THE DYNAMIC CHARACTERISTICS OF TAPERED FLUID LINES WITH VISCOELASTIC PIPE WALLS
(TRANSFER MATRIX AND FREQUENCY RESPONSE)

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Abstract

For convenience in investigating the dynamic responses of a liquid-filled tapered line with a viscoelastic pipe wall, a transfer matrix equation, relating pressure to volumetric flow, is derived. In this derivation, it was assumed that the rate of divergence (or convergence) of the line is comparatively small. The fluid line model employed in the analysis is one of an unsteady viscous flow; that is, the frequency-dependent effect of viscosity is taken into consideration. The viscoelastic pipe wall model is a modified version of the Voigt mechanical model, and it is distributed along the pipeline. The frequency response curves are calculated from the matrix, and the accuracy of the curves is evaluated by comparing them with the response curves obtained without assuming the small taper angle. The results verify that the transfer matrix is accurate enough for practical applications.

Keywords: fluid power systems, tapered fluid lines, transfer matrix equation, viscoelastic pipe wall, frequency response

1 Introduction

This study deals with the dynamic characteristics of a liquid-filled tapered pipeline with a viscoelastic pipe wall. Such characteristics are important in investigating not only hydraulic control systems (Nakano, 1970; Muto, 1981; Oldenburger, 1964; Goodson, 1972; Rouleau, 1965), but also blood flow in arteries (Belardinelli, 1992; Womersley, 1957; McDonald, 1960; Pontrelli, 2000; Misra, 1987; Ursino, 1992). In the field of arterial hemodynamics, a keen topic in recent years (Belardinelli, 1992; Pontrelli, 2000) has been the development of a mathematical model of arterial flow through complex geometries (tapering, bifurcations, curvature) that account for the effect of a viscoelastic pipe wall.

Nakano (Nakano, 1970) derived an accurate transfer matrix equation of a straight fluid line with a viscoelastic pipe wall. In that analysis, he proposed a novel method whose equations account for the effect of slightly compressible fluid, though assuming incompressible flow. Muto (Muto, 1981) derived a transfer matrix of a tapered fluid line with a rigid pipe wall. In that analysis, it was assumed that the rate of divergence (or convergence) of the pipeline - that is, the non-dimensional taper angle ε - is comparatively small, and it was confirmed that the transfer matrix was sufficiently accurate under the condition of $|\varepsilon| < 0.1$.

The purpose of this study is to derive the transfer matrix equation of a tapered fluid line with a viscoelastic pipe wall so that the dynamic responses of such lines can be conveniently obtained. In the derivation, it was assumed that the non-dimensional taper angle ε is comparatively small. In this study, we also assume that the axial motion of the pipe wall has no effect on the pressure and flow fluctuations in the fluid.

The fluid line model employed in the analysis is one of an unsteady viscous flow, that is, the frequency-dependent effect of viscosity is taken into consideration. The viscoelastic pipe wall is modelled in the same way as proposed by Nakano (Nakano, 1970). This model corresponds to a slightly modified version of the Voigt mechanical model, and it is distributed along the pipeline.

The frequency response curves are calculated from the matrix, and then the accuracy of the curves is evaluated by comparing them with the response curves obtained without assuming that the taper angle is small. As a result, it is verified that the transfer matrix is accurate enough for practical applications.

This manuscript was received on 17 Mai 2001 and was accepted after revision for publication on 5 October 2001

2 Derivation of Transfer Matrix

2.1 Basic Equations

In dealing with the dynamic characteristics of fluid in a flexible tube such as a rubber hose or blood vessel, the effect of the viscoelastic wall can be more dominant than that of fluid compressibility, as pointed out by Nakano (Nakano, 1970). Accordingly, the present analysis first assumes that the fluid is incompressible, allowing us to derive the basic equations; thereafter we take the effect of fluid compressibility into account by incorporating it into the model of the viscoelastic wall.

For a laminar, incompressible and newtonian flow through a tapered fluid line with a viscoelastic pipe wall, as is shown in Fig. 1, the basic equations for small amplitude disturbances are derived as follows, under the assumptions that the rate of divergence (or convergence) of the line is comparatively small.

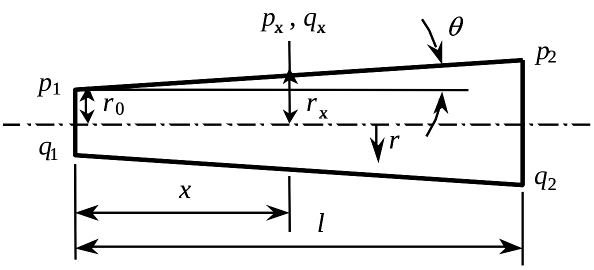

Fig. 1: *Tapered fluid line*

Considering the cylindrical coordinates of the line, as shown in Fig. 2, the momentum equations in the axial and circumferential directions are written, respectively, as follows under the assumption that the nonlinear convective acceleration terms may be neglected:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{1}$$

$$\frac{1}{\nu}\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{w}{r^2} + \frac{\partial^2 w}{\partial x^2} \tag{2}$$

Taking rotation of Eq. 2 yields:

$$\frac{1}{\nu}\frac{\partial w_\varphi}{\partial t} = \frac{\partial^2 w_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial w_\varphi}{\partial r} - \frac{w_\varphi}{r^2} + \frac{\partial^2 w_\varphi}{\partial x^2} \tag{3}$$

where w_φ = rot w .

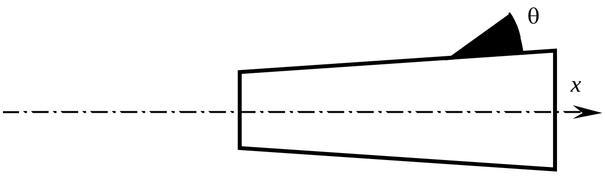

Fig. 2: *Cylindrical coordinate of fluid line*

The equation of continuity is:

$$\frac{\partial v}{\partial r} + \frac{v}{r} + \frac{\partial u}{\partial x} = 0 \tag{4}$$

For a slightly tapered line, a pipe radius r_x at axial distance x relates to r_0 as follows:

$$r_x = r_0 + x\tan\theta \cong r_0 + \theta x \tag{5}$$

In non-dimensional form, Eq. 1, 3, 4 and 5 are re-written as follows, respectively:

$$\frac{\partial U}{\partial T} = -\frac{\partial P}{\partial X} + D\left(\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R}\right) \tag{6}$$

$$\frac{\partial^2 W_\varphi}{\partial R^2} + \frac{1}{R}\frac{\partial W_\varphi}{\partial R} - \frac{W_\varphi}{R^2} + \frac{1}{L^2}\frac{\partial^2 W_\varphi}{\partial X^2} - \frac{1}{D}\frac{\partial W_\varphi}{\partial T} = 0 \tag{7}$$

$$\frac{\partial V}{\partial R} + \frac{V}{R} + \frac{1}{L}\frac{\partial U}{\partial X} = 0 \tag{8}$$

$$\alpha = 1 + \varepsilon X \tag{9}$$

where

$$\left.\begin{aligned} &X = x/l,\ R = r/r_0,\ T = t/t_0,\ U = u/u_0, \\ &V = v/u_0,\ W_\varphi = w_\varphi/u_0,\ P = p/p_0, \\ &p_0 = \rho c u_0,\ D = \nu t_0/r_0^2,\ t_0 = l/c, \\ &L = l/r_0,\ \alpha = r_x/r_0,\ \varepsilon = \theta l/r_0 \end{aligned}\right\} \tag{10}$$

Here, u_0 is the representative velocity (an average velocity across the pipe cross-section at x = 0).

The solution to Eq. 7 is (Nakano, 1970):

$$\hat{W}_\varphi = \sum_n \left(a_n e^{\Omega_n X} + b_n e^{-\Omega_n X} \right) J_1\left(\sqrt{\Omega_n^2 - \frac{S}{D}}\,R\right) \tag{11}$$

The determination of coefficients a_n, b_n and Ω_n (here, Ω_n is propagation constant of order n) is discussed later.

In the following analysis, Ω_1, that is Ω_n for n=1, is used instead of Ω_n as the first approximation as treated by Nakano (Nakano, 1970). Similarly, a_1 and b_1 are used instead of a_n and b_n, respectively. The expression for P can thus be obtained by using Eq. 6, 8 and 11, as follows: (Appendix A)

$$\hat{P}_X = \frac{S}{\Omega_1}\left(a_1 e^{\Omega_1 X} - b_1 e^{-\Omega_1 X} \right)\frac{J_0(\chi)}{\chi_0} \tag{12}$$

where

$$\chi = j\alpha\sqrt{S/D},$$
$$\chi_0 = j\sqrt{S/D} \qquad (\alpha = 1)$$

The solution to Eq. 6 for velocity U and then for Q is as follows (Muto, 1981):

$$\hat{U} = f(X)\,J_0\left(jR\sqrt{S/D}\right) - \frac{1}{S}\frac{\partial \hat{P}}{\partial X} \tag{13}$$

$$\hat{Q}_X = f(X)\,\alpha^2 J_2(\chi) \tag{14}$$

In the derivation of Eq. 13 and 14, the Bessel function $J_n(\chi)$ are expanded in Taylor series about the point χ_0 and all terms higher than 2nd power are rejected.

Equations 12 and 14 are the fundamental equations for deriving the transfer matrix of the fluid line.

2.2 Model of Viscoelastic Pipe Wall

The viscoelastic pipe wall in a fluid transmission line is basically a version of the Voigt mechanical model, and it is distributed along the pipeline. Taking a slight compressibility of fluid into account, a modified version of the Voigt model can be schematically expressed as shown in Fig. 3 (Nakano, 1970).

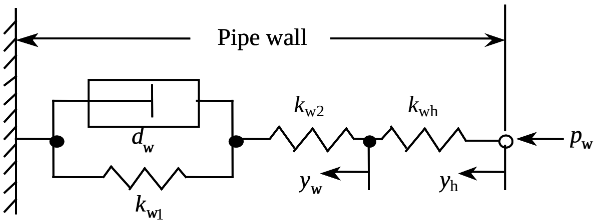

Fig. 3: *Modified model of viscoelastic pipe wall taking slightly compressible fluid into account*

In the figure, y_w: displacement of the pipe wall in radial direction and y_h: virtual displacement of the pipe wall. In this modified model, the effect of a slight compressibility of fluid is treated in such a manner that the compressibility affects the expansion of the pipe wall. This effect is included in the Voigt model by an equivalent spring modulus k_{wh}. According to this model, the relation between the pressure p_w and the virtual displacement y_h is expressed in Laplace domain as follows (Nakano, 1970):

$$\hat{P}_w = H(s)\cdot\hat{Y}_h \tag{15}$$

where,

$$\left.\begin{aligned} &H(s) = k_w G(s), \quad G(s) = \frac{\tau_1 s+1}{\tau_2 s+1}, \\ &\frac{1}{k_w} = \frac{1}{k_{w1}} + \frac{1}{k_{w2}} + \frac{1}{k_{wh}}, \quad \tau_1 = \frac{d_w}{k_{w2}}, \\ &\tau_2 = \frac{d_w}{k_{w2} + k_{w1}k_{wh}/(k_{w1}+k_{wh})} \end{aligned}\right\} \tag{16}$$

2.3 Transfer Matrix

The transfer matrix equation of a tapered line consisting of a viscoelastic pipe wall is derived under the assumption that the non-dimensional taper angle ε in Eq. 10 is of the first order of smallness, and then quantities smaller than the first order are neglected.

To derive the transfer matrix, the characteristic equation of the fluid line has to be obtained by considering the same approximation as mentioned in section 2.1, and then has to be solved for variable Ω_1. Here, the characteristic equation is derived by using Eq. 8, Eq. 11 and 15 as follows: (Appendix B)

$$\frac{2J_1\left(\alpha\sqrt{\Omega_1^2 - S/D}\right)}{\left(\alpha\sqrt{\Omega_1^2 - S/D}\right) J_0\left(\alpha\sqrt{\Omega_1^2 - S/D}\right)} = \frac{2J_1(\Omega_1\alpha)}{(\Omega_1\alpha)J_0(\Omega_1\alpha)} - \frac{2S^2}{\Omega_1^2\alpha H(S)} \tag{17}$$

By calculating the eigenvalue from Eq. 17, the variable Ω_1 is obtained as,

$$\Omega_1 = \frac{S}{c_{ve}(S)}\left\{1 - \frac{2J_1(\chi)}{\chi J_0(\chi)}\right\}^{-\frac{1}{2}} \tag{18}$$

where

$$\left.\begin{aligned} &c_{ve}(S) = \sqrt{\frac{\alpha H(S)}{2}}, \quad \chi_0 = j\sqrt{\frac{S}{D}}, \\ &J_0(\chi) = J_0(\chi_0) - \varepsilon X \chi_0 J_1(\chi_0), \\ &J_1(\chi) = J_1(\chi_0) \\ &\qquad + \frac{\varepsilon}{2} X\chi_0\{J_0(\chi_0) - J_2(\chi_0)\}, \\ &J_2(\chi) = J_2(\chi_0) \\ &\qquad + \frac{\varepsilon}{2} X\chi_0\{J_1(\chi_0) - J_3(\chi_0)\} \end{aligned}\right\} \tag{19}$$

Equation 18 can be changed into the next form, from an expression with Bessel functions such as $J_0(\chi_0)$ into ones with $J_0(\chi_0)$:

$$\Omega_1 = \frac{S\gamma_0}{\sqrt{G(S)}}\{1 + \varepsilon(1-\lambda_0)X\} \tag{20}$$

where

$$\left.\begin{aligned} &\lambda_0 = \frac{1}{2}\left[1 - \left\{\frac{\chi_0(\gamma_0^2 - 1)}{2\gamma_0}\right\}^2\right], \\ &\gamma_0 = \sqrt{-\frac{J_0(\chi_0)}{J_2(\chi_0)}} \end{aligned}\right\} \tag{21}$$

With respect to the fundamental equations of Eq. 12 and 14, the unknown function $f(X)$ in Eq. 14 can be determined as follows, by applying to Eq. 13 the boundary condition that $U = 0$ at $R = \alpha$:

$$f(X) = \frac{1}{S}\frac{\partial\hat{P}}{\partial X}\frac{1}{J_0(\chi)} \tag{22}$$

Substituting Eq. 22 into Eq. 14 yields:

$$\hat{Q}_X = \frac{\alpha^2 J_2(\chi)}{S J_0(\chi)}\frac{\partial\hat{P}}{\partial X} \tag{23}$$

Differentiating Eq. 12 with respect to X, we get:

$$\frac{\partial\hat{P}}{\partial X} = \frac{S}{\chi_0}\left(a_1 e^{\Omega_1 X} + b_1 e^{-\Omega_1 X}\right)J_0(\chi)$$

$$-\frac{S\varepsilon}{\Omega_1\chi_0}\left(a_1 e^{\Omega_1 X} - b_1 e^{-\Omega_1 X} \right) J_1(\chi) \tag{24}$$

where a_1 and b_1 are arbitrary constants.

By using Eq. 24 into Eq. 23, we have:

$$\hat{Q}_X = \frac{\alpha^2 J_2(\chi)}{\chi_0}\left\{ \left(a_1 e^{\Omega_1 X} + b_1 e^{-\Omega_1 X} \right) \right.$$

$$\left. -\frac{\varepsilon}{\Omega_1}\left(a_1 e^{\Omega_1 X} - b_1 e^{-\Omega_1 X} \right)\frac{J_1(\chi)}{J_0(\chi)} \right\} \tag{25}$$

Equations 12 and 25 can be rewritten as follows:

$$\hat{P}_X = \frac{S J_0(\chi)}{\Omega_1 \chi_0}\left[(a_1 - b_1)\cosh\Omega_1 X \right.$$

$$\left. + (a_1 + b_1)\sinh\Omega_1 X \right] \tag{26}$$

$$\hat{Q}_X = \frac{\alpha^2 J_2(\chi)}{\chi_0}\left[\left\{ (a_1 + b_1)\cosh\Omega_1 X \right. \right.$$

$$+ (a_1 - b_1)\sinh\Omega_1 X \Big\} - \frac{\varepsilon}{\Omega_1}\Big\{ (a_1 - b_1)\cosh\Omega_1 X$$

$$\left. \left. + (a_1 + b_1)\sinh\Omega_1 X \right\} \frac{J_1(\chi)}{J_0(\chi)} \right] \tag{27}$$

where

$$\left.\begin{aligned} \cosh\Omega_1 X &= \frac{1}{2}\left(e^{\Omega_1 X} + e^{-\Omega_1 X}\right), \\ \sinh\Omega_1 X &= \frac{1}{2}\left(e^{\Omega_1 X} - e^{-\Omega_1 X}\right) \end{aligned}\right\} \tag{28}$$

The arbitrary constants a_1, b_1 in Eq. 26 and 27 are determined by using the boundary conditions that $P_X = P_1$ and $Q_X = Q_1$ at $X = 0$.

Finally, by substituting the thus-obtained arbitrary constants into Eq. 26 and 27, and then doing some rearranging, the transfer matrix is derived as follows:

$$\begin{bmatrix} \dfrac{\hat{P}_X}{1-\varepsilon\lambda_0 X} \\ \dfrac{\hat{Q}_X}{1+\varepsilon\lambda_0 X} \end{bmatrix} = \begin{bmatrix} \cosh\Gamma_X + \dfrac{\varepsilon\lambda_0\sqrt{G(S)}}{S\gamma_0}\sinh\Gamma_X & -\gamma_0\sqrt{G(S)}\sinh\Gamma_X \\ -\dfrac{1}{\gamma_0\sqrt{G(S)}}\sinh\Gamma_X & \cosh\Gamma_X - \dfrac{\varepsilon\lambda_0\sqrt{G(S)}}{S\gamma_0}\sinh\Gamma_X \end{bmatrix} \begin{bmatrix} \hat{P}_1 \\ \hat{Q}_1 \end{bmatrix} \tag{29}$$

where Γ_x is the propagation operator at arbitrary pipe position X, and is expressed as follows:

$$\Gamma_X = \Omega_1 X = \frac{S\gamma_0 X}{\sqrt{G(S)}}\left\{ 1 + \varepsilon\left(1-\lambda_0\right) X \right\} \tag{30}$$

The transfer matrix Eq. 29 is represented as a relation between the upstream end and the section of distance X of the tapered line in non-dimensional form.

Putting $X = 1$ in Eq. 29 yields the relation between the upstream end and the downstream end, as follows:

$$\begin{bmatrix} \hat{P}_1 \\ \hat{Q}_1 \end{bmatrix} = \begin{bmatrix} \cosh\Gamma - \dfrac{\varepsilon\lambda_0\sqrt{G(S)}}{S\gamma_0}\sinh\Gamma & \gamma_0\sqrt{G(S)}\sinh\Gamma \\ \dfrac{1}{\gamma_0\sqrt{G(S)}}\sinh\Gamma & \cosh\Gamma + \dfrac{\varepsilon\lambda_0\sqrt{G(S)}}{S\gamma_0}\sinh\Gamma \end{bmatrix} \begin{bmatrix} \dfrac{\hat{P}_2}{1-\varepsilon\lambda_0} \\ \dfrac{\hat{Q}_2}{1+\varepsilon\lambda_0} \end{bmatrix} \tag{31}$$

where Γ is the propagation operator for a pipeline of length l, and is expressed as follows:

$$\Gamma = \Omega_1 l = \frac{S\gamma_0}{\sqrt{G(S)}}\left\{ 1 + \varepsilon\left(1-\lambda_0\right) \right\} \tag{32}$$

In dimensional form, Eq. 31 is expressed as:

$$\begin{bmatrix} \hat{p}_1 \\ \hat{q}_1 \end{bmatrix} = \begin{bmatrix} \cosh\Gamma - \dfrac{c\lambda_0\theta\sqrt{G(s)}}{r_0\, s\,\gamma_0}\sinh\Gamma & Z_c\sinh\Gamma \\ \dfrac{1}{Z_c}\sinh\Gamma & \cosh\Gamma + \dfrac{c\lambda_0\theta\sqrt{G(s)}}{r_0\, s\,\gamma_0}\sinh\Gamma \end{bmatrix} \begin{bmatrix} \dfrac{\hat{p}_2}{1-\lambda_0\theta l/r_0} \\ \dfrac{\hat{q}_2}{1+\lambda_0\theta l/r_0} \end{bmatrix} \tag{33}$$

where

$$Z_c = Z_0\,\gamma_0\sqrt{G(S)}, \qquad Z_0 = \frac{\rho c}{a_0} \tag{34}$$

Putting $\theta = 0$, or $\tau_1 = \tau_2 = 0$, in the matrix Eq. 33 leads to the equation for a straight pipeline with a viscoelastic pipe wall, or one for a tapered line with a rigid pipe wall, respectively. From this fact, it is confirmed that the former or the latter of the resultant matrix equations coincides with one obtained by Nakano (Nakano, 1970) or by Muto (Muto, 1981), respectively.

The fluid line model employed in the present analysis is one of an unsteady viscous flow that is, the frequency-dependent effect of viscosity is taken into account. As a consequence of this adaptation, the propagation operator Γ and the characteristic impedance Z_c are, as seen in Eq. 32 and 34, respectively, represented as functions of Bessel and hyperbolic functions. Since such properties of Γ and Z_c are the essential criteria for an "exact" model of a fluid transmission line (Goodson, 1972), Eq. 31 may be called the exact transfer matrix of a tapered line with a viscoelastic pipe wall, as far as the assumption of smallness of the tapered line parameter ε is permitted.

3 Frequency Responses

In this chapter, the characteristics of frequency response are investigated in terms of the transfer matrix of Eq. 31 under some representative pipeline conditions. Whole the parameters used for calculating the frequency responses are listed in Table 1.

For the first line condition, the pressure at the upstream end P_1 is selected as the input signal of the line system, and the pressure at the downstream end P_2 is selected as the output signal. As for the pipe-end condition, first, the downstream end is considered to be closed-end, that is, $Q_2 = 0$. Hereafter, such a line system, containing both a P_1-input and a closed end, is called a P_1-closed system for brevity's sake.

An example of frequency response curves for a P_1-closed system is shown in Fig. 4, taking the rate of divergence ε as a parameter. These curves are calculated under the condition that the input and output pressures P_1 and P_2 are given as $P_1 = P_{10} \sin\omega t$, and $P_2 = P_{20} \sin(\omega t - \phi)$, respectively.

Table 1: Parameters of the pipeline

Pipeline	l = 9.5 m , r_0 = 5.0x10^{-3} m
Other Parameters	ν = 0.82x10^{-4} m^2/s , ρ = 856 kg/m^3 c = 805 m/s , τ_1 = 1.25x10^{-3} s τ_2 = 0.95x10^{-3} s, D = 0.01

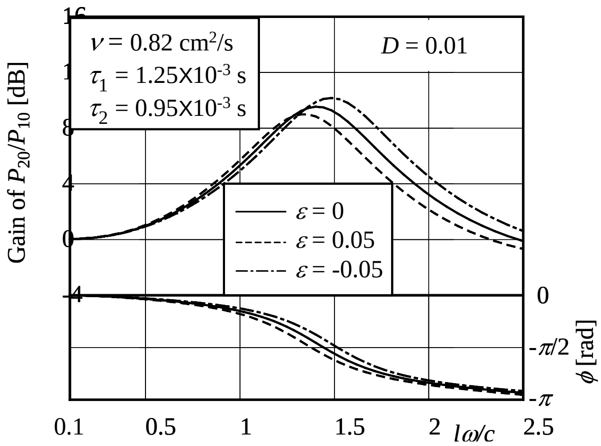

Fig. 4: *Frequency response curves for tapered line with a viscoelastic pipe wall*

The figure is represented with the gain of pressure amplitude P_{20}/P_{10} and the phase shift ϕ in the ordinate against the non-dimensional frequency of the fluid line $l\omega/c$. Furthermore, in a similar manner to that in Fig. 4, Fig. 5 represents the frequency response curves calculated for a tapered line with a rigid pipe wall (that is, $\tau_1 = \tau_2 = 0$) under the same pipe-end condition as treated in Fig. 4. By observing these two figures, it is recognized that both response curves are similar to each other in their tendency for a positive (or a negative) value of the parameter ε to bring about a lower (or a higher) resonance frequency than that in $\varepsilon = 0$. It is also recognized that the pressure amplitude ratios of flexible tapered lines are smaller than those of rigid tapered lines in the corresponding frequency range.

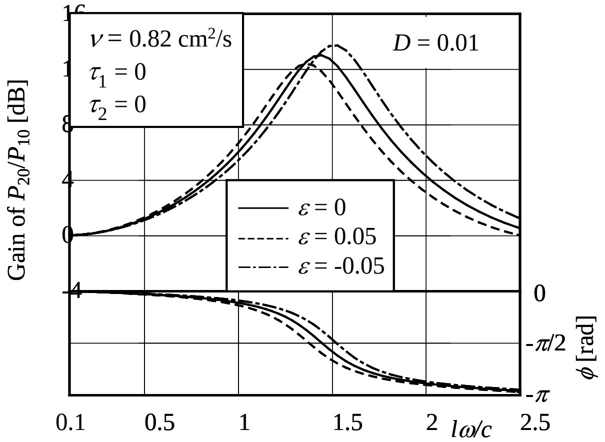

Fig. 5: *Frequency response curves for tapered line with a rigid pipe wall*

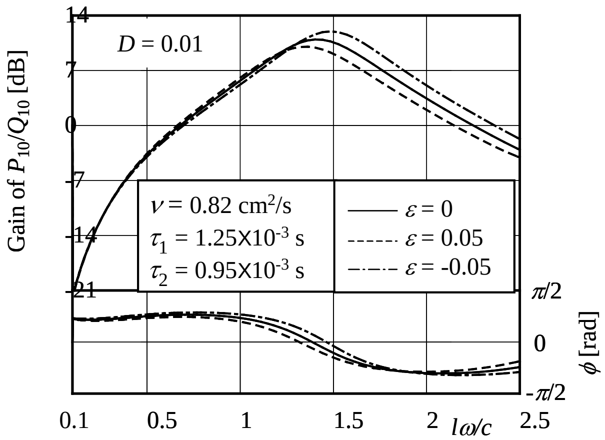

Fig. 6: *Frequency response curves for a Q_1-open system considering ε as a parameter*

For the second line condition, by changing the pipe-end condition from the closed end to the open end, that is, $P_2 = 0$, a Q_1-open system is treated by taking the pressure P_1 as the output signal. The response curves for the system are shown in Fig. 6 in a manner similar to that in Fig. 4. From this figure it is seen that the resonance frequency of the system decreases with an increase in the parameter ε. Such a tendency of resonance frequency is also seen in Fig. 4 for the P_1-closed system.

4 Accuracy of Transfer Matrix

In order to investigate the accuracy of the transfer matrix Eq. 31, in this chapter the errors induced in frequency response curves are examined for the P_1-closed system treated in Fig. 4. Here, the transfer function of the system is defined as $F_M(S) = \hat{P}_2 / \hat{P}_1$, since the pressure P_2 is considered as the output signal. The function $F_M(S)$ is derived analytically from Eq. 31 as follows:

$$F_{\mathrm{M}}\left(S\right)=\frac{1}{\cosh\Gamma}+\frac{\varepsilon\,\lambda_0\sqrt{G(S)}}{S\gamma_0}\frac{\sinh\Gamma}{\cosh^2\Gamma}-\frac{\varepsilon\,\lambda_0}{\cosh\Gamma} \quad (35)$$

The function $F_{\mathrm{M}}(S)$ is an approximate one since it includes errors smaller than the first order smallness of ε. Corresponding to this function, the strict function, defined as $F_{\mathrm{R}}(S)$, is derived analytically without assuming the smallness of the rate of divergence ε. The function $F_{\mathrm{R}}(S)$ is obtained as follows from Eq. 12 and 25, by applying two boundary conditions, namely, that $P_{\mathrm{X}} = P_1$ at $X = 0$ and that $P_{\mathrm{X}} = P_2$ and $Q_{\mathrm{X}} = Q_2$ at $X = 1$:

$$F_{\mathrm{R}}\left(S\right)=\left\{2S\gamma_0\,J_0\left(\chi_0\right)\Big/\sqrt{G\left(S\right)}\,J_0\left(\chi_0\right)\right\}\cdot$$

$$\Bigg/\left[e^{-\Omega'}\left\{\frac{\varepsilon\,J_1\left(\chi_1\right)}{J_0\left(\chi_1\right)}+\Omega'\right\}-e^{\Omega'}\left\{\frac{\varepsilon\,J_1\left(\chi_1\right)}{J_0\left(\chi_1\right)}-\Omega'\right\}\right] \quad (36)$$

where

$$\left.\begin{aligned}\Omega'&=\frac{S}{\sqrt{G(S)}}\left\{1-\frac{2J_1\left(\chi_1\right)}{\chi_1\,J_0\left(\chi_1\right)}\right\}^{-1/2},\\ \chi_1&=j\left(1+\varepsilon\right)\sqrt{S/D}\end{aligned}\right\} \quad (37)$$

Based on the thus-obtained strict function $F_{\mathrm{R}}(S)$, the estimated errors of the approximate function $F_{\mathrm{M}}(S)$ are shown in Fig. 7. The thick and thin lines in the figure represent the errors estimated for positive and negative value of ε, respectively. In the figure, the relative errors of amplitude ratio, as well as the phase shift errors, are shown against the nondimensional frequency. The figure shows that the relative error concerning the amplitude is 10% at most when the rate $|\varepsilon|$ is smaller than 0.05. By a similar error estimation for other pipeline systems, such as a Q_1-open system, a similar result was obtained. From these facts, it can be concluded that the matrix Eq. 31 is one with considerable accuracy.

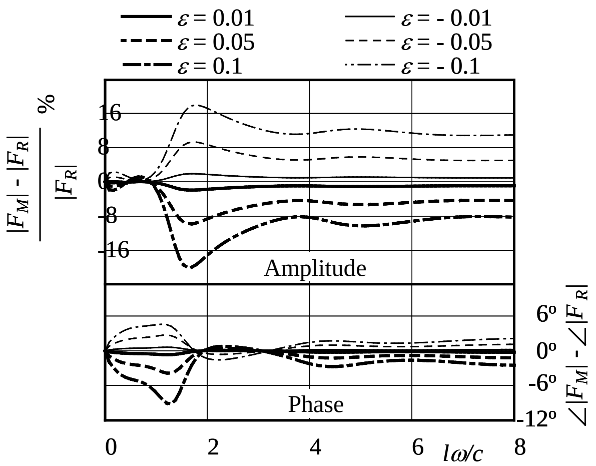

Fig. 7: *Accuracy of the transfer matrix*

On the other hand, the application of the matrix to those cases in which the rate ε is not so small, may necessarily be accompanied by a large degree of error. As a way to improve the accuracy in such cases, Muto (1981) proposed to divide a line with length l into many small segments of the number N (here, the length of each segment Δl is equal to l/N). In this method, the rate of divergence $\Delta\varepsilon_{\mathrm{i}}$ corresponding to the segment is expressed by

$$\Delta\varepsilon_{\mathrm{i}}=\frac{\theta\,\Delta l}{r_{0\mathrm{i}}} \quad (\mathrm{i}=1,2,\cdots,N) \quad (37)$$

where $r_{0\mathrm{i}}$ is the pipe radius at the upstream end of the i-th segment. The value $\Delta\varepsilon_{\mathrm{i}}$ in Eq. 37 can be decreased by increasing the number N. Thus the matrix relation for the total line system, in such a case, is expressed in the form of a cascade connection for each matrix i= 1 to i= N, by applying Eq. 31.

5 Conclusions

The objective of this paper is to derive a transfer matrix equation of a tapered fluid line with a viscoelastic pipe wall. In the derivation, a modified version of the Voigt mechanical model was used to accommodate the viscoelastic behavior in the pipe wall. As for the fluid line model, on the other hand, the unsteady viscous flow model was adopted, thus the frequency-dependent effects of viscosity were considered in the model.

Under the assumption that the non-dimensional taper angle ε is of the first order of smallness, and neglecting quantities smaller than the first order, the transfer matrix was derived. Frequency response curves were calculated by using the derived matrix in order to see the effects of tapered lines on the system dynamics under some representative line conditions. The accuracy of the transfer matrix was investigated by a method of error estimation, and it then confirmed that the matrix is of considerably high accuracy, at least in the region $|\varepsilon| < 0.05$.

For a future application of the transfer matrix derived in this study, the authors intend to develop a simulation program for calculating transient responses (Tahmeen, 2001) of a tapered fluid line with a viscoelastic pipe wall. Such a program may greatly contribute not only to the field of hydraulic control systems, but also to the study of blood flow in arteries.

Nomenclature

a_0	Cross-sectional area of line at $x = 0$	[m^2]
c	Velocity of pressure wave	[m/s]
D	Dimensionless dissipation number	[-]
d_{w}	Viscosity of the dashpot	[Pa s/m]
J_0, J_1	Bessel function of first kind of order 0 and 1	[-]
J_2, J_3	Bessel function of first kind of order 2 and 3	[-]
j	$j=\sqrt{-1}$	
$k_{\mathrm{w1}}, k_{\mathrm{w2}}$	Moduli of spring	[Pa/m]
k_{wh}	Spring modulus	[Pa/m]
l	Length of tapered line	[m]
L	Length in non-dimensional form	[-]

p	Pressure of fluid	[Pa]
p_w	Pressure in pipe	[Pa]
p_s	Supply pressure	[Pa]
p_0	Average pressure at $x = 0$	[Pa]
P	Pressure in non-dimensional form	[-]
q_s	Supply flow rate	[m^3/s]
Q	Flow rate in non-dimensional form	[-]
r_0	Inner pipe radius at $x = 0$	[m]
r_x	Pipe radius at axial distance x	[m]
r	Pipe radius	[m]
R	Radius in non-dimensional form	[-]
s	Laplace operator	[1/s]
S	Normalized Laplace operator	[-]
t	Time	[s]
t_0	Nominal value of time	[s]
T	Time in non-dimensional form	[-]
u	Velocity in axial-direction	[m/s]
u_0	Average velocity at $x = 0$	[m/s]
U	Non-dimensional velocity of u	[-]
v	Velocity in radial-direction	[m/s]
V	Non-dimensional velocity of v	[-]
w	Velocity in circumferential-direction	[m/s]
W	Non-dimensional velocity of w	[-]
x	Coordinate in axial-direction	
X	Non-dimensional form of x	[-]
Z_c	Characteristic impedance of line	[Pa s/m^3]
θ	Taper angle	[rad]
ε	Divergence or convergence parameter of line	[-]
φ	Coordinate in circumferential-direction	
Ω_n	Propagation constant of order n	[-]
Ω_1	Propagation constant of order 1	[-]
Ω_n'	Propagation constant defined in Eq. 37	[-]
Γ	Propagation operator of fluid line	[-]
ν	Kinematic viscosity	[m^2/s]
σ	Density of fluid	[kg/m^3]

The variables marked with ˆ are those in the Laplace domain.

References

Belardinelli, E. and Cavalcanti, S. 1992. Theoretical analysis of pressure pulse propagation in arterial vessels. *Comput. Biol. Med.*, pp. 1337-1349.

Goodson, R. E. and Leonard, R. G. 1972. A Survey of Modeling Techniques for Fluid Transmission Line Transients. *Journal of Basic Engineering, Trans. ASME*, Ser. D, Vol. 94, No. 2, pp. 474-482.

McDonald, D. A. 1960. Blood flow in arteries. *Edward Arnold*.

Misra, J. C. and Singh, S. I. 1987. A Study on the Non-linear Flow of Blood Through Arteries. *Bull. of Mechanical Biology*, Vol. 49, No. 3, pp. 257-277.

Muto, T., Kinoshita, Y. and Yoneda, R. 1981. Dynamic Response of Tapered Fluid Lines (1st Report, Transfer Matrix and Frequency Response). *Bull. of JSME*, Vol. 24, No. 191, pp. 809-815.

Muto, T., Yamada, H. and Kato, H. 1996 A Fast and Convenient Method for Simulating Transient Response of Fluid Lines with Viscoelastic Pipe Walls. *12. Aachener Fluidtechnisches Kolloquium*, Band 2, pp. 257-268

Nakano, K. and Yoshimoto, M. 1970. The Dynamic Characteristics of a Hydraulic Pipe Line with a Viscoelastic Pipe Wall. *Trans. of the SICE (in Japanese)*, Vol. 6, No. 3, pp. 201-209.

Oldenburger, R. and Goodson, R. E. 1964. Simplification of Hydraulic Line Dynamics by Use of Infinite Products. *Journal of Basic Engineering, Trans. ASME*, Ser. D, Vol. 86, pp. 1-10.

Pontrelli, G. 2000. Blood Flow through a Circular Pipe with an Impulsive Pressure Gradient. *Mathematical* Models and Methods in Applied Sciences, Vol. 10, No. 2, pp.187-202.

Rouleau, W. T. and Young, F. J. 1965. Distortion of Short Pulses in Tapered Tube Pulse Transformers. Part I-Inviscid Liquid. *Trans. ASME*, Ser. D, Vol. 87, No. 2, pp. 465-470.

Tahmeen, M., Muto, T. and Yamada, H. 2001. Simulation of Dynamic Responses of Tapered Fluid Lines. *JSME Int. Journal*, Ser. B, Vol. 44, No. 2.

Ursino, M. and Artioli, E. 1992. High Frequency Pressure Propagation in Viscoelastic Tubes: A New Experimental Approach. *Bio-Medical Materials and Engineering*, Vol. 2, pp. 19-31.

Womersley, J. R. 1957. An Elastic Tube Theory of Pulse Transmission and Oscillatory Flow in Mammalian Arteries. *McGregor and Midwest Corp. Dayton, Ohio*.

Appendix A: Derivation of Eq. 12

The following relations (in non-dimensional form) are obtained by using Eq. 8 and 11 (Nakano, 1970),

$$\frac{\partial^2 V}{\partial R^2} + \frac{1}{R}\frac{\partial V}{\partial R} - \frac{V}{R^2} + \frac{1}{L^2}\frac{\partial^2 V}{\partial X^2} = \sum_{n} \Omega_n \left(a_n e^{\Omega_n X} - b_n e^{-\Omega_n X} \right) J_1\left(\sqrt{\Omega_n^2 - \frac{S}{D}}\, R \right) \quad \text{(A1)}$$

$$\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} + \frac{1}{L^2}\frac{\partial^2 U}{\partial X^2} = -\sum_{n} \left\{ \sqrt{\Omega_n^2 - \frac{S}{D}} \times \left(a_n e^{\Omega_n X} + b_n e^{-\Omega_n X} \right) J_0\left(\sqrt{\Omega_n^2 - \frac{S}{D}}\, R \right) \right\} \quad \text{(A2)}$$

The above expressions are obtained by considering the following relations,

$$\frac{\partial U}{\partial R} = \frac{\partial W}{\partial R} \tag{A3}$$

The solutions of Eq. A1 and A2 are,

$$V = \sum_n \left(c_n e^{\Omega_n X} + d_n e^{-\Omega_n X} \right) J_1(\Omega_n R) + \sum_n \frac{\Omega_n D}{S}$$

$$\times \left(a_n e^{\Omega_n X} - b_n e^{-\Omega_n X} \right) J_1\left(\sqrt{\Omega_n^2 - \frac{S}{D}}\, R \right) \tag{A4}$$

$$U = \sum_n \left(-c_n e^{\Omega_n X} + d_n e^{-\Omega_n X} \right) J_0(\Omega_n R) - \sum_n \frac{\sqrt{\Omega_n^2 - S/D}}{S/D}$$

$$\times \left(a_n e^{\Omega_n X} + b_n e^{-\Omega_n X} \right) J_0\left(\sqrt{\Omega_n^2 - S/D}\, R \right) \tag{A5}$$

The expression for P can be obtained by using Eq. 6, A4 and A5 (Nakano, 1970) as follows,

$$\widehat{P}_X = \frac{S}{\Omega_n} \left(c_n e^{\Omega_n X} + d_n e^{-\Omega_n X} \right) J_0(\Omega_n R) \tag{A6}$$

Considering the same approximation for Ω_n as mentioned in section 2.1, the above equation becomes,

$$\widehat{P}_X = \frac{S}{\Omega_1} \left(c_1 e^{\Omega_1 X} + d_1 e^{-\Omega_1 X} \right) J_0(\Omega_1 R) \tag{A7}$$

and using the boundary condition that $U \approx 0$ at $R = \alpha$ we have the following relation,

$$\frac{c_1}{a_1} = -\frac{d_1}{b_1} = \frac{J_0(\chi)}{\chi_0 \; J_0(\Omega_1 \alpha)} \tag{A8}$$

Using Eq. A6, finally the expression for P becomes as follows,

$$\widehat{P}_X = \frac{S}{\Omega_1} \left(a_1 e^{\Omega_1 X} - b_1 e^{-\Omega_1 X} \right) \frac{J_0(\chi)}{\chi_0} \tag{A9}$$

Appendix B: Derivation of Characteristic Equation in Eq. 17

Considering the same approximation for Ω_n as mentioned in section 2.1 and using the boundary conditions (in non-dimensional form) that $P_X \approx P_W(S,X)$ and $V \approx S\, Y_h$ at $R = \alpha$, we have the following relation from Eq. A1, A3 and 15,

$$\frac{S}{H(S)} \left(c_1 e^{\Omega_1 X} + d_1 e^{-\Omega_1 X} \right) J_0(\Omega_1 \alpha) =$$

$$\left(c_1 e^{\Omega_1 X} + d_1 e^{-\Omega_1 X} \right) J_1(\Omega_1 \alpha) +$$

$$\frac{\Omega_1}{S/D} \left(a_1 e^{\Omega_1 X} - b_1 e^{-\Omega_1 X} \right) J_1\left(\sqrt{\Omega_1^2 - S/D}\, \alpha \right) \tag{B1}$$

Using the boundary condition that $U \approx 0$ at $R = \alpha$ Eq. B1 becomes,

$$\frac{1}{Z} \frac{\Omega_1}{S/D} J_1\left(\sqrt{\Omega_1^2 - S/D}\, \alpha \right) =$$

$$J_1(\Omega_1 \alpha) - \frac{S^2}{\Omega_1 H(S)} J_0(\Omega_1 \alpha) \tag{B2}$$

where

$$Z = \frac{\sqrt{\Omega_1^2 - S/D}}{S/D} \; \frac{J_0\left(\sqrt{\Omega_1^2 - S/D}\, \alpha \right)}{J_0(\Omega_1 \alpha)} \tag{B3}$$

Using the relation of Eq. B3 into Eq. B2, we finally get the characteristic equation as follows,

$$\frac{2\; J_1\left(\alpha \sqrt{\Omega_1^2 - S/D} \right)}{\left(\alpha \sqrt{\Omega_1^2 - S/D} \right) \; J_0\left(\alpha \sqrt{\Omega_1^2 - S/D} \right)}$$

$$= \frac{2\, J_1(\Omega_1 \alpha)}{(\Omega_1 \alpha)\; J_0(\Omega \alpha)} - \frac{2\; S^2}{\Omega_1^2\; \alpha\; H(S)} \tag{B4}$$

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