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ON SOME CURRENT RESULTS OF GRAPH THEORY FOR AD-HOC NETWORKS

GIUSEPPE DE MARCO
Department of Communication and Information Engineering, Fukuoka Institute of Technology (FIT), 3-30-1 Wajiro-higashi, Higashi-ku, Fukuoka 811-0295, Japan
demarco@fit.ac.jp

LEONARD BAROLLI
Department of Communication and Information Engineering, Fukuoka Institute of Technology (FIT), 3-30-1 Wajiro-higashi, Higashi-ku, Fukuoka 811-0295, Japan
barolli@fit.ac.jp

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The goal of this paper is twofold. Firstly, we present results from graph-theory which can be used to understand the fundamental properties of ad-hoc networks and wireless sensor networks. Graph-theory is a well studied branch of discrete mathematics, and it has been applied in many knowledge fields, e.g. social network, Internet tomography and epidemiology. We review literature results from the point of view of the designer of an ad-hoc network, who must set simulation parameters in order to predict the behaviour of the real network. Secondly, we study the impact of the asymmetries of radio links on the connectivity properties of an ad-hoc network. To the best knowledge of the author, this further hypothesys has been addressed in the case of geometric random graph only, but not for radio models with randomnesses. As expected, we found that randomness in the radio model directly affects the distribution of the asymmetries and the connectivity properties. This result can be very useful in the understanding of more complicated aspects of ad-hoc nets, like routing and coordinated wake-up in power saving techniques.

Keywords: Wireless networks, ad-hoc networks, graph theory, sensor networks.
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1 Introduction

Ad hoc networks are the research frontier of networking community. Such a networks have been promising new and attractive applications in many fields. We can cite some of them, as vehicles networking, communications support to rescue platoons, infrastructure-less communications and Wireless Sensor Networks (WSNs), which are a subset of ad-hoc nets. If mobility is of concern, we have a Mobile Ad hoc NETwork (MANET), where every node is supposed to be mobile. Compared with infrastructure based wired- and wireless-networks, these networks need different solutions to standard problems, like routing, addressing, fault-tolerance and Quality of Service (QoS). Besides the huge amount of technical proposals arose in recent years, it has been envisioned that these problems can be solved by looking at the behaviour of biological systems, which are inherently distributed, decentralized and time-varying systems

[5]. In fact, these properties are present also in ad-hoc nets. In biological systems, some very interesting properties seem to arise whenever the number of nodes increases. In other words, complex systems composed of thousand of nodes can find optimal solutions in a cheap and elegant way. Thus, solutions to standard problems of ad-hoc nets can be found by exploiting the similarities with natural systems. An example of this method is surely the distributed synchronization of nodes clock in a large-scale ad-hoc net [22]. On the other hand, simulation of ad-hoc networks is a key tool to understand the performance of applications and protocols designed for such networks. It is in fact prohibitive a trial-and-test approach to these networks. Ad-hoc networks are very complicated. So far, few works have been discussing the interaction of different aspects of the communication stack of an ad-hoc node. For example, in [15] the authors shown the intimate relation between connectivity and MAC operations. As in other problems of complex systems simulation, sometimes embedding all physical details in the simulation model is not advantageous to the understanding of the laws which rule the behaviour of the system. Another approach is searching for invariant or scaling properties of such large networks. In fact, this method benefits also optimizing simulation techniques.

One of the most important properties of these networks is the connectivity, that is the probability that two any given nodes of the network can establish a multi-hop connection. Connectivity affects other related performance variables, such as the transmission latency of packets, interference level inside the network, the goodput and, for power limited nodes, the energy consumption, as in the case of WSNs. The properties of large-scale networks can be well understood by means of graph theory, which has been recently re-discovered to study the fundamental properties of MANETs. Hereinafter, we will suppose that nodes of the network can be thought as nodes of a graph, where an edge between two nodes of the graph means that a communication link exists between the corresponding nodes in the network. In general, this event (the existence of link) depends on the hardware: modulation, coding, probability of bit error (or BER) and Signal to Interference-Noise Ratio (SINR). Many researchers studied the fundamental properties of ad-hoc networks by means of the theory of Geometric Random Graphs (GRG), which have been widely studied in the past [19, 9]. The seminal work of such method of analysis is that of Gupta and Kumar [10], which furnished some bounds on the transmission range and transport capacity of an ideal plane network in order to achieve full connectivity. However, sometimes the literature is dispersive. Here, we review in a concise fashion results on GRGs from recent works as well as the more realistic case of ξ-GRG, that is a GRG with the addition of the shadowing phenomena in the pathloss of links. The main parameter of this model is indeed the $\xi \triangleq \frac{\sigma}{\alpha}$, where σ^2 is the variance of the pathloss [21, 11]. Actually, several measurements on test-beds have shown that the presence of asymmetric links is not infrequent [6]. Furthermore, the presence of asymmetric links affects other important performance metrics of ad hoc networks and application specific protocols as well. For example, ad hoc routing protocols usually make extensive use of route-discovery and route-reply messages. In the presence of asymmetric links, it is not immediately clear how well these protocols perform, because the existence of a forward path does not necessarily imply the existence of a reverse path. For instance, the reverse path is the path used to send back the information of the previously discovered forward path. Many routing protocols encompass this problem by using multiple routes between the source and destination node. However, the effectiveness of this technique depends on the density of nodes which in turn modifies the connectivity properties of the network. Motivated by recent experimental results, we are interested in the connectivity of ξ-GRG by adding the assumption that links now are asymmetric. In this case the ξ-GRG becomes directed.

The paper is structured as follows. In Section 2, we present main results on connectivity of Pure Random Graphs (PRGs) and GRGs. In the same section, we show the extension of these results to the more general model of ξ-GRGs. In Section 3, we summarize the methods useful for compute the connectivity property of a graph via simulation. These results are used in Section 4, where we study the differences of undirected graphs and directed graphs. In Section 5, we conclude the paper by discussing some further improvements of the work.

2 Preliminaries

Notation

Hereinafter, we will use the following notation.

- $\mathbf{A}$, area of the network, $a = |\mathbf{A}|$
- ρ, density of nodes
- r_0, radio range of a transmitter
- P_t, the transmitting power of a node
- P_r, the receiving power of a node
- $\beta(r) = \frac{P_t}{P_r(r)}$, the pathloss at distance r
- α, pathloss exponent (Friis-like formula)

In this paper, we assume that nodes are not moving. In the case of mobile nodes, we could apply our results to a snapshot of the network. In other words, at every instant, the network is supposed stationary [11]. A common assumption concerning the deployment of nodes within an area $\mathbf{A}$ is that the number of nodes N in $\mathbf{A}$ is a poissonian r.v., i.e.:

$$P(N = n) = e^{-\lambda}\frac{\lambda^n}{n!}\,, \quad \lambda = \rho A\,.$$

Moreover, the number of nodes in non overlapping subsets of $\mathbf{A}$ are independent. In the context of WSNs, the poissonian distribution is a good approximation for networks built up by dropping sensor nodes from an airplane. In this work, we will analyze also the opposite case to random deployment, that is the lattice case.

2.1 Pure Random Graphs (PRGs)

A random graph $G_u(p)$ is defined as a set of vertexes $V(G_u(p))$ and a set of links or edges $E(G_u(p))$ connecting pairs of nodes, $i, j \in V(G_u(p))$. For simplicity, we will use also V and E for the vertexes set and edges set, respectively. The subscript u in G_u stands for "undirected": The graph has not directed links. We drop the subscript whenever we refer to more general directed graphs, or digraphs, i.e. $G(p)$. Given a pair of nodes (i, j), the probability that a link exists between i and j is p. In random graph, p does not depend on the geometric distance between nodes. In other words, the exact relative distance (in some metric space) of nodes is irrelevant. Examples of PRGs are the well known Erdös-Rény random graph [8], the Barabási-Albert graph [2]. The degree D_i of a node i is the number of links or edges of that node. Usually, to represent a graph we use an adjacency matrix, $A = (a_{ij})_{ij \leq N}$, where $n = |V|$ and $a_{ij} = 1$ iff there is a link between nodes i and j. By this way, $D_i = \sum_j a_{ij}$. For digraphs, the adjacency matrix is not necessarily symmetric, and we distinguish the in- and the out-degree of a node. The in-degree is $D_i^- = \sum_j a_{ji}$, while the out-degree is $D_i^+ = \sum_j a_{ij}$. The graph $G_u(p)$ is said to be connected if there is at least a path between any pair of nodes in V. Generally, $G_u(p)$ is $k-$connected if there are at least k disjoint paths connecting any pair of nodes. An equivalent definition of the connectivity is given by considering a fault-tolerance argument: $G_u(p)$ is said k-connected if there is a path between any two nodes after randomly removing $k-1$ nodes in $G_u(p)$. The connectedness is a r.v. and its probability depends on the distribution of D or rather the $D_{\min} = \min_i(D_i)$. We name with $P(conn)$ the probability that $G_u(p)$ is 1-connected. Intuitively, one can say that if $D_{\min}$ is greater than 1 for every node, the graph is connected. But this happens asymptotically only, i.e. for $n \to +\infty$. For example, we can have a graph with 4 nodes and 2 links: The minimum degree is 1 but the network is clearly not connected at all. Moreover, generally a graph can be considered as composed of several connected sub-graphs which we call components of $G_u(p)$. If there is one component spanning all the nodes of $G_u(p)$, the graph is connected. Other measures of connectivity can make use of the size of the “giant” component. If there are few components and the giant component has size $S \approx |V|$, we can say that $G_u(p)$ is *quasi* connected. We will see in Section 3, that connectivity and giant component size are intimately tight to the structure of the adjacency matrix. The same properties can be defined for digraphs, which are the main target of this work. Given $G(p)$, if there is a path between any two nodes, we say that $G(p)$ is strongly connected. Otherwise, if only the undirected version of $G(p)$ is connected, we say that $G(p)$ is weakly connected. In what follows, we briefly list the main results on PRGs.

✣ The probability of connectivity is [8]:

$$P(G_p \text{ is } 1-\text{conn}) = 1, \quad \text{a.s.} \qquad \text{iff } p \geq \frac{\log(n)}{n}\,,\ n \to \infty\,. \tag{1}$$

Under the same hypotheses, we have the well known result [4]:

$$P(D_{\min} \geq 1) = P(1-\text{conn})\,, \quad \text{a.s.} \tag{2}$$

✣ The hop count h between two nodes is defined as the number of nodes of the shortest path connecting them, if it exists. For disconnected networks, one can define the mean hop count inside the components of G_p. For PRGs, we have that [2]:

$$E\{h\} \sim \frac{\ln(n)}{\ln(E[D])}\,.$$

In general, the hop count distribution of the graph representing the physical network is of concern, because it can serve as a good tool to understand important metrics of ad-hoc networks, as packet delay, end-to-end throughput and power consumption [23].

✥ The probability of the degree D is:[a]

$$P(D=k)=\begin{pmatrix} n-1 \\ k \end{pmatrix} p^k(1-p)^{n-1-k} \simeq \frac{\overline{D}^k e^{-\overline{D}}}{k!}\,, n\to\infty\,, \tag{3}$$

where we used the Poisson approximation of the binomial distribution, and $\overline{D}=(n-1)p$. We can easily prove (3), if we consider the number of ways we can set a link among a given node and the other $(n-1)$ nodes.

✥ By using (2) and (3), and by assuming D_i be i.i.d., we have:

$$P(D_{\min}\geq 1)=P\left(\min_{i\in V}(D_i)\geq 1\right)=\left[\sum_{k=1}^{n-1}\begin{pmatrix} n-1 \\ k \end{pmatrix} p^k(1-p)^{n-1-k}\right]^n=[1-(1-p)^{n-1}]^n\,. \tag{4}$$

✥ By using the approximation in (3), the distribution of D is:

$$P(D\leq k)=e^{-\overline{D}}\sum_{i=0}^{k}\frac{\overline{D}^i}{i!}=\frac{\Gamma(k+1,\overline{D})}{k!}\,, \tag{5}$$

where $\Gamma(a,b)$ is the incomplete gamma function, i.e. $\Gamma(a,b)=(a-1)!e^{-b}\sum_{i=0}^{a-1}\frac{b^i}{i!}$ [1].

2.2 Geometric Random Graphs (GRGs)

In PRGs, links are uncorrelated. For example, the link probability between two nodes does not affect the link probability with another geometrically closed node. This is not the case of a real network, where we remember that a link can be set up whenever the transmission power is greater than a specific threshold. This power sets up in turn the transmission range r_0 of a node. Consequently, links become correlated. Because of this simple fact, we obtain a GRG, where the link probability depends on distance r between nodes. For simplicity we use the notation G only for GRGs. This model is called also Boolean or Bernoulli model. In fact, the link probability between $i,j\in V$ is the event that two nodes fall inside a sphere of radius r. This event induces a bernoullian r.v. for which the probability is $p(r)=\mathbf{1}_{r\geq r_0}$. Let us note that in general the link probability is not a commutative property of the graph. In other words, having a link between nodes i and j does not imply a link between j and i. If we omit considering the direction of links, we have an undirected graph we call with G_u. If not otherwise stated, we assume always an undirected graph and we study the properties of G in Section 3. Here, we consider first the poissonian deployment.

Deterministic Path Loss Model (α-GRG)

The first-order model for the pathloss of the radiated signal is the classic Friis formula. By expressing in decibel all variables, we have[b]:

$$P_r(r)=P_t-\beta=P_t-\beta_0-10\alpha\log\frac{r}{d_0}\quad(\mathrm{dB})\,, \tag{6}$$

[a]Whenever it does not make confusion, we drop the subscript in order to simplify notation.
[b]When the variable of the equations are expressed in dB, we put the unit in parentheses at the end of the equation.

where β_0 and d_0 are constants.[c] In other words, the pathloss is:

$$\beta(r) = cr^{\alpha}\,, \tag{7}$$

and $c = \frac{10^{\beta_0/10}}{d_0^{\alpha}}$. To correctly receive the transmitted signal, we lower bound the received power by a threshold, $P_{\mathrm{r,th}}$. Accordingly, the radio range is defined as:

$$r_0 : P_r(r_0) \geq P_{\mathrm{r,th}}\,, \tag{8}$$

and by using (7), $\beta_{\mathrm{th}} \triangleq \beta(r_0)$.

✣ We have that also for GRGs but not in one dimension [18]:

$$P(k\text{-conn}) = P(D_{\min} \geq k) \quad \text{a.s.}, n \to \infty\,. \tag{9}$$

In a GRG, if a node has degree $D = d$, this means that d nodes can hear its signal in an area of πr_0^2 square meters. By taking the mean, we have $\overline{D} = \rho\pi r_0^2$. If n is fixed, we have:

$$P(D_{\min} \geq k) = \left(1 - P(D_{\min} < k)\right)^n\,. \tag{10}$$

If also the number of nodes N is a r.v. with mean $E\{N\} = \rho a$, then:

$$P(D_{\min} \geq k) = \sum_{n=1}^{+\infty} \left(1 - P(D < k)\right)^n P(N = n)\,.$$

In the next paragraph, we show that the distribution of D is poissonian[d]. We note that this expression is in the form $E\{g(N)\}$, with $g(n) = e^{-P(D<k)n}$. Thus, by using Jensen's inequality and by borrowing results of (5), we have:

$$P(D_{\min} \geq k) \geq e^{-\rho A P(D \leq k-1)} = e^{-\rho A \frac{\Gamma(k,\overline{D})}{(k-1)!}}\,. \tag{11}$$

For example, for $k = 1$, the lower bound is:

$$P(D_{\min} \geq 1) = e^{-\rho A e^{-\overline{D}}} = e^{-\rho A e^{-\pi\rho r_0^2}}\,. \tag{12}$$

A plot of this function is given in Fig 1. As shown, given a ρ, the critical transmission range acts as a phase transition threshold, after which a sharp increase of $D_{\min}$ arises. We show in Section 4, that this behaviour well approximate the connectivity property of a simulated network. By using (9), we can solve (12) with respect to r_0:

$$r_0 = \sqrt{\frac{\ln\left(\frac{\ln(P(1\text{-conn})}{-\rho A}\right)}{-\rho\pi}}\,. \tag{13}$$

[c] The pathloss β_0 depends on the hardware, for instance the antenna shape, the insertion pathloss and the frequency of radiated signal.

[d] Under our notation, a GRG is simply a 0-GRG, for which the results of (20) and (22) are easily adapted.

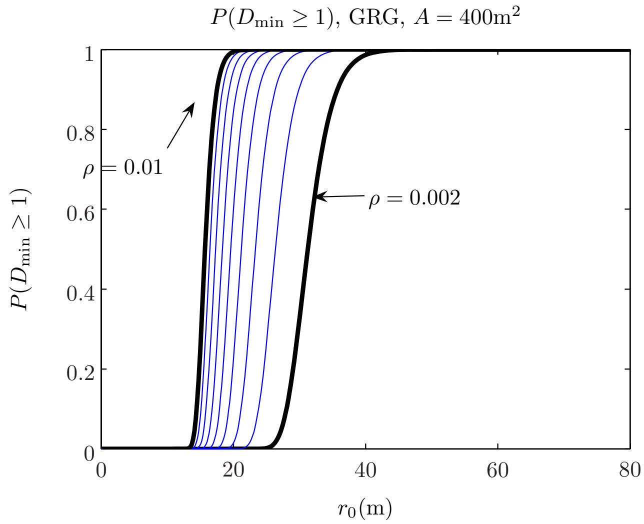


Fig. 1. Probability of the minimum degree. The density is expressed in nodes/m^2.

This value can be thought as the minimum transmission range of nodes of a large WSN, in order to guarantee at least 1-connectivity. The value of r_0 fixes the transmission power by means of (8). Let us note that this result do not take into account the border effects, that is the fact that border nodes of the network can have lower degree. When the area $a \rightarrow +\infty$, the impact of the border effects is negligible. Otherwise, in simulation one has to use a toroidal distance. A simple artifact to keep using (13) is making the transmission range of border nodes greater than r_0.

✣ Another definition of degree is as follows. Consider the number of links, L, in a random geometric network, G_u. Then,

$$L = \sum_{i=1}^{n} \sum_{j=i+1}^{n} p(r_{ij}) \, ,$$

where r_{ij} is the distance between node i and j. If we divide the total number of links $2L$ by the number of nodes n, we obtain the degree. By taking the average, we have [11]:

$$\overline{D} \triangleq E\{D\} = \frac{2E\{L\}}{n} \, .$$

However, the computation of $E\{L\}$ is not always simple.

Random Path Loss Model (ξ-GRG)

Here, the pathloss is a r.v., i.e.:

$$\beta(r) = \beta_0 + 10\alpha \log \frac{r}{d_0} + X_\sigma \quad (\mathrm{dB}) \, , \tag{14}$$

where X_σ is a r.v. Because of this fact, the transmission range of a node is now a r.v. In the shadowing model, X_σ is a normal distributed r.v., with mean 0 and variance σ^2. Let assume a shadowing model with parameter $\xi \triangleq \frac{\sigma}{\alpha}$.

The link probability depends on the threshold. By using (8), we can define again r_0 by using the mean value of $P_r(r)$, that is:

$$r_0 : \overline{P_r(r_0)} = P_{\mathrm{r,th}} \,. \tag{15}$$

Similarly, a definition for β_{th} as above holds. Since X_σ is log-normal, by using (14), it is straightforward to show that:

$$P\left(\beta(r) \leq \beta_{\mathrm{th}}\right) = P\left(P_r(r) \geq P_{\mathrm{r,th}}\right) = \frac{1}{2}\mathrm{erfc}\left(\frac{P_{\mathrm{r,th}} - \overline{P_r(r)}}{\sqrt{2}\sigma}\right) \quad (\mathrm{dB})\,, \tag{16}$$

where $\overline{P_r(r)}$ is the value computed by means of (6) and erfc is the standard complementary error function [1]. Accordingly, we have the following results.

✣ The link probability between two nodes is:

$$p_r(\widehat{r}) = P\left(P_r(r) \geq P_{\mathrm{r,th}}\right) = \frac{1}{2}\mathrm{erfc}\left(\frac{10\log\widehat{r}}{\sqrt{2}\xi}\right)\,, \quad \widehat{r} \triangleq \frac{r}{r_0}\,, \tag{17}$$

where we used a normalized distance $\widehat{r}$.

✣ According to simulation results of [11], we have:

$$P(D_{\min} \geq 1) = P(1-\mathrm{conn})\,, \quad \text{if } \xi < 3,\ D_{\min} \geq 15 \tag{18}$$

$$P(D_{\min} \geq 1) = P(1-\mathrm{conn})\,, \qquad \text{a.s., if } \xi \geq 3\,. \tag{19}$$

This mean that, for connectivity properties, a graph G_u can be considered a random graph for high values of the shadowing parameter ξ. However, other properties such as $\overline{D}$ and $E\{L\}$ can be different.

✣ Under the poissonian assumption for the nodes deployment, and by assuming an unbounded area ($a \to +\infty$), the number of nodes that can hear the signal of a given node is also a poissonian r.v. [17][3]:

$$P(D=d) = \frac{\overline{D}^d}{d!}e^{-\overline{D}}\,, \quad \overline{D} = \rho\pi r^2\,. \tag{20}$$

Furthermore, by adapting the value of constants in [17], we obtain the following definitions:

$$r = 10^{\eta}e^{\zeta}\,, (\text{in meters})$$

$$\eta = \frac{\beta_{\mathrm{th}} - \beta_0}{10\alpha}\,, \quad \zeta = \left(\tfrac{\ln(10)\xi}{10}\right)^2 \tag{21}$$

$$\beta_{\mathrm{th}} = 10\log\left(\tfrac{P_t}{P_{\mathrm{r,th}}}\right)\,. \tag{22}$$

In (20), we can recognize a similarity with (3). In other words, the degree distribution of the network resembles that of a random graph, with different mean.

✣ For the minimum degree, we have an expression similar to (11), but with a mean $\overline{D}$ defined as in (20). By using the expression of β_0 in (21), we obtain the very important result:

$$\overline{D} = \pi \rho r_0^2 e^{2\zeta} . \tag{23}$$

With this value for $\overline{D}$, we have again:

$$P(D_{\min} \geq 1) = e^{-\rho A e^{-\overline{D}}} , \tag{24}$$

and, for the critical range r_0:

$$r_0 = \sqrt{\frac{\ln\left(\frac{\ln P(conn)}{-\rho A}\right)}{-\pi \rho e^{2\zeta}}} . \tag{25}$$

We plot this probability in Fig. 2. As shown, for a given density, the presence of shadowing decreases the transmission range r_0 in order to achieve the asymptotic connectivity. On the other hand, if we are given a certain r_0, the critical density for connectivity decreases as ξ increases.

There are not other results on connectivity for other radio models. However, a generalization of the methodology in [17] can be found in [13].

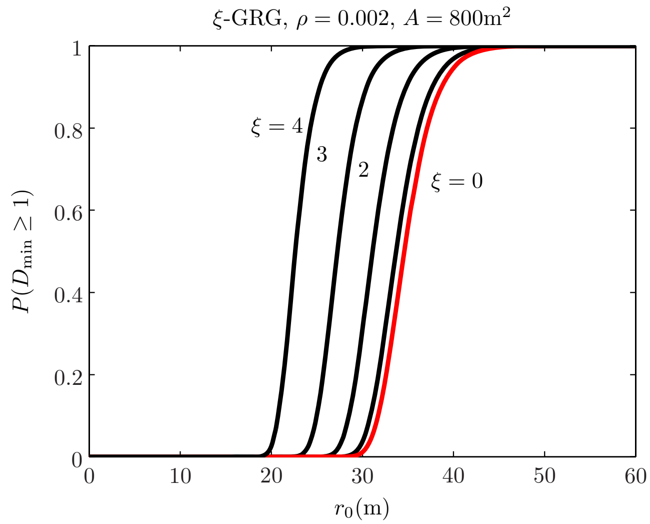
2.3 Lattice Graphs

In general, in a lattice graph nodes are vertexes of some geometric structure. In other words, the minimum distance between adjacent nodes is fixed and it is a parameter of the lattice. We call lattice step, s, the minimum distance. The simplest lattice graph in one dimension is a a line of nodes. For particular applications, the topology of ad-hoc networks can be arranged as a lattice graph. To say the truth, the lattice structure is more suitable for WSNs, where the position of nodes can be computed *a priori*. Another simple structure in two dimensions is the square lattice, which is simply a grid. As in the case of poissonian deployment, we can have both GRG and ξ-GRG for lattice graphs. For example, for a square lattice, if we suppose to set the transmission power of every node in such a way that the transmission range of nodes equals s, we obtain $D_{\min} = 2$. If one discards the border nodes,[e] $D_{\min} = 4$. When the nodes of the graph G have the same degree d, we say that G is d-regular. If we consider also the effects of border nodes which have degree lower than d, we call G a quasi-d-regular graph. For example, with the above assumptions on the transmission power, a lattice graph with $D_{\min} = 4$ is a quasi 4-regular graph. For this kind of graphs, the degree distribution is $P(D = d) = \delta(k - d) + \sum_{i \in V_b} \delta(k_i^{'} - d)$, where V_b is a partition of the set of border nodes and k_i' depends on the type of lattice. For example, in a square lattice $k_i^{'} = 2, 3$. For ξ-GRG, the degree of nodes changes and G is not regular anymore, since every node can establish a link with nodes far away the step size s. However, we will see that when ξ increases, the lattice connectivity resembles that of a ξ-GRG with poissonian degree distribution. We do not derive a distribution of D for lattice ξ-GRG, but we study the connectivity via simulation.

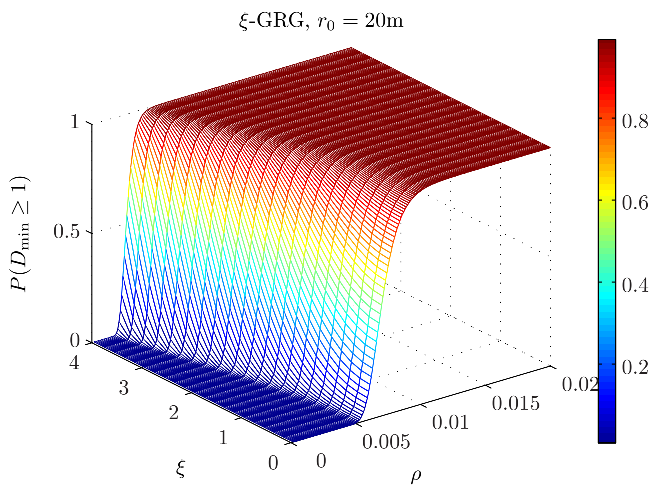
✣ The hop count in quasi 4-regular graph is:

$$E\{h\} = \frac{m + n}{3} , \tag{26}$$

[e]This can be done by considering a smaller service area $\mathbf{A}^{'} \subset \mathbf{A}$, or by assuming a toroidal distance.



(a) fixed density



(b) fixed r_0

Fig. 2. ξ-GRG.

where $m = |E|$. Obviously, this is true for GRG; things change if we consider ξ-GRG, for which the mean hop count can be lower than the value in (26).

3 Connectivity

To compute the connectivity of a graph G, one can use the probability of minimum degree of G, as a lower bound. In fact, $P(k - \text{conn}) = P(D_{\min} \geq k)$, a.s. To test the validity of formulas above discussed, we need a method to check the connectivity of the network during simulations. Given the network or a graph G, a brutal-force method could be the enumeration of all paths between any pair of nodes, and stop whenever we discover a pair which does not have a path at all. In this case, we declare the graph G disconnected. This procedure works for few nodes, but its complexity time explodes after $|V| \simeq 24$. In fact, the complexity is not polynomial but factorial. Another approach is to use a Depth-First Search (DFS) algorithm which has complexity $O(n + m)$. A clever procedure to check the connectivity of a graph G relies on exploiting a theorem of graph theory. Let us remember that a matrix A is said irreducible if there is not any permutation P such that PAP^T has a non trivial block triangular form, e.g.:

$$PAP^T = \begin{pmatrix} A_1 & A_2 \\ \mathbf{0} & A_3 \end{pmatrix}.$$

Theorem 1 (Dulmage-Mendelsohn [12]) *Given a graph G with adjacency matrix A, the following relations hold.*

$$\text{if } A \text{ is irreducible} \Leftrightarrow G \text{ is strongly connected}$$
$$\text{if } A \text{ is reducible} \Leftrightarrow G \text{ is not strongly connected}$$

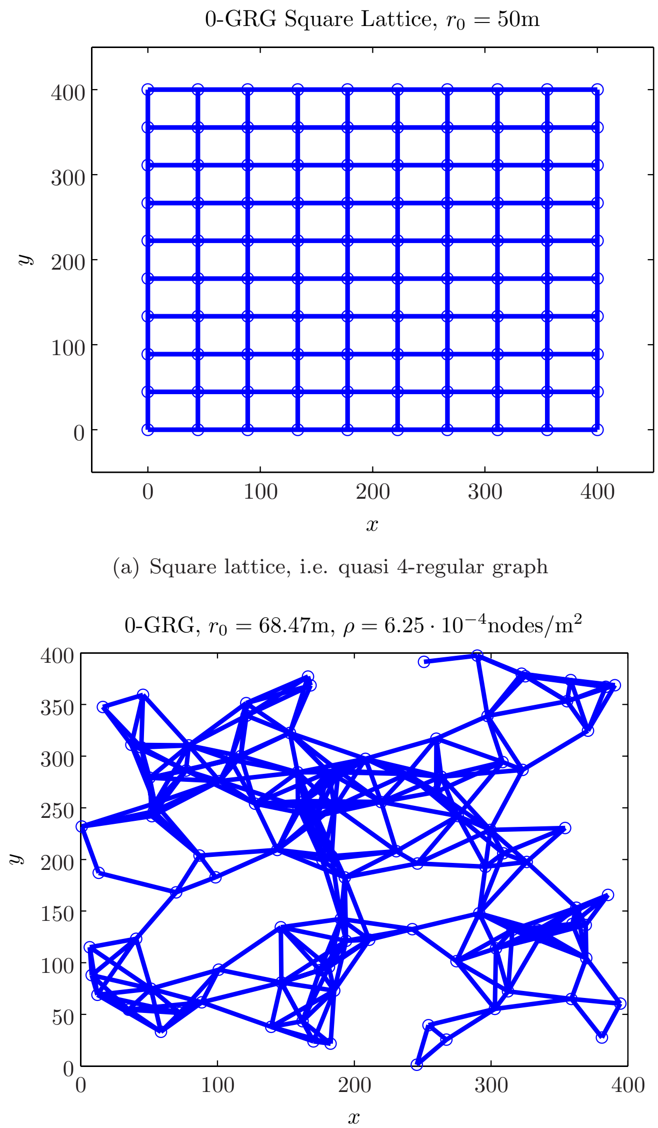
The theorem is valid also for G_u. For other details and for the implementation of the algorithm, which is promptly done in MATLAB, see [20]. In summary, to check the connectivity of G we have to check the irreducibility of the matrix A. The algorithm furnishes also the sizes of the (strongly) connected components of G. It is worth noting that for G_u, another procedure exists. It exploits the laplacian of the G_u, which is defined as $L(G) = R(G) - A$, where $R(G)$ is the diagonal matrix of degrees, i.e. $R_{ii} = D_i,\ i \in V$. If we order the eigenvalues λ_i of $L(G)$ in increasing order, it has been proved that always $\lambda_1 = 0$, and $\lambda_2 = 0$ iff G_u is connected. Moreover, the multiplicity of 0 as eigenvalues of G_u is the number of component of G_u [14].

4 Examples

In this section, we give some examples, by showing also the impact of asymmetries on the connectivity properties of an ad-hoc network. This fact is important because it affects the performance of routing protocols. In fact, a routing protocol does use forward and reverse links in order to find a suitable path between any two given nodes.

If we use a deterministic pathloss model, i.e. a GRG, we have topologies as in Fig. 3. In particular, for the GRG of Fig. 3(b), we set $\rho = 6.25 \cdot 10^{-4}$ and by using (13), $r_0 = 68.47$m with P(1-conn) $= 0.99$. For the square lattice, with $s = 50$m, we have $r_0 = s$. In the figure, links among border nodes are not shown. For ξ-GRG, we first try an experiment with $\xi = 2$, i.e. $\alpha = 3$ and $\sigma = 6$. Accordingly to (25), $r_0 = 55.38$m, i.e. the transmission range now is decreased, as expected. At this point we must note that the expressions for $D_{\min}$ have been computed by assuming undirected graphs, that is, given a pair of nodes, there is no distinction between forward link and reverse link at all. It has been documented in some works that this is far from reality. Several measurements on test-beds have shown that the presence of asymmetric links is not infrequent [6]. The asymmetry means that the given two nodes, i and j, the characteristics of link $i \to j$ are different from those of link $j \to i$. For example, one can consider the BER in the two directions. We can inspect more in details this phenomenon by generating a topology where link events are independent in the two directions. A realization of such a network, i.e. a digraph, is shown in Fig. 4, where asymmetric links are shown with a black arrow. As we can see, if we neglect the asymmetry of links, the topology is connected. On the other hand, if we consider the asymmetries, we see the emergence of components in G. In the figure, two of them are marked with a big circle. There are also two components which are composed by a single node only. These components are isolated because they have out-links only. To stress more this fact, by considering G_u, a visual inspection tell us that $D_{\min} = 1$. However, if we take into account asymmetric links too, the result changes. In fact, the (real) topology now is disconnected, as pointed out by the presence of nodes that have not a path towards other nodes. The same happens also for the lattice graph, as shown in Fig. 4(b). The behaviour of G, i.e. the digraph, depends on the distribution of in- and out-degree of nodes. Furthermore, although a link between components of G can be present, in general also this link is a directed link. This means that it might happens that we can go towards a component of G but we cannot come back from it. In other words, we should distinguish the giant component of G_u from the strongly connected giant component of G. In general, the strongly connected component of G arises after the emergence of the giant component of G[f]. We can see this fact also by analysing the distribution of minimum degree in G and G_u. For G, we plot the distribution of $D^+_{\min}$, while for G_u we plot the distribution of $D_{\min}$. To obtain the plots shown in Fig. 5, we used the following procedure. We first generated a ξ-GRG with parameters shown in the figure. For every topology, we computed several graph properties, as the minimum degree within the graph, the connectivity by means of results of Section 3. Then, we repeat the experiment 500 more times. These steps are then repeated for increasing values of the density ρ. In Fig. 5, for a better inspection, we plot the kernel density estimate of the underlying density function (i.e. the minimum degree distribution), even if the true density is a discrete probability function. As shown, the density functions are shifted with each other, in particular the density function of $D^+_{\min}$ is always at the left side of the density of $D_{\min}$. This mean that, for G, in order to achieve connectivity we have to increase the density with respect to the case of G_u.

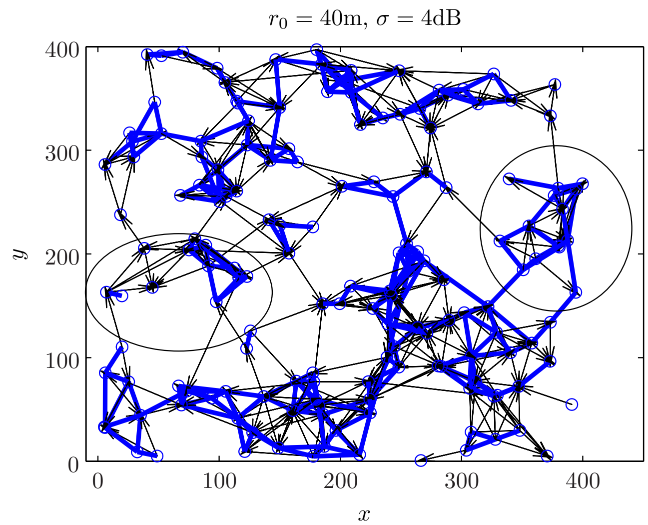
[f]Other details can be found in [7] and [16].



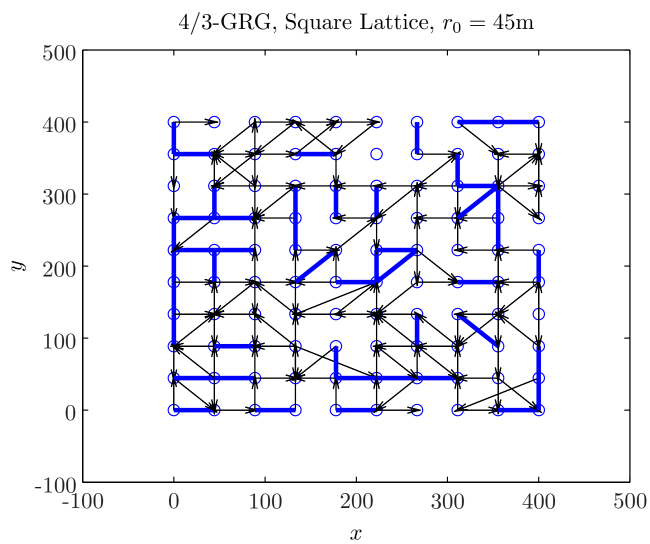
(a) Square lattice, i.e. quasi 4-regular graph

(b) GRG (poissonian deployment)

Fig. 3. Deterministic pathloss model.



(a) Poissonian deployment



(b) Square Lattice

Fig. 4. 4/3-GRG with asymmetries.

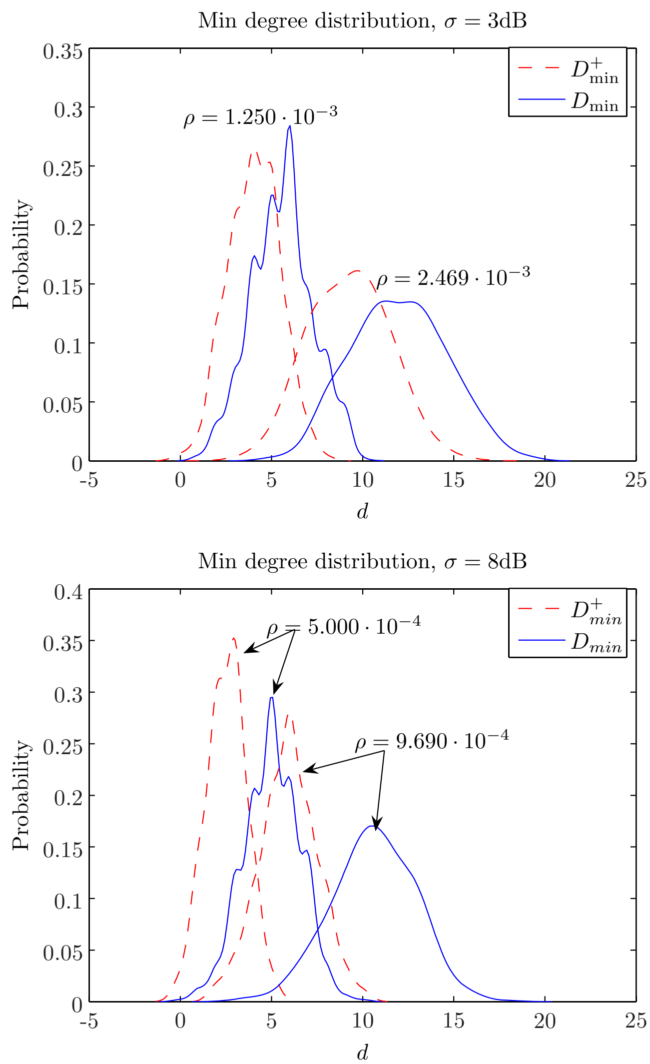
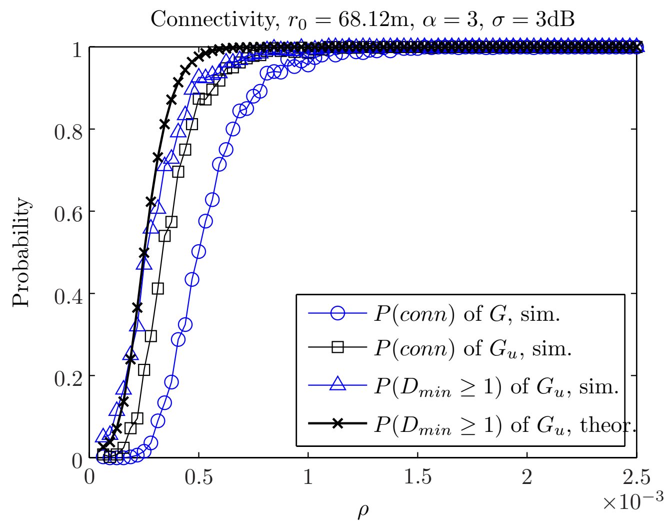
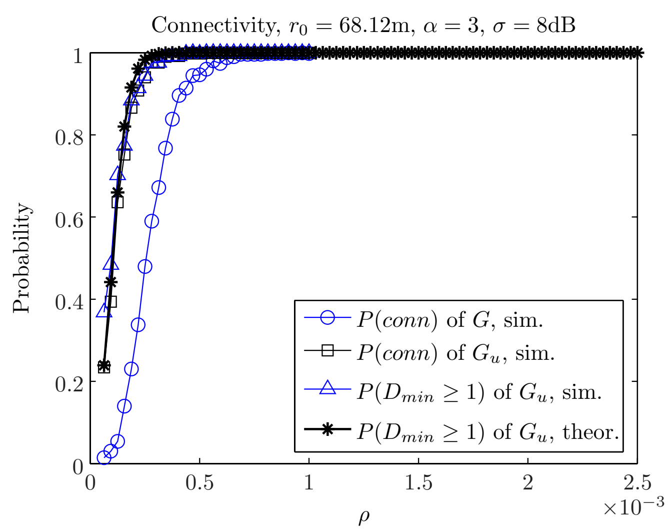


Fig. 5. Minimum degree distribution for σ/3-GRGs. Densities are expressend in nodes/m^2.

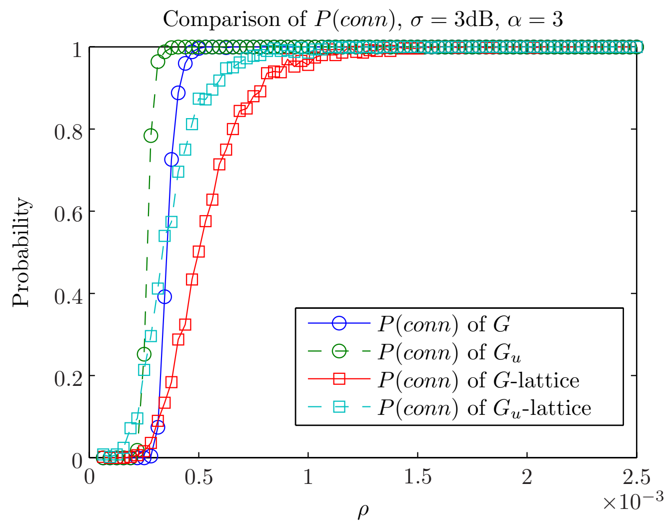


(a)

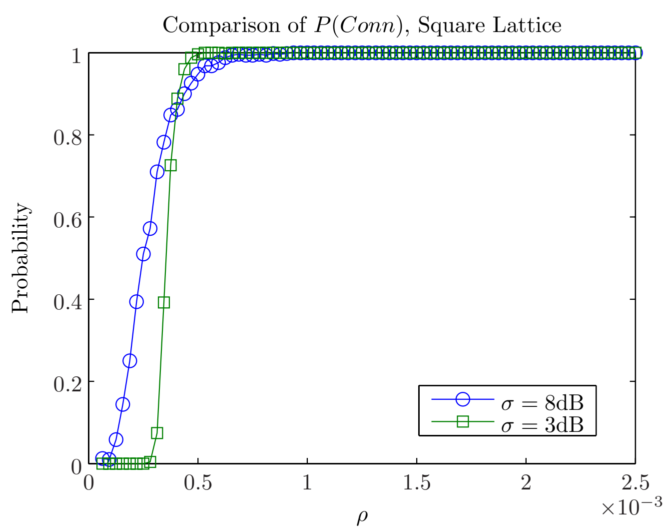


(b)

Fig. 6. Connectivity property for ξ-GRGs.



(a)



(b)

Fig. 7. Comparison of connectivity of ξ-GRGs with lattice graphs.

We can see this phenomenon also from the point of view of the connectivity. By using the same steps above explained, we estimate the connectivity probability by its average, i.e. we divide the number of connected topologies by 500. It is worth noting that we always inherently assumed a toroidal distance. In Fig. 6, we plotted the the estimates of the connectivity probability as a function of ρ for $\xi = 1$ and $\xi = 2.6667$, respectively. For the undirected graph G_u, we obtain the results explained in the previous sections, that is $P(conn)$ is well approximated by $P(D_{min} \geq 1)$ as ξ increases. This means that, as ξ increases, the critical density point above which the net is connected a.s. (i.e. $P(conn) = 99\%$) can be computed by using $P(D_{\min} \geq 1)$, and this value can be computed by means of (24). However, if we compare $P(conn)$ for G and G_u, there is a noticeable distance between the two curves. In particular, we have that: (1) The critical density value rougly doubles; (2) increasing ξ does not reduce the differences. For square lattice, the same differences hold. In this case, we note also two more things. In the square lattice, the connectivity probability resembles a step function for low values of ξ, as shown in Fig. 7(a), because of the structure of $P(D_{\min})$, as explained in Section 2.3. However, as the variance of the pathloss increases, the square lattice is more similar to a GRG, i.e. the presence of the severe shadowing destroys the regularity of the lattice. Accordingly, the connectivity probability is not a step function any longer, as clearly shown in Fig. 7(b).

5 Concluding Remarks

In this paper, we reviewed some basic properties of random graphs as model of ad-hoc networks. We insisted on the distribution of minimum degree of the network, which directly affects the connectivity. We have shown the dependence of these properties on system parameters, as the density of nodes and the transmission range, and environment parameters, as the pathloss coefficient α and the variance σ, which we embodied in one parameter, ξ. By means of extensive simulations we presented the importance of considering asymmetric links, for instance links which do not have the same communication characteristics in both directions. For simplicity, for the transmission range of nodes we consider a threshold model by disregarding the presence of other important factors, such as the ambient noise and the overall interference due to neighbouring nodes. However, the extension of these results to more complete scenario is straightforward. One lesson we can learn from these results is that connectivity of real networks directly affect the performance of routing and MAC protocols. In fact, the routing protocol needs to discover a path between two nodes. Although the performance of the protocols depends on other particular mechanisms, in general for on-demand protocols, there is a common routing discovery procedure during which the source node broadcasts special routing request packets. When these packets reach the destination node, a similar procedure is used in order to acknowledge the previously discovered path. The asymmetries affect some routing parameters, as timeouts which in turn trigger the searching for alternate path. The overall effect of these asymmetries is on the transmission delay of a packet between two given nodes. On the other hand, at the MAC layer, nodes perform some coordinated access to the channel. This access depends on the contention level on the channel, which in turn depends on the degree of the nodes.

References

1. M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover Publications, 1970.
2. R. Albert and A.-L. Barabási. Statistical mechanics of complex networks. *Reviews of Modern Physics*, 74:47–97, 2002.
3. C. Bettstetter. Failure-resilient ad hoc and sensor networks in a shadow fading environment. In *Proceedings IEEE/IFIP Intern. Conf. on Dependable Systems and Networks (DSN), Workshop on Dependability Issues in Wireless Ad Hoc Networks and Sensor Networks (DIWANS)*, page 900, June 2004.
4. B. Bollobas. *Random Graph*. Academic Press, 1985.
5. I. Carreras, I. Chlamtac, H. Woesner, and C. Kiraly. Bionets: Bio-inspired next generation networks. In *Lecture Notes in Computer Science*, volume 3457, pages 245 – 252, July 2005.
6. A. Cerpa, J. L. Wong, L. Kuang, M. Potkonjak, and Deborah Estrin. Statistical model of lossy links in wireless sensor networks. In *ACM/IEEE Fourth International Conference on Information Processing in Sensor Networks (IPSN05)*, pages 81–88, April 2005.
7. S.N. Dorogovtsev and J.F.F. Mendes. *Evolution of Networks: From Biological Nets to the Internet and WWW*. Oxford University Press, Oxford, January 2003.
8. P. Erdös and A. Rény. On the evolution of random graphs. *Pubblication of the Mathematical Institute of the Hungarian Academy of Science*, 5:17–61, 1960.
9. E. N. Gilbert. Random plane networks. *Journal of SIAM*, 9(4):533–543, December 1961.
10. P. Gupta and P. R. Kumar. Critical power for asymptotic connectivity in wireless networks. *Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W.H. Fleming*, pages 547–566, 1998.
11. R. Hekmat and P. Van Mieghem. Study of connectivity in wireless ad-hoc networks with an improved radio model. In *Proceedings of the Second Workshop on Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks (WiOpt'04)*.
12. D. M. Johnson, A. L. Dulmage, and N. S. Mendelsohn. Connectivity and reducibility of graph. *Canad. J. Math.*, 14:529–539, 1962.
13. D. Miorandi and E. Altman. Coverage and connectivity of ad hoc networks in presence of channel randomness. In *Proceedings of IEEE INFOCOM 2005*, volume 1, pages 491–502, 2005.
14. Bojan Mohar. *The laplacian spectrum of graphs*, volume 2 of *Graph Theory, Combinatorics, and Applications*. Wiley, 1991.
15. T. Moscibroda and R. Wattenhofer. The complexity of connectivity in wireless networks. In *Proceedings of INFOCOM 2006*, 2006.
16. M. E. J. Newman, S. H. Strogatz, and D. J. Watts. Random graphs with arbitrary degree distribution and their applications. *Phys. Rev.*, 2001.
17. J. Orriss and S. K. Barton. Probability distributions for the number of radio tranceivers which can communicate with one another. *IEEE Transaction on Communications*, 51(4):676–681, April 2003.
18. M. D. Penrose. On k-connectivity for a geometric random graph. *Random Structures and Algorithms*, 15:145–164, 1999.
19. M. D. Penrose. *Random Geometric Graphs*. Oxford University Press, Oxford, UK, 2003.
20. A. Pothen and C.-J. Fan. Computing the block triangular form of a sparse matrix. *ACM Transactions on Mathematical Software*, 16(4):303–324, 1990.
21. T.S. Rappaport. *Wireless Communications*. Prentice Hall PTR, 2001.
22. G. W.-Allen, G. Tewari, A. Patel, M. Welsh, and R. Nagpal. Firefly-inspired sensor network synchronicity with realistic radio effects. In *SenSys '05: Proceedings of the 3rd international conference on embedded networked sensor systems*, pages 142–153, New York, NY, USA, 2005. ACM Press.
23. Q. Zhao and L. Tong. Energy efficiency of large-scale wireless networks: Proactive versus reactive networking. *IEEE Journal on Selected Areas in Communications*, 23(5):1100–1112, May 2005.