EUROPEAN JOURNAL OF COMPUTATIONAL MECHANICS, 2016
VOL. 25, NOS. 1–2, 109–128
http://dx.doi.org/10.1080/17797179.2016.1181039

Taylor & Francis
Taylor & Francis Group

A comparison of three evaluation methods for Green's function and its derivatives for 3D generally anisotropic solids

Longtao Xie[a], Chuanzeng Zhang[a], Chyanbin Hwu[b], Jan Sladek[c] and Vladimir Sladek[c]

[a]Department of Civil Engineering, University of Siegen, Siegen, Germany; [b]Institute of Aeronautics and Astronautics, National Cheng Kung University, Tainan, Taiwan; [c]Institute of Construction and Architecture, Slovak Academy of Sciences, Bratislava, Slovakia

ABSTRACT

A comparison of three different methods for the numerical evaluation of three-dimensional (3D) anisotropic Green's function and its first and second derivatives is presented. The line integral expressions of the Green's function and its derivatives are the starting point of this investigation. The conventional line integral expressions are rewritten in terms of three different kinds of line integrals. In the first method, the numerical integration is applied to the line integrals. In the second method, the residue calculus is used, which results in explicit expressions of the Green's function and its derivatives in non-degenerate cases. In the third method, the three line integrals are expressed in terms of two elementary line integrals, and after a rewritten of the explicit expressions evaluated by the simple pole residue calculus, the final explicit expressions are applicable in both degenerate and non-degenerate cases. The three methods are implemented in FORTRAN to make a direct comparison. Using the analytical solutions, the three expressions of the Green's function and its derivatives are proved to be correct. The numerical phenomenon of the three methods near a degenerate point is studied numerically. Besides, the efficiency of the three methods is compared through the computing CPU times.

ARTICLE HISTORY
Received 13 October 2015
Accepted 11 April 2016

KEYWORDS
Elasticity; three-dimensional; general anisotropy; Green's function; derivatives of Green's function

1. Introduction

The Green's function and its derivatives play an important role in the boundary integral equation or boundary element method (BEM). In homogeneous, isotropic and linear elasticity, these functions have a simple analytical form. They can be evaluated directly in a BEM program. However, in generally anisotropic linear elasticity, the Green's function and its derivatives are much more complicated. Though Wilson and Cruse (1978) proposed a practical algorithm by employing a cubic interpolation from tabulated pre-calculated values for the evaluation of the Green's function and its derivatives in BEM programs, the

CONTACT Chuanzeng Zhang zhang@bauwesen.uni-siegen.de

direct evaluation of the anisotropic Green's function and its derivatives was preferred and hence investigated by many researchers.

Let us consider an infinite static linear elastic homogeneous three-dimensional (3D) anisotropic solid. In this paper, a vector or tensor is represented by either a bold letter or a letter in indices notation. The Green's function $G_{ij}(\boldsymbol{x})$ satisfies the following partial differential equation

$$c_{ijkl}G_{km,lj}(\boldsymbol{x}) + \delta_{im}\delta(\boldsymbol{x}) = 0, \tag{1}$$

where c_{ijkl} is the elasticity tensor, δ_{im} is the Kronecker delta, and $\delta(\boldsymbol{x})$ is the three-dimensional Dirac delta function which is zero everywhere, except at the point $\boldsymbol{x} = 0$. The elasticity tensor c_{ijkl} is a symmetrical tensor

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}. \tag{2}$$

By applying either Fourier transforms (Fredholm, 1900) or Radon transforms (Wang, 1997) to Equation (1) followed by some elementary manipulations, the Green's function for 3D anisotropic materials can be deduced to a contour integral

$$G_{ij}(\boldsymbol{x}) = \frac{1}{8\pi^2 r}\oint_S N_{ij}(\boldsymbol{\xi})D^{-1}(\boldsymbol{\xi})\mathrm{d}S(\boldsymbol{\xi}), \tag{3}$$

where $r = |\boldsymbol{x}|$, S is a unit circle in a plane whose normal vector is along $\boldsymbol{x}$, $\boldsymbol{\xi}$ is a vector located on S, and $N_{ij}(\boldsymbol{\xi})$ and $D(\boldsymbol{\xi})$ are, respectively, the cofactors and determinant of $K_{ik}(\boldsymbol{\xi}) = c_{ijkl}\xi_j\xi_l$. As will be shown in the next section, with proper change of the variables, the Green's function in line integral form can be written as

$$G_{ij}(\boldsymbol{x}) = \frac{1}{4\pi^2 r}\int_{-\infty}^{+\infty} N_{ij}(p)D^{-1}(p)\mathrm{d}p, \tag{4}$$

where $N_{ij}(p)$ and $D(p)$ are deduced from $N_{ij}(\boldsymbol{\xi}^*)$ and $D(\boldsymbol{\xi}^*)$ with the substitution of $\boldsymbol{\xi}^* = \boldsymbol{n}+p\boldsymbol{m}$, and $\boldsymbol{n}$ and $\boldsymbol{m}$ are any two mutually orthogonal unit vectors on the oblique plane perpendicular to $\boldsymbol{x}$. In particular, $D(p)$ is a sixth-order polynomial. As long as the elastic strain energy is positive, the roots of $D(p)$ are three pairs of complex conjugates known as Stroh eigenvalues. By applying the Cauchy residue theorem to the line integrals with the assumption that p_i $(i = 1, 2, 3)$ are three distinct roots of $D(p)$ with a positive imaginary part, Equation (4) becomes

$$G_{ij}(\boldsymbol{x}) = \frac{i}{2\pi r}\sum_{v=1}^{3}\frac{N_{ij}(p_v)}{D'(p_v)}, \tag{5}$$

where $D'(p) = \mathrm{d}D(p)/\mathrm{d}p$ and $i = \sqrt{-1}$.

Equation (5) is simple but not applicable when any two of p_i are identical. To deal with the so-called degenerate situations with repeated roots, Phan, Gray, and Kaplan (2004) and Buroni, Ortiz, and Sáez (2011) presented the explicit expressions of the Green's function by applying the multiple pole residue calculus to the line integral. Although these explicit expressions of the Green's function in the degenerate and non-degenerate cases are correct, the results become unstable in the numerical calculation when the Stroh eigenvalues are distinct, but very close to each other (nearly degenerate cases). After a magical rewritten of the explicit expression of the Green's function in the non-degenerate case, Ting and Lee (1997) found a novel explicit expression of the Green's function applicable in the non-degenerate and degenerate cases. Moreover, the numerical results of these explicit expression were stable in the nearly degenerate case, which has not been emphasised in the literature, and will be confirmed in the following of this paper.

The derivatives of the Green's function were investigated also by many researchers (Barnett, 1972; Buroni et al., 2011; Lee, 2003; Phan, Gray, and Kaplan, 2005; Sales & Gray, 1998). Although the numerical integration method (NIM) for the evaluation of the Green's function and its derivatives was suggested many years ago (Barnett, 1972), researchers are still interested in the explicit expressions of the Green's function and its derivatives, which should be advantageous in the BEM programming. Phan et al. (2004, 2005) used the Cauchy residue theorem to derive explicit expressions of the Green's function and its first derivative in terms of the Stroh eigenvalues. However, their expressions were different for three different cases, namely, the non-degenerate case (three distinct eigenvalues), the partially degenerate case (two identical eigenvalues) and the degenerate case (three identical eigenvalues). Lee (2009) also derived the explicit expression of the first derivative of the Green's function for three different cases based on the novel explicit expression of the Green's function proposed by Ting and Lee (1997). She mentioned the way to obtain the second derivative, but no final expressions and examples for the second derivative of the Green's function were given. Buroni and Sáez (2013) presented novel unified explicit expressions for the first and second derivatives of the Green's function in the spherical coordinate system with the help of the expression proposed by Ting and Lee (1997). These expressions shared the same character of Ting and Lee's expression, i.e. applicable in degenerate, non-degenerate and nearly degenerate cases. Recently, Xie, Zhang, Wan, and Zhong (2013) suggested a new way to obtain the explicit expressions for the Green's function and its first and second derivatives, which are applicable in all cases. Different from the work of Buroni and Sáez (2013), partial derivatives of the Green's function in Xie et al. (2013) were performed in the Cartesian coordinate system, which are more attractive in the applications. It is expected that the explicit expressions are applicable in all cases and much more convenient for the numerical implementation because programmers needn't to distinguish different cases in the programming. Besides, for the implementation of the Green's function and its derivatives in the BEM programs, Shiah, Tan, and Wang (2012) and Tan, Shiah, and Wang (2013) expressed the Green's function and its derivatives as Fourier series, and they demonstrated that their method was very efficient from the numerical point of view. Since the Green's function and its derivatives can be expressed as three different formulae, it is useful to investigate the accuracy and efficiency of the numerical evaluation of these different formulae. However, to the authors' best knowledge, a direct and detailed comparison between the different expressions of the Green's function and its derivatives has not yet been reported in the literature. Besides the above-mentioned methods, interested readers may be refered to Malén (1971), Lavagnino (1995), Ting (1996), Hwu (2010) and Pan and Chen (2015) for methods constructing the Green's function and its derivatives using the so-called Stroh formalism.

In this paper, we mainly focus our attention to the three different formulae to evaluate the Green's function and its derivatives for 3D generally anisotropic materials. Specifically we investigate the numerical implementations of the three formulae. In the first method, the numerical integration is applied to the line integral expressions of the Green's function and its derivatives, while in the second method the residue calculus with distinctness assumption of Stroh eigenvalues is applied to the line integrals which leads to explicit expressions of the Green's function and its derivatives. In the third method, the Green's function and its derivatives are first expressed in terms of two elementary line integrals. Then, the residue calculus with the distinctness assumption is applied to the elementary line integrals, and thereafter a rewritten of the resulting expressions leads to explicit expressions applicable in both non-degenerate and degenerate cases. The three methods are implemented in FORTRAN programs. Since the nearly degenerate case is involved in the second and third methods, a transversely isotropic material is chosen to investigate these two methods in the nearly degenerate case. Besides, the accuracy and the efficiency of the three methods are compared and discussed.

2. Three different formulae for the Green's function and its derivatives

2.1. Line integral expressions of the Green's function and its derivatives

The line integral expression of the Green's function was firstly investigated by Fredholm (1900), Lifshitz and Rozenzweig (1947) and Synge (1957). While the line integral expression of the derivatives of the Green's function was derived by Barnett (1972) and Mura (1987).

Here, the Green's function and its first and second derivatives in terms of the line integrals over a unit circle presented by Mura (1987) are extracted as our starting point. For the details of the derivation, interested readers may find them in the work of Mura (1987). The expressions are given by

$$G_{ij}(\boldsymbol{x}) = \frac{1}{8\pi^2 r} \oint_S K_{ij}^{-1}(\boldsymbol{\xi})\mathrm{d}\psi, \tag{6}$$

$$\begin{aligned} G_{ij,k}(\boldsymbol{x}) = \frac{1}{8\pi^2 r^2} \oint_S \Big[& -\bar{x}_k K_{ij}^{-1}(\boldsymbol{\xi}) \\ & + \xi_k c_{lpmq}(\bar{x}_p \xi_q + \xi_p \bar{x}_q) K_{li}^{-1}(\boldsymbol{\xi}) K_{mj}^{-1}(\boldsymbol{\xi}) \Big] \mathrm{d}\psi, \end{aligned} \tag{7}$$

$$
\begin{aligned}
G_{ij,kl}(\boldsymbol{x}) = \frac{1}{8\pi^2 r^3} \oint_S \Big\{ & 2\bar{x}_k\bar{x}_l K_{ij}^{-1}(\boldsymbol{\xi}) - 2[(\bar{x}_k\xi_l + \xi_k\bar{x}_l)(\bar{x}_p\xi_q + \xi_p\bar{x}_q) + \xi_k\xi_l\bar{x}_p\bar{x}_q] \\
& \times c_{hpmq} K_{ih}^{-1}(\boldsymbol{\xi}) K_{jm}^{-1}(\boldsymbol{\xi}) + \xi_k\xi_l c_{hpmq}(\bar{x}_p\xi_q + \xi_p\bar{x}_q) c_{satb}(\bar{x}_a\xi_b + \xi_a\bar{x}_b) \\
& \times [K_{jm}^{-1}(\boldsymbol{\xi}) K_{is}^{-1}(\boldsymbol{\xi}) K_{ht}^{-1}(\boldsymbol{\xi}) + K_{ih}^{-1}(\boldsymbol{\xi}) K_{js}^{-1}(\boldsymbol{\xi}) K_{mt}^{-1}(\boldsymbol{\xi})] \Big\} \, \mathrm{d}\psi, \qquad (8)
\end{aligned}
$$

where $\bar{\boldsymbol{x}}$ is the unit vector of $\boldsymbol{x}$ and ψ is a parameter on the unit circle.

Now, we present an alternative form of the Green's function and its derivatives based on Equations (6)–(8). Using

$$
\boldsymbol{\xi} = \boldsymbol{n}\cos\psi + \boldsymbol{m}\sin\psi, \qquad (9)
$$

where $\boldsymbol{n}$ and $\boldsymbol{m}$ are any two mutually orthogonal unit vectors in the oblique plane, the line integrals over the unit circle are transformed to line integrals over ($-\frac{\pi}{2}, \frac{\pi}{2}$), or $(0, \pi)$ if necessary, because the period of the integrands in Equations (6)-(8) after substituting Equation (9) is π. The three newly introduced line integrals are

$$
A_{ij}(\bar{\boldsymbol{x}}) = \frac{1}{4\pi} \oint_S N_{ij}(\boldsymbol{\xi}) D^{-1}(\boldsymbol{\xi}) \mathrm{d}\psi = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} N_{ij}(\psi) D^{-1}(\psi) \mathrm{d}\psi, \qquad (10)
$$

$$
P_{ijk}(\bar{\boldsymbol{x}}) = \frac{1}{4\pi} \oint_S \xi_k H_{ij}(\boldsymbol{\xi}) D^{-2}(\boldsymbol{\xi}) \mathrm{d}\psi = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \xi_k H_{ij}(\psi) D^{-2}(\psi) \mathrm{d}\psi, \qquad (11)
$$

$$
Q_{ijkl}(\bar{\boldsymbol{x}}) = \frac{1}{4\pi} \oint_S \xi_k \xi_l M_{ij}(\boldsymbol{\xi}) D^{-3}(\boldsymbol{\xi}) \mathrm{d}\psi = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \xi_k \xi_l M_{ij}(\psi) D^{-3}(\psi) \mathrm{d}\psi, \qquad (12)
$$

where N_{ij} and D are, respectively, cofactors and determinant of the matrix K_{ij}, and H_{ij} and M_{ij} are defined as

$$
H_{ij} = F_{im} N_{jm}, \quad M_{ij} = L_{ij} - R_{ij} D, \qquad (13)
$$

with

$$
\begin{aligned}
F_{im} &= E_{hm} N_{ih}, & E_{hm} &= c_{phmq}(\bar{x}_p\xi_q + \bar{x}_q\xi_p), \\
L_{ij} &= F_{jh} H_{ih}, & R_{ij} &= \bar{x}_p\bar{x}_q c_{phmq} N_{ih} N_{jm}. \qquad (14)
\end{aligned}
$$

Note that the argument in Equations (13) and (14) could be $\boldsymbol{\xi}$ or ψ or even p introduced in the following. Then the reformulated line integral expressions of the Green's function and its derivatives are given by

$$G_{ij}(\boldsymbol{x}) = \frac{1}{2\pi r} A_{ij}(\bar{\boldsymbol{x}}), \tag{15}$$

$$G_{ij,k}(\boldsymbol{x}) = \frac{1}{2\pi r^2}\left[-\bar{x}_k A_{ij}(\bar{\boldsymbol{x}}) + P_{ijk}(\bar{\boldsymbol{x}})\right], \tag{16}$$

$$G_{ij,kl}(\boldsymbol{x}) = \frac{1}{\pi r^3}\left\{A_{ij}(\bar{\boldsymbol{x}})\bar{x}_k\bar{x}_l - \left[P_{ijk}(\bar{\boldsymbol{x}})\bar{x}_l + P_{ijl}(\bar{\boldsymbol{x}})\bar{x}_k\right] + Q_{ijkl}(\bar{\boldsymbol{x}})\right\}. \tag{17}$$

These integral expressions are equivalent to those proposed by Barnett (1972) and Mura (1987). It should be mentioned that the symmetry of c_{ijkl} and K_{ij} is used to deduce Equations (16) and (17).

The line integrals presented in Equations (10)–(12) ranging from $-\pi/2$ to $\pi/2$ are expressed in terms of the matrices in a unified form which are independent of their eigenvalue features. In other words, these basic direct line integrals can be applied to any kinds of anisotropic materials no matter whether their Stroh's eigenvalues are distinct or repeated. Quadrature rules such as the standard Gaussian quadrature can be applied on Equations (10)–(12) to calculate the Green's function and its derivatives by Equations (15)–(17). The numerical implementation of the reformulated line integral expressions of the Green's function and its derivatives associated with other methods is discussed in Section 3. Note that the direct line integral method could be particularly useful when dealing with the Green's function issues in half or bimaterial spaces. For example, by virtue of the direct line integral, the half-space Green's functions can be expressed in terms of the Stroh matrices and Stroh eigenvalues in a unified form Pan and Chen (2015).

2.2. Explicit expressions of the Green's function and its derivatives for non-degenerate cases

In the following, we investigate the explicit Green's function and its derivatives in terms of Stroh eigenvalues. Here, *explicit expressions* have mainly two meanings: firstly, they have no integrals; and secondly, they become algebraically analytical as long as the Stroh eigenvalues and/or the Stroh eigenvectors are algebraically analytical.

For the easy use of the residue calculus, the interval of the line integrals in Equations (10)–(12) is further transformed to ($-\infty, +\infty$). To illustrate the procedure, we take $A_{ij}(\bar{\boldsymbol{x}})$ as an example.

By setting $p = \tan\psi$, we have

$$\boldsymbol{\xi} = \cos\psi(\boldsymbol{n} + p\boldsymbol{m}), \quad \mathrm{d}p = \frac{1}{\cos^2\psi}\mathrm{d}\psi. \tag{18}$$

Note that due to the definition of $K_{ij}(\boldsymbol{\xi})$ we have

$$N_{ij}(\boldsymbol{\xi}) = \cos^4\psi N_{ij}(\boldsymbol{n} + p\boldsymbol{m}), \quad D(\boldsymbol{\xi}) = \cos^6\psi D(\boldsymbol{n} + p\boldsymbol{m}). \tag{19}$$

Then we can obtain

$$N_{ij}(\psi)D^{-1}(\psi) = N_{ij}(\boldsymbol{\xi})D^{-1}(\boldsymbol{\xi}) = \frac{1}{\cos^2\psi}N_{ij}(p)D^{-1}(p), \tag{20}$$

where $\boldsymbol{n}$ and $\boldsymbol{m}$ are omitted in the last term for simplicity, and $N_{ij}(p)$ and $D(p)$ are cofactors and determinant of the matrix $K_{ij}(p) = c_{ikjl}\xi_k^*\xi_l^*$, where $\boldsymbol{\xi}^* = \boldsymbol{n} + p\boldsymbol{m}$. Substitution of Equation (20) into Equation (10) leads to

$$A_{ij}(\bar{\boldsymbol{x}}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} N_{ij}(p)D^{-1}(p)\mathrm{d}p. \tag{21}$$

Similarly, $P_{ijk}(\bar{\boldsymbol{x}})$ and $Q_{ijkl}(\bar{\boldsymbol{x}})$ become

$$P_{ijk}(\bar{\boldsymbol{x}}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \xi_k^* H_{ij}(p)D^{-2}(p)\mathrm{d}p, \tag{22}$$

$$Q_{ijkl}(\bar{\boldsymbol{x}}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \xi_k^*\xi_l^* M_{ij}(p)D^{-3}(p)\mathrm{d}p, \tag{23}$$

where $H_{ij}(p)$ and $M_{ij}(p)$ are determined by Equations (13) and (14), in which $\boldsymbol{\xi}$ is replaced by $\boldsymbol{\xi}^*$. All $\boldsymbol{\xi}^*$, $N_{ij}(p)$, $H_{ij}(p)$ and $M_{ij}(p)$ are polynomials in p.

Suppose $f(p)$ is a rational polynomial function of the following form

$$f(p) = \frac{P(p)}{Q(p)}, \tag{24}$$

where $P(p)$ and $Q(p)$ are polynomials in p, and the order of $Q(p)$ is higher than that of $P(p)$. Then using the Cauchy residue theorem, it is easy to conclude that if there are n different poles $p_k,\ k = 1, 2, \ldots, n$ with $\mathrm{Im}\,(p_k) > 0$ among the poles of $f(p)$, we have

$$\int_{-\infty}^{+\infty} f(p)\mathrm{d}p = 2\pi i\sum_{k=1}^{n}\mathrm{Res}(p_k). \tag{25}$$

If p_k is a pole of mth order, then

$$\mathrm{Res}(p_k) = \frac{1}{(m-1)!}\lim_{p\to p_k}\frac{\mathrm{d}^{m-1}}{\mathrm{d}p^{m-1}}[(p-p_k)^m f(p)]. \tag{26}$$

Under the assumption that the Stroh eigenvalues, which are zeros of the polynomial $D(p)$ with positive imaginary parts, are distinct, $A_{ij}(\bar{\boldsymbol{x}})$, $P_{ijk}(\bar{\boldsymbol{x}})$ and $Q_{ijkl}(\bar{\boldsymbol{x}})$ have the same 3 poles. The orders of the 3 poles are the same in $A_{ij}(\bar{\boldsymbol{x}})$, $P_{ijk}(\bar{\boldsymbol{x}})$ or $Q_{ijkl}(\bar{\boldsymbol{x}})$, but the orders of each p_k in $A_{ij}(\bar{\boldsymbol{x}})$, $P_{ijk}(\bar{\boldsymbol{x}})$ and $Q_{ijkl}(\bar{\boldsymbol{x}})$ are, respectively, 1, 2 and 3. In virtue of Equations (25) and (26), Equations (21)–(23) become

$$A_{ij}(\bar{\boldsymbol{x}}) = -\operatorname{Im}\sum_{n=1}^{3}\frac{N_{ij}(p_n)}{D'(p_n)}, \tag{27}$$

$$P_{ijk}(\bar{\boldsymbol{x}}) = -\operatorname{Im}\sum_{n=1}^{3}\frac{D'(p_n)\hat{H}'_{ijk}(p_n) - D''(p_n)\hat{H}_{ijk}(p_n)}{D'^3(p_n)}, \tag{28}$$

$$\begin{aligned} Q_{ijkl}(\bar{\boldsymbol{x}}) = -\operatorname{Im}\sum_{n=1}^{3}\frac{1}{2D'^5(p_n)}\Big\{&D'^2(p_n)\hat{M}''_{ijkl}(p_n) - 3D'(p_n)D''(p_n)\hat{M}'_{ijkl}(p_n) \\ &+ \left[3D''^2(p_n) - D'''(p_n)D'(p_n)\right]\hat{M}_{ijkl}(p_n)\Big\}, \end{aligned} \tag{29}$$

where

$$\hat{H}_{ijk}(p) = \xi_k^* H_{ij}(p), \quad \hat{M}_{ijkl}(p) = \xi_k^*\xi_l^* M_{ij}(p), \tag{30}$$

in which $\hat{H}_{ijk}(p)$ and $\hat{M}_{ijkl}(p)$ are polynomials of 10th and 16th order, respectively.

Substitution of Equation (27) into Equation (15) yields the following explicit Green's function

$$G_{ij}(\boldsymbol{x}) = -\frac{1}{2\pi r}\operatorname{Im}\sum_{n=1}^{3}\frac{N_{ij}(p_n)}{D'(p_n)}. \tag{31}$$

Equation (31) is equivalent to Equation (5) and known as Fredholm's formula in the early literature (Dederichs & Liebfried, 1969). Sales and Gray (1998) firstly gave explicit derivatives of the Green's function in terms of the Stroh's eigenvalues. The starting point of Sales and Gray (1998) was a modulation function like $A_{ij}(\bar{\boldsymbol{x}})$ in Equation (21). The explicit derivatives of the Green's function were obtained after the differentiation of the modulation function with respect to two angles, namely polar angle and azimuthal angle in the spherical coordinate system, which determine the orientation of $\boldsymbol{x}$. Based on the three integrals, Lee (2003) presented explicit derivatives of the Green's function with respect to Cartesian coordinates. Note that the Fredholm's formula, explicit expressions presented by Sales and Gray (1998) and Lee (2003), as well as Equations (27)–(29) are only applicable when the Stroh eigenvalues are distinct. For a general evaluation, a small perturbation on the material constants is suggested to keep the Stroh eigenvalues distinct. Using multiple pole residue calculus, Phan et al. (2004, 2005) extended the work of Sales and Gray (1998) by giving explicit Green's function and its derivatives for the repeated or degenerated Stroh eigenvalues. Buroni et al. (2011) extended the work of Lee (2003). The explicit expressions by Sales and Gray (1998) and Lee (2003) were either with respect to spherical coordinates or contained tensors of the orders higher than 4. Our newly proposed explicit derivatives of the Green's function have two beneficial features: they are given in Cartesian coordinates and contain only low-order tensors.

2.3. Explicit expressions of the Green's function and its derivatives for non-degenerate and degenerate cases

In this section, we present unified explicit expressions of the Green's function and its derivatives which are applicable in both non-degenerate and degenerate cases. The word *unified* is used to emphasise the difference from the explicit expressions by using multiple pole residue calculus. It should be mentioned that the authors derived recently novel unified explicit expressions of the Green's function and its derivatives, (Xie et al., 2013; Xie et al., 2016) which are briefly described in the following for the completeness and comparison purposes.

The determinant $D(p)$ is a sixth-order polynomial in p. Because the elasticity tensor c_{ijkl} is positive definite, the roots of the determinant $D(p)$ are three pairs of complex conjugates. So $D(p)$ can be written as

$$\begin{aligned} D(p) &= \alpha(p-p_1)(p-p_2)(p-p_3)(p-\bar{p}_1)(p-\bar{p}_2)(p-\bar{p}_3) \\ &= \alpha\prod_{i=1}^{3}(p-p_i)(p-\bar{p}_i), \end{aligned} \tag{32}$$

where α is the coefficient of p^6 in $D(p)$, and the overbar denotes the complex conjugate. Since $D(p)$ is the determinant of $K_{ij}(p) = c_{ikjl}\xi_k^*\xi_l^*$, it can be concluded that the Stroh eigenvalues depend on the material constants, the direction of the observation point $\boldsymbol{x}$ and the chosen coordinates $\boldsymbol{n}$ and $\boldsymbol{m}$ in the oblique plane.

Since $N_{ij}(p)$, $\hat{H}_{ijk}(p)$ and $\hat{M}_{ijkl}(p)$ are polynomials with the highest order 4, 10 and 16, respectively, we can express them as

$$N_{ij}(p) = \sum_{n=0}^{4} a_{ij}^n p^n, \tag{33}$$

$$\hat{H}_{ijk}(p) = \sum_{n=0}^{10} a_{ijk}^n p^n, \tag{34}$$

$$\hat{M}_{ijkl}(p) = \sum_{n=0}^{16} a_{ijkl}^n p^n, \tag{35}$$

where a_{ij}^n, a_{ijk}^n and a_{ijkl}^n are independent of p. Substituting Equations (33)–(35) and Equation (32) into Equations (21)–(23), the three integrals can be rewritten as

$$A_{ij}(\bar{\boldsymbol{x}}) = \frac{1}{\alpha}\sum_{n=0}^{4} a_{ij}^n I_3^n, \tag{36}$$

$$P_{ijk}(\bar{\boldsymbol{x}}) = \frac{1}{\alpha^2}\sum_{n=0}^{10} a_{ijk}^n I_6^n, \tag{37}$$

$$Q_{ijkl}(\bar{\boldsymbol{x}}) = \frac{1}{\alpha^3}\sum_{n=0}^{16} a_{ijkl}^n I_9^n, \tag{38}$$

where

$$I_3^n = \int_{-\infty}^{+\infty} \frac{p^n}{f(p)}\mathrm{d}p, \quad 0 \le n \le 4, \tag{39}$$

$$I_6^n = \int_{-\infty}^{+\infty} \frac{p^n}{f^2(p)}\mathrm{d}p, \quad 0 \le n \le 10, \tag{40}$$

$$I_9^n = \int_{-\infty}^{+\infty} \frac{p^n}{f^3(p)}\mathrm{d}p, \quad 0 \le n \le 16, \tag{41}$$

with

$$f(p) = \prod_{i=1}^{3}(p - p_i)(p - \bar{p}_i). \tag{42}$$

Although the coefficients a_{ij}^n, a_{ijk}^n and a_{ijkl}^n are complicated, they can be obtained nearly exactly in a program by polynomial algorithms (Press et al., 2007). Besides it is not difficult to show that both the coefficients and the integrals I_3^n, I_6^n and I_9^n are real-valued.

If p_1, p_2 and p_3 are distinct, the orders of the poles in Equations (40) and (41) are, respectively, 2 and 3, which makes the resulting explicit expressions by residue calculus complicated. Therefore, instead of Equations (40) and (41) we consider

$$I_6^n = \int_{-\infty}^{+\infty} \frac{p^n}{\prod_{i=1}^{6}(p - p_i)(p - \bar{p}_i)}\mathrm{d}p, \quad 0 \le n \le 10, \tag{43}$$

$$I_9^n = \int_{-\infty}^{+\infty} \frac{p^n}{\prod_{i=1}^{9}(p - p_i)(p - \bar{p}_i)}\mathrm{d}p, \quad 0 \le n \le 16, \tag{44}$$

which are identical to Equations (40) and (41) when p_4 and p_7, p_5 and p_8, and p_6 and p_9 are, respectively, set to p_1, p_2 and p_3. Further, I_3^n, I_6^n and I_9^n can be expressed by the following two elementary integrals

$$I_m^0 = \int_{-\infty}^{+\infty} \frac{1}{\prod_{i=1}^{m}(p - p_i)(p - \bar{p}_i)}\mathrm{d}p, \tag{45}$$

$$I_m^1 = \int_{-\infty}^{+\infty} \frac{p}{\prod_{i=1}^{m}(p - p_i)(p - \bar{p}_i)}\mathrm{d}p. \tag{46}$$

It can be shown that I_m^0 $(1 \le m \le 3)$ and I_m^1 $(2 \le m \le 3)$ are needed for the calculation of I_3^n which is required by the Green's function, I_m^0 $(4 \le m \le 6)$ and I_m^1 $(4 \le m \le 6)$ are needed for the calculation of I_6^n which is required by the first derivative of the Green's function, and I_m^0 $(7 \le m \le 9)$ and I_m^1 $(7 \le m \le 9)$ are needed for the calculation of I_9^n which is required by the second derivative of the Green's function. So unified explicit expressions of I_m^0 $(1 \le m \le 9)$

and I_m^1 ($2 \le m \le 9$) are required by unified explicit Green's function and its derivatives.

The expressions of I_3^n in terms of I_m^0 and I_m^1 are given by

$$\begin{aligned}
I_3^2 &= I_2^0 + 2\,\mathrm{Re}\,(p_3)I_3^1 - |p_3|^2 I_3^0,\\
I_3^3 &= I_2^1 + 2\,\mathrm{Re}\,(p_3)I_3^2 - |p_3|^2 I_3^1,\\
I_3^4 &= I_1^0 + 2\,\mathrm{Re}\,(p_2+p_3)I_3^3 - \left[|p_2|^2 + |p_3|^2 + 4\,\mathrm{Re}\,(p_2)\,\mathrm{Re}\,(p_3)\right] I_3^2\\
&\quad + 2\left[\mathrm{Re}\,(p_2)|p_3|^2 + \mathrm{Re}\,(p_3)|p_2|^2\right] I_3^1 - |p_2|^2|p_3|^2 I_3^0,
\end{aligned} \tag{47}$$

while for $n = 0, 1$,

$$\begin{aligned}
I_1^0 &= \frac{\pi}{\beta_1},\\
I_2^n &= -\frac{\pi}{\beta_1\beta_2}\,\mathrm{Im}\left(\frac{p_1^n}{p_1 - \bar{p}_2}\right),\\
I_3^n &= -\frac{\pi}{2\beta_1\beta_2\beta_3}\,\mathrm{Re}\left[\frac{p_1^n}{(p_1-\bar{p}_2)(p_1-\bar{p}_3)} + \frac{p_2^n}{(p_2-\bar{p}_1)(p_2-\bar{p}_3)}\right.\\
&\quad \left. + \frac{p_3^n}{(p_3-\bar{p}_1)(p_3-\bar{p}_2)}\right],
\end{aligned} \tag{48}$$

in which β_i is the imaginary part of p_i. It should be mentioned here that I_m^n are real-valued.

The explicit expressions of I_6^n and I_9^n required by the derivatives of the Green's function can be found in the Appendix 1.

The most important advantage of Equation (48) is that the explicit expressions are applicable not only when p_i are distinct but also when some p_i are identical or any two of p_i are very close to each other. This advantage will be verified by the following numerical evaluation. Besides, the Green's function in terms of the unified explicit I_m^n ($n = 0, 1, m = 1, 2, 3$) can be easily proved to be equivalent to the explicit expressions derived by Ting and Lee (1997).

3. Numerical implementations and results

In the previous sections, we presented three formulae of the 3D anisotropic elastic Green's function and its derivatives. In this section, we discuss and describe their implementations. For convenience, the three methods are, respectively, named as the NIM, the residue calculus method (RCM), and the improved residue calculus method (iRCM).

In contrast to the RCM and the iRCM, the NIM avoids the need of the Stroh eigenvalues. Therefore, it is applicable in both non-degenerate and degenerate cases. The Gaussian quadrature is used for the numerical integration. The number of the Gaussian points is 25 to ensure a comparable accuracy to the other two methods. The Stroh eigenvalues required by the RCM and iRCM can be obtained by finding the roots of $D(p)$, or finding the eigenvalues of the fundamental elasticity matrix $\mathbf{N}$ (Hwu, 2010; Ting, 1996)

$$\mathbf{N} = \begin{pmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^T \end{pmatrix}, \qquad (49)$$
$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^T,\ \mathbf{N}_2 = \mathbf{T}^{-1} = \mathbf{N}_2^T,\ \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q},$$

in which

$$Q_{ij} = c_{kijl}n_k n_l, \quad R_{ij} = c_{kijl}n_k m_l, \quad T_{ij} = c_{kijl}m_k m_l, \qquad (50)$$

where $\boldsymbol{n}$ and $\boldsymbol{m}$ are two mutually orthogonal unit vectors on the oblique plane perpendicular to $\boldsymbol{x}$.

The material for the numerical examples is taken as the transversely isotropic material Mg. When the symmetry axis of the material is along the x_3-axis of the Cartesian coordinate system, the non-zero components of the elasticity tensor c_{ijkl} in Voigt notation are

$$C_{11} = C_{22} = 59.7\text{GPa}, \quad C_{33} = 61.7\text{GPa}, \quad C_{13} = C_{23} = 21.7\text{GPa},$$
$$C_{12} = 26.2\text{GPa}, \quad C_{44} = C_{55} = 16.4\text{GPa}, \quad C_{66} = 16.75\text{GPa}. \qquad (51)$$

In order to check the correctness of the three formulae, the numerical results of the three methods for the Green's function and its derivatives at the point $(1, 2, 3)$ are compared with the analytical expressions (Pan & Chou, 1976). The three methods are implemented by FORTRAN while the analytical expressions are evaluated by MATHEMATICA. Tables 1 and 2 are the numerical results of the NIM, the RCM and the iRCM as well as the analytical results. The underlined digits agree perfectly with the analytical results. Therefore, we can conclude that all the three formulae are correct.

It is easy to prove that for a transversely isotropic material whose symmetry axis is along the x_3-axis, there are usually three distinct Stroh eigenvalues, except at the points on the x_3-axis where there is only one Stroh eigenvalue $p_i = i$, i.e. we have a fully degenerate case. So the three formulae are evaluated around the point $(0, 0, 1)$ to investigate their ability to deal with nearly degenerate case. In particular, the chosen evaluation points are $\boldsymbol{x} = (0, \sin\theta, \cos\theta)$ around $\theta = 0$. Figures 1–3 are the numerical results by the three methods evaluated near the degenerate point $(0, 0, 1)$. The results of the Green's function and its derivatives evaluated by the NIM and the iRCM agree well with each other, and are stable. But the results evaluated by the RCM become unstable near the degenerate point. Besides, the NIM and the iRCM can calculate the results at the fully degenerate point, while the RCM cannot. It is observed that in the RCM, the Green's function has the smallest unstable area while the second derivative of the Green's function has the largest one.

The FORTRAN programs of the three different methods are implemented under the same computing environment. Figure 4 is a comparison of the computing time required by the three methods. The material is Mg, and the evaluated

Table 1. Components of the Green's function and its derivatives by the NIM with 25 Gaussian points and analytical solutions for transversely isotropic material Mg at point $(1, 2, 3)$.

	NIM	Pan and Chou (1976)	Unit
G_{11}	$8.3782981337130575 \times 10^{-4}$	$8.3782981337130640 \times 10^{-4}$	
G_{12}	$6.0007221557104541 \times 10^{-5}$	$6.0007221557104690 \times 10^{-5}$	
G_{13}	$8.0163881274800664 \times 10^{-5}$	$8.0163881274800705 \times 10^{-5}$	$10^{-9}\mathrm{m}$
G_{22}	$9.2784064570696238 \times 10^{-4}$	$9.2784064570696325 \times 10^{-4}$	
G_{23}	$1.6032776254960144 \times 10^{-4}$	$1.6032776254960141 \times 10^{-4}$	
G_{33}	$1.0578889644135979 \times 10^{-3}$	$1.0578889644135983 \times 10^{-3}$	
$G_{11,1}$	$-5.9117971638181820 \times 10^{-6}$	$-5.9117971638178034 \times 10^{-6}$	
$G_{12,1}$	$4.7768458294984388 \times 10^{-5}$	$4.7768458294984570 \times 10^{-5}$	
$G_{13,1}$	$6.1775349906881231 \times 10^{-5}$	$6.1775349906880784 \times 10^{-5}$	1
$G_{22,1}$	$-8.4277163614102321 \times 10^{-5}$	$-8.4277163614102660 \times 10^{-5}$	
$G_{23,1}$	$-3.6777062735840222 \times 10^{-5}$	$-3.6777062735839842 \times 10^{-5}$	
$G_{33,1}$	$-1.2597090883943352 \times 10^{-4}$	$-1.2597090883943393 \times 10^{-4}$	
$G_{11,11}$	$-1.5317520185559242 \times 10^{-5}$	$-1.5317520185560475 \times 10^{-5}$	
$G_{12,11}$	$-3.2300147619148905 \times 10^{-5}$	$-3.2300147619146344 \times 10^{-5}$	
$G_{13,11}$	$-4.7740579505162444 \times 10^{-5}$	$-4.7740579505166856 \times 10^{-5}$	$10^{9}\mathrm{m}^{-1}$
$G_{22,11}$	$-6.2581146860783276 \times 10^{-5}$	$-6.2581146860784102 \times 10^{-5}$	
$G_{23,11}$	$-2.1927033538652794 \times 10^{-5}$	$-2.1927033538654044 \times 10^{-5}$	
$G_{33,11}$	$-8.3977103691749630 \times 10^{-5}$	$-8.3977103691742474 \times 10^{-5}$	

Table 2. Components of the Green's function and its derivatives by the residue calculus method (RCM) and the improved residue calculus method (iRCM) for transversely isotropic materials Mg at point $(1, 2, 3)$.

	RCM	iRCM	Unit
G_{11}	$8.3782981337141092 \times 10^{-4}$	$8.3782981337130402 \times 10^{-4}$	
G_{12}	$6.0007221556859634 \times 10^{-5}$	$6.0007221557104867 \times 10^{-5}$	
G_{13}	$8.0163881274806153 \times 10^{-5}$	$8.0163881274800475 \times 10^{-5}$	$10^{-9}\mathrm{m}$
G_{22}	$9.2784064570683217 \times 10^{-4}$	$9.2784064570696130 \times 10^{-4}$	
G_{23}	$1.6032776254963158 \times 10^{-4}$	$1.6032776254960092 \times 10^{-4}$	
G_{33}	$1.0578889644135944 \times 10^{-3}$	$1.0578889644135964 \times 10^{-3}$	
$G_{11,1}$	$-5.9117971643756500 \times 10^{-6}$	$-5.9117971638177865 \times 10^{-6}$	
$G_{12,1}$	$4.7768458294935781 \times 10^{-5}$	$4.7768458294984333 \times 10^{-5}$	
$G_{13,1}$	$6.1775349906918487 \times 10^{-5}$	$6.1775349906880689 \times 10^{-5}$	1
$G_{22,1}$	$-8.4277163613431945 \times 10^{-5}$	$-8.4277163614102633 \times 10^{-5}$	
$G_{23,1}$	$-3.6777062735798426 \times 10^{-5}$	$-3.6777062735839781 \times 10^{-5}$	
$G_{33,1}$	$-1.2597090883938349 \times 10^{-4}$	$-1.2597090883943341 \times 10^{-4}$	
$G_{11,11}$	$-1.5317520228453641 \times 10^{-5}$	$-1.5317520185560367 \times 10^{-5}$	
$G_{12,11}$	$-3.2300147602260451 \times 10^{-5}$	$-3.2300147619146337 \times 10^{-5}$	
$G_{13,11}$	$-4.7740579727232704 \times 10^{-5}$	$-4.7740579505166646 \times 10^{-5}$	$10^{9}\mathrm{m}^{-1}$
$G_{22,11}$	$-6.2581147470664643 \times 10^{-5}$	$-6.2581146860783736 \times 10^{-5}$	
$G_{23,11}$	$-2.1927033545174070 \times 10^{-5}$	$-2.1927033538653885 \times 10^{-5}$	
$G_{33,11}$	$-8.3977104326935273 \times 10^{-5}$	$-8.3977103691742231 \times 10^{-5}$	

point is $(1, 2, 3)$. The bottom box of each method represents the computing time for the Green's function which includes the time for determining the Gaussian points and the weights in the NIM, and for finding the Stroh eigenvalues in

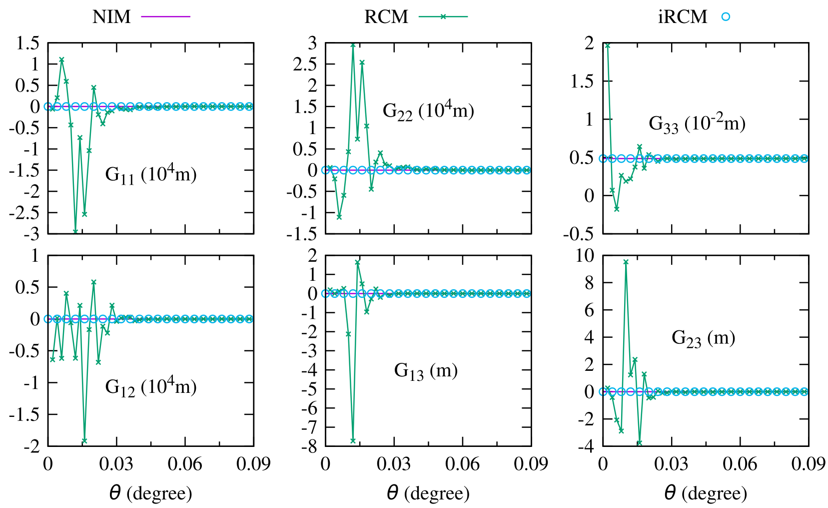

Figure 1. Numerical evaluation of the Green's function near the degenerate point (0, 0, 1) by the three methods.

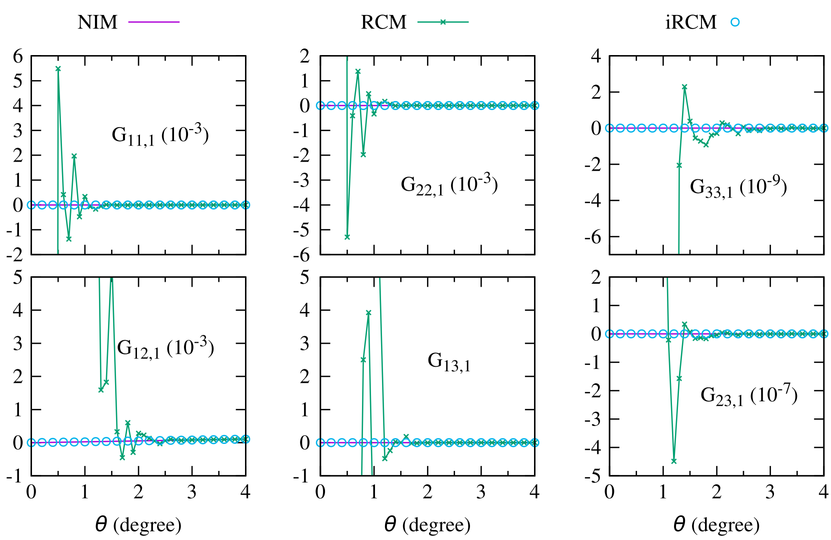

Figure 2. Numerical evaluation of the first derivative of the Green's function near the degenerate point (0, 0, 1) by the three methods.

the RCM and the iRCM. The middle box represents the additional computing time for the first derivative of the Green's function excluding the time for the Green's function. The total computing time for the second derivative of the Green's function is represented by the three boxes, i.e. the stacked column. From Figure 4, the explicit methods, namely the RCM and the iRCM, have a higher efficiency for computing the Green's function and its first derivative compared to the NIM, but may lose the advantage for computing the second derivative

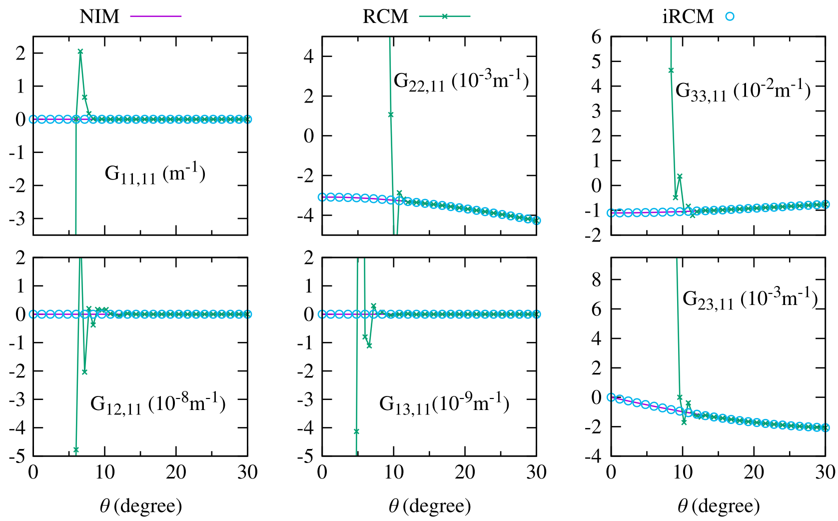

Figure 3. Numerical evaluation of the second derivative of the Green's function near the degenerate point (0, 0, 1) by the three methods.

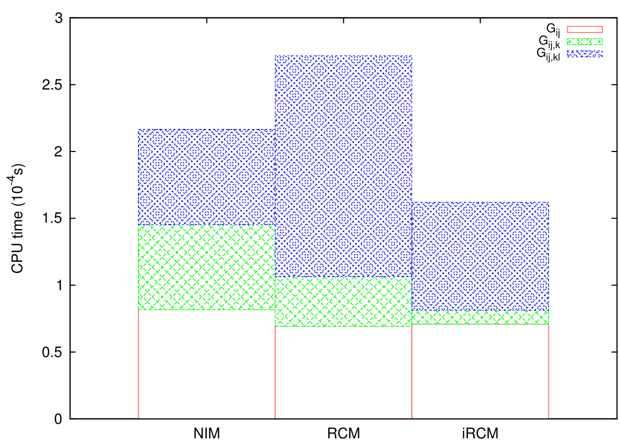

Figure 4. Comparison of the computing time for the Green's function and its derivatives by the three methods.

of the Green's function, especially, in the RCM. Besides, the iRCM has great advantage in computing the first derivative of the Green's function and is the most efficient one among the three different methods for computing the first and second derivatives of the Green's function.

4. Conclusions

Three different methods for computing the Green's function and its first and second derivatives are presented in this paper. The Green's function and its derivatives are expressed in terms of three different kinds of line integrals. The first method is based on the direct numerical integration of the line integrals. The second method is based on the explicit expressions derived by applying the residue calculus with the distinctness assumption on the Stroh eigenvalues to the line integrals, which are valid for non-degenerate cases. The used line integrals in the first and second methods are the same. In the third method, the original line integrals are expressed in terms of two elementary line integrals first. Then, they are evaluated by simple pole residue calculus to obtain explicit expressions which are recast into the novel unified explicit expressions. Although the unified explicit expressions in the third method are derived with the distinctness assumption on the Stroh eigenvalues, after the rewritten they are applicable also for degenerate cases with repeated Stroh's eigenvalues. The correctness of the expressions in the three methods is confirmed by the numerical results of the Green's function and its derivatives for a transversely isotropic material at an arbitrary point, and validated by the analytical results. The numerical results of the second method near a degenerate point may become unstable, while the third method remains applicable near the degenerate point as well as at the degenerate point. According to the CPU times used by the three different methods for calculating the Green's function and its derivatives, the third method seems to be the most efficient one.

Acknowledgements

The first author would like to thank the financial support by the China Scholarship Council (CSC, Project No. 2011626148).

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was financially supported by the China Scholarship Council [CSC, Project No. 2011626148].

References

Barnett, D. (1972). The precise evaluation of derivatives of the anisotropic elastic Green's functions. *Physica status solidi (b), 49*, 741–748.

Buroni, F. C., Ortiz, J. E., & Sáez, A. (2011). Multiple pole residue approach for 3D BEM analysis of mathematical degenerate and non-degenerate materials. *International Journal for Numerical Methods in Engineering, 86*, 1125–1143.

Buroni, F. C., & Sáez, A. (2013). Unique and explicit formulas for Green's function in three-dimensional anisotropic linear elasticity. *ASME Journal of Applied Mechanics*, *80*, 051018–1–14.

Dederichs, P., & Liebfried, G. (1969). Elastic Green's function for anisotropic cubic crystals. *Physical Review, 188*, 1175–1183.

Fredholm, I. (1900). Sur les équations de l'équilibre d'un corps solide élastique [On the equations of equilibrium of an elastic solid body]. *Acta Mathematica, 23*(1), 1–42.

Hwu, C. (2010). *Anisotropic Elastic Plates*. New York: Springer.

Lavagnino, A. M. (1995). *Selected static and dynamic problems in anisotropic linear elasticity* (PhD thesis). Stanford University, Stanford.

Lee, V.-G. (2003). Explicit expression of derivatives of elastic Green's functions for general anisotropic materials. *Mechanics Research Communications, 30*, 241–249.

Lee, V.-G. (2009). Derivatives of the three-dimensional Green's functions for anisotropic materials. *International Journal of Solids and Structures, 46*, 3471–3479.

Lifshitz, I., & Rozenzweig, L. (1947). On the construction of the Green tensor for the fundamental equation of elasticity theory in the case of unbounded elastically anisotropic medium. *Zhurnal Éksperimental'noĭ i Teoreticheskoĭ Fiziki, 17*, 783–791.

Malén, K. (1971). A unified six-dimensional treatment of elastic Green's functions and dislocations. *Physica Status Solidi (b), 44*, 661–672.

Mura, T. (1987). *Micromechanics of Defects in Solids*. Dordrecht: Kluwer Academic Publishers.

Pan, E., & Chen, W. (2015). *Static Green's Functions in Anisotropic Media*. New York, NY: Cambridge University Press.

Pan, Y.-C., & Chou, T.-W. (1976). Point force solution for an infinite transversely isotropic solid. *Journal of Applied Mechanics, 43*, 608–612.

Phan, A.-V., Gray, L., & Kaplan, T. (2004). On the residue calculus evaluation of the 3D anisotropic elastic Green's function. *Communications in Numerical Methods in Engineering, 20*, 335–341.

Phan, A.-V., Gray, L., & Kaplan, T. (2005). Residue approach for evaluating the 3D anisotropic elastic Green's function: Multiple roots. *Engineering Analysis with Boundary Elements, 29*, 570–576.

Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flannery, B. P. (2007). *Numerical Recipes: The Art of Scientific Computing* (3rd ed.). New York, NY: Cambridge University Press.

Sales, M., & Gray, L. (1998). Evaluation of the anisotropic Green's function and its derivatives. *Computers and Structures, 69*, 247–254.

Shiah, Y. C., Tan, C. L., & Wang, C. Y. (2012). Efficient computation of the Green's function and its derivatives for three-dimensional anisotropic elasticity in BEM analysis. *Engineering Analysis with Boundary Elements, 36*, 1746–1755.

Synge, J. L. (1957). *The Hypercircle in Mathematical Physics*. Cambridge: Cambridge University Press.

Tan, C., Shiah, Y., & Wang, C. (2013). Boundary element elastic stress analysis of 3d generally anisotropic solids using fundamental solutions based on Fourier series. *International Journal of Solids and Structures, 50*, 2701–2711.

Ting, T. C. T. (1996). *Anisotropic Elasticity: Theory and Applications*. Oxford: Oxford University Press.

Ting, T. C. T., & Lee, V.-G. (1997). The three-dimensional elastostatic Green's function for general anisotropic linear elastic solids. *The Quarterly Journal of Mechanics and Applied Mathematics, 50*, 407–426.

Wang, C. (1997). Elastic fields produced by a point source in solids of general anisotropy. *Journal of Engineering Mathematics, 32*, 41–52.

Wilson, R. B., & Cruse, T. A. (1978). Efficient implementation of anisotropic three dimensional boundary-integral equation stress analysis. *International Journal for Numerical Methods in Engineering, 12*, 1383–1397.

Xie, L., Zhang, C., Wan, Y., & Zhong, Z. (2013). On explicit expressions of 3D elastostatic Green's functions and their derivatives for anisotropic solids. In A. Sellier & M. H. Aliabadi (Eds.), *Advances in Boundary Element & Meshless Techniques XIV* (pp. 286–291). Eastleigh: EC Ltd.

Xie, L., Zhang, C.h., Sladek, J. & Sladek, V. (2016). Unified analytical expressions of the three-dimensional fundamental solutions and their derivatives for linear elastic anisotropic materials. *Proceedings of the Royal Society of London A*, *472*, 20150272.

Appendix 1. Explicit expressions of the auxiliary integrals for the Green's function and its derivatives

In this appendix, the explicit expressions of the auxiliary integrals required by the Green's function and its derivatives are given. More details to the explicit expressions can be found in the recent works by Xie et al. (2013, 2016).

A.1. Auxiliary integrals for the Green's function

For computing the Green's function, the auxiliary integrals I_3^n $(n = 0, 1, \ldots, 4)$ are given by Equations (47) and (48), which are not repeated here.

A.2. Auxiliary integrals for the first derivative of the Green's function

For computing the first derivative of the Green's function, the auxiliary integrals I_6^n $(n = 0, 1, \ldots, 10)$ in terms of I_m^0 $(m = 1, 2, \ldots, 6)$ and I_m^1 $(m = 2, 3, \ldots, 6)$ are needed, which are determined by

$$I_6^{2k} = I_{6-k}^0 - \sum_{i=1}^{2k} (-1)^i E_i^{(6(7-k))} I_6^{2k-i}, \quad (k = 1, 2, \ldots, 5), \tag{A1}$$

$$I_6^{2k+1} = I_{6-k}^1 - \sum_{i=1}^{2k} (-1)^i E_i^{(6(7-k))} I_6^{2k+1-i}, \quad (k = 1, 2, \ldots, 4), \tag{A2}$$

while for $n = 0, 1$ we have

$$I_4^n = \frac{\pi}{4\beta_1^2\beta_2\beta_3} \operatorname{Im}\left[\frac{-ip_1^n}{\beta_1(p_1-\bar{p}_2)(p_1-\bar{p}_3)} + \frac{p_2^n}{(p_2-\bar{p}_1)^2(p_2-\bar{p}_3)} \right.$$
$$\left. + \frac{p_3^n}{(p_3-\bar{p}_1)^2(p_3-\bar{p}_2)} + 2F_0^{(n)}(1,2,\bar{1},\bar{3}) + F_0^{(n)}(1,1,\bar{2},\bar{3})\right],$$

$$I_5^n = \frac{\pi}{8\beta_1^2\beta_2^2\beta_3} \operatorname{Re}\left[\frac{-p_1^n i}{\beta_1(p_1-\bar{p}_2)^2(p_1-\bar{p}_3)} + \frac{-p_2^n i}{\beta_2(p_2-\bar{p}_1)^2(p_2-\bar{p}_3)} \right.$$
$$+ \frac{p_3^n}{(p_3-\bar{p}_1)^2(p_3-\bar{p}_2)^2} + 4F_1^{(n)}(1,2,\bar{1},\bar{2},\bar{3}) + 2F_1^{(n)}(1,3,\bar{1},\bar{2},\bar{2})$$
$$\left. + 2F_1^{(n)}(2,3,\bar{1},\bar{1},\bar{2}) + F_1^{(n)}(1,1,\bar{2},\bar{2},\bar{3}) + F_1^{(n)}(2,2,\bar{1},\bar{1},\bar{3})\right],$$

$$I_6^n = \frac{-\pi}{16\beta_1^2\beta_2^2\beta_3^2} \operatorname{Im}\left[\frac{-p_1^n i}{\beta_1(p_1-\bar{p}_2)^2(p_1-\bar{p}_3)^2} + \frac{-p_2^n i}{\beta_2(p_2-\bar{p}_1)^2(p_2-\bar{p}_3)^2} \right.$$
$$+ \frac{-p_3^n i}{\beta_3(p_3-\bar{p}_1)^2(p_3-\bar{p}_2)^2} + 4F_2^{(n)}(1,2,\bar{1},\bar{2},\bar{3},\bar{3}) + 4F_2^{(n)}(1,3,\bar{1},\bar{2},\bar{2},\bar{3})$$
$$+ 4F_2^{(n)}(2,3,\bar{1},\bar{1},\bar{2},\bar{3}) + F_2^{(n)}(1,1,\bar{2},\bar{2},\bar{3},\bar{3}) + F_2^{(n)}(2,2,\bar{1},\bar{1},\bar{3},\bar{3})$$
$$+ F_2^{(n)}(3,3,\bar{1},\bar{1},\bar{2},\bar{2}) + 4F_3^{(n)}(1,2,3,\bar{1},\bar{2},\bar{3}) + 2F_3^{(n)}(1,1,2,\bar{2},\bar{3},\bar{3})$$
$$\left. + 2F_3^{(n)}(1,2,2,\bar{1},\bar{3},\bar{3}) + 2F_3^{(n)}(1,1,3,\bar{2},\bar{2},\bar{3})\right]. \tag{A3}$$

In Equations (A1) and (A2),

$$E_i^{(kl)} = \begin{cases} e_i(p_k, \bar{p}_k, \ldots, p_l, \bar{p}_l), & l < k, \\ e_i(p_k, \bar{p}_k), & l = k, \end{cases} \tag{A4}$$

where $e_i(x_1, \ldots, x_n)$ is the elementary symmetric polynomial defined by

$$\begin{aligned} e_1(x_1, \ldots, x_n) &= \sum_{i=1}^{n} x_i, \\ e_2(x_1, \ldots, x_n) &= \sum_{1 \le i_1 < i_2 \le n} x_{i_1} x_{i_2}, \\ &\vdots \\ e_m(x_1, \ldots, x_n) &= \sum_{1 \le i_1 < \cdots < i_m \le n} x_{i_1} \ldots x_{i_m}, \\ &\vdots \\ e_n(x_1, \ldots, x_n) &= x_1 x_2 \ldots x_n. \end{aligned} \tag{A5}$$

In Equation (A3), the abbreviations p_k=k and $\bar{p}_k$=$\bar{k}$ are introduced for the variables of the functions $F_m^{(n)}(\ldots)$ $(n = 0, 1, m = 1, 2, 3)$, which are given by

$$\begin{aligned} F_0^{(0)}(x_1, \ldots, x_4) &= \left[\prod_{i=3}^{4} (x_1 - x_i)(x_2 - x_i) \right]^{-1} \times \left(x_1 + x_2 - x_3 - x_4 \right), \\ F_1^{(0)}(x_1, \ldots, x_5) &= \left[\prod_{i=3}^{5} (x_1 - x_i)(x_2 - x_i) \right]^{-1} \\ &\quad \times [(x_1 - x_3)(x_1 - x_4) + (x_1 - x_3)(x_2 - x_5) + (x_2 - x_4)(x_2 - x_5)], \\ F_2^{(0)}(x_1, \ldots, x_6) &= \left[\prod_{i=3}^{6} (x_1 - x_i)(x_2 - x_i) \right]^{-1} \\ &\quad \times [(x_1 - x_3)(x_1 - x_4)(x_1 - x_5) + (x_1 - x_3)(x_1 - x_4)(x_2 - x_6) \\ &\quad + (x_1 - x_3)(x_2 - x_5)(x_2 - x_6) + (x_2 - x_4)(x_2 - x_5)(x_2 - x_6)], \\ F_3^{(0)}(x_1, \ldots, x_6) &= \left[\prod_{i=4}^{6} (x_1 - x_i)(x_2 - x_i)(x_3 - x_i) \right]^{-1} \\ &\quad \times [y_2^2 - y_1 y_3 + y_3 y_4 + y_2(-y_1 y_4 + y_4^2 - 2y_5) \\ &\quad + (y_1 - y_4) y_6 + y_5(y_1^2 - y_1 y_4 + y_5)], \\ F_0^{(1)}(x_1, \ldots, x_4) &= \left[(x_1 - x_3)(x_2 - x_3) \right]^{-1} + x_4 F_0^{(0)}(x_1, \ldots, x_4), \\ F_1^{(1)}(x_1, \ldots, x_5) &= F_0^{(0)}(x_1, \ldots, x_4) + x_5 F_1^{(0)}(x_1, \ldots, x_5), \\ F_2^{(1)}(x_1, \ldots, x_6) &= F_1^{(0)}(x_1, \ldots, x_5) + x_6 F_2^{(0)}(x_1, \ldots, x_6), \\ F_3^{(1)}(x_1, \ldots, x_6) &= F_1^{(0)}(x_4, x_5, x_1, x_2, x_3) + x_6 F_3^{(0)}(x_1, \ldots, x_6). \end{aligned} \tag{A6}$$

In Equation (A6), the elementary symmetric polynomials y_i are defined by

$$y_i = \begin{cases} e_i(x_1, x_2, x_3), & i = 1, 2, 3, \\ e_{i-3}(x_4, x_5, x_6), & i = 4, 5, 6. \end{cases} \tag{A7}$$

A.3. Auxiliary integrals for the second derivative of the Green's function

For computing the second derivative of the Green's function, the auxiliary integrals I_9^n ($n = 0, 1, \ldots, 6$) in terms of I_m^0 ($m = 1, 2, \ldots, 9$) and I_m^1 ($m = 2, 3, \ldots, 9$) are required, which are determined by

$$I_9^{2k} = I_{9-k}^0 - \sum_{i=1}^{2k} (-1)^i E_i^{(9(10-k))} I_9^{2k-i}, \quad (k = 1, 2, \ldots, 8), \tag{A8}$$

$$I_9^{2k+1} = I_{9-k}^1 - \sum_{i=1}^{2k} (-1)^i E_i^{(9(10-k))} I_9^{2k+1-i}, \quad (k = 1, 2, \ldots, 7). \tag{A9}$$

Similarly to the integrals I_4^n, I_5^n and I_6^n ($n = 0, 1$) given by Equation (A3), the other auxiliary integrals I_7^n, I_8^n and I_9^n ($n = 0, 1$) can be also expressed as regular functions of the Stroh eigenvalues p_i ($i = 1, 2, 3$). However, they are not listed here for the sake of brevity because they are quite lengthy.

It should be mentioned here that the key idea to obtain the unified explicit expressions of the auxiliary integrals I_3^n ($n = 0, 1, \ldots, 4$) for the Green's function, I_6^n ($n = 0, 1, \ldots, 10$) for the first derivative, and I_9^n ($n = 0, 1, \ldots, 16$) for the second derivative of the Green's function is the elimination of the terms like $(p_i - p_j)$ and $(\bar{p}_i - \bar{p}_j)$ in the denominators of the auxiliary integrals by proper rearrangements of the explicit expressions obtained by the simple pole Cauchy residue calculus. Thus, they are valid for both non-degenerate and degenerate cases.