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# Bayesian and MLE of $R = P(Y > X)$ for Exponential Distribution Based on Varied L Ranked Set Sampling

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## Abstract

The ranked set sampling (RSS) is an effective scheme popularly used to produce more precisely estimators. Despite its popularity, RSS suffers from some drawbacks which includes high sensitivity to outliers and it cannot sometimes be applicable when the population is relatively small. To overcome these limitations, varied L ranked set sampling (VLRSS) is recently introduced. It is shown that VLRSS scheme enjoys with many interesting properties over RSS and also encompasses several existing RSS schemes. In addition, it is also helpful for providing precise estimates of several population parameters. To fill this gap, this article extends the work and address the estimation of based  $\mathcal{R}$  on VLRSS when the strength and stress both follow exponential distribution. Maximum likelihood approach as well as Bayesian method are considered for estimating  $\mathcal{R}$ . The Bayes estimators are obtained by using gamma distribution under general entropy loss function and LINEX loss function. The performance of the estimators based on VLRSS are investigated

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by a simulation study as well as a real dataset relevant to industrial field. The results reveal that the proposed estimators are more efficient relative to their analogues estimators under L ranked set sampling given that the quality of ranking is fairly good.

**Keywords:** Varied L ranked set sampling, maximum likelihood estimation, Bayes estimation, Lindley approximation, relative efficiency.

## 1 Introduction

The stress-strength reliability can be defined as the probability that the strength of the item ( $Y$ ) is greater than the stress ( $X$ ) imposed on it. This can be expressed as  $\mathcal{R} = P(X < Y)$ . It is clear that if  $X$  is greater than  $Y$ , then the system will fail, otherwise it will continue to work. Therefore  $\mathcal{R}$  is commonly called as system reliability. Birnbaum (1956) applied firstly the idea of  $\mathcal{R}$  in physics and engineering, since then it has been applied in many other fields such as agriculture, quality control, statistics, medicine and so on. For more detailed applications, one can refer to the monograph of Kotz et al. (2003) and Vexler and Hutson (2018). Investigating the difference between two populations (say control group and treatment group) is an attractive practical example of  $\mathcal{R}$ , see Zamanzade et al. (2020).

Ranked set sampling (RSS), first proposed by McIntyre (1952), is one of the most significant sampling technique in statistics. Subsequently, RSS has been received extensive academic attention and concern by many scholars from various domains for its satisfactory performance and nice properties when making statistical inferences. Bouza and Al-Omari (2019) and references cited therein addressed the applicability of RSS in a wide range of disciplines. Since the introduction of RSS, several modifications of the RSS have been suggested to raise the efficiency of RSS-based inference, among which Al-nasser (2007) proposed LRSS as a variation of RSS to reduce the negative effect of the outliers and the extreme units to the information produced by RSS. The steps of LRSS can be summarized as:

1. Randomly select  $m$  sets each of size  $m$  items from the interested population.
2. Without any certain quantification, rank the units within each set with respect to virtual comparisons, expert opinion or auxiliary variables.

3. Identify the LRSS coefficient,  $k = [m\alpha]$ , such that  $0 \leq \alpha < .5$ , where  $[x]$  is the largest integer value less than or equal to  $x$ .
4. For each of the first  $k + 1$  ranked sets, exactly quantify the  $k + 1$  ordered sampling unit.
5. For each of the last  $k + 1$  ranked sets, exactly quantify the  $m - k$  ordered sampling unit.
6. For  $j = k + 2, \dots, m - k - 1$ , exactly quantify the item with rank  $j$  in the  $j^{th}$  ranked item.
7. Repeat the preceding steps, if needed,  $r$  cycles to get a sample of size  $n = rm$  for actual quantification.

Note that the ranking process may be imprecise and contains errors, this situation is known as imperfect ranking. In contradiction, perfect ranking is a situation in which the ranking process is free from errors. Additionally, if  $k = 0$ , then the LRSS will be reduced to the traditional RSS. Al-Omari (2015) considered the estimation of the distribution function under LRSS. Göçoğlu and Demirel (2019) used LRSS and other sampling variations for estimating the population proportion. Recently, Haq et al. (2015) introduced a generalization variation of LRSS as proposing a new sampling design termed as varied LRSS (VLRSS).

To get VLRSS, draw randomly  $k$  sets each of size  $m_1$  from the parent population, where  $m_1 \geq m$  or  $m_1 < m$  is a determined value based on cost- or budget-constraints. Rank these items separately within each set via any inexpensive way and then select the  $v^{th}$  smallest ranked item, where  $v = 1, 2, \dots, [m_1/2]$ . Draw again another  $k$  sets each of size  $m_1$  from the parent population and rank them, then select this time the  $(m_1 - v + 1)^{th}$  smallest ranked item. Finally, draw  $(m - 2k)$  sets each of size  $m$  items from the parent population and also rank them, then select  $i^{th}$  smallest ranked item from each set, where  $i = k + 1, \dots, m - k$ . This completes one cycle of VLRSS of size  $m$ . The whole procedure can be repeated  $r$  times to obtain VLRSS of size  $n = rm$ .

The beauty of VLRSS is that it encompasses several variations of RSS schemes, thus studying VLRSS implies considering many popular RSS designs. Therefore, VLRSS can be adopted for estimating much many interested parameters such as proportion parameter (Frey and Zhang 2019), dynamic reliability measure parameter (Mahdizadeh and Zamanzade 2018b), Quantile parameter (Morabbi and Razmkhah 2020), mean past lifetime parameter (Zamanzade et al. 2022) . . . etc. Nevertheless, not too much studies are published in this concern. Haq et al. (2015) proved that the mean estimator

under VLRSS is an unbiased estimator and better than its analog in RSS under certain conditions. Recently, Al-Omari (2021) investigated the superiority of VLRSS in estimating the location and scale parameters of a location-scale family. Abdallah (2022) used VLRSS in estimating the population distribution function. These findings encourage us to study the properties of MLE and Bayesian estimators of  $\mathcal{R}$  under exponentiality in the case of adopting VLRSS.

Numerous works have studied the estimation  $\mathcal{R}$  under RSS and its variations using either parametric or nonparametric approach. For instance, Mahdizadeh and Zamanzade (2016, 2018a, 2020) adopted the nonparametric approach, while Akgul et al. (2018, 2020), Esemien et al. (2021) and Hassan et al. (2021) considered the parametric approach. In the same context, Dong and Zhang (2019) estimated  $\mathcal{R}$  under LRSS in the case that  $X$  and  $Y$  follow the exponential distribution. This motivated us to extent this particular study to estimate  $\mathcal{R}$  based on VLRSS using Maximum likelihood estimation (MLE) as well as Bayesian approach.

A continuous random variable  $X$  has the exponential distribution if its probability density function (pdf) and cumulative distribution function (CDF) are given, respectively, by:

$$f(x; \lambda) = \lambda e^{-\lambda x}$$

and

$$F(x; \lambda) = 1 - e^{-\lambda x},$$

Kotz et al. (2003) obtained  $\mathcal{R}$  under the assumption that  $X$  and  $Y$  are two independent exponential random variables with parameters  $\mu$  and  $\lambda$ , respectively as follows:

$$\mathcal{R} = P(Y > X) = \int_0^{\infty} F(x, \mu) f(x, \lambda) dx = \frac{\mu}{\mu + \lambda}. \quad (1)$$

The rest of the paper is organized as follows: Section 2 presents the estimates of  $\mathcal{R}$  under LRSS using MLE and Bayesian method. Section 3 proposes the MLE and Bayesian estimators for  $\mathcal{R}$  under VLRSS. Section 4 gives the numerical comparisons between the considered estimators in terms of relative efficiency criterion. In Section 5, an application to an empirical dataset is addressed. Eventually, some concluding remarks and some potential directions for future studies are presented in Section 6. Finally, R code are postponed to an appendix.

## 2 Estimators of $\mathcal{R}$ Under LRSS

In this section, the MLE and the Bayes estimators of  $R$  under LRSS data are derived.

### 2.1 MLE Estimators of $\mathcal{R}$

Let  $X_{i(s:m)j}$ , ( $i = 1, \dots, m, j = 1, \dots, r$ ) denote the  $s^{th}$  order statistics from the  $i^{th}$  set of size  $m$  in the  $j^{th}$  cycle. Then  $X_{i(h_{ix}: m_x)j}$ , ( $i = 1, \dots, m_x, j = 1, \dots, r_x$ ) and  $Y_{i(h_{iy}: m_y)j}$ , ( $i = 1, \dots, m_y, j = 1, \dots, r_y$ ) denote the LRSS samples from the exponential distribution with parameters  $\mu$  and  $\lambda$  at sample sizes  $n_x = m_x r_x$  and  $n_y = m_y r_y$  respectively, where  $m_x$  and  $m_y$  are the set sizes,  $r_x$  and  $r_y$  are the number of cycles and  $h_{iz}$  can be expressed as:

$$h_{iz} = \begin{cases} k + 1 & 1 \leq i \leq k + 1 \\ i & k + 2 \leq i \leq m_z - k - 1 \\ m_z - k & m_z - k \leq i \leq m_z \end{cases}$$

Assuming the ranking is perfect, the pdf of  $X_{i(h_{ix}: m_x)j}$  and  $Y_{i(h_{iy}: m_y)j}$  are respectively given by:

$$g_x(x_{i(h_{ix}: m_x)j}) = \frac{m_x!}{(h_{ix} - 1)!(m_x - h_{ix})!} [F(x_{i(h_{ix}: m_x)j}, \mu)]^{h_{ix}-1} \times [1 - F(x_{i(h_{ix}: m_x)j}, \mu)]^{m_x - h_{ix}} f(x_{i(h_{ix}: m_x)j}, \mu) \quad (2)$$

and

$$g_y(y_{i(h_{iy}: m_y)j}) = \frac{m_y!}{(h_{iy} - 1)!(m_y - h_{iy})!} [F(y_{i(h_{iy}: m_y)j}, \lambda)]^{h_{iy}-1} \times [1 - F(y_{i(h_{iy}: m_y)j}, \lambda)]^{m_y - h_{iy}} f(y_{i(h_{iy}: m_y)j}, \lambda). \quad (3)$$

Using (2) and (3), the likelihood function  $L_1(\mu, \lambda)$  can be written as:

$$L_1(\mu, \lambda) = \left[ \prod_{j=1}^{r_x} \prod_{i=1}^{m_x} g_x(x_{i(h_{ix}: m_x)j}) \right] \times \left[ \prod_{j=1}^{r_y} \prod_{i=1}^{m_y} g_y(y_{i(h_{iy}: m_y)j}) \right] \\ = (\mu^{n_x} \lambda^{n_y}) \left( \left[ \prod_{j=1}^{r_x} \prod_{i=1}^{m_x} \frac{m_x!}{(h_{ix} - 1)!(m_x - h_{ix})!} \right] \right)$$

$$\begin{aligned} & \times \left. \left( 1 - e^{-\mu x_{(h_{ix}:m_x)j}} \right)^{h_{ix}-1} e^{-(m_x-h_{ix}+1)\mu x_{(h_{ix}:m_x)j}} \right] \\ & \times \left[ \prod_{j=1}^{r_y} \prod_{i=1}^{m_y} \frac{m_y!}{(h_{iy}-1)!(m_y-h_{iy})!} (1 - e^{-\lambda y_{(h_{iy}:m_y)j}})^{h_{iy}-1} \right. \\ & \left. \times e^{-(m_y-h_{iy}+1)\lambda y_{(h_{iy}:m_y)j}} \right]. \end{aligned}$$

According to Dong and Zhang (2019), the MLE estimators of the parameters for  $\mu$  and  $\lambda$ , say  $\hat{\mu}_1$  and  $\hat{\lambda}_1$ , can be obtained by solving simultaneously the following equations:

$$\begin{aligned} \frac{\partial \ln(L_2(\mu, \lambda))}{\partial \mu} &= \frac{n_x}{\mu} - \sum_{j=1}^{r_x} \sum_{i=1}^{m_x} (m_{ix} - h_{ix} + 1) x_{(h_{ix}:m_{ix})j} \\ &\quad - \sum_{j=1}^{r_x} \sum_{i=1}^{m_x} (h_{ix} - 1) \frac{x_{(h_{ix}:m_{ix})j} e^{-\mu x_{(h_{ix}:m_{ix})j}}}{1 - e^{-\mu x_{(h_{ix}:m_{ix})j}}} = 0 \quad (4) \end{aligned}$$

$$\begin{aligned} \frac{\partial \ln(L_2(\mu, \lambda))}{\partial \lambda} &= \frac{n_y}{\lambda} - \sum_{j=1}^{r_y} \sum_{i=1}^{m_y} (m_{iy} - h_{iy} + 1) y_{(h_{iy}:m_{iy})j} \\ &\quad - \sum_{j=1}^{r_y} \sum_{i=1}^{m_y} (h_{iy} - 1) \frac{y_{(h_{iy}:m_{iy})j} e^{-\lambda y_{(h_{iy}:m_{iy})j}}}{1 - e^{-\lambda y_{(h_{iy}:m_{iy})j}}} = 0 \quad (5) \end{aligned}$$

By using the invariance property, the estimated value of  $R$  given by:

$$\hat{\mathcal{R}}_1 = \frac{\hat{\mu}_1}{\hat{\mu}_1 + \hat{\lambda}_1}.$$

Additionally, Dong and Zhang (2019) proved that  $\hat{\mathcal{R}}_1$  has existence and uniqueness property and it is asymptotically normally distributed.

## 2.2 Bayesian Estimators of $\mathcal{R}$

In this part, we consider the Bayesian approach to estimate  $R$  under LRSS. This implies treating  $\mu$  and  $\lambda$  as random variables instated of fixed constants. This idea is believable since the parameters of any population may

be change throughout the study. This variation in the parameters can be taken into account by assuming prior distributions of unknown parameters. This explains the reason that the Bayesian approach often provides better and more precision estimates than MLE method. Here, we assume that the parameters  $\mu$  and  $\lambda$  follow independent gamma prior distributions of the following forms:

$$\pi(\mu, \lambda) \propto \mu^{a_1-1} \lambda^{\lambda_1-1} \exp(-a_2\mu - \lambda_2\lambda),$$

where  $a_1, \lambda_1, a_2$  and  $\lambda_2$  are called the hyper-parameters.  $L_1(\mu, \lambda)$  can be expressed as:

$$\begin{aligned} L_1(\mu, \lambda) &= \left[ (\mu^{n_x}) \prod_{j=1}^{r_x} \prod_{i=1}^{m_x} \sum_{s=0}^{h_{ix}-1} (-1)^s h_{ix} \binom{m_x}{h_{ix}} \binom{h_{ix}-1}{s} \right. \\ &\quad \left. \times e^{-(m_x-h_{ix}+s+1)\mu x_{(h_{ix};m_x)j}} \right] \\ &\quad \times (\lambda^{n_y}) \left[ \prod_{j=1}^{r_y} \prod_{i=1}^{m_y} \sum_{s=0}^{h_{iy}-1} (-1)^s h_{iy} \binom{m_y}{h_{iy}} \binom{h_{iy}-1}{s} \right. \\ &\quad \left. \times e^{-(m_y-h_{iy}+s+1)\lambda y_{(h_{iy};m_y)j}} \right]. \\ &= \left[ (\mu^{n_x}) \sum_{j=1}^{r_x} \left[ \sum_{s_1^j=0}^{h_{1x}-1} \sum_{s_2^j=0}^{h_{2x}-1} \cdots \sum_{s_{m_x}^j=0}^{h_{m_x}-1} \right. \right. \\ &\quad \left. \times \left[ \prod_{i=1}^{m_x} (-1)^{s_i^j} h_{ix} \binom{m_x}{h_{ix}} \binom{h_{ix}-1}{s_i^j} \right] \right. \\ &\quad \left. \left. \times e^{-\mu \sum_{i=1}^{m_x} (m_x-h_{ix}+s_i^j+1)x_{(h_{ix};m_x)j}} \right] \right] \\ &\quad \times \left[ (\lambda^{n_y}) \sum_{j=1}^{r_y} \left[ \sum_{s_1^j=0}^{h_{1y}-1} \sum_{s_2^j=0}^{h_{2y}-1} \cdots \sum_{s_{m_y}^j=0}^{h_{m_y}-1} \right. \right. \\ &\quad \left. \left. \times \left[ \prod_{i=1}^{m_y} (-1)^{s_i^j} h_{iy} \binom{m_y}{h_{iy}} \binom{h_{iy}-1}{s_i^j} \right] \right. \right. \\ &\quad \left. \left. \times e^{-\lambda \sum_{i=1}^{m_y} (m_y-h_{iy}+s_i^j+1)y_{(h_{iy};m_y)j}} \right] \right]. \end{aligned}$$

$$\times \left[ \prod_{i=1}^{m_y} (-1)^{s_i^j} h_{iy} \binom{m_y}{h_{iy}} \binom{h_{iy}-1}{s_i^j} \right] e^{-\lambda \sum_{i=1}^{m_y} (m_y - h_{iy} + s_i^j + 1) y_{(h_{iy}; m_y)j}} \Bigg].$$

Consequently, the posterior density can be obtained as:

$$\begin{aligned} &\pi(\mu, \lambda | X, Y) \\ &= \frac{\sum_{j=1}^{r_x} \left[ \sum_{s_1^j=0}^{h_{1x}-1} \sum_{s_2^j=0}^{h_{2x}-1} \dots \sum_{s_{m_x}^j=0}^{h_{m_x}-1} \right. \\ &\quad \times \left[ \prod_{i=1}^{m_x} (-1)^{s_i^j} h_{ix} \binom{m_x}{h_{ix}} \binom{h_{ix}-1}{s_i^j} \right] (\mu^{m_x+a_1-1}) \\ &\quad \times e^{-\mu \sum_{i=1}^{m_x} (m_x - h_{ix} + s_i^j + a_2 + 1) x_{(h_{ix}; m_x)j}} \Bigg]}{\sum_{j=1}^{r_x} \left[ \sum_{s_1^j=0}^{h_{1x}-1} \sum_{s_2^j=0}^{h_{2x}-1} \dots \sum_{s_{m_x}^j=0}^{h_{m_x}-1} \right. \\ &\quad \times \left[ \prod_{i=1}^{m_x} (-1)^{s_i^j} h_{ix} \binom{m_x}{h_{ix}} \binom{h_{ix}-1}{s_i^j} \right] \\ &\quad \times \frac{\Gamma(m_x+a_1)}{(\sum_{i=1}^{m_x} (m_x - h_{ix} + s_i^j + a_2 + 1) x_{(h_{ix}; m_x)j})^{m_x+a_1}} \Bigg]} \\ &\times \frac{\sum_{j=1}^{r_y} \left[ \sum_{s_1^j=0}^{h_{1y}-1} \sum_{s_2^j=0}^{h_{2y}-1} \dots \sum_{s_{m_y}^j=0}^{h_{m_y}-1} \right. \\ &\quad \times \left[ \prod_{i=1}^{m_y} (-1)^{s_i^j} h_{iy} \binom{m_y}{h_{iy}} \binom{h_{iy}-1}{s_i^j} \right] (\lambda^{m_y+\lambda_1-1}) \\ &\quad \times e^{-\lambda \sum_{i=1}^{m_y} (m_y - h_{iy} + s_i^j + \lambda_2 + 1) y_{(h_{iy}; m_y)j}} \Bigg]}{\sum_{j=1}^{r_y} \left[ \sum_{s_1^j=0}^{h_{1y}-1} \sum_{s_2^j=0}^{h_{2y}-1} \dots \sum_{s_{m_y}^j=0}^{h_{m_y}-1} \right. \\ &\quad \times \left[ \prod_{i=1}^{m_y} (-1)^{s_i^j} h_{iy} \binom{m_y}{h_{iy}} \binom{h_{iy}-1}{s_i^j} \right] \\ &\quad \times \frac{\Gamma(m_y+\lambda_1)}{(\sum_{i=1}^{m_y} (m_y - h_{iy} + s_i^j + \lambda_2 + 1) y_{(h_{iy}; m_y)j})^{m_y+\lambda_1}} \Bigg]} \end{aligned}$$

Although squared error loss function is the most popular loss function used in the Bayesian analysis, it is not preferable for estimating reliability



function (see Singh et al. (2014)). Hereafter, general entropy loss (GEL) function and LINEX loss function are adopted. The Bayesian estimates of  $\mu$  and  $\lambda$  based on GEL can be expressed respectively as:

$$\hat{\mu}_2 = (E(\mu^{-p}))^{\frac{-1}{p}}$$

$$= \frac{\left[ \sum_{j=1}^{r_x} \left[ \sum_{s_1^j=0}^{h_{1x}-1} \sum_{s_2^j=0}^{h_{2x}-1} \dots \sum_{s_n^j=0}^{h_{mx}-1} \right. \right. \right.}{\left. \left. \left. \sum_{j=1}^{r_x} \left[ \sum_{s_1^j=0}^{h_{1x}-1} \sum_{s_2^j=0}^{h_{2x}-1} \dots \sum_{s_n^j=0}^{h_{mx}-1} \right. \right. \right. \right.}^{\frac{-1}{p}}$$

$$\left. \left. \left. \begin{aligned} &\times \left[ \prod_{i=1}^{m_x} (-1)^{s_i^j} h_{ix} \binom{m_x}{h_{ix}} \binom{h_{ix}-1}{s_i^j} \right] \\ &\times \frac{\Gamma(m_x+a_1-p)}{\left( \sum_{i=1}^{m_x} (m_x-h_{ix}+s_i^j+a_2+1)x_{(h_{ix};m_x)j} \right)^{m_x+a_1-p}} \end{aligned} \right] \right. \right. \right. \\ \left. \left. \left. \begin{aligned} &\times \left[ \prod_{i=1}^{m_x} (-1)^{s_i^j} h_{ix} \binom{m_x}{h_{ix}} \binom{h_{ix}-1}{s_i^j} \right] \\ &\times \frac{\Gamma(m_x+a_1)}{\left( \sum_{i=1}^{m_x} (m_x-h_{ix}+s_i^j+a_2+1)x_{(h_{ix};m_x)j} \right)^{m_x+a_1}} \end{aligned} \right] \right. \right. \right. \\ \hat{\lambda}_2 = (E(\lambda^{-p}))^{\frac{-1}{p}}$$

$$= \frac{\left[ \sum_{j=1}^{r_y} \left[ \sum_{s_1^j=0}^{h_{1y}-1} \sum_{s_2^j=0}^{h_{2y}-1} \dots \sum_{s_n^j=0}^{h_{my}-1} \right. \right. \right.}{\left. \left. \left. \sum_{j=1}^{r_y} \left[ \sum_{s_1^j=0}^{h_{1y}-1} \sum_{s_2^j=0}^{h_{2y}-1} \dots \sum_{s_n^j=0}^{h_{my}-1} \right. \right. \right. \right.}^{\frac{-1}{p}}$$

$$\left. \left. \left. \begin{aligned} &\times \left[ \prod_{i=1}^{m_y} (-1)^{s_i^j} h_{iy} \binom{m_y}{h_{iy}} \binom{h_{iy}-1}{s_i^j} \right] \\ &\times \frac{\Gamma(m_y+\lambda_1-p)}{\left( \sum_{i=1}^{m_y} (m_y-h_{iy}+s_i^j+\lambda_2+1)y_{(h_{iy};m_y)j} \right)^{m_y+\lambda_1-p}} \end{aligned} \right] \right. \right. \right. \\ \left. \left. \left. \begin{aligned} &\times \left[ \prod_{i=1}^{m_y} (-1)^{s_i^j} h_{iy} \binom{m_y}{h_{iy}} \binom{h_{iy}-1}{s_i^j} \right] \\ &\times \frac{\Gamma(m_y+\lambda_1)}{\left( \sum_{i=1}^{m_y} (m_y-h_{iy}+s_i^j+\lambda_2+1)y_{(h_{iy};m_y)j} \right)^{m_y+\lambda_1}} \end{aligned} \right] \right. \right. \right.$$

Then the Bayesian estimator of  $\mathcal{R}$  under GEL given by:

$$\widehat{\mathcal{R}}_2 = \frac{\widehat{\mu}_2}{\widehat{\mu}_2 + \widehat{\lambda}_2}.$$

Under LINEX loss function, the Bayesian estimates of  $\mu$  and  $\lambda$  can be described respectively as:

$$\widehat{\mu}_3 = \frac{-1}{p} \ln(E(e^{-p\mu}))$$

$$= \frac{-1}{p} \ln \left[ \frac{\sum_{j=1}^{r_x} \left[ \sum_{s_1^j=0}^{h_{1x}-1} \sum_{s_2^j=0}^{h_{2x}-1} \dots \sum_{s_{m_x}^j=0}^{h_{m_x}-1} \right. \right. \\ \times \left. \left. \left[ \prod_{i=1}^{m_x} (-1)^{s_i^j} h_{ix} \binom{m_x}{h_{ix}} \binom{h_{ix}-1}{s_i^j} \right] \right. \right. \\ \times \left. \left. \frac{\Gamma(m_x+a_1)}{\left( \sum_{i=1}^{m_x} (m_x-h_{ix}+s_i^j+a_2+1)x_{(h_{ix}:m_x)_j}+p \right)^{m_x+a_1}} \right]}{\sum_{j=1}^{r_x} \left[ \sum_{s_1^j=0}^{h_{1x}-1} \sum_{s_2^j=0}^{h_{2x}-1} \dots \sum_{s_{m_x}^j=0}^{h_{m_x}-1} \right. \right. \\ \times \left. \left. \left[ \prod_{i=1}^{m_x} (-1)^{s_i^j} h_{ix} \binom{m_x}{h_{ix}} \binom{h_{ix}-1}{s_i^j} \right] \right. \right. \\ \times \left. \left. \frac{\Gamma(m_x+a_1)}{\left( \sum_{i=1}^{m_x} (m_x-h_{ix}+s_i^j+a_2+1)x_{(h_{ix}:m_x)_j} \right)^{m_x+a_1}} \right]} \right].$$

$$\widehat{\lambda}_3 = \frac{-1}{p} \ln(E(e^{-p\lambda}))$$

$$= \frac{-1}{p} \ln \left[ \frac{\sum_{j=1}^{r_y} \left[ \sum_{s_1^j=0}^{h_{1y}-1} \sum_{s_2^j=0}^{h_{2y}-1} \dots \sum_{s_{m_y}^j=0}^{h_{m_y}-1} \right. \right. \\ \times \left. \left. \left[ \prod_{i=1}^{m_y} (-1)^{s_i^j} h_{iy} \binom{m_y}{h_{iy}} \binom{h_{iy}-1}{s_i^j} \right] \right. \right. \\ \times \left. \left. \frac{\Gamma(m_y+\lambda_1)}{\left( \sum_{i=1}^{m_y} (m_y-h_{iy}+s_i^j+\lambda_2+1)y_{(h_{iy}:m_y)_j}+p \right)^{m_y+\lambda_1}} \right]}{\sum_{j=1}^{r_y} \left[ \sum_{s_1^j=0}^{h_{1y}-1} \sum_{s_2^j=0}^{h_{2y}-1} \dots \sum_{s_{m_y}^j=0}^{h_{m_y}-1} \right. \right. \\ \times \left. \left. \left[ \prod_{i=1}^{m_y} (-1)^{s_i^j} h_{iy} \binom{m_y}{h_{iy}} \binom{h_{iy}-1}{s_i^j} \right] \right. \right. \\ \times \left. \left. \frac{\Gamma(m_y+\lambda_1)}{\left( \sum_{i=1}^{m_y} (m_y-h_{iy}+s_i^j+\lambda_2+1)y_{(h_{iy}:m_y)_j} \right)^{m_y+\lambda_1}} \right]} \right].$$

Hence the Bayesian estimator of  $\mathcal{R}$  under LINEX loss function given by:

$$\widehat{\mathcal{R}}_3 = \frac{\widehat{\mu}_3}{\widehat{\mu}_3 + \widehat{\lambda}_3}.$$

Note that by setting  $a_1 = a_2 = \lambda_1 = \lambda_2 = 0$ , the  $\widehat{\mathcal{R}}_2$  and  $\widehat{\mathcal{R}}_3$  are then corresponding to the Jeffreys prior.

### 3 Estimators of $\mathcal{R}$ Under VLRSS

In this section, the MLE and the Bayes estimators of  $\mathcal{R}$  under VLRSS data are considered.

#### 3.1 MLE Estimators of $\mathcal{R}$

Let  $X_{i(t_{ix}: m'_{ix})j}$  and  $Y_{i(t_{iy}: m'_{iy})j}$  denote a perfect VLRSS drawn from the exponential distribution, where:

$$t_{iz} = \begin{cases} v & 1 \leq i \leq k+1 \\ m_1 - v + 1 & k+2 \leq i \leq 2k \\ i - k & 2k+1 \leq i \leq m_z \end{cases}.$$

and

$$m'_{iz} = \begin{cases} m_{1z} & 1 \leq i \leq 2k \\ m_z & 2k+1 \leq i \leq m_z \end{cases}.$$

Assuming the ranking is perfect, the pdf of  $X_{i(t_{ix}: m'_{ix})j}$  and  $Y_{i(t_{iy}: m'_{iy})j}$  are respectively given by:

$$\begin{aligned} w_x(x_{i(t_{ix}: m'_{ix})j}) &= \frac{m'_{ix}!}{(t_{ix} - 1)!(m'_{ix} - t_{ix})!} [F(x_{i(t_{ix}: m'_{ix})j}, \mu)]^{t_{ix}-1} \\ &\times [1 - F(x_{i(t_{ix}: m'_{ix})j}, \mu)]^{m'_{ix} - t_{ix}} f(x_{i(t_{ix}: m'_{ix})j}, \mu) \end{aligned} \quad (6)$$

and

$$\begin{aligned} w_y(y_{i(t_{iy}: m'_{iy})j}) &= \frac{m'_{iy}!}{(t_{iy} - 1)!(m'_{iy} - t_{iy})!} [F(y_{i(t_{iy}: m'_{iy})j}, \lambda)]^{t_{iy}-1} \\ &\times [1 - F(y_{i(t_{iy}: m'_{iy})j}, \lambda)]^{m'_{iy} - t_{iy}} f(y_{i(t_{iy}: m'_{iy})j}, \lambda). \end{aligned} \quad (7)$$

Using (6) and (7), the likelihood function  $L_2(\mu, \lambda)$  can be expressed as:

$$\begin{aligned}
 L_2(\mu, \lambda) &= \left[ \prod_{j=1}^{r_x} \prod_{i=1}^{m_x} w_x(x_{i(t_{ix}:m'_{ix})j}) \right] \times \left[ \prod_{j=1}^{r_y} \prod_{i=1}^{m_y} w_y(y_{i(t_{iy}:m'_{iy})j}) \right] \cdot \\
 &= (\mu^{n_x} \lambda^{n_y}) \left( \left[ \prod_{j=1}^{r_x} \prod_{i=1}^{m_x} \frac{m'_{ix}!}{(t_{ix}-1)!(m'_{ix}-t_{ix})!} \right. \right. \\
 &\quad \left. \left. \times (1 - e^{-\mu x_{(t_{ix}:m'_{ix})j}})^{t_{ix}-1} e^{-(m'_{ix}-t_{ix}+1)\mu x_{(t_{ix}:m'_{ix})j}} \right] \right. \\
 &\quad \left. \times \left[ \prod_{j=1}^{r_y} \prod_{i=1}^{m_y} \frac{m'_{iy}!}{(t_{iy}-1)!(m'_{iy}-t_{iy})!} (1 - e^{-\lambda y_{(t_{iy}:m'_{iy})j}})^{t_{iy}-1} \right. \right. \\
 &\quad \left. \left. \times e^{-(m'_{iy}-t_{iy}+1)\lambda y_{(t_{iy}:m'_{iy})j}} \right] \right).
 \end{aligned}$$

The MLEs of  $\mu$  and  $\lambda$  can be obtained by solving the following likelihood equations:

$$\begin{aligned}
 \frac{\partial \ln(L_2(\mu, \lambda))}{\partial \mu} &= \frac{n_x}{\mu} - \sum_{j=1}^{r_x} \sum_{i=1}^{m_x} (m'_{ix} - t_{ix} + 1) \mu x_{(t_{ix}:m'_{ix})j} \\
 &\quad + \sum_{j=1}^{r_x} \sum_{i=1}^{m_x} \frac{x_{(t_{ix}:m'_{ix})j} e^{-\mu x_{(t_{ix}:m'_{ix})j}} (t_{ix} - 1)}{e^{-\mu x_{(t_{ix}:m'_{ix})j}} - 1} = 0 \\
 \frac{\partial \ln(L_2(\mu, \lambda))}{\partial \lambda} &= \frac{n_y}{\lambda} - \sum_{j=1}^{r_y} \sum_{i=1}^{m_y} (m'_{iy} - t_{iy} + 1) \lambda y_{(t_{iy}:m'_{iy})j} \\
 &\quad + \sum_{j=1}^{r_y} \sum_{i=1}^{m_y} \frac{y_{(t_{iy}:m'_{iy})j} e^{-\lambda y_{(t_{iy}:m'_{iy})j}} (t_{iy} - 1)}{e^{-\lambda y_{(t_{iy}:m'_{iy})j}} - 1} = 0
 \end{aligned}$$

The MLEs of  $\mu$  and  $\lambda$  based on VLRSS are denoted by  $\hat{\mu}_4$  and  $\hat{\lambda}_4$  respectively. By using the invariance property of the MLEs, the MLE of  $\mathcal{R}$

under VLRSS is given by:

$$\widehat{\mathcal{R}}_4 = \frac{\widehat{\mu}_4}{\widehat{\mu}_4 + \widehat{\lambda}_4}.$$

**Proposition 1:** For any a given VLRSS scheme,  $\widehat{R}_4$  has existence and uniqueness.

**Proof:** It is clear that when  $\mu \rightarrow 0$ , the first and the third terms of  $\frac{\partial \ln(L_2(\mu, \lambda))}{\partial \mu}$  tend to  $\infty$  while the second term tend to 0 yielding  $\frac{\partial \ln(L_2(\mu, \lambda))}{\partial \mu}$  to be positive. On the other hand, as  $\mu \rightarrow \infty$ , the second and the third terms of  $\frac{\partial \ln(L_2(\mu, \lambda))}{\partial \mu}$  tend to  $-\infty$  while the first term tend to 0 yielding  $\frac{\partial \ln(L_2(\mu, \lambda))}{\partial \mu}$  to be negative. By using the first derivative test, see Sinha et al. (2016), this is to say that  $\frac{\partial \ln(L_2(\mu, \lambda))}{\partial \mu}$  has exactly one root which is  $\widehat{\mu}_4$ . Likewise, the same argument can also be extended in the case of  $\frac{\partial \ln(L_2(\mu, \lambda))}{\partial \lambda}$ . The proposition is proved.

In order to get the asymptotic distribution of  $\widehat{\mathcal{R}}_4$ , we have to derive the fisher information matrix of  $I(\widehat{\mu}_4, \widehat{\lambda}_4)$  expressed as:

$$I(\widehat{\mu}_4, \widehat{\lambda}_4) = \begin{bmatrix} I_{11} & 0 \\ 0 & I_{22} \end{bmatrix},$$

where:

$$\begin{aligned} I_{11} &= -E \left( \frac{\partial^2 \ln(L_2(\mu, \lambda))}{\partial \mu^2} \right) \\ &= -E \left( \frac{-n_x}{\mu^2} - \sum_{j=1}^{r_x} \sum_{i=1}^{m_x} (m'_{ix} - t_{ix} + 1) x_{(t_{ix}:m'_{ix})j} \right. \\ &\quad + \sum_{j=1}^{r_x} \sum_{i=1}^{m_x} \frac{x_{(t_{ix}:m'_{ix})j}^2 e^{-2\mu x_{(t_{ix}:m'_{ix})j}} (t_{ix} - 1)}{(1 - e^{-\mu x_{(t_{ix}:m'_{ix})j}})^2} \\ &\quad \left. + \frac{x_{(t_{ix}:m'_{ix})j}^2 e^{-\mu x_{(t_{ix}:m'_{ix})j}} (t_{ix} - 1)}{1 - e^{-\mu x_{(t_{ix}:m'_{ix})j}}} \right) \\ &= \frac{n_x}{\mu^2} + F_{1x}(\mu) - F_{2x}(\mu) - F_{3x}(\mu), \end{aligned}$$

where:

$$\begin{aligned}
 F_{1x}(\mu) &= E \left( \sum_{j=1}^{r_x} \sum_{i=1}^{m_x} (m'_{ix} - t_{ix} + 1) x_{(t_{ix}:m'_{ix})j} \right) \\
 &= \sum_{j=1}^{r_x} \sum_{i=1}^{m_x} (m'_{ix} - t_{ix} + 1) E(x_{(t_{ix}:m'_{ix})j}). \quad (8)
 \end{aligned}$$

Since:

$$\begin{aligned}
 E(x_{(t_{ix}:m'_{ix})j}) &= \int_0^\infty x_{(t_{ix}:m'_{ix})j} w_x(x_{(t_{ix}:m'_{ix})j}) dx_{(t_{ix}:m'_{ix})j} \\
 &= \int_0^\infty x_{(t_{ix}:m'_{ix})j} \frac{m'_{ix}!}{(t_{ix} - 1)!(m'_{ix} - t_{ix})!} \\
 &\quad \times (1 - e^{-\mu x_{(t_{ix}:m'_{ix})j}})^{t_{ix}-1} \\
 &\quad \times e^{-(m'_{ix}-t_{ix}+1)\mu x_{(t_{ix}:m'_{ix})j}} dx_{(t_{ix}:m'_{ix})j}. \\
 &= \frac{m'_{ix}!}{(t_{ix} - 1)!(m'_{ix} - t_{ix})!} \sum_{s=0}^{t_{ix}-1} \binom{t_{ix} - 1}{s} (-1)^s \\
 &\quad \times \int_0^\infty x_{(t_{ix}:m'_{ix})j} \mu e^{-(m'_{ix}-t_{ix}+1+s)\mu x_{(t_{ix}:m'_{ix})j}} dx_{(t_{ix}:m'_{ix})j}. \\
 &= \frac{m'_{ix}!}{(t_{ix} - 1)!(m'_{ix} - t_{ix})!} \sum_{s=0}^{t_{ix}-1} \binom{t_{ix} - 1}{s} \\
 &\quad \times \frac{(-1)^s}{(m'_{ix} - t_{ix} + 1 + s)^2 \mu}. \quad (9)
 \end{aligned}$$

By substituting (9) in (8), we will get:

$$\begin{aligned}
 F_{1x}(\mu) &= \sum_{j=1}^{r_x} \sum_{i=1}^{m_x} (m'_{ix} - t_{ix} + 1) \frac{m'_{ix}!}{(t_{ix} - 1)!(m'_{ix} - t_{ix})!} \\
 &\quad \times \sum_{s=0}^{t_{ix}-1} \binom{t_{ix} - 1}{s} \frac{(-1)^s}{(m'_{ix} - t_{ix} + 1 + s)^2 \mu}.
 \end{aligned}$$

By using the same procedure, one can investigate that:

$$F_{2x}(\mu) = E \left( \sum_{j=1}^{r_x} \sum_{i=1}^{m_x} \frac{x_{(t_{ix}:m'_{ix})j}^2 e^{-2\mu x_{(t_{ix}:m'_{ix})j}} (t_{ix} - 1)}{(1 - e^{-\mu x_{(t_{ix}:m'_{ix})j}})^2} \right)$$

$$\begin{aligned}
 &= \sum_{j=1}^{r_x} \sum_{i=1}^{m_x} \frac{m'_{ix}!(t_{ix}-1)}{(t_{ix}-1)!(m'_{ix}-t_{ix})!} \sum_{s=0}^{t_{ix}-3} \binom{t_{ix}-3}{s} \\
 &\quad \times \frac{2(-1)^s}{(m'_{ix}-t_{ix}+3+s)^3 \mu^2} \quad \text{and} \\
 F_{3x}(\mu) &= E \left( \sum_{j=1}^{r_x} \sum_{i=1}^{m_x} \frac{x^2_{(t_{ix}:m'_{ix})j} e^{-\mu x_{(t_{ix}:m'_{ix})j}} (t_{ix}-1)}{1 - e^{-\mu x_{(t_{ix}:m'_{ix})j}}} \right) \\
 &= \sum_{j=1}^{r_x} \sum_{i=1}^{m_x} \frac{m'_{ix}!(t_{ix}-1)}{(t_{ix}-1)!(m'_{ix}-t_{ix})!} \sum_{s=0}^{t_{ix}-2} \binom{t_{ix}-2}{s} \\
 &\quad \times \frac{2(-1)^s}{(m'_{ix}-t_{ix}+2+s)^3 \mu^2}.
 \end{aligned}$$

Similarly,

$$I_{22} = -E \left( \frac{\partial^2 \ln(L_2(\mu, \lambda))}{\partial \lambda^2} \right) = \frac{n_y}{\lambda^2} + F_{1y}(\lambda) - F_{2y}(\lambda) - F_{3y}(\lambda).$$

**Proposition 2:** As  $n_x \rightarrow \infty$  and  $n_y \rightarrow \infty$ , then:

$$(\sqrt{n_x}(\hat{\mu}_4 - \mu), \sqrt{n_y}(\hat{\lambda}_4 - \lambda)) \xrightarrow{d} N \left( 0, \begin{bmatrix} \frac{I_{11}}{n_x} & 0 \\ 0 & \frac{I_{22}}{n_y} \end{bmatrix}^{-1} \right).$$

**Proof.** The proof follows from the asymptotic properties of the ML estimators under regularity conditions.

**Proposition 3:** As  $n_x \rightarrow \infty$  and  $n_y \rightarrow \infty$ , the asymptotic distribution of  $\hat{\mathcal{R}}_4$  is obtained as:

$$\frac{\hat{\mathcal{R}}_4 - \mathcal{R}}{\sqrt{\frac{d_1}{n_x I_{11}} + \frac{d_2}{n_y I_{22}}}} \xrightarrow{d} N(0, 1),$$

where  $d_1 = \frac{\partial \mathcal{R}}{\partial \mu} = \frac{\lambda}{(\mu+\lambda)^2}$  and  $d_2 = \frac{\partial \mathcal{R}}{\partial \lambda} = -\frac{\mu}{(\mu+\lambda)^2}$ .

**Proof.** The proof follows from proposition 2 and the delta method.

### 3.2 Bayesian Estimators of $\mathcal{R}$

Here the unknown parameter  $\mathcal{R}$  is estimated under Bayesian set-up based on VLRSS. Similar to what is done in the preceding section,  $L_2(\mu, \lambda)$  can be rewritten as:

$$\begin{aligned}
 L_2(\mu, \lambda) &= (\mu^{n_x} \lambda^{n_y}) \left( \left[ \prod_{j=1}^{r_x} \prod_{i=1}^{m_x} \frac{m'_{ix}!}{(t_{ix} - 1)!(m'_{ix} - t_{ix})!} \right. \right. \\
 &\quad \left. \left. \times (1 - e^{-\mu x_{(t_{ix}:m'_{ix})j}}) t_{ix}^{-1} e^{-(m'_{ix} - t_{ix} + 1)\mu x_{(t_{ix}:m'_{ix})j}} \right] \right. \\
 &\quad \left. \times \left[ \prod_{j=1}^{r_y} \prod_{i=1}^{m_y} \frac{m'_{iy}!}{(t_{iy} - 1)!(m'_{iy} - t_{iy})!} (1 - e^{-\lambda y_{(t_{iy}:m'_{iy})j}}) t_{iy}^{-1} \right. \right. \\
 &\quad \left. \left. \times e^{-(m'_{iy} - t_{iy} + 1)\lambda y_{(t_{iy}:m'_{iy})j}} \right] \right). \\
 &= \left[ (\mu^{n_x}) \sum_{j=1}^{r_x} \left[ \sum_{s_1^j=0}^{t_{1x}-1} \sum_{s_2^j=0}^{t_{2x}-1} \cdots \sum_{s_{m_x}^j=0}^{t_{m_x}-1} \right. \right. \\
 &\quad \left. \left. \times \left[ \prod_{i=1}^{m_x} (-1)^{s_i^j} t_{ix} \binom{m'_{ix}}{t_{ix}} \binom{t_{ix} - 1}{s_i^j} \right] \right. \right. \\
 &\quad \left. \left. \times e^{-\mu \sum_{i=1}^{m_x} (m'_{ix} - t_{ix} + s_i^j + 1) x_{(t_{ix}:m'_{ix})j}} \right] \right] \\
 &\quad \times \left[ (\lambda^{n_y}) \sum_{j=1}^{r_y} \left[ \sum_{s_1^j=0}^{t_{1y}-1} \sum_{s_2^j=0}^{t_{2y}-1} \cdots \sum_{s_{m_y}^j=0}^{t_{m_y}-1} \right. \right. \\
 &\quad \left. \left. \times \left[ \prod_{i=1}^{m_y} (-1)^{s_i^j} t_{iy} \binom{m'_{iy}}{t_{iy}} \binom{t_{iy} - 1}{s_i^j} \right] \right. \right. \\
 &\quad \left. \left. \times e^{-\lambda \sum_{i=1}^{m_y} (m'_{iy} - t_{iy} + s_i^j + 1) y_{(t_{iy}:m'_{iy})j}} \right] \right].
 \end{aligned}$$



Consequently, the posterior density can be obtained as:

$$\pi(\mu, \lambda | X, Y) = \frac{\pi(\mu, \lambda)L_2(\mu, \lambda)}{\iint \pi(\mu, \lambda)L_2(\mu, \lambda)d\mu d\lambda}$$

$$= \left[ \frac{\sum_{j=1}^{r_x} \left[ \sum_{s_1^j=0}^{t_{1x}-1} \sum_{s_2^j=0}^{t_{2x}-1} \dots \sum_{s_{m_x}^j=0}^{t_{m_x}-1} \right. \right. \\ \times \left. \left[ \prod_{i=1}^{m_x} (-1)^{s_i^j} t_{ix} \binom{m'_{ix}}{t_{ix}} \binom{t_{ix}-1}{s_i^j} \right] (\mu^{m_x+a_1-1}) \right. \\ \left. \times e^{-\mu \sum_{i=1}^{m_x} (m'_{ix}-t_{ix}+s_i^j+a_2+1)x_{(t_{ix}:m'_{ix})j}} \right]}{\sum_{j=1}^{r_x} \left[ \sum_{s_1^j=0}^{t_{1x}-1} \sum_{s_2^j=0}^{t_{2x}-1} \dots \sum_{s_{m_x}^j=0}^{t_{m_x}-1} \right. \\ \times \left[ \prod_{i=1}^{m_x} (-1)^{s_i^j} t_{ix} \binom{m'_{ix}}{t_{ix}} \binom{t_{ix}-1}{s_i^j} \right] \\ \times \frac{\Gamma(m_x+a_1)}{\left( \sum_{i=1}^{m_x} (m'_{ix}-t_{ix}+s_i^j+a_2+1)x_{(t_{ix}:m'_{ix})j} \right)^{m_x+a_1}} \left. \right]} \right]$$

$$\times \left[ \frac{\sum_{j=1}^{r_y} \left[ \sum_{s_1^j=0}^{t_{1y}-1} \sum_{s_2^j=0}^{t_{2y}-1} \dots \sum_{s_{m_y}^j=0}^{t_{m_y}-1} \right. \right. \\ \times \left[ \prod_{i=1}^{m_y} (-1)^{s_i^j} t_{iy} \binom{m'_{iy}}{t_{iy}} \binom{t_{iy}-1}{s_i^j} \right] \\ \times (\lambda^{m_y+\lambda_1-1}) \\ \left. \times e^{-\lambda \sum_{i=1}^{m_y} (m'_{iy}-t_{iy}+s_i^j+\lambda_2+1)y_{(t_{iy}:m'_{iy})j}} \right]}{\sum_{j=1}^{r_y} \left[ \sum_{s_1^j=0}^{t_{1y}-1} \sum_{s_2^j=0}^{t_{2y}-1} \dots \sum_{s_{m_y}^j=0}^{t_{m_y}-1} \right. \\ \times \left[ \prod_{i=1}^{m_y} (-1)^{s_i^j} t_{iy} \binom{m'_{iy}}{t_{iy}} \binom{t_{iy}-1}{s_i^j} \right] \\ \times \frac{\Gamma(m_y+\lambda_1)}{\left( \sum_{i=1}^{m_y} (m'_{iy}-t_{iy}+s_i^j+\lambda_2+1)y_{(t_{iy}:m'_{iy})j} \right)^{m_y+\lambda_1}} \left. \right]} \right].$$

The Bayesian estimates of  $\mu$  and  $\lambda$  based on GEL function can be expressed respectively as:

$$\hat{\mu}_5 = (E(\mu^{-p}))^{-\frac{1}{p}}$$

$$= \frac{\left[ \sum_{j=1}^{r_x} \left[ \sum_{s_1^j=0}^{t_{1x}-1} \sum_{s_2^j=0}^{t_{2x}-1} \dots \sum_{s_n^j=0}^{t_{mx}-1} \right. \right. \\ \times \left. \left[ \prod_{i=1}^{m_x} (-1)^{s_i^j} t_{ix} \binom{m'_{ix}}{t_{ix}} \binom{t_{ix}-1}{s_i^j} \right] \right. \\ \times \left. \frac{\Gamma(m_x+a_1-p)}{\left( \sum_{i=1}^{m_x} (m'_{ix}-t_{ix}+s_i^j+a_2+1)x_{(t_{ix}:m'_{ix})j} \right)^{m_x+a_1-p}} \right]}{\left[ \sum_{j=1}^{r_x} \left[ \sum_{s_1^j=0}^{t_{1x}-1} \sum_{s_2^j=0}^{t_{2x}-1} \dots \sum_{s_n^j=0}^{t_{mx}-1} \right. \right. \\ \times \left. \left[ \prod_{i=1}^{m_x} (-1)^{s_i^j} t_{ix} \binom{m'_{ix}}{t_{ix}} \binom{t_{ix}-1}{s_i^j} \right] \right. \\ \times \left. \frac{\Gamma(m_x+a_1)}{\left( \sum_{i=1}^{m_x} (m'_{ix}-t_{ix}+s_i^j+a_2+1)x_{(t_{ix}:m'_{ix})j} \right)^{m_x+a_1}} \right]} \\ \widehat{\lambda}_5 = (E(\lambda^{-p}))^{\frac{-1}{p}}$$

$$= \frac{\left[ \sum_{j=1}^{r_y} \left[ \sum_{s_1^j=0}^{t_{1y}-1} \sum_{s_2^j=0}^{t_{2y}-1} \dots \sum_{s_n^j=0}^{t_{my}-1} \right. \right. \\ \times \left. \left[ \prod_{i=1}^{m_y} (-1)^{s_i^j} t_{iy} \binom{m'_{iy}}{t_{iy}} \binom{t_{iy}-1}{s_i^j} \right] \right. \\ \times \left. \frac{\Gamma(m_y+\lambda_1-p)}{\left( \sum_{i=1}^{m_y} (m'_{iy}-t_{iy}+s_i^j+\lambda_2+1)y_{(t_{iy}:m'_{iy})j} \right)^{m_y+\lambda_1-p}} \right]}{\left[ \sum_{j=1}^{r_y} \left[ \sum_{s_1^j=0}^{t_{1y}-1} \sum_{s_2^j=0}^{t_{2y}-1} \dots \sum_{s_n^j=0}^{t_{my}-1} \right. \right. \\ \times \left. \left[ \prod_{i=1}^{m_y} (-1)^{s_i^j} t_{iy} \binom{m'_{iy}}{t_{iy}} \binom{t_{iy}-1}{s_i^j} \right] \right. \\ \times \left. \frac{\Gamma(m_y+\lambda_1)}{\left( \sum_{i=1}^{m_y} (m'_{iy}-t_{iy}+s_i^j+\lambda_2+1)y_{(t_{iy}:m'_{iy})j} \right)^{m_y+\lambda_1}} \right]} \\ \widehat{\lambda}_5 = (E(\lambda^{-p}))^{\frac{-1}{p}}$$

Then the Bayesian estimator of  $\mathcal{R}$  under GEL given by:

$$\widehat{\mathcal{R}}_5 = \frac{\widehat{\mu}_5}{\widehat{\mu}_5 + \widehat{\lambda}_5}.$$

Under LINEX loss function, the Bayesian estimates of  $\mu$  and  $\lambda$  can be described respectively as:

$$\hat{\mu}_6 = \frac{-1}{p} \ln(E(e^{-p\mu}))$$

$$= \frac{-1}{p} \ln \left[ \frac{\sum_{j=1}^{r_x} \left[ \sum_{s_1^j=0}^{t_{1x}-1} \sum_{s_2^j=0}^{t_{2x}-1} \dots \sum_{s_{m_x}^j=0}^{t_{m_x}-1} \right. \right. \\ \times \left. \left. \left[ \prod_{i=1}^{m_x} (-1)^{s_i^j} t_{ix} \binom{m'_{ix}}{t_{ix}} \binom{t_{ix}-1}{s_i^j} \right] \right. \right. \\ \times \left. \left. \frac{\Gamma(m_x+a_1)}{\left( \sum_{i=1}^{m_x} (m'_{ix} - t_{ix} + s_i^j + a_2 + 1) x_{(t_{ix}:m'_{ix})j} + p \right)^{m_x+a_1}} \right]}{\sum_{j=1}^{r_x} \left[ \sum_{s_1^j=0}^{t_{1x}-1} \sum_{s_2^j=0}^{t_{2x}-1} \dots \sum_{s_{m_x}^j=0}^{t_{m_x}-1} \right. \right. \\ \times \left. \left. \left[ \prod_{i=1}^{m_x} (-1)^{s_i^j} t_{ix} \binom{m'_{ix}}{t_{ix}} \binom{t_{ix}-1}{s_i^j} \right] \right. \right. \\ \times \left. \left. \frac{\Gamma(m_x+a_1)}{\left( \sum_{i=1}^{m_x} (m'_{ix} - t_{ix} + s_i^j + a_2 + 1) x_{(t_{ix}:m'_{ix})j} \right)^{m_x+a_1}} \right]} \right].$$

$$\hat{\lambda}_6 = \frac{-1}{p} \ln(E(e^{-p\lambda}))$$

$$= \frac{-1}{p} \ln \left[ \frac{\sum_{j=1}^{r_y} \left[ \sum_{s_1^j=0}^{h_{1y}-1} \sum_{s_2^j=0}^{h_{2y}-1} \dots \sum_{s_n^j=0}^{h_{m_y}-1} \right. \right. \\ \times \left. \left. \left[ \prod_{i=1}^{m_y} (-1)^{s_i^j} t_{iy} \binom{m'_{iy}}{t_{iy}} \binom{t_{iy}-1}{s_i^j} \right] \right. \right. \\ \times \left. \left. \frac{\Gamma(m_y+\lambda_1)}{\left( \sum_{i=1}^{m_y} (m'_{iy} - t_{iy} + s_i^j + \lambda_2 + 1) y_{(t_{iy}:m'_{iy})j} + p \right)^{m_y+\lambda_1}} \right]}{\sum_{j=1}^{r_y} \left[ \sum_{s_1^j=0}^{h_{1y}-1} \sum_{s_2^j=0}^{h_{2y}-1} \dots \sum_{s_n^j=0}^{h_{m_y}-1} \right. \right. \\ \times \left. \left. \left[ \prod_{i=1}^{m_y} (-1)^{s_i^j} t_{iy} \binom{m'_{iy}}{t_{iy}} \binom{t_{iy}-1}{s_i^j} \right] \right. \right. \\ \times \left. \left. \frac{\Gamma(m_y+\lambda_1)}{\left( \sum_{i=1}^{m_y} (m'_{iy} - t_{iy} + s_i^j + \lambda_2 + 1) y_{(t_{iy}:m'_{iy})j} \right)^{m_y+\lambda_1}} \right]} \right].$$

Hence the Bayesian estimator of  $\mathcal{R}$  under LINEX loss function given by:

$$\widehat{\mathcal{R}}_6 = \frac{\widehat{\mu}_6}{\widehat{\mu}_6 + \widehat{\lambda}_6}.$$

#### 4 Comparison of the Proposed Estimators

In this part, a comparison study between the suggested estimators for  $\mathcal{R}$  based on VLRSS and their LRSS analogues is made under varying sample sizes, different values of  $\mathcal{R}$  and several quality ranking configurations. Two levels of the set sizes and the cycle sizes are considered:  $m_x = m_y = m \in \{3, 5\}$  and  $r_x = r_y = r \in \{5, 10\}$  which enables us to assess the effect of the set size as well as the sample size. For studying the effect of  $m_1, k$  and  $v$ , we considered  $m_1 \in \{3, 4, \dots, 10\}$  and  $(k, v) \in \{1, 2\}$ . We set  $\mu = 1$  and  $\lambda$  is selected such that  $\mathcal{R} \in \{0.25, 0.5, 0.75\}$ . Aiming better analyzing the simulated results, Dell and Clutter (1972)'s imperfect ranking model with correlation coefficient  $\rho$  is considered to take the effect of the quality of ranking process into account. In this case, a sample from the bivariate normal distribution with linear correlation  $\rho$  is first generated. One of these variables is considered as the variable of interest and then it is transformed to the exponential distribution, while the other one is taken as a concomitant variable and it is used for ranking process. Three different configurations of  $\rho$  are considered:  $\rho = 1$  for perfect ranking,  $\rho = 0.9$  for imperfect ranking with reasonable good accuracy, and  $\rho = 0.5$  for imperfect ranking. We compute the relative efficiency (RE) measure as the comparison characteristic in the light of the following formula:

$$RE_i = \frac{MSE(\widehat{\mathcal{R}}_i)}{MSE(\widehat{\mathcal{R}}_{3+i})}, \quad i \in \{1, 2, 3\}.$$

With the above definition, a  $RE_i$  larger (less) than one implies that  $\widehat{\mathcal{R}}_{3+i}$  asymptotically more (less) efficient than  $\widehat{\mathcal{R}}_i$ . Under Bayesian set up, we let hyper-parameters  $a_1 = a_2 = \lambda_1 = \lambda_2 = 0.01$  and also  $p = 1$ . For each combination of  $m, r, m_1, k, v, \rho$  and  $\mathcal{R}$ , 5,000 samples are generated based on LRSS, and VLRSS schemes. Finally, for each generated sample the aforesaid estimators are computed, then the  $REs$  are estimated and listed in Tables 1–3.

It can be noted that the  $REs$  are increasing as the values of either  $m_1$  or  $v$  are increasing provided that the quality of ranking process is not weak. In contrast, in the case of the imperfect ranking, this pattern is generally

**Table 1** The estimates of REs at  $\rho = 1$

$RE_i$	$\mathcal{R}$	$(m, r)$	$M_1$																
			3	4	5	6	7	8	9	10									
			1	1	1	2	1	2	1	2	3	1	2	3	1	2	3		
			$k = 1$																
$RE_1$	0.25	(3,5)	1.17	1.23	1.44	2.01	1.47	2.12	1.50	2.27	1.55	2.31	2.39	1.61	2.41	2.51	1.65	2.58	2.66
		(3,10)	1.15	1.43	1.48	1.54	1.59	1.78	1.66	1.85	1.71	1.92	2.01	1.75	2.07	2.11	1.76	2.15	2.17
		(5,5)	0.82	0.86	0.90	1.01	0.94	0.74	0.96	1.03	0.99	1.04	1.13	1.01	1.04	1.24	1.03	1.05	1.45
		(5,10)	0.87	0.87	0.92	0.97	0.87	0.87	0.89	1.01	0.89	1.01	1.10	0.97	1.05	1.23	1.04	1.06	1.35
	0.50	(3,5)	1.04	1.11	1.34	1.39	1.59	1.61	1.65	1.74	1.71	2.10	2.01	1.79	2.21	2.22	1.87	2.35	2.41
		(3,10)	1.12	1.44	1.50	1.18	1.58	1.28	1.61	1.37	1.70	1.72	1.80	1.77	1.88	1.99	1.81	2.01	2.15
		(5,5)	0.89	0.87	0.92	1.03	0.99	1.06	1.02	1.14	1.04	1.20	1.21	1.05	1.30	1.34	1.08	1.352	1.39
		(5,10)	0.88	0.89	0.96	1.02	0.99	1.13	1.01	1.25	1.03	1.25	1.19	1.03	1.26	1.30	1.09	1.40	1.35
	0.75	(3,5)	0.90	1.21	1.41	1.66	1.54	1.78	1.63	1.81	1.77	1.99	2.01	1.81	2.16	2.22	1.84	2.40	2.51
		(3,10)	0.98	1.33	1.41	1.45	1.49	1.52	1.53	1.62	1.68	1.78	1.87	1.75	1.89	2.01	1.80	2.03	2.22
		(5,5)	0.91	0.92	0.95	1.12	0.99	1.17	1.02	1.19	1.02	1.22	1.34	1.02	1.23	1.38	1.01	1.24	1.42
		(5,10)	0.79	0.93	0.98	1.11	1.01	1.12	1.02	1.18	1.03	1.25	1.36	1.04	1.26	1.40	1.03	1.31	1.45
$RE_2$	0.25	(3,5)	1.08	1.46	2.01	1.99	2.09	2.13	2.16	2.36	2.30	2.63	2.66	2.40	2.81	2.89	2.48	2.99	3.10
		(3,10)	1.03	1.35	1.51	1.23	1.61	1.34	1.99	1.76	2.23	2.65	2.70	2.43	2.98	2.96	2.81	3.20	3.65
		(5,5)	0.76	0.79	0.94	0.97	1.11	0.99	1.12	1.11	1.14	1.13	1.17	1.16	1.14	1.18	1.14	1.15	1.20
		(5,10)	0.79	0.82	0.96	0.89	1.08	1.02	1.22	1.13	1.15	1.13	1.17	1.18	1.21	1.21	1.20	1.24	1.26
	0.50	(3,5)	1.26	1.35	1.43	1.69	1.58	1.81	1.94	1.90	2.10	2.15	2.33	2.54	2.49	2.61	2.91	2.99	3.11
		(3,10)	1.19	1.11	1.76	1.54	1.89	1.98	2.21	2.42	2.65	2.78	2.98	2.87	2.91	3.10	3.01	3.01	3.23
		(5,5)	0.91	0.93	0.98	1.06	1.11	1.12	1.13	1.15	1.15	1.16	1.03	1.17	1.20	1.05	1.18	1.21	1.11
		(5,10)	0.82	0.92	0.97	1.05	1.13	1.14	1.10	1.16	1.16	1.20	1.02	1.21	1.18	1.04	1.20	1.25	1.13
	0.75	(3,5)	1.07	1.26	1.46	1.71	1.60	2.14	1.74	2.43	1.91	2.65	2.69	2.13	2.87	2.99	2.31	3.21	3.51
		(3,10)	1.21	2.41	2.78	1.73	2.99	3.12	3.20	3.31	3.65	3.81	3.90	3.87	3.99	4.20	4.19	3.92	4.32
		(5,5)	0.94	0.93	1.02	1.03	1.04	1.05	1.08	1.09	1.11	1.13	1.05	1.14	1.17	1.22	1.16	1.22	1.32
		(5,10)	0.89	0.94	1.04	1.08	1.05	1.07	1.07	1.11	1.15	1.09	1.19	1.18	1.23	1.24	1.23	1.29	1.34
$RE_3$	0.25	(3,5)	1.07	1.15	1.54	1.16	1.89	2.18	2.12	2.31	2.25	2.41	2.47	2.37	2.77	2.80	2.42	2.88	2.99
		(3,10)	1.09	1.30	1.80	1.47	1.76	1.65	1.89	1.79	1.99	2.32	2.46	2.25	2.77	2.98	2.54	3.82	4.05
		(5,5)	0.78	0.89	0.87	0.99	1.16	1.06	1.45	1.15	1.64	1.27	1.79	1.75	1.35	1.89	1.85	1.45	2.12

(Continued)

Table 1 Continued

$RE_i$	$\mathcal{R}$	$(m, \tau)$	$M_1$															
			3	4	5	6	7	8	9	10	1	2	3	1	2	3		
0.50	(5,10)	0.84	0.92	0.97	0.99	1.19	1.12	1.54	1.20	1.69	1.32	1.83	1.88	1.43	1.97	1.90	1.54	2.09
		1.21	1.27	1.32	1.57	1.48	1.71	1.65	1.99	1.78	2.17	2.22	2.12	2.30	2.28	2.45	2.39	2.41
		1.13	1.15	1.45	1.50	1.59	1.56	1.71	1.79	1.98	2.12	2.33	2.34	2.69	2.82	2.51	2.91	3.09
		0.94	0.95	1.02	1.03	1.08	1.12	1.11	1.15	1.14	1.19	1.06	1.16	1.23	1.09	1.19	1.27	1.19
		0.86	0.91	1.03	1.04	1.09	1.13	1.10	1.18	1.20	1.25	1.16	1.26	1.29	1.19	1.27	1.37	1.34
		1.08	1.19	1.48	1.61	1.54	1.81	1.68	2.07	2.072	2.39	2.41	2.16	2.69	2.78	2.26	3.25	3.31
		1.23	1.70	2.01	1.75	2.20	2.54	2.54	2.69	2.76	2.84	2.99	2.89	3.32	3.63	3.12	3.85	3.99
		0.88	0.97	1.05	1.08	0.99	1.16	1.09	1.24	1.25	1.33	1.04	1.30	1.45	1.14	1.38	1.61	1.24
		0.918	0.93	1.07	1.10	1.02	1.22	1.12	1.27	1.29	1.39	1.09	1.38	1.49	1.19	1.43	1.69	1.29
		$k = 2$																
0.25	(5,5)	0.68	0.70	0.78	0.88	0.83	0.89	0.85	0.99	1.02	1.05	1.10	1.08	1.14	1.27	1.12	1.20	1.30
		0.79	0.82	0.89	0.95	0.90	0.99	1.03	1.10	1.10	1.10	1.15	1.23	1.18	1.26	1.33	1.29	1.33
		0.59	0.64	0.76	0.83	0.89	0.99	1.01	1.02	1.02	1.06	1.04	1.04	1.09	1.15	1.04	1.14	1.22
		0.63	0.68	0.82	0.91	0.96	0.99	1.03	1.05	1.04	1.10	1.07	1.06	1.12	1.18	1.08	1.23	1.43
		0.52	0.66	0.73	0.81	0.94	0.98	0.96	0.98	0.99	1.02	1.06	1.01	1.05	1.09	1.02	1.06	1.12
		0.59	0.70	0.79	0.85	0.90	0.99	0.99	0.99	1.02	1.04	1.07	1.03	1.07	1.17	1.05	1.10	1.21
		0.61	0.67	0.75	0.89	0.94	1.019	1.13	1.10	1.17	1.21	1.26	1.27	1.33	1.49	1.34	1.45	1.54
		0.67	0.74	0.84	0.94	0.92	1.028	1.26	1.13	1.35	1.24	1.38	1.47	1.36	1.54	1.54	1.61	1.73
		0.55	0.66	0.74	0.88	0.96	1.018	1.04	1.12	1.08	1.25	1.30	1.11	1.27	1.49	1.14	1.35	1.54
		0.52	0.62	0.78	0.82	0.91	1.018	1.06	1.16	1.10	1.31	1.34	1.17	1.31	1.54	1.23	1.42	1.63
0.75	(5,5)	0.57	0.69	0.73	0.88	0.97	0.98	1.07	1.09	1.13	1.19	1.26	1.23	1.32	1.42	1.34	1.40	1.52
		0.51	0.73	0.79	0.85	0.90	0.99	1.09	1.12	1.16	1.22	1.32	1.31	1.39	1.51	1.39	1.52	1.61
		0.54	0.66	0.70	0.81	0.91	0.97	1.10	1.15	1.29	1.32	1.39	1.33	1.49	1.51	1.41	1.56	1.70
		0.58	0.73	0.79	0.87	0.93	0.99	1.14	1.19	1.33	1.39	1.46	1.43	1.50	1.64	1.52	1.66	1.88
		0.58	0.79	0.87	0.93	0.97	1.038	1.01	1.051	1.02	1.09	1.15	1.03	1.21	1.32	1.03	1.27	1.40
		0.59	0.84	0.90	0.98	0.99	0.98	1.02	1.06	1.02	1.13	1.19	1.04	1.28	1.43	1.05	1.36	1.54
		0.63	0.89	0.96	1.01	0.90	0.96	1.05	1.09	1.09	1.14	1.17	1.16	1.29	1.37	1.25	1.43	1.60
		0.55	0.94	0.99	1.00	0.94	0.95	1.10	1.12	1.14	1.18	1.23	1.26	1.32	1.43	1.33	1.49	1.73

**Table 2** The estimates of REs at  $\rho = 0.90$

RE <sub>i</sub>	R	(m, r)	v	M <sub>1</sub>																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																			
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				1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3																																																																																																																																																																																																																																																																																																																																																																																																																																																													
RE <sub>1</sub>	0.25	(3,5)	1.16	1.18	1.19	1.37	1.23	1.43	1.251	1.58	1.26	1.71	1.72	1.27	1.78	1.84	1.28	1.84	1.88	(3,10)	1.25	1.31	1.35	1.45	1.41	1.55	1.44	1.68	1.48	1.86	1.98	1.98	2.10	1.53	2.01	2.19	(5,5)	0.79	0.84	0.89	0.85	0.96	0.94	1.02	0.99	1.05	1.01	1.05	1.07	1.02	1.09	1.10	1.04	1.13	(5,10)	0.65	0.79	0.82	0.89	0.88	0.91	0.95	0.94	0.99	0.97	1.01	1.03	0.99	1.04	1.09	1.03	1.10	(3,5)	1.14	1.17	1.19	1.28	1.21	1.36	1.25	1.44	1.29	1.48	1.58	1.32	1.51	1.66	1.33	1.54	1.70	(3,10)	1.503	1.56	1.60	1.72	1.63	1.48	1.65	1.50	1.69	1.61	1.69	1.76	1.81	1.88	1.78	1.86	1.93	(5,5)	0.73	0.85	0.90	0.91	1.026	0.959	1.06	0.99	1.11	1.03	0.99	1.15	1.04	1.03	1.18	1.05	1.06	(5,10)	0.80	0.90	0.96	0.93	1.04	0.96	1.07	1.02	1.15	1.05	0.97	1.18	1.05	1.01	1.23	1.06	1.04	(3,5)	1.23	1.26	1.32	1.35	1.38	1.422	1.45	1.47	1.53	1.59	1.61	1.64	1.61	1.64	1.66	1.65	1.67	(3,10)	1.26	1.29	1.36	1.39	1.41	1.45	1.49	1.55	1.58	1.61	1.64	1.61	1.64	1.66	1.66	1.66	1.6871	1.703	(5,5)	0.76	0.82	0.92	0.89	0.99	0.92	1.04	0.97	1.09	0.97	0.98	1.16	0.99	1.01	1.22	0.99	1.03	(5,10)	0.79	0.80	0.96	0.90	1.02	0.931	1.06	0.98	1.10	1.01	1.02	1.14	1.01	1.03	1.20	1.02	1.05	(3,5)	1.37	1.55	1.69	1.66	1.74	1.70	1.79	1.75	1.85	1.79	1.88	1.83	1.80	1.80	1.91	1.846	1.86	(3,10)	1.44	1.63	1.76	1.749	1.80	1.75	1.85	1.77	1.90	1.84	1.79	1.97	1.88	1.85	2.11	1.92	1.89	(5,5)	0.83	0.88	0.98	0.82	1.06	0.92	1.17	1.07	1.29	1.12	0.993	1.41	1.15	1.03	1.53	1.17	1.05	(5,10)	0.77	0.86	0.94	0.99	1.02	1.01	1.10	1.04	1.20	1.09	1.01	1.24	1.13	1.04	1.27	1.18	1.07	(3,5)	1.24	1.29	1.35	1.44	1.42	1.54	1.51	1.62	1.60	1.74	1.80	1.64	1.82	1.94	1.71	1.91	2.051	(3,10)	1.33	1.39	1.45	1.47	1.51	1.57	1.54	1.68	1.65	1.78	1.83	1.70	1.89	1.99	1.79	2.04	2.14	(5,5)	0.79	0.84	0.99	0.96	1.03	0.99	1.06	1.01	1.09	1.02	0.97	1.12	1.03	1.00	1.14	1.03	1.02	(5,10)	0.84	0.88	0.99	0.98	1.06	1.01	1.13	1.03	1.17	1.04	0.99	1.207	1.06	1.00	1.27	1.04	1.03	(3,5)	1.42	1.54	1.66	1.58	1.70	1.65	1.84	1.80	1.98	1.89	1.86	2.12	1.89	1.87	2.24	1.930	1.91	(3,10)	1.46	1.50	1.73	1.68	1.80	1.76	1.85	1.84	2.02	1.96	1.90	2.20	2.05	2.00	2.31	2.25	2.072	(5,5)	0.80	0.89	1.02	0.97	1.09	1.01	1.127	1.09	1.16	1.15	1.02	1.1920	1.1707	1.08	1.21	1.20	1.10	(5,10)	0.82	0.95	1.05	0.99	1.11	1.04	1.15	1.07	1.19	1.194	1.04	1.21	1.24	1.06	1.25	1.28	1.08	(3,5)	1.40	1.45	1.51	1.548	1.57	1.603	1.64	1.679	1.69	1.739	1.75	1.71	1.74	1.79	1.75	1.8045	1.82	(3,10)	1.53	1.69	1.76	1.781	1.80	1.84	1.89	1.910	2.03	2.07	2.10	2.16	2.11	2.15	2.17	2.14	2.19	(5,5)	0.80	0.87	0.949	0.92	1.04	0.99	1.15	1.04	1.25	1.072	1.02	1.37	1.10	1.04	1.50	1.12	1.06

(Continued)

Table 2 Continued

$RE_i$	$\mathcal{R}$	$(m, r)$	$M_1$																																																	
			3			4			5			6			7			8			9			10																												
0.50		(5,10)	0.75	0.80	0.92	0.97	1.01	0.99	1.10	1.041	1.22	1.09	0.95	1.30	1.10	0.99	1.02	1.30	1.02	1.03	1.32	0.75	0.81	0.93	0.98	1.02	1.06	1.10	1.14	1.18	1.22	1.26	1.30	1.34	1.38	1.42	1.46	1.50	1.54	1.58												
			1.19	1.26	1.31	1.38	1.43	1.43	1.48	1.54	1.60	1.64	1.70	1.74	1.777	1.84	1.90	1.95	1.99	2.03	2.07	1.19	1.26	1.31	1.36	1.43	1.47	1.51	1.54	1.60	1.64	1.70	1.74	1.777	1.84	1.90	1.95	1.99	2.03	2.07												
			1.26	1.31	1.36	1.39	1.43	1.47	1.51	1.54	1.60	1.64	1.70	1.74	1.777	1.84	1.90	1.95	1.99	2.03	2.07	1.26	1.31	1.36	1.39	1.43	1.47	1.51	1.54	1.60	1.64	1.70	1.74	1.777	1.84	1.90	1.95	1.99	2.03	2.07												
			0.77	0.89	1.01	0.902	1.02	0.994	1.07	1.01	1.119	1.03	0.99	1.12	1.06	1.02	1.15	1.03	1.01	1.119	1.03	0.99	1.12	1.06	1.02	1.15	1.03	1.01	1.119	1.03	0.99	1.12	1.06	1.02	1.15	1.03	1.01	1.119	1.03	0.99	1.12	1.06	1.02	1.15	1.03	1.01						
			0.75	0.94	1.04	0.93	1.05	0.98	1.10	1.02	1.15	1.02	0.98	1.191	1.04	1.01	1.26	1.05	1.04	1.119	1.03	0.99	1.12	1.06	1.02	1.15	1.03	1.01	1.119	1.03	0.99	1.12	1.06	1.02	1.15	1.03	1.01	1.119	1.03	0.99	1.12	1.06	1.02	1.15	1.03	1.01						
			(3,5)	1.49	1.54	1.63	1.60	1.71	1.66	1.78	1.75	1.84	1.80	1.70	1.90	1.84	1.74	1.96	1.86	1.76	1.84	1.80	1.70	1.90	1.84	1.74	1.96	1.86	1.76	1.84	1.80	1.70	1.90	1.84	1.74	1.96	1.86	1.76	1.84	1.80	1.70	1.90	1.84	1.74	1.96	1.86	1.76					
			(3,10)	1.54	1.70	1.79	1.64	1.83	1.70	1.89	1.80	1.94	1.85	1.75	1.99	1.87	1.80	2.07	1.94	1.83	1.94	1.85	1.75	1.99	1.87	1.80	2.07	1.94	1.83	1.94	1.85	1.75	1.99	1.87	1.80	2.07	1.94	1.83	1.94	1.85	1.75	1.99	1.87	1.80	2.07	1.94	1.83					
			(5,5)	0.86	0.96	1.03	0.99	1.06	1.04	1.09	1.08	1.13	1.11	1.05	1.16	1.16	1.08	1.22	1.19	1.12	1.13	1.11	1.05	1.16	1.16	1.08	1.22	1.19	1.12	1.13	1.11	1.05	1.16	1.16	1.08	1.22	1.19	1.12	1.13	1.11	1.05	1.16	1.16	1.08	1.22	1.19	1.12					
			(5,10)	0.90	0.99	1.05	1.001	1.09	1.05	1.10	1.10	1.15	1.14	1.07	1.218	1.20	1.097	1.28	1.26	1.13	1.14	1.07	1.218	1.20	1.097	1.28	1.26	1.13	1.14	1.07	1.218	1.20	1.097	1.28	1.26	1.13	1.14	1.07	1.218	1.20	1.097	1.28	1.26	1.13	1.14	1.07	1.218	1.20	1.097	1.28	1.26	1.13
			$RE_1$	0.25	(5,5)	0.65	0.69	0.79	0.80	0.84	0.86	0.87	0.94	0.95	0.92	1.29	1.02	0.99	1.02	1.30	1.02	1.03	1.32	0.65	0.71	0.81	0.84	0.88	0.89	0.90	0.92	0.98	0.97	1.31	1.03	1.05	1.36	1.03	1.05	1.37												
						0.71	0.73	0.81	0.84	0.88	0.89	0.90	0.92	0.98	0.97	1.31	1.03	1.05	1.36	1.03	1.05	1.37	0.67	0.721	0.779	0.82	0.92	1.01	0.94	1.08	0.99	1.15	1.24	1.03	1.20	1.34	1.03	1.25	1.45													
						0.67	0.721	0.779	0.82	0.92	1.01	0.94	1.08	0.99	1.15	1.24	1.03	1.20	1.34	1.03	1.25	1.45	0.72	0.741	0.749	0.852	0.95	1.03	0.984	1.108	1.0299	1.195	1.254	1.043	1.220	1.364	1.02	1.29	1.51													
						0.75	0.781	0.79	0.872	0.942	1.02	0.974	1.048	0.99	1.075	1.194	1.03	1.09	1.24	1.02	1.10	1.39	0.627	0.771	0.89	0.892	0.975	1.04	0.984	1.06	1.03	1.09	1.25	1.04	1.12	1.26	1.03	1.14	1.41													
						(5,10)	0.627	0.771	0.89	0.892	0.975	1.04	0.984	1.06	1.03	1.09	1.25	1.04	1.12	1.26	1.03	1.14	1.41	0.57	0.731	0.9179	0.92	0.9149	1.03	0.98	1.0246	1.02	1.06	1.0938	1.03	1.12	1.15	1.04	1.23	1.13												
						$RE_2$	0.25	(5,5)	0.697	0.87881	0.89	0.902	0.93	1.04	0.99	1.03	1.03	1.07	1.10	1.04	1.14	1.17	1.05	1.26	1.15	0.697	0.87881	0.89	0.902	0.93	1.04	0.99	1.03	1.03	1.07	1.10	1.04	1.14	1.17	1.05	1.26	1.15										
0.50	(5,10)	0.70				0.8571	0.869	0.872	0.91	0.93	0.94	0.96	0.96	1.17	1.079	0.97	1.21	1.24	1.00	1.23	1.35	0.70	0.8571	0.869	0.872	0.91	0.93	0.94	0.96	0.96	1.17	1.079	0.97	1.21	1.24	1.00	1.23	1.35														
0.75	(5,5)	0.71				0.91	0.839	0.922	0.92	0.94	0.95	0.97	0.99	1.19	1.08	0.99	1.24	1.26	0.99	1.28	1.31	0.71	0.91	0.839	0.922	0.92	0.94	0.95	0.97	0.99	1.19	1.08	0.99	1.24	1.26	0.99	1.28	1.31														
$RE_3$	0.25	(5,5)				0.647	0.851	0.819	0.852	0.91	0.94	0.94	0.96	0.96	1.09	1.09	0.97	1.12	1.13	0.98	1.17	1.150	0.647	0.851	0.819	0.852	0.91	0.94	0.94	0.96	0.96	1.09	1.09	0.97	1.12	1.13	0.98	1.17	1.150													
0.50	(5,10)	0.67				0.891	0.8889	0.862	0.941	0.974	0.964	0.986	0.996	1.129	1.19	1.09	0.97	1.17	1.23	1.01	1.18	1.21	0.67	0.891	0.8889	0.862	0.941	0.974	0.964	0.986	0.996	1.129	1.19	1.09	0.97	1.17	1.23	1.01	1.18	1.21												
0.75	(5,5)	0.607				0.911	0.899	0.872	0.88	0.91	0.95	0.94	0.99	1.07	1.081	1.03	1.139	1.14	1.03	1.18	1.15	0.607	0.911	0.899	0.872	0.88	0.91	0.95	0.94	0.99	1.07	1.081	1.03	1.139	1.14	1.03	1.18	1.15														
(5,10)	0.617	0.921				0.769	0.92	0.908	0.931	0.975	0.964	1.0199	1.087	1.0981	1.053	1.1539	1.174	1.04	1.195	1.21	1.21	0.617	0.921	0.769	0.92	0.908	0.931	0.975	0.964	1.0199	1.087	1.0981	1.053	1.1539	1.174	1.04	1.195	1.21	1.21													
(5,5)	0.627	0.761				0.929	0.9582	0.929	0.959	0.946	1.0199	0.961	1.03	1.189	1.01	1.04	1.289	1.02	1.04	1.34	1.34	0.627	0.761	0.929	0.9582	0.929	0.959	0.946	1.0199	0.961	1.03	1.189	1.01	1.04	1.289	1.02	1.04	1.34	1.34													
(5,10)	0.67	0.861				0.95	0.9382	0.94	0.96	0.97	1.02	0.95	1.05	1.197	1.02	1.05	1.297	1.03	1.06	1.35	1.35	0.67	0.861	0.95	0.9382	0.94	0.96	0.97	1.02	0.95	1.05	1.197	1.02	1.05	1.297	1.03	1.06	1.35	1.35													

$k = 2$



**Table 3** The estimates of REs at  $\rho = 0.5$

$RE_i$	$\mathcal{R}$	$(m, r) v$	$M_1$																								
			3			4			5			6			7			8			9			10			
			1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	
$k = 1$																											
$RE_1$	0.25	(3,5)	0.95	1.091	1.11	1.05	1.15	1.04	1.10	1.03	1.02	1.02	1.02	0.99	1.01	1.00	1.01	1.00	1.01	1.00	1.01	1.01	1.01	1.01	0.98	1.02	
		(3,10)	0.97	1.05	1.13	1.08	1.14	1.07	1.08	1.05	1.05	1.05	1.04	1.04	0.97	0.99	0.98	1.04	0.98	0.98	0.98	1.04	0.98	1.04	0.98	0.99	1.03
	0.50	(5,5)	1.13	1.11	1.02	0.89	1.01	0.94	1.01	0.99	0.98	0.99	0.97	0.92	0.96	0.92	0.96	0.92	0.92	0.96	0.92	0.92	0.96	0.92	0.88	0.96	0.86
		(5,10)	1.17	1.15	1.04	0.93	1.03	0.96	0.98	0.98	0.99	0.98	0.93	0.93	0.99	0.95	0.90	0.93	0.90	0.95	0.90	0.90	0.93	0.94	0.89	0.89	0.89
	0.75	(3,5)	1.06	1.07	1.07	1.05	1.08	1.07	1.04	1.05	1.03	1.03	1.04	1.12	1.01	1.01	0.98	1.02	1.01	1.01	0.99	1.04	1.01	0.93	0.91	0.91	0.91
		(3,10)	1.07	1.08	1.09	1.06	1.10	1.04	1.04	1.05	1.03	1.03	1.02	1.11	1.03	0.98	1.08	1.02	1.03	0.98	1.08	1.02	1.02	0.97	0.97	0.97	0.97
	0.25	(5,5)	1.03	0.99	0.92	0.98	0.90	1.01	0.88	0.99	0.87	0.87	0.97	1.03	0.86	0.93	1.03	0.88	0.93	1.03	0.88	0.87	1.03	0.88	0.87	1.03	1.03
		(5,10)	1.01	0.98	0.94	1.01	0.92	1.02	0.90	0.97	0.85	0.85	0.96	1.02	0.84	0.91	1.02	0.84	0.91	1.02	0.85	0.89	1.02	0.85	0.89	1.02	1.02
	0.50	(3,5)	1.129	1.05	1.03	1.02	0.97	1.04	0.94	1.06	0.90	1.04	1.03	1.02	0.88	1.02	1.03	0.88	1.02	1.03	0.89	1.01	1.04	1.01	1.04	1.04	1.04
		(3,10)	1.10	1.04	1.03	1.03	0.99	1.05	0.95	1.08	0.91	1.03	1.02	1.02	0.87	1.02	1.03	0.95	1.01	1.02	0.95	1.01	1.02	0.95	1.01	1.02	1.02
0.25	(5,5)	1.04	1.02	1.01	0.92	0.99	0.96	0.96	0.99	0.93	0.96	1.03	0.96	1.03	0.92	1.03	0.92	0.94	1.02	0.90	0.95	1.03	0.95	1.03	1.03	1.03	
	(5,10)	1.06	1.04	1.02	0.99	0.98	0.98	0.98	1.01	0.949	0.91	0.949	0.91	1.02	0.92	0.91	1.02	0.92	0.91	0.97	0.91	0.99	1.02	0.99	1.02	1.02	
0.50	(3,5)	1.10	1.14	1.16	1.14	1.172	1.12	1.20	1.09	1.22	1.05	1.07	1.10	1.07	1.10	1.03	1.03	1.03	1.03	1.05	1.05	1.027	0.91	0.91	0.91	0.91	
	(3,10)	1.17	1.16	1.19	1.10	1.20	1.08	1.25	1.06	1.30	1.03	1.11	1.12	1.11	1.12	1.01	1.02	1.01	1.02	1.11	1.01	1.01	1.01	0.98	0.98	0.98	
0.75	(5,5)	1.05	1.02	0.98	1.12	0.97	1.03	0.95	0.993	0.93	1.03	1.02	0.92	1.02	0.92	1.03	1.01	0.90	1.01	0.90	0.82	1.02	0.82	1.02	1.02	1.02	
	(5,10)	1.07	1.03	0.92	1.09	0.96	1.02	0.97	0.989	0.90	1.02	1.04	0.91	1.02	0.91	1.02	1.04	0.91	1.02	1.04	0.89	0.85	1.03	1.03	1.03	1.03	
0.25	(3,5)	0.99	1.02	1.03	1.13	1.045	1.15	1.07	1.17	1.04	1.13	1.19	1.19	1.02	1.10	1.05	1.03	1.062	0.88	0.88	0.88	0.88	0.88	0.88	0.88	0.88	
	(3,10)	1.01	1.03	1.04	1.10	1.05	1.14	1.08	1.11	1.03	1.10	1.14	1.02	1.02	1.09	1.08	1.01	1.10	0.92	1.01	0.92	0.91	0.91	0.91	0.91	0.91	
0.50	(5,5)	1.10	0.99	0.94	1.10	0.93	1.06	0.92	1.04	0.90	0.98	0.99	0.90	0.98	0.90	0.98	1.01	0.90	0.97	0.91	0.90	0.97	0.91	0.91	0.91	0.91	
	(5,10)	1.08	0.98	0.97	1.11	0.96	1.03	0.94	1.01	0.93	0.91	0.96	0.92	0.91	0.96	0.92	0.91	1.03	0.92	0.99	0.96	0.99	0.96	0.96	0.96	0.96	
0.75	(3,5)	1.02	1.04	1.06	1.14	1.08	1.13	1.05	1.10	1.03	1.08	1.21	1.02	1.02	1.05	1.06	0.98	1.04	0.99	1.04	0.99	1.04	0.99	0.99	0.99	0.99	
	(3,10)	1.08	1.06	1.08	1.16	1.10	1.15	1.07	1.13	1.05	1.09	1.22	1.01	1.04	1.04	1.03	0.99	1.03	0.99	1.03	0.99	1.03	0.99	0.98	0.98	0.98	
0.25	(5,5)	1.21	1.16	1.12	1.05	1.04	1.04	1.03	1.01	1.02	0.97	0.90	1.01	0.97	0.90	1.01	0.97	1.01	0.97	1.01	0.97	1.01	1.04	0.927	0.927	0.927	
	(5,10)	1.20	1.19	1.14	1.07	1.09	1.04	1.05	1.024	1.03	0.980	0.980	1.02	0.980	1.02	0.958	0.980	0.99	1.06	0.980	0.99	1.06	0.980	0.980	0.980	0.980	
0.50	(3,5)	1.08	1.10	1.11	1.05	1.19	1.06	1.24	1.07	1.30	1.09	1.05	1.25	1.13	1.13	1.13	1.06	1.10	1.10	1.10	1.10	1.10	1.10	1.10	1.10	1.10	
	(3,10)	1.10	1.12	1.13	1.07	1.24	1.08	1.27	1.09	1.32	1.10	1.06	1.26	1.10	1.06	1.10	1.08	1.11	1.11	1.11	1.11	1.11	1.11	1.11	1.11	1.11	1.11

(Continued)

Table 3 Continued

$RE_i$	$\mathcal{R}$	$(m, r) v$	$M_1$																				
			3			4			5			6			7			8			9		
0.50		(5,5)	1.03	0.97	0.98	1.09	0.99	1.068	0.97	1.036	0.95	0.99	0.91	0.94	0.99	0.94	0.93	0.94	0.99	0.94	0.93	0.83	0.859
		(5,10)	1.05	0.99	0.96	1.12	0.99	1.039	0.99	1.013	0.97	0.96	0.946	0.95	0.96	0.956	0.95	0.88	0.829				
		(3,5)	0.99	1.01	1.02	1.06	1.03	1.09	1.02	1.06	1.01	1.04	1.07	1.00	1.03	1.04	1.02	1.02	1.02	1.02	1.02	1.02	0.92
		(3,10)	1.01	1.02	1.03	1.07	1.04	1.05	1.03	1.04	1.02	1.02	1.02	1.09	1.01	1.02	1.01	1.03	1.03	1.03	1.03	1.03	0.94
		(5,5)	1.11	1.06	0.99	1.09	0.96	1.08	1.08	0.95	1.05	0.94	0.89	0.93	0.92	0.89	0.93	0.89	0.93	0.89	0.90	0.99	0.96
0.75		(5,10)	1.14	1.04	0.98	1.12	0.97	1.10	0.96	1.06	0.95	0.92	0.99	0.94	0.92	0.99	0.90	0.99	0.99				
		(3,5)	1.18	1.11	1.08	1.10	1.05	1.12	1.04	1.10	1.00	1.07	1.14	1.10	1.05	1.09	0.96	1.04	1.02				
		(3,10)	1.19	1.12	1.09	1.12	1.06	1.11	1.039	1.07	0.99	1.05	1.17	0.93	1.06	1.11	0.97	1.04	1.04				
		(5,5)	1.19	1.16	1.16	1.06	1.11	1.04	1.09	1.01	1.06	0.85	0.90	1.03	0.89	0.93	0.98	1.02	0.829				
		(5,10)	1.18	1.16	1.15	1.05	1.12	1.01	1.07	0.99	1.05	0.972	0.92	1.04	0.92	0.99	0.97	1.03	0.92				
$k = 2$																							
$RE_1$	0.25	(5,5)	0.97	0.99	1.01	1.01	1.021	1.03	1.00	1.06	0.99	1.09	0.99	0.98	1.03	0.89	0.96	1.013	0.89				
		(5,10)	0.99	0.99	1.01	1.05	1.042	1.07	1.02	1.09	0.99	1.10	0.97	0.99	1.07	0.89	0.98	1.016	0.87				
		(5,5)	1.11	1.13	1.18	1.02	1.121	1.09	1.00	1.11	0.99	1.12	0.92	0.98	1.06	0.82	0.97	1.018	0.72				
		(5,10)	1.10	1.15	1.20	1.03	1.094	1.04	1.02	1.07	0.99	1.09	0.99	0.99	1.04	0.89	0.93	1.020	0.79				
		(5,5)	0.95	0.97	0.98	1.08	0.021	1.10	1.00	1.12	0.99	1.12	0.96	0.98	1.10	0.83	0.95	1.08	0.76				
$RE_2$	0.25	(5,10)	0.927	0.99	1.01	1.08	1.021	1.10	1.00	1.12	0.99	1.12	0.96	0.98	1.10	0.82	0.91	1.013	0.716				
		(5,10)	0.969	0.99	1.01	1.11	1.042	1.13	1.02	1.15	0.99	1.15	0.94	0.99	1.111	0.84	0.98	1.06	0.744				
		(5,5)	1.08	1.08	1.10	1.01	1.021	1.05	1.00	1.06	0.99	1.08	0.99	0.98	1.05	0.92	0.97	1.08	0.809				
		(5,10)	1.07	1.11	1.12	1.05	1.042	1.07	1.02	1.09	0.99	1.09	0.97	0.99	1.06	0.87	0.93	1.020	0.717				
		(5,5)	0.95	0.97	0.98	1.02	1.021	1.04	1.00	1.06	0.99	1.07	0.92	0.98	1.04	0.85	0.95	1.08	0.72				
$RE_3$	0.25	(5,10)	0.97	0.99	1.01	1.08	1.021	1.10	1.00	1.12	0.99	1.13	0.96	0.98	1.09	0.90	0.94	1.07	0.79				
		(5,10)	0.97	0.99	1.01	1.04	1.021	1.10	1.00	1.12	0.99	1.13	0.96	0.98	1.09	0.90	0.96	1.03	0.86				
		(5,10)	0.99	0.99	1.01	1.04	1.042	1.06	1.02	1.09	0.99	1.09	0.98	0.99	1.06	0.92	0.98	1.06	0.78				
		(5,5)	1.11	1.13	1.18	1.08	1.021	1.10	1.00	1.12	0.99	1.13	0.96	0.98	1.10	0.96	0.97	1.08	0.76				
		(5,10)	1.10	1.15	1.20	1.11	1.042	1.13	1.02	1.15	0.99	1.16	0.94	0.99	1.12	0.90	0.98	1.020	0.704				
(5,5)	0.95	0.97	0.98	1.06	1.021	1.08	1.00	1.11	0.99	1.10	0.99	0.98	1.08	0.91	0.95	1.08	0.749						
(5,10)	0.97	0.99	1.00	1.05	1.042	1.07	1.02	1.11	0.99	1.09	0.97	0.99	1.07	0.91	0.94	1.07	0.737						

reversed. This is not surprising as all the studied estimators are derived under the perfectness, thus when the ranking quality is low, increasing  $m_1$  or  $v$  yields to increasing the error ranking which leads to decreasing the efficiency of VLRSS-based estimators. It is also observed that the VLRSS-based estimators have better performance at small  $m$  rather than large  $m$  at fixed  $n$ . However, for a fixed  $m$  the effect of  $n$  is not pronounced. Furthermore, at many considered cases increasing the values of  $k$  has a negative effect to  $REs$ . Generally speaking, it is to say that VLRSS becomes more efficient than the LRSS for both MLE and Bayesian methods as improving the quality of the ranking, increasing the values of  $m_1$  or  $v$  and decreasing the values of  $m$  or  $k$ .

## 5 Real Data Analysis

In what follows, two empirical data set are used to explore the efficiency of estimating  $\mathcal{R}$  under exponentiality based on VLRSS. The first one is related to the industrial field reported by Bader and Priest (1982) and can be found at Esemien et al. (2021). The data reflect the strength measured in general purpose amplifier (GPA) for single carbon fibers and 1000-carbon fiber tows at different levels of gauge lengths. Here, we restrict our consideration on single carbon fibers tested under tension of 20 mm gauge ( $X$ ) and 10 mm ( $Y$ ). Suppose that we are interested in comparing with these two different levels of gauge lengths. One can easily to investigate that  $Y^5$  and  $X^5$  fit approximately the exponential distribution with  $\tilde{\lambda}_1 = 0.25\%$  and  $\tilde{\mu}_1 = 0.79\%$ . Consequently, our analysis will be based on  $Y^5$  and  $X^5$  rather than  $Y$  and  $X$ . Hence, the single carbon fibers data set is considered as the hypothetical population with the value of the  $\mathcal{R} = \frac{\tilde{\mu}_1}{\tilde{\mu}_1 + \tilde{\lambda}_1} = 76\%$  which is close to that was estimated by Bader and Priest (1982).

On the other hand, the second data set is related to the medical field well-known as Veterans' Administration Lungn Cancer Study (WALCS) data and it can be found in survival package of R statistical software. In this data set, sample of patients with lung cancer of size 137 data set include on eight variables. Here, we restrict our attention on the time of death of these patients who are divided into control, a standard therapy, group and treatment, chemotherapy, group. Suppose that our target is to assess if the survival time of the patients treated with chemotherapy ( $Y$ ) is greater than those treated with a standard therapy ( $X$ ). The exponentiality of  $Y$  and  $X$  was well-investigated by Zamanzade (2019) and Abdallah et al. (2021) with

**Table 4** The estimates of REs based on single carbon fibers data set

$RE_i$	$(m, r)$	$M_1$																		
		3	4	5	6	7	8	9	10	2	3									
$RE_1$	$(3,5)$	1	1	1	1.15	1.36	1.18	1.40	1.20	1.45	1.22	1.52	1.27	1.25	1.62	1.32	1.28	1.82	1.39	
		2	1	1	1.17	1.40	1.22	1.43	1.24	1.49	1.26	1.56	1.31	1.29	1.68	1.35	1.32	1.86	1.42	
		3	1	1	1.17	1.21	1.19	1.29	1.20	1.35	1.20	1.42	1.15	1.21	1.46	1.26	1.21	1.52	1.32	
		4	1	1	1.04	1.24	1.07	1.32	1.09	1.37	1.12	1.46	1.12	1.15	1.49	1.19	1.18	1.54	1.22	
	$(5,5)$	1	1	1	0.79	0.89	0.99	1.02	1.05	1.10	1.13	1.19	1.15	1.17	1.16	1.19	1.21	1.22	1.24	
		2	1	1	0.84	0.94	0.97	1.05	1.09	1.12	1.15	1.17	1.12	1.15	1.19	1.20	1.18	1.24	1.22	
		3	1	1	0.85	0.84	0.94	0.97	1.05	1.09	1.12	1.15	1.17	1.12	1.15	1.19	1.20	1.18	1.24	
		4	1	1	0.82	0.72	0.79	0.89	0.99	1.02	1.05	1.10	1.13	1.19	1.15	1.17	1.16	1.19	1.21	1.22
	$RE_2$	$(3,5)$	1	1	1	1.05	1.36	1.05	1.06	1.04	1.08	1.06	1.12	1.42	1.02	1.15	1.49	1.02	1.19	1.52
			2	1	1	1.02	1.40	1.06	1.09	1.03	1.10	1.03	1.15	1.47	1.03	1.17	1.52	1.03	1.20	1.60
			3	1	1	1.02	1.12	1.03	1.15	1.02	1.20	1.02	1.24	1.17	1.02	1.27	1.21	1.05	1.29	1.23
			4	1	1	1.01	1.10	1.03	1.12	1.02	1.18	1.02	1.26	1.20	1.02	1.30	1.22	1.06	1.32	1.24
$(5,5)$		1	1	1	1.07	1.16	1.07	1.26	1.09	1.32	1.08	1.49	1.58	1.09	1.62	1.60	1.12	1.72	1.62	
		2	1	1	1.02	1.15	1.06	1.29	1.08	1.39	1.07	1.52	1.59	1.09	1.61	1.62	1.10	1.75	1.69	
		3	1	1	1.03	1.02	1.15	1.06	1.29	1.08	1.39	1.07	1.52	1.59	1.09	1.61	1.62	1.10	1.75	
		4	1	1	1.02	1.06	1.18	1.05	1.22	1.02	1.25	1.06	1.27	1.47	1.09	1.32	1.49	1.07	1.35	
$RE_3$		$(3,5)$	1	1	1	1.09	1.20	1.07	1.25	1.05	1.28	1.09	1.29	1.50	1.11	1.36	1.52	1.09	1.39	1.50
			2	1	1	1.03	1.15	1.02	1.18	1.02	1.20	1.02	1.26	1.12	1.12	1.29	1.16	1.05	1.33	1.19
			3	1	1	1.05	1.11	1.02	1.12	1.02	1.19	1.02	1.28	1.13	1.10	1.30	1.17	1.04	1.35	1.22
			4	1	1	1.02	1.05	1.11	1.02	1.12	1.02	1.19	1.02	1.28	1.13	1.10	1.30	1.17	1.04	1.35
	$(5,5)$	1	1	1	1.01	1.17	1.02	1.21	1.04	1.32	1.06	1.42	1.58	1.07	1.55	1.62	1.10	1.72	1.69	
		2	1	1	1.03	1.14	1.04	1.24	1.03	1.36	1.05	1.49	1.52	1.08	1.59	1.68	1.12	1.66	1.72	
		3	1	1	1.03	1.03	1.14	1.04	1.24	1.03	1.36	1.05	1.49	1.52	1.08	1.59	1.68	1.12	1.66	
		4	1	1	1.01	1.03	1.14	1.04	1.24	1.03	1.36	1.05	1.49	1.52	1.08	1.59	1.68	1.12	1.66	

**Table 5** The estimates of REs based on WALCS data set

$RE_i$	$(m, r) v$	$M_1$																
		3	4	5	6	7	8	9	10	2	3							
$RE_1$	(3,5)	1.23	1.24	1.27	1.50	1.37	1.62	1.39	1.68	1.42	1.76	1.47	1.50	1.94	1.58	1.53	2.18	1.66
	(3,10)	1.27	1.22	1.29	1.54	1.42	1.66	1.44	1.73	1.46	1.81	1.52	1.54	2.01	1.61	1.58	2.22	1.70
	(5,5)	1.12	1.12	1.29	1.33	1.38	1.50	1.39	1.57	1.39	1.65	1.33	1.45	1.75	1.59	1.45	1.82	1.58
	(5,10)	1.01	1.03	1.14	1.36	1.24	1.53	1.26	1.59	1.30	1.69	1.30	1.38	1.78	1.42	1.41	1.84	1.46
		$k = 2$																
	(5,5)	0.90	0.79	0.87	0.98	1.15	1.18	1.22	1.25	1.31	1.38	1.33	1.40	1.39	1.42	1.45	1.46	1.48
	(5,10)	0.90	0.94	0.92	1.03	1.13	1.22	1.26	1.30	1.33	1.36	1.30	1.38	1.42	1.44	1.41	1.48	1.46
$RE_2$	(3,5)	1.12	1.14	1.16	1.50	1.22	1.23	1.21	1.25	1.23	1.30	1.65	1.22	1.38	1.78	1.22	1.42	1.82
	(3,10)	1.13	1.13	1.12	1.54	1.23	1.26	1.19	1.28	1.19	1.33	1.71	1.20	1.40	1.82	1.23	1.44	1.91
	(5,5)	1.01	1.12	1.12	1.23	1.19	1.33	1.18	1.39	1.18	1.44	1.36	1.22	1.52	1.45	1.26	1.54	1.47
	(5,10)	1.12	1.12	1.11	1.21	1.19	1.30	1.18	1.37	1.18	1.46	1.39	1.22	1.55	1.46	1.27	1.58	1.48
		$k = 2$																
	(5,5)	1.01	1.16	1.18	1.28	1.24	1.46	1.26	1.53	1.25	1.73	1.83	1.30	1.94	1.91	1.34	2.06	1.94
	(5,10)	1.11	1.13	1.12	1.27	1.23	1.50	1.25	1.61	1.24	1.76	1.84	1.30	1.93	1.94	1.32	2.09	2.02
$RE_3$	(3,5)	1.16	1.12	1.17	1.30	1.22	1.42	1.18	1.45	1.23	1.47	1.71	1.30	1.58	1.78	1.28	1.61	1.85
	(3,10)	1.17	1.10	1.20	1.32	1.24	1.45	1.22	1.48	1.26	1.50	1.74	1.33	1.63	1.82	1.30	1.66	1.79
	(5,5)	1.06	1.08	1.13	1.27	1.18	1.37	1.18	1.39	1.18	1.46	1.30	1.34	1.54	1.39	1.26	1.59	1.42
	(5,10)	1.12	1.12	1.16	1.22	1.18	1.30	1.18	1.38	1.18	1.48	1.31	1.32	1.55	1.40	1.24	1.61	1.46
		$k = 2$																
	(5,5)	1.06	1.09	1.11	1.29	1.18	1.40	1.21	1.53	1.23	1.65	1.83	1.28	1.85	1.94	1.32	2.06	2.02
	(5,10)	1.08	1.09	1.13	1.25	1.21	1.44	1.19	1.58	1.22	1.73	1.76	1.29	1.90	2.01	1.34	1.99	2.06

$\tilde{\lambda}_2 = 0.78\%$  and  $\tilde{\mu}_2 = 0.86\%$ . Similar to what is done with the single carbon fibers data set, WALCS is considered as the hypothetical population with the value of the  $\mathcal{R} = \frac{\tilde{\mu}_2}{\tilde{\mu}_2 + \tilde{\lambda}_2} = 52\%$ .

For the same values of  $m, r, m_1, k$  and  $v$  shown in Section 4, 5,000 samples with replacement are drawn using perfect LRSS and perfect VLRSS separately from the two aforementioned datasets, then the three different *REs* are computed and reported in Tables 4 and 5. It appears that all the *REs* mostly exceed unity, and the improvement happens as increasing the values of  $m_1$  or  $v$ , yet the effect of the value of  $v$  is more pronounced. It is to be noted that the VLRSS-based estimators have better performance at small values of  $m$  relative to large ones. All of these remarks are consistent with what we noted in the preceding section.

## 6 Conclusion

In this study, we have considered the MLE and Bayesian methods of the system reliability  $\mathcal{R} = P(X < Y)$  using LRSS and VLRSS, where  $X$  and  $Y$  are independent one-parameter exponential distribution. The asymptotic statistical properties of the MLE estimators under VLRSS are analytically proved. The Bayes estimators are obtained by Jeffreys prior under general entropy loss function and LINEX loss function. It turns out from the numeric results based on simulated data and practical application that the estimates of reliability using VLRSS are more precise than LRSS analogues given that the quality of ranking is fairly good. Moreover, increasing the efficiency of the VLRSS-based estimators can be achieved by increasing  $m_1, v$  or decreasing the set size. In a subsequent work, it may be of interest to study interval estimation of the system reliability parameter under the VLRSS scheme. Also, it is important to examine the superiority of VLRSS under other parametric distributions (see for instance Ali et al. (2020)). Further, using Upper Record Values based on VLRSS can be considered as an attractive future point (see Yousaf et al. (2019)).

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## **Biography**



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