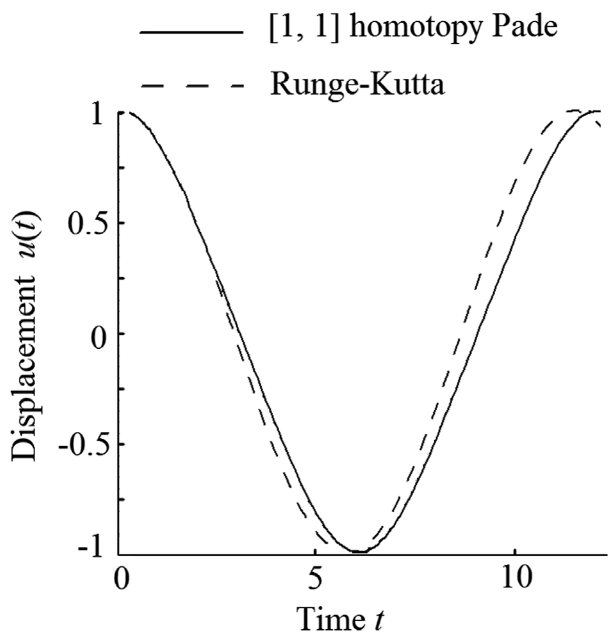

Figure 3. Displacement, $u(t)$, vs. time, t, for $\lambda = .95$ and $a = 1$.

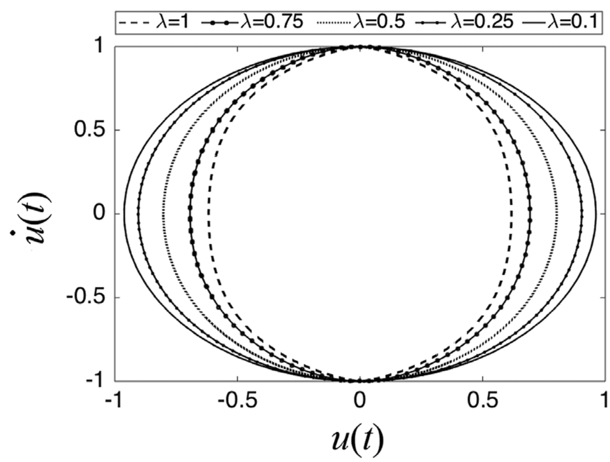

Figure 4. Phase-space curves ($\dot{u}(t)$ vs. $u(t)$ curve) for different λ values.

4. Conclusions

The HAM and homotopy Pade technique has been used to obtain analytical solutions for oscillation of a mass attached to a stretched elastic wire. Present study shows that the results of HAM and homotopy Pade technique are valid on a wide range of considered system parameter. The accuracy of the results reveals that this method can be considered as viable alternative for conventional methods to solve highly nonlinear oscillatory systems. Beside all the advantages of the HAM, there are no rigorous theories to direct us to choose the initial approximations, auxiliary linear operators, auxiliary functions and auxiliary parameter. However, further research is needed to better understand the effect of these parameters on the solution quality.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendix 1

$$L_1 = 320a^2 + 384 - 64\sqrt{\lambda^2\left(30a^2 + 36\right)}$$

$$L_2 = 30720\left(5a^2 + 6\right)^2\left(6\lambda^2 - 5a^2 - 6\right)^2$$

$$L_3 = -\lambda^2\hbar^2 a^5\Big(-1480800\lambda^2 a^4 - 1911564\lambda^2 a^2 + 1267092\lambda^4 a^2 - 803520\lambda^2 + 803520\lambda^4 - 371525\lambda^2 a^6\sqrt{\lambda^2\left(30a^2 + 36\right)}\Big)\cos(t)$$

$$L_4 = 15\lambda^2\hbar^2 a^5\Big(-68590\lambda^2 a^4 - 91152\lambda^2 a^2 + 58416\lambda^4 a^2 - 39168\lambda^2 + 39168\lambda^4 - 16525\lambda^2 a^6 - 15184a^2 + 6528\lambda^2\sqrt{\lambda^2\left(30a^2 + 36\right)}\Big)\cos(3t)$$

$$L_5 = 5\lambda^2\hbar^2 a^5\Big(-74872\lambda^2 a^4 - 92952\lambda^2 a^2 + 68472\lambda^4 a^2 - 38016\lambda^2 + 38016\lambda^4 - 19685\lambda^2 a^6 + 27066\lambda^4 a^4 + \left(4798\lambda^2 a^4 - 3520a^6 - 6336\right)\sqrt{\lambda^2\left(30a^2 + 36\right)}\Big)\cos(5t)$$

$$L_6 = 10\lambda^2\hbar^2 a^5\left(5a^2 + 6\right)\Big(-455\lambda^2 a^4 - 906\lambda^2 a^2 + 678\lambda^4 a^2 - 432\lambda^2 + 432\lambda^4 + \left(124\lambda^2 a^2 - 85a^4 - 162a^2 + 72\lambda^2 - 72\right)\sqrt{\lambda^2\left(30a^2 + 36\right)}\Big)\cos(7t)$$

$$L_7 = 9\lambda^2\hbar^2 a^5\left(5a^2 + 6\right)\left(-55\lambda^2 a^2 - 66\lambda^2 + 78\lambda^4 + \left(14\lambda^2 - 10a^2 - 12\right)\sqrt{\lambda^2\left(30a^2 + 36\right)}\right)\cos(t)$$

$$L_8 = 151340\hbar^2\omega_0^8 a^{10}\lambda^2 + 511040\hbar^2\omega_0^8 a^8\lambda^2 - 229440\hbar^2\omega_0^6 a^{10}\lambda^2 + 539148\hbar^2\omega_0^8 a^6\lambda^2 - 769520\hbar^2\omega_0^6 a^8\lambda^2 - 112080\hbar^2\omega_0^2 a^4\lambda^4 - 36880\hbar^2 a^8\lambda^2 - 42369\hbar^2 a^6\lambda^4 - 9420\hbar^2 a^4\lambda^6$$

$$L_9 = 61440\omega_0^5\left(5a^2 + 6\right)^3\left(\omega_0^2 - 1\right)$$