A Generalized Class of Estimators for Finite Population Mean Using Two Auxiliary Variables in Sample Surveys

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Abstract

In this paper we have suggested a generalized class of estimators for estimating the finite population mean $\bar{Y}$ of the study variable y using information on two auxiliary variables x and z. We have studied the properties of the proposed generalized class of estimators in simple random sampling without replacement scheme and in stratified random sampling up to the first order of approximation. It is shown that the suggested class of estimators is more efficient than the conventional unbiased estimator, ratio estimator, product estimator, traditional difference estimator, Srivastava (1967) estimator, Ray et al. (1979) estimator, Vos (1980) estimator, Upadhyaya et al. (1985) estimator, Rao (1991) estimator and Gupta and Shabbir (2008) estimator. Theoretical results are well supported through an empirical study.

Keywords: Auxiliary variable, study variable, bias, mean squared error, simple random sampling, stratified random sampling.

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1 Introduction

In sample surveys, the use of auxiliary variable(s) at the estimation stage played a prominent role in improving the precision of an estimate of the population mean. Various authors have paid their attention towards the estimation of population mean $\bar{Y}$ of the study variable y using information on a single auxiliary variable x and suggested a large number of estimators along with their properties in simple random sampling without (or with) replacement schemes for instance, see Murthy (1967), Singh (1986, 2003) and the references cited therein. In many survey situations of practical importance, adequate information on more than one of auxiliary variables is available. In such a situation Olkin (1958) was first to introduce multivariate ratio estimator for population mean $\bar{Y}$ of the study variable y using information on $p(>1)$ auxiliary variables. Later many authors including Raj (1965), Rao and Mudholkar (1967), Singh (1967, 1969), Shukla (1966), Srivastava (1965, 1967, 1971), Singh and Tailor (2005), Gupta and Shabbir (2008), Singh et al. (2009), Swain (2013) and Sharma and Singh (2014, 2015) etc. have developed estimators which utilize data from $p(>1)$ auxiliary variables. The properties of the estimators studied under simple random sampling with (or without) replacement i.e. *SRSWR* (or *SRSWOR*) scheme.

It is well established fact that the simple random sampling (*SRS*) procedure is employed when the population is homogeneous. However in practice, the populations encountered are not homogeneous (i.e. populations are heterogeneous). Thus in such a situation *SRS* procedure does not provide a sample which is good representative of the entire population. Hence we can say that when the population is heterogeneous, *SRS* procedure does not provide better estimate of the population mean $\bar{Y}$. To cope up with this situation, we use stratified random sampling for selecting a good sample from the target population. Thus when the population is heterogeneous stratified random sampling is more appropriate and gives better estimate of the population mean. In a stratified random sampling design, we divide the population into groups known as strata and samples are selected from each group with pre-determined sample size. Several authors including Diana (1993), Kadilar and Cingi (2003), Shabbir and Gupta (2005), Singh and Vishwakarma (2008), Singh et al. (2008), Koyuncu and Kadilar (2009, 2010), Koyuncu (2013), Yadav et al. (2015a, 2015b) and Koyuncu (2016) etc. have suggested estimators for population mean $\bar{Y}$ of y using information on single auxiliary variable x in stratified random sampling. It is further noticed that various authors including Koyuncu and Kadilar (2009), Tailor et al. (2012), Singh and Kumar (2012), Olufadi (2013), Tailor and Chouhan (2014), Verma et al. (2015), Shabbir and Gupta (2015, 2016), Muneer et al. (2016), Malik and Singh (2017), Mishra et al. (2017) and Shabbir (2018) etc. have suggested several estimators for population mean $\bar{Y}$ of *y* using two auxiliary variables *x* and *z* in stratified random sampling.

In this paper we have made an effort to develop a generalized class of estimators for population mean $\bar{Y}$ of *y* using information on two auxiliary variables *x* and *z*. The properties of the suggested class of estimators are studied up to the first order of approximation in *SRSWOR* scheme as well as in stratified random sampling. Numerical examples are given in support of the present study.

2 Some Existing Estimators of *SRS*

Consider a finite population $\Omega = \{\Omega_1, \Omega_2, \ldots \Omega_N\}$ of *N* units. Let *y* and (*x*, *z*) be study variable and auxiliary variables respectively. Let y_i and (x_i, z_i) be the values of study variable *y* and auxiliary variables (*x*, *z*) on the *i*th unit Ω_i of the population Ω. Suppose a sample of size *n* is drawn by using *SRSWOR* scheme from the population Ω for estimating the population mean $\bar{Y}$ of the study variable *y*. Let $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $(\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i,\ \bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i)$ be the unbiased estimators of the population means $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$ and $(\bar{X} = \frac{1}{N}\sum_{i=1}^{N} x_i,\ \bar{Z} = \frac{1}{N}\sum_{i=1}^{N} z_i)$ respectively.

It is assumed that the population means $(\bar{X}, \bar{Z})$ of (*x*, *z*) are known. Further we denote

$C_y = \frac{S_y}{\bar{Y}}$: the population coefficient of variation of *y*,
$C_x = \frac{S_x}{\bar{X}}$: the population coefficient of variation of *x*,
$C_z = \frac{S_z}{\bar{Z}}$: the population coefficient of variation of *z*,
$\rho_{yx} = \frac{S_{yx}}{S_y S_x}$: the population correlation coefficient between *y* and *x*,
$\rho_{yz} = \frac{S_{yz}}{S_y S_z}$: the population correlation coefficient between *y* and *z*,
$\rho_{xz} = \frac{S_{xz}}{S_x S_z}$: the population correlation coefficient between *x* and *z*,
$S_{yx} = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i - \bar{X})$: the population covariance between *y* and *x*,
$S_{yz} = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})(z_i - \bar{Z})$: the population covariance between *y* and *z*,
$S_{xz} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{X})(z_i - \bar{Z})$: the population covariance between *x* and *z*,

$S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2$: the population mean square of *y*,
$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{X})^2$: the population mean square of *x*,
$S_z^2 = \frac{1}{N-1}\sum_{i=1}^{N}(z_i - \bar{Z})^2$: the population mean square of *z*.
$K_{yx} = \frac{\rho_{yx}C_y}{C_x}$, $K_{yz} = \frac{\rho_{yz}C_y}{C_z}$, $K_{xz} = \frac{\rho_{xz}C_x}{C_z}$, $K_{zx} = \frac{\rho_{zx}C_z}{C_x}$ and $f = \frac{n}{N}$: is the sampling fraction.

Now we review some existing estimators.
The usual unbiased estimator for population mean $\bar{Y}$ is given by

$$\hat{\bar{Y}}_0 = \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \tag{1}$$

The variance/mean squared error (*MSE*) under *SRSWOR* is given by

$$Var(\hat{\bar{Y}}_0) = MSE(\hat{\bar{Y}}_0) = \left(\frac{1-f}{n}\right)\bar{Y}^2 C_y^2 = \left(\frac{1-f}{n}\right)S_y^2 \tag{2}$$

The usual ratio and product estimators for population mean $\bar{Y}$ are respectively defined by

$$\hat{\bar{Y}}_R = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right) \tag{3}$$

and

$$\hat{\bar{Y}}_P = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right) \tag{4}$$

To the first degree of approximation (*fda*), the *MSE*s of the estimators $\hat{\bar{Y}}_R$ and $\hat{\bar{Y}}_P$ are respectively given by

$$MSE(\hat{\bar{Y}}_R) = \left(\frac{1-f}{n}\right)\bar{Y}^2[C_y^2 + C_x^2(1 - 2K_{yx})] \tag{5}$$

$$MSE(\hat{\bar{Y}}_P) = \left(\frac{1-f}{n}\right)\bar{Y}^2[C_y^2 + C_x^2(1 + 2K_{yx})] \tag{6}$$

The generalized version of the estimators $\hat{\bar{Y}}_0$, $\hat{\bar{Y}}_R$ and $\hat{\bar{Y}}_P$ due to Srivastava (1967) is given by

$$\hat{\bar{Y}}_{\alpha_1} = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)^{\alpha_1} \tag{7}$$

where α_1 is a suitably chosen constant.

To the *fda*, the *MSE*$(\hat{\bar{Y}}_{\alpha_1})$ is given by

$$MSE(\hat{\bar{Y}}_{\alpha_1}) = \frac{(1-f)}{n}\bar{Y}^2[C_y^2 + \alpha_1 C_x^2(\alpha_1 + 2K_{yx})] \tag{8}$$

which is minimum when

$$\alpha_1 = -K_{yx}\alpha_{1(opt)}, \text{ say} \tag{9}$$

Substitution of (9) in (8) yields the minimum *MSE* of $\hat{\bar{Y}}_{\alpha_1}$ as

$$\begin{aligned} MSE_{\min}(\hat{\bar{Y}}_{\alpha_1}) &= \frac{(1-f)}{n}\bar{Y}^2[C_y^2 - K_{yx}^2 C_x^2] \\ &= \frac{(1-f)}{n}\bar{Y}^2 C_y^2[1-\rho_{yx}^2] = \frac{(1-f)}{n}S_y^2[1-\rho_{yx}^2] \end{aligned} \tag{10}$$

The traditional difference estimator for $\bar{Y}$ is defined by

$$\hat{\bar{Y}}_{D1} = \bar{y} + d_0(\bar{X} - \bar{x}), \tag{11}$$

where d_0 is a suitably chosen constant to be determined such that the *MSE* of $\hat{\bar{Y}}_{D_1}$ is minimum.

The minimum *MSE* of $\hat{\bar{Y}}_{D1}$ is given by

$$MSE_{\min}(\hat{\bar{Y}}_{D1}) = \left(\frac{1-f}{n}\right)\bar{Y}^2 C_y^2(1-\rho_{yx}^2) = \left(\frac{1-f}{n}\right)S_y^2(1-\rho_{yx}^2) \tag{12}$$

Upadhyaya et al. (1985) suggested a class of estimators for population mean $\bar{Y}$ as

$$\hat{\bar{Y}}_{USV} = w_0\bar{y} + w_1\bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)^{\alpha_1} \tag{13}$$

where w_0 and w_1 are suitably chosen weights whose sum need not be ‘unity’ and α_1 is a design parameter.

The *MSE* of $\hat{\bar{Y}}_{USV}$ to the *fda* is given by

$$\begin{aligned} MSE(\hat{\bar{Y}}_{USV}) = \bar{Y}^2[&1 + w_0^2 A_{0(srs)} + w_1^2 A_{1(srs)} \\ &+ 2w_0w_1A_{3(srs)} - 2w_0 - 2w_1A_{6(srs)}] \end{aligned} \tag{14}$$

where

$$A_{0(srs)} = \left[1 + \left(\frac{1-f}{n}\right) C_y^2\right]$$

$$A_{1(srs)} = \left[1 + \left(\frac{1-f}{n}\right) \{C_y^2 + 4\alpha_1 \rho_{yx} C_y C_x + \alpha_1(2\alpha_1 - 1) C_x^2\}\right]$$

$$A_{3(srs)} = \left[1 + \left(\frac{1-f}{n}\right) \{C_y^2 + 2\alpha_1 \rho_{yx} C_y C_x + \frac{\alpha_1(\alpha_1 - 1)}{2} C_x^2\}\right]$$

$$A_{6(srs)} = \left[1 + \left(\frac{1-f}{n}\right) \{\alpha_1 \rho_{yx} C_y C_x + \frac{\alpha_1(\alpha_1 - 1)}{2} C_x^2\}\right]$$

The best values of (w_0, w_1) for which the *MSE* of $\hat{\hat{Y}}_{USV}$ is minimized, are given by

$$w_0 = \frac{\Delta_0^*}{\Delta^*}, \quad w_1 = \frac{\Delta_1^*}{\Delta^*} \tag{15}$$

where

$$\Delta^* = \begin{vmatrix} A_{0(srs)} & A_{3(srs)} \\ A_{3(srs)} & A_{1(srs)} \end{vmatrix} = (A_{0(srs)} A_{1(srs)} - A_{3(srs)}^2)$$

$$\Delta_0^* = \begin{vmatrix} 1 & A_{3(srs)} \\ A_{6(srs)} & A_{1(srs)} \end{vmatrix} = (A_{1(srs)} - A_{3(srs)} A_{6(srs)})$$

$$\Delta_1^* = \begin{vmatrix} A_{0(srs)} & 1 \\ A_{3(srs)} & A_{6(srs)} \end{vmatrix} = (A_{0(srs)} A_{6(srs)} - A_{3(srs)})$$

Thus the minimum *MSE* of $\hat{\hat{Y}}_{USV}$ is given by

$$MSE_{\min}(\hat{\hat{Y}}_{USV}) = \bar{Y}^2 \left[1 - \frac{\{A_{1(srs)} - 2A_{3(srs)} A_{6(srs)} + A_{0(srs)} A_{6(srs)}^2\}}{(A_{0(srs)} A_{1(srs)} - A_{3(srs)}^2)}\right] \tag{16}$$

If we set $w_0 + w_1 = 1 \Rightarrow w_1 = (1 - w_0)$ in (13) we get an estimator for population mean $\bar{Y}$ of *y* as

$$\hat{\hat{Y}}^* = w_0 \bar{y} + (1 - w_0) \bar{y} \left(\frac{\bar{x}}{\bar{X}}\right)^{\alpha_1} \tag{17}$$

which includes the estimators due to Srivastava (1967), Ray et al. (1979) and Vos (1980).

Putting $w_1 = (1 - w_0)$ in (14) we get the *MSE* of $\hat{\hat{Y}}^*$ to the *fda* as

$$MSE(\hat{\hat{Y}}^*) = \bar{Y}^2[1 + A_{1(srs)} - 2A_{6(srs)} + w_0^2(A_{0(srs)} + A_{1(srs)} - 2A_{3(srs)}) - 2w_0(1 + A_{1(srs)} - A_{3(srs)} - A_{6(srs)})] \tag{18}$$

which is minimized for

$$w_0 = \frac{(1 + A_{1(srs)} - A_{3(srs)} - A_{6(srs)})}{(A_{0(srs)} + A_{1(srs)} - 2A_{3(srs)})} = \left(1 + \frac{K_{yx}}{\alpha_1}\right) = w_{0(opt)}, \text{ say} \tag{19}$$

Thus the minimum *MSE* of $\hat{\hat{Y}}^*$ is given by

$$\begin{aligned} MSE_{\min}(\hat{\hat{Y}}^*) &= \bar{Y}^2\left[1 + A_{1(srs)} - 2A_{6(srs)} - \frac{(1 + A_{1(srs)} - A_{3(srs)} - A_{6(srs)})^2}{(A_{0(srs)} + A_{1(srs)} - 2A_{3(srs)})}\right] \\ &= \left(\frac{1-f}{n}\right)\bar{Y}^2C_y^2\left(1 - \rho_{yx}^2\right) = \left(\frac{1-f}{n}\right)S_y^2(1 - \rho_{yx}^2) \end{aligned} \tag{20}$$

Rao (1991) suggested difference-type estimator for population mean $\bar{Y}$ as

$$\hat{\hat{Y}}_{Rao} = \alpha_1\bar{y} + \alpha_2(\bar{X} - \bar{x}), \tag{21}$$

where α_1 and α_2 are constants to be determined such that *MSE* of $\hat{\hat{Y}}_{Rao}$ is minimum.

The bias and *MSE* of $\hat{\hat{Y}}_{Rao}$ are respectively given by

$$B(\hat{\hat{Y}}_{Rao}) = \bar{Y}(\alpha_1 - 1), \tag{22}$$

$$MSE(\hat{\hat{Y}}_{Rao}) = \bar{Y}^2\left[1 + \alpha_1^2\left\{1 + \left(\frac{1-f}{n}\right)C_y^2\right\} + \alpha_2^2\left(\frac{1-f}{n}\right)\frac{C_x^2}{R^2} - 2\alpha_1\alpha_2\left(\frac{1-f}{n}\right)\frac{K_{yx}C_x^2}{R} - 2\alpha_1\right] \tag{23}$$

where $R = \frac{\bar{Y}}{\bar{X}}$.

The *MSE*$(\hat{\bar{Y}}_{Rao})$ at (23) is minimized for

$$\alpha_1 = \left\{1 + \left(\frac{1-f}{n}\right)(C_y^2 - K_{yx}^2 C_x^2)\right\}^{-1} = \alpha_{1(opt)}, \text{ say}$$

$$\alpha_2 = -RK_{yx}\left\{1 + \left(\frac{1-f}{n}\right)\left(C_y^2 - K_{yx}^2 C_x^2\right)\right\}^{-1} = \alpha_{2(opt)}, \text{ say}$$

Thus the minimum *MSE* of $\hat{\bar{Y}}_{Rao}$ is given by

$$MSE_{\min}(\hat{\bar{Y}}_{Rao}) = \frac{\left(\frac{1-f}{n}\right)\bar{Y}^2\left(C_y^2 - K_{yx}^2 C_x^2\right)}{1 + \left(\frac{1-f}{n}\right)\left(C_y^2 - K_{yx}^2 C_x^2\right)} \tag{24}$$

Gupta and Shabbir (2008) proposed the following class of estimators for population mean $\bar{Y}$ as

$$\hat{\bar{Y}}_{GS} = [\alpha_3 \bar{y} + \alpha_4(\bar{X} - \bar{x})]\left(\frac{\bar{X}}{\bar{x}}\right) \tag{25}$$

where (α_3, α_4) are suitably chosen constants such that the *MSE* of $\hat{\bar{Y}}_{GS}$ is minimum.

To the *fda*, the bias and *MSE* of $\hat{\bar{Y}}_{GS}$ are respectively given by

$$B(\hat{\bar{Y}}_{GS}) = \bar{Y}\left[\alpha_3\left\{1 + \left(\frac{1-f}{n}\right)C_x^2(1 - K_{yx})\right\} + \frac{\alpha_4}{R}C_x^2 - 1\right] \tag{26}$$

$$MSE(\hat{\bar{Y}}_{GS}) = \bar{Y}^2\begin{bmatrix} 1 + \alpha_3^2\left\{1 + \frac{(1-f)}{n}\left[C_y^2 + C_x^2(3 - 4K_{yx})\right]\right\} \\ +\alpha_4^2\frac{(1-f)}{n}\frac{C_x^2}{R^2} + \frac{2\alpha_3\alpha_4}{R}\frac{(1-f)}{n}C_x^2(2 - K_{yx}) \\ -2\alpha_3\left\{1 + \frac{(1-f)}{n}C_x^2(1 - K_{yx})\right. \\ \left. -2\alpha_4\frac{(1-f)}{n}\frac{C_x^2}{R}\right\} \end{bmatrix} \tag{27}$$

The $MSE(\hat{\hat{Y}}_{GS})$ at (27) is minimum when

$$\alpha_3 = \frac{(a_2 a_4 - a_3 a_5)}{\left(a_1 a_2 - a_3^2\right)} = \alpha_{3(opt)}, \text{ say}$$

$$\alpha_4 = \frac{(a_1 a_5 - a_3 a_4)}{\left(a_1 a_2 - a_3^2\right)} = \alpha_{4(opt)}, \text{ say}$$

where

$$a_1 = \left[1 + \left(\frac{1-f}{n}\right)\left\{C_y^2 + C_x^2\left(3 - 4K_{yx}\right)\right\}\right]$$

$$a_2 = \left(\frac{1-f}{n}\right)\frac{C_x^2}{R^2}$$

$$a_3 = \left(\frac{1-f}{n}\right)\frac{C_x^2}{R}(2 - K_{yx})$$

$$a_4 = \left[1 + \left(\frac{1-f}{n}\right)C_x^2\left(1 - K_{yx}\right)\right]$$

$$a_5 = \left(\frac{1-f}{n}\right)\frac{C_x^2}{R}$$

Thus the minimum *MSE* of $\hat{\hat{Y}}_{GS}$ is given by

$$MSE_{\min}\left(\hat{\hat{Y}}_{GS}\right) = \bar{Y}^2\left\{1 - \frac{\left(a_2 a_4^2 - 2a_3 a_4 a_5 + a_1 a_5^2\right)}{\left(a_1 a_2 - a_3^2\right)}\right\} \tag{28}$$

The traditional difference estimator for population mean $\bar{Y}$ using two auxiliary variables *x* and *z* is defined by

$$\hat{\hat{Y}}_{D2} = \left\{\bar{y} + k_1\left(\bar{X} - \bar{x}\right) + k_2\left(\bar{Z} - \bar{z}\right)\right\} \tag{29}$$

where k_1 and k_2 are constants to be determined such that $MSE(\hat{\hat{Y}}_{D2})$ is minimum.

It is obvious from (29) that the difference estimator $\hat{\hat{Y}}_{D2}$ is unbiased for population mean $\bar{Y}$.

The variance/*MSE* of the estimator $\hat{\hat{Y}}_{D2}$ is given by

$$
\begin{aligned}
Var(\hat{\hat{Y}}_{D2}) &= MSE(\hat{\hat{Y}}_{D2}) \\
&= \left(\frac{1-f}{n}\right)\bar{Y}^2\left[C_y^2 + k_1^2\frac{C_x^2}{R^2} + k_2^2\frac{C_z^2}{R^2} + 2k_1k_2\frac{K_{xz}C_z^2}{RR^*}\right. \\
&\quad \left. -2k_1\frac{K_{yx}C_x^2}{R} - 2k_2\frac{K_{yz}C_z^2}{R^*}\right]
\end{aligned}
\tag{30}
$$

where $R^* = \frac{\bar{Y}}{\bar{Z}}$.

The $MSE(\hat{\hat{Y}}_{D2})$ at (30) is minimized for

$$k_1 = \frac{RC_y\left(\rho_{yx} - \rho_{yz}\rho_{xz}\right)}{C_x\left(1-\rho_{xz}^2\right)} = k_{1(opt)}, \text{ say}$$

$$k_2 = \frac{R^*C_y\left(\rho_{yz} - \rho_{yx}\rho_{xz}\right)}{C_z\left(1-\rho_{xz}^2\right)} = k_{2(opt)}, \text{ say}$$

Thus the minimum *MSE* of $\hat{\hat{Y}}_{D2}$ is given by

$$MSE_{\min}(\hat{\hat{Y}}_{D2}) = \left(\frac{1-f}{n}\right)\bar{Y}^2C_y^2(1-R_{y.xz}^2) = \left(\frac{1-f}{n}\right)S_y^2(1-R_{y.xz}^2) \tag{31}$$

where $R_{y.xz}^2 = \frac{\rho_{yx}^2+\rho_{yz}^2-2\rho_{yx}\rho_{yz}\rho_{xz}}{1-\rho_{xz}^2}$ is the multiple correlation coefficient.

3 Suggested Generalized Class of Estimators in Simple Random Sampling

Motivated by Upadhyaya et al. (1985) we propose a generalized class of estimators based on two auxiliary variables *x* and *z* for population mean $\bar{Y}$ of *y* as

$$t = w_0\bar{y} + w_1\bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)^{\alpha_1} + w_2\bar{y}\left(\frac{\bar{z}}{\bar{Z}}\right)^{\alpha_2} \tag{32}$$

where (w_0, w_1, w_2) are weights whose sum need not be 'unity' and (α_1, α_2) are design parameters. The constants (α_1, α_2) may take positive $(+,+)$ or negative $(-,-)$ or positive-negative $(+,-)$ or negative-positive $(-,+)$ values to form product-type or ratio-type or product-cum-ratio-type or ratio-cum-product-type estimator.

To obtain the bias and *MSE* of the propounded estimator *t*, we write

$$\bar{y} = \bar{Y}(1+e_0),\ \bar{x} = \bar{X}(1+e_1) \quad \text{and} \quad \bar{z} = \bar{Z}(1+e_2)$$

such that $E(e_0) = E(e_1) = E(e_2) = 0$

$$E(e_0^2) = \left(\frac{1-f}{n}\right)C_y^2, \quad E\left(e_1^2\right) = \left(\frac{1-f}{n}\right)C_x^2,$$

$$E\left(e_2^2\right) = \left(\frac{1-f}{n}\right)C_z^2$$

$$E(e_0e_1) = \left(\frac{1-f}{n}\right)\rho_{yx}C_yC_x = \left(\frac{1-f}{n}\right)K_{yx}C_x^2,$$

$$E(e_0e_2) = \left(\frac{1-f}{n}\right)\rho_{yz}C_yC_z = \left(\frac{1-f}{n}\right)K_{yz}C_z^2$$

$$E(e_1e_2) = \left(\frac{1-f}{n}\right)\rho_{xz}C_xC_z = \left(\frac{1-f}{n}\right)K_{xz}C_x^2 = \left(\frac{1-f}{n}\right)K_{zx}C_z^2.$$

Expressing (32) in terms of *e*'s we have

$$t = \bar{Y}[w_0(1+e_0) + w_1(1+e_0)(1+e_1)^{\alpha_1} + w_2(1+e_0)(1+e_2)^{\alpha_2}] \tag{33}$$

We suppose that $|e_i| \ll 1$ so that $(1+e_i)^{\alpha_i}, i = 1, 2$ are expandable.

Expanding the right hand side of (33), multiplying out and ignoring terms of *e*'s having power greater than two, we have

$$t \cong \bar{Y}\left[w_0(1+e_0) + w_1\left\{1 + e_0 + \alpha_1e_1 + \alpha_1e_0e_1 + \frac{\alpha_1(\alpha_1-1)}{2}e_1^2\right\}\right.$$
$$\left. + w_2\left\{1 + e_0 + \alpha_2e_2 + \alpha_2e_0e_2 + \frac{\alpha_2(\alpha_2-1)}{2}e_2^2\right\}\right]$$

or

$$\left(t - \bar{Y}\right) \cong \bar{Y}\begin{bmatrix} w_0(1+e_0) \\ \quad + w_1\left\{1 + e_0 + \alpha_1e_1 + \alpha_1e_0e_1 + \frac{\alpha_1(\alpha_1-1)}{2}e_1^2\right\} \\ \quad + w_2\left\{1 + e_0 + \alpha_2e_2 + \alpha_2e_0e_2 + \frac{\alpha_2(\alpha_2-1)}{2}e_2^2\right\} - 1 \end{bmatrix} \tag{34}$$

Taking expectation of both sides of (34) we get the bias of *t* to the *fda* as

$$
\begin{aligned}
B(t) = \bar{Y} \Bigg[w_0 + w_1 \left\{ 1 + \left(\frac{1-f}{n} \right) \frac{\alpha_1}{2} (\alpha_1 + 2K_{yx} - 1) C_x^2 \right\} \\
+ w_2 \left\{ 1 + \left(\frac{1-f}{n} \right) \frac{\alpha_2}{2} (\alpha_2 + 2K_{yz} - 1) C_z^2 \right\} - 1 \Bigg] \\
= \bar{Y} \left[w_0 + w_2 A_{6(srs)} + w_3 A_{7(srs)} - 1 \right] \qquad (35)
\end{aligned}
$$

where

$$
\begin{aligned}
A_{6(srs)} &= \left[1 + \left(\frac{1-f}{n} \right) \frac{\alpha_1}{2} (\alpha_1 + 2K_{yx} - 1) C_x^2 \right] \\
A_{7(srs)} &= \left[1 + \left(\frac{1-f}{n} \right) \frac{\alpha_2}{2} (\alpha_2 + 2K_{yz} - 1) C_z^2 \right]
\end{aligned}
$$

Squaring both sides of (34) and ignoring terms of *e*'s having power greater than two we have

$$
(t - \bar{Y})^2 = \bar{Y}^2 \left[\begin{array}{l}
1 + w_0^2 (1 + 2e_0 + e_0^2) \\
+ w_1^2 \left\{ \begin{array}{l} 1 + 2e_0 + 2\alpha_1 e_1 \\ + e_0^2 + 4\alpha_1 e_0 e_1 \\ + \alpha_1 (2\alpha_1 - 1) e_1^2 \end{array} \right\} \\
+ w_2^2 \left\{ \begin{array}{l} 1 + 2e_0 + 2\alpha_2 e_2 \\ + e_0^2 + 4\alpha_2 e_0 e_2 \\ + \alpha_2 (2\alpha_2 - 1) e_2^2 \end{array} \right\} \\
+ 2w_0 w_1 \left\{ \begin{array}{l} 1 + 2e_0 + \alpha_1 e_1 + e_0^2 \\ + 2\alpha_1 e_0 e_1 + \frac{\alpha_1 (\alpha_1 - 1)}{2} e_1^2 \end{array} \right\} \\
+ 2w_0 w_2 \left\{ \begin{array}{l} 1 + 2e_0 + \alpha_2 e_2 + e_0^2 + 2\alpha_2 e_0 e_2 \\ + \frac{\alpha_2 (\alpha_2 - 1)}{2} e_2^2 \end{array} \right\} \\
+ 2w_1 w_2 \left\{ \begin{array}{l} 1 + 2e_0 + \alpha_1 e_1 + \alpha_2 e_2 \\ + e_0^2 + 2\alpha_1 e_0 e_1 \\ + 2\alpha_2 e_0 e_2 + \alpha_1 \alpha_1 e_1 e_0 \\ + \frac{\alpha_1 (\alpha_1 - 1)}{2} e_1^2 \\ + \frac{\alpha_2 (\alpha_2 - 1)}{2} e_2^2 \end{array} \right\} - 2w_0 \\
- 2w_1 \left\{ 1 + e_0 + \alpha_1 e_1 + \alpha_1 e_0 e_1 + \frac{\alpha_1 (\alpha_1 - 1)}{2} e_1^2 \right\} \\
- 2w_2 \left\{ 1 + e_0 + \alpha_2 e_2 + \alpha_2 e_0 e_2 + \frac{\alpha_2 (\alpha_2 - 1)}{2} e_2^2 \right\}
\end{array} \right] \qquad (36)
$$

Taking expectation of both sides of (36) we get the *MSE* of *t* to the *fda* as

$$MSE\,(t) = \bar{Y}^2 \left[\begin{array}{l} 1 + w_0^2 A_{0(srs)} + w_1^2 A_{1(srs)} + w_2^2 A_{2(srs)} \\ \quad +2w_0 w_1 A_{3(srs)} + 2w_0 w_2 A_{4(srs)} + 2w_1 w_2 A_{5(srs)} \\ \quad -2w_0 - 2w_1 A_{6(srs)} - 2w_2 A_{7(srs)} \end{array} \right] \tag{37}$$

where

$$A_{2(srs)} = \left[1 + \left(\frac{1-f}{n}\right)\left\{C_y^2 + 4\alpha_2 \rho_{yz} C_y C_z + \alpha_2 \left(2\alpha_2 - 1\right) C_z^2\right\}\right]$$

$$A_{4(srs)} = \left[1 + \left(\frac{1-f}{n}\right)\left\{C_y^2 + 2\alpha_2 \rho_{yz} C_y C_z + \frac{\alpha_2 \left(\alpha_2 - 1\right)}{2} C_z^2\right\}\right]$$

$$A_{5(srs)} = \left[1 + \left(\frac{1-f}{n}\right)\left\{ \begin{array}{l} C_y^2 + 2\alpha_1 \rho_{yx} C_y C_x + 2\alpha_2 \rho_{yz} C_y C_z \\ \quad +\alpha_1 \alpha_2 \rho_{xz} C_x C_z \\ \quad +\frac{\alpha_1 \left(\alpha_1 - 1\right)}{2} C_x^2 + \frac{\alpha_2 \left(\alpha_2 - 1\right)}{2} C_z^2 \end{array} \right\}\right]$$

$(A_{0(srs)},\ A_{1(srs)},\ A_{3(srs)},\ A_{6(srs)}\text{and } A_{7(srs)})$ are same as defined earlier.

Minimization of $MSE(t)$ at (37) with respect to (w_0, w_1, w_2) yields

$$\begin{bmatrix} A_{0(srs)} & A_{3(srs)} & A_{4(srs)} \\ A_{3(srs)} & A_{1(srs)} & A_{5(srs)} \\ A_{4(srs)} & A_{5(srs)} & A_{2(srs)} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ A_{6(srs)} \\ A_{7(srs)} \end{bmatrix} \tag{38}$$

After simplification of (38), we get the optimum values of (w_0, w_1, w_2) respectively as

$$w_{00} = \frac{\Delta_0}{\Delta}, \quad w_{10} = \frac{\Delta_1}{\Delta}, \quad w_{20} = \frac{\Delta_2}{\Delta}; \tag{39}$$

where

$$\begin{aligned} \Delta &= \begin{vmatrix} A_{0(srs)} & A_{3(srs)} & A_{4(srs)} \\ A_{3(srs)} & A_{1(srs)} & A_{5(srs)} \\ A_{4(srs)} & A_{5(srs)} & A_{2(srs)} \end{vmatrix} \\ &= A_{0(srs)}(A_{1(srs)} A_{2(srs)} - A_{5(srs)}^2) - A_{3(srs)}(A_{2(srs)} A_{3(srs)} \\ &\quad - A_{4(srs)} A_{5(srs)}) + A_{4(srs)}(A_{3(srs)} A_{5(srs)} - A_{1(srs)} A_{4(srs)}) \end{aligned}$$

$$\Delta_0 = \begin{vmatrix} 1 & A_{3(srs)} & A_{4(srs)} \\ A_{6(srs)} & A_{1(srs)} & A_{5(srs)} \\ A_{7(srs)} & A_{5(srs)} & A_{2(srs)} \end{vmatrix}$$

$$= (A_{1(srs)}A_{2(srs)} - A_{5(srs)}^2) - A_{3(srs)}(A_{2(srs)}A_{6(srs)}$$

$$- A_{5(srs)}A_{7(srs)}) + A_{4(srs)}(A_{5(srs)}A_{6(srs)} - A_{1(srs)}A_{7(srs)})$$

$$\Delta_1 = \begin{vmatrix} A_{0(srs)} & 1 & A_{4(srs)} \\ A_{3(srs)} & A_{6(srs)} & A_{5(srs)} \\ A_{4(srs)} & A_{7(srs)} & A_{2(srs)} \end{vmatrix}$$

$$= A_{0(srs)}(A_{2(srs)}A_{6(srs)} - A_{5(srs)}A_{7(srs)}) - (A_{2(srs)}A_{3(srs)}$$

$$- A_{4(srs)}A_{5(srs)}) + A_{4(srs)}(A_{3(srs)}A_{7(srs)} - A_{4(srs)}A_{6(srs)})$$

$$\Delta_2 = \begin{vmatrix} A_{0(srs)} & A_{3(srs)} & 1 \\ A_{3(srs)} & A_{1(srs)} & A_{6(srs)} \\ A_{4(srs)} & A_{5(srs)} & A_{7(srs)} \end{vmatrix}$$

$$= A_{0(srs)}(A_{1(srs)}A_{7(srs)} - A_{5(srs)}A_{6(srs)}) - A_{3(srs)}(A_{3(srs)}A_{7(srs)}$$

$$- A_{4(srs)}A_{6(srs)}) + (A_{3(srs)}A_{5(srs)} - A_{1(srs)}A_{4(srs)})$$

Substitution of (39) in (37) yields the minimum *MSE* of *t* as

$$\mathit{MSE}_{\min}(t) = \bar{Y}^2 \left[1 - \frac{\Delta_0}{\Delta} - \frac{A_{6(srs)}\Delta_1}{\Delta} - \frac{A_{7(srs)}\Delta_2}{\Delta}\right] \tag{40}$$

Now we state the following theorem.

Theorem-3.1 – The minimum *MSE* of *t* is greater than or equal to *MSE*(*t*) i.e.

$$MSE(t) \geq \mathit{MSE}_{\min}(t) = \bar{Y}^2 \left[1 - \frac{(\Delta_0 + A_{6(srs)}\Delta_1 + A_{7(srs)}\Delta_2)}{\Delta}\right] \tag{41}$$

with equality holding if

$$w_{0i} = \frac{\Delta_i}{\Delta}, \quad i = 0, 1, 2.$$

Remark-3.1 – It is to be mentioned that the class of estimators '*t*' at (32) will attained its minimum *MSE* at (40) only when the optimum values (w_{00}, w_{10}, w_{20}) at (39) of the weights (w_0, w_1, w_2) are known exactly, but in practice the exact values of the population parameters $(C_y, C_x, C_z, \rho_{yx}, \rho_{yz}, \rho_{xz})$ are rarely available. However in repeated surveys or studies based on multiphase sampling, where information regarding the same variates is gathered on several occasions, it is possible to guess quite precisely the values of certain population parameters such as $(C_y, C_x, C_z, \rho_{yx}, \rho_{yz}, \rho_{xz})$. Further we mention that the good guess values of these population parameters can also be obtained from the past data or the experience gathered in due course of time or through a pilot sample survey. This problem has been discussed among others by Murthy (1967, pp. 96–99), Searls (1964), Srivastava (1966), Gleser and Healy (1976), Das and Tripathi (1978), Reddy (1978), Tripathi et al. (1983) and Srivenkataramana and Tracy (1984). Thus the values of such population parameters $(C_y, C_x, C_z, \rho_{yx}, \rho_{yz}, \rho_{xz})$ can be known exactly. We recall that the scalars (α_1, α_2) are real. The values of the scalars (α_1, α_2) are known (or can be known by the experimental practitioner) as the values of (α_1, α_2) yield the form of the estimator. Thus the optimum values (w_{00}, w_{10}, w_{20}) of the corresponding constants (w_0, w_1, w_2) can be obtained quite accurately. Hence we conclude that in practice, an operational estimator can be derived from the suggested class of estimators '*t*' with mean squared error smaller than the conventional estimators.

On the other hand if the values of the population parameters such as $(C_y, C_x, C_z, \rho_{yx}, \rho_{yz}, \rho_{xz})$ are not known (or cannot be made known) at all. In such situations, the practical utility of such estimators is limited. So in such circumstances one can estimate the value of these population parameters by their corresponding sample statistics. Hence the estimates $(\hat{w}_{00}, \hat{w}_{10}, \hat{w}_{20})$ say, of the corresponding optimum values (w_{00}, w_{10}, w_{20}) can be obtained. Thus this also suggests that one can also obtain the operational (feasible) estimator from the proposed class of estimators 't' having mean squared error fewer than the usual estimators.

4 Efficiency Comparison

It is observed from (10), (12) and (20) that the common minimum *MSE*s of the estimators $\hat{\hat{Y}}_{\alpha_1}, \hat{\hat{Y}}_{D1}$ and $\hat{\hat{Y}}^*$ is same, i.e.

$$\mathit{MSE}_{\min}\left(\hat{\hat{Y}}_{\alpha_1}\right) = \mathit{MSE}_{\min}\left(\hat{\hat{Y}}_{D1}\right) = \mathit{MSE}_{\min}\left(\hat{\hat{Y}}^*\right) = \left(\frac{1-f}{n}\right)\bar{Y}^2\left(C_y^2 - K_{yx}^2 C_x^2\right) \quad (42)$$

Now we compare the efficiency of traditional difference estimator with usual unbiased estimator $\hat{\bar{Y}}_0 = \bar{y}$, ratio estimator $\hat{\bar{Y}}_R$ and product estimator $\hat{\bar{Y}}_P$.

From (2), (5), (6) and (12), we have,

$$MSE\left(\hat{\bar{Y}}_0 = \bar{y}\right) - MSE_{\min}\left(\hat{\bar{Y}}_{D1}\right) = \left(\frac{1-f}{n}\right)\bar{Y}^2 K_{yx}^2 C_x^2 \geq 0 \tag{43}$$

$$MSE\left(\hat{\bar{Y}}_R\right) - MSE_{\min}\left(\hat{\bar{Y}}_{D1}\right) = \left(\frac{1-f}{n}\right)\bar{Y}^2 C_x^2 \left(1 - K_{yx}\right)^2 \geq 0 \tag{44}$$

$$MSE\left(\hat{\bar{Y}}_P\right) - MSE_{\min}\left(\hat{\bar{Y}}_{D1}\right) = \left(\frac{1-f}{n}\right)\bar{Y}^2 C_x^2 \left(1 + K_{yx}\right)^2 \geq 0 \tag{45}$$

It follows from (42), (43), (44) and (45) that the estimators $\hat{\bar{Y}}_{\alpha_1}, \hat{\bar{Y}}_{D1}$ and $\hat{\bar{Y}}^*$ are more efficient than usual unbiased estimator $\bar{y}$, ratio estimator $\hat{\bar{Y}}_R$ and product estimator $\hat{\bar{Y}}_P$.

From (23) and (42), we have,

$$\begin{aligned}
&\left[MSE_{\min}\left(\hat{\bar{Y}}_{\alpha_1}\right) = MSE_{\min}\left(\hat{\bar{Y}}_{D1}\right) = MSE_{\min}\left(\hat{\bar{Y}}^*\right)\right] - MSE_{\min}\left(\hat{\bar{Y}}_{Rao}\right) \\
&\quad = \left(\frac{1-f}{n}\right)^2 \bar{Y}^2 \frac{\left(C_y^2 - K_{yx}^2 C_x^2\right)^2}{\left\{1 + \left(\frac{1-f}{n}\right)\left(C_y^2 - K_{yx}^2 C_x^2\right)\right\}} \\
&\quad \geq 0
\end{aligned} \tag{46}$$

It follows from (42), (43), (44), (45) and (46) that the estimator $\hat{\bar{Y}}_{Rao}$ due to Rao (1991) is more efficient than $\bar{y}, \hat{\bar{Y}}_R, \hat{\bar{Y}}_P, \hat{\bar{Y}}_{\alpha_1}, \hat{\bar{Y}}_{D1}, \hat{\bar{Y}}_{\alpha_1}$ and $\hat{\bar{Y}}^*$.

The minimum *MSE* of the difference estimator $\hat{\bar{Y}}_{D1}$ given by (12) can be expressed as

$$\begin{aligned}
&MSE_{\min}\left(\hat{\bar{Y}}_{D1}\right) \\
&\quad = \bar{Y}^2\left[1 + A_{1(srs)} - 2A_{1(srs)} - \frac{\left(1 + A_{1(srs)} - A_{3(srs)} - A_{6(srs)}\right)^2}{\left(A_{0(srs)} + A_{1(srs)} - 2A_{3(srs)}\right)}\right]
\end{aligned} \tag{47}$$

From (16) and (47), we have

$$MSE_{\min}\left(\hat{\bar{Y}}_{D1}\right) - MSE_{\min}\left(\hat{\bar{Y}}_{USV}\right)$$
$$= \bar{Y}^2 \frac{\begin{bmatrix} A_{1(srs)}\left(1 - A_{0(srs)}\right) + A_{3(srs)}\left(A_{3(srs)} - 1\right) \\ +A_{6(srs)}\left(A_{0(srs)} - A_{3(srs)}\right) \end{bmatrix}^2}{\left(A_{0(srs)}A_{1(srs)} - A^2_{3(srs)}\right)\left(A_{0(srs)} + A_{1(srs)} - 2A_{3(srs)}\right)} \geq 0 \tag{48}$$

From (12) and (31) we have

$$MSE_{\min}\left(\hat{\bar{Y}}_{D1}\right) - MSE_{\min}\left(\hat{\bar{Y}}_{D2}\right) = \left(\frac{1-f}{n}\right)\bar{Y}^2 C_y^2 \frac{\left(\rho_{yz} - \rho_{yx}\rho_{xz}\right)^2}{\left(1 - \rho^2_{xz}\right)} \tag{49}$$

which shows that the traditional estimator $\hat{\bar{Y}}_{D2}$ is better than $\hat{\bar{Y}}_{D1}$.

From (16) and (40) we have

$$MSE_{\min}\left(\hat{\bar{Y}}_{USV}\right) - MSE_{\min}\left(t\right)$$
$$= \frac{\bar{Y}^2}{\Delta\Delta_1}\begin{bmatrix} A_{7(srs)}\left(A_{0(srs)}A_{1(srs)} - A^2_{3(srs)}\right) \\ +\left(A_{3(srs)}A_{5(srs)} - A_{1(srs)}A_{4(srs)}\right) \\ +A_{6(srs)}\left(A_{3(srs)}A_{4(srs)} - A_{0(srs)}A_{5(srs)}\right) \end{bmatrix}^2 \geq 0 \tag{50}$$

It shows that the proposed class of estimators *t* is more efficient than the Upadhyaya et al. (1985) estimator $\hat{\bar{Y}}_{USV}$.

Hence from (42), (43), (44), (45), (46), (48) and (49) it is observed that the suggested generalized class of estimators *t* is better than the estimators $\bar{y}$, $\hat{\bar{Y}}_R$, $\hat{\bar{Y}}_P$, $\hat{\bar{Y}}_{\alpha_1}$, $\hat{\bar{Y}}_{D1}$, $\hat{\bar{Y}}^*$ and $\hat{\bar{Y}}_{USV}$.

From (24) and (40) we have that $MSE_{\min}(t) < MSE_{\min}(\hat{\bar{Y}}_{Rao})$ if

$$\left[1 - \frac{\left(\Delta_0 + A_{6(srs)}\Delta_1 + A_{7(srs)}\Delta_2\right)}{\Delta}\right] < \frac{\left(\frac{1-f}{n}\right)\left(C_y^2 - K^2_{yx}C_x^2\right)}{\left\{1 + \left(\frac{1-f}{n}\right)\left(C_y^2 - K^2_{yx}C_x^2\right)\right\}} \tag{51}$$

From (28) and (40) it is observed that $MSE_{\min}(t) < MSE_{\min}\left(\hat{\bar{Y}}_{GS}\right)$ if

$$\frac{a_2a_4^2 - 2a_3a_4a_5 + a_1a_5^2}{\left(a_1a_2 - a_3^2\right)} < \frac{\left(\Delta_0 + A_{6(srs)}\Delta_1 + A_{7(srs)}\Delta_2\right)}{\Delta} \tag{52}$$

Further from (31) and (40) we note that $MSE_{\min}(t) < MSE_{\min}(\hat{\hat{Y}}_{D2})$ if

$$\left[1 - \frac{\left(\Delta_0 + A_{6(srs)}\Delta_1 + A_{7(srs)}\Delta_2\right)}{\Delta}\right] < \left(\frac{1-f}{n}\right) C_y^2(1 - R_{y.xz}^2) \tag{53}$$

Thus from (51), (52) and (53) it is observed that the proposed generalized class of estimators t is more efficient than $\hat{\hat{Y}}_{Rao}, \hat{\hat{Y}}_{GS}$ and $\hat{\hat{Y}}_{D2}$ as long as the conditions (51), (52) and (53) are satisfied respectively.

5 Empirical Study

For numerical comparisons of different estimators, we use the following data sets.

Data I: [Source: Singh and Chaudhary (1986), page 177]
Data II: [Source: Abu-Dayyeh et al. (2003)]
Data III: [Source: Steel and Torrie (1960)]
Data IV: [Source: Cochran (1977)]
Data V: [Source: Ahmed (1997)]
Data VI: [Source: PCR (1998)]

Data	I	II	III	IV	V	VI
N	34	332	30	34	376	424
N	20	80	6	15	159	169
$\bar{Y}$	856.41	1093.1	0.6860	4.92	316.65	646.215
$\bar{X}$	208.88	181.57	4.6537	2.59	141.13	4533.981
$\bar{Z}$	199.44	143.37	0.8077	2.91	1075.31	325.0325
C_y	0.86	0.7626	0.4803	1.01232	0.7721	1.509
C_x	0.72	0.7684	0.2295	1.23187	0.845	1.342
C_z	0.75	0.7616	0.7493	1.05351	0.7746	1.335
ρ_{yx}	0.45	0.973	0.7194	0.7326	0.9106	0.623
ρ_{yz}	0.45	0.862	0.04996	0.643	0.9094	0.907
ρ_{xz}	0.98	0.842	0.4074	0.6837	0.8614	0.682

Table 1 gives the *PRE*'s of $\hat{\hat{Y}}_R, \hat{\hat{Y}}_P, \hat{\hat{Y}}_{D1}, \hat{\hat{Y}}_{Rao}, \hat{\hat{Y}}_{GS}$ and $\hat{\hat{Y}}_{D2}$ estimators with respect to $\bar{y}$ for six data sets respectively.

Table 2 gives the *PRE* of $\hat{\hat{Y}}_{USV}$ with respect to $\bar{y}$ for $\alpha_1 = (-1, 1)$, for six data sets.

Table 1 *PRE*'s of different estimators of population mean $\bar{Y}$ with respect to $\bar{y}$

Estimator	Data I	Data II	Data III	Data IV	Data V	Data VI
$\hat{\bar{Y}}_R$	105.55	1835.92	94.62	143.30	488.77	146.46
$\hat{\bar{Y}}_P$	40.74	25.15	71.44	23.45	23.86	34.49
$\hat{\bar{Y}}_{D1}$	125.39	1877.19	103.33	215.84	585.45	163.43
$\hat{\bar{Y}}_{Rao}$	126.92	1877.75	106.40	219.66	585.67	164.24
$\hat{\bar{Y}}_{Gs}$	126.93	1877.75	106.42	219.89	585.67	164.25
$\hat{\bar{Y}}_{D2}$	125.71	2127.83	174.04	235.09	907.16	563.97

Table 2 *PRE*'s of $\hat{\bar{Y}}_{USV}$ with respect to $\bar{y}$

Data I		Data II		Data III		Data IV		Data V		Data VI	
α_1	*PRE*	α_1	*PRE*	α_1	*PRE*	α_1	*PRE*	α_1	*PRE*	α_1	*PRE*
−1	127.66	−1	1878.66	−1	106.76	−1	227.99	−1	586.30	−1	164.74
1	126.24	1	2049.28	1	106.20	1	215.95	1	588.70	1	163.54

Table 3(a) depicts the *PRE*s of proposed class of estimators t with respect to $\bar{y}$ at different values of (α_1, α_2) for data sets I, II, III.

Table 3(a) *PRE*'s of proposed class of estimators *t* for population mean $\bar{Y}$ with respect to $\bar{y}$ (for data sets I, II, III)

Data I			Data II			Data III		
α_1	α_2	*PRE*	α_1	α_2	*PRE*	α_1	α_2	*PRE*
−16.56	−16.56	**17654.39**	−6.5	−6.9	**772384**	−4	−4	1286.97
−16.55	−16.55	11434.21	−6.5	−6.8	234504.2	−3	−3	240.98
−16.54	−16.54	8469.43	−6.5	−6.7	139357.5	−2	−2	194.41
−16.53	−16.53	6734.46	−6.5	−6.6	99699.33	−1	−1	180.01
−16.52	−16.52	5595.56	−6.5	−6.5	77951.88	1	1	174.25
−16.51	−16.51	4790.57	−6.4	−6.4	34176.73	2	2	177.14
−16.5	−16.5	4191.43	−6.3	−6.3	22046.63	3	3	183.83
−16	−16	657.28	−6.2	−6.2	16359.33	4	4	196.20
−10	−10	153.84	−6.1	−6.1	13060.19	5	5	219.66
−5	−5	133.88	−6	−6	10906.9	6	6	273.01
−4	−4	131.95	−5	−5	4435.19	7	7	484.92
−3	−3	130.37	−4	−4	3039.74	−4	6	**9201.75**
−2	−2	129.08	−3	−3	2477.50	−3	5	341.57
−1	−1	128.03	−2	−2	2220.59	−2	4	230.89
1	1	126.56	−1	−1	2130.16	−1	3	195.37
2	2	126.10	1	1	2358.74	1	2	178.42
3	3	125.82	2	2	2796.94	2	1	174.06
4	4	125.72	3	3	3848.96	−4	5	386.31
5	5	125.81	4	4	7795.28	−4	4	253.76
8	8	127.58	4.5	4.5	20478.73	−4	3	209.95

(*Continued*)

Table 3(a) Continued

Data I			Data II			Data III		
α_1	α_2	*PRE*	α_1	α_2	*PRE*	α_1	α_2	*PRE*
10	10	130.80	4.6	4.6	31462.35	−4	2	189.32
12	12	137.87	4.7	4.7	69729.23	−4	1	178.73
15	15	187.23	1	2	2404.66	−3	6	1354.17
16	16	328.47	2	3	2902.67	−3	5	341.57
16.1	16.1	378.07	4	5	9706.82	−3	4	241.09
16.2	16.2	456.93	−1	−2	2135.10	−3	3	204.27
16.3	16.3	601.84	−3	−4	2539.23	−3	2	186.40
16.4	16.4	955.32	−4	−5	3182.99	−2	6	788.06
16.5	16.5	3122.77	−5	−6	4877.70	−2	5	310.28
*	*	*	−6	−7	15453.26	−2	4	230.89

Table 3(b) indicates the *PRE*s of proposed class of estimators t with respect to $\bar{y}$ at different values of (α_1, α_2) for data sets IV, V, VI.

Table 3(b) *PRE*'s of proposed class of estimators *t* with respect to $\bar{y}$ (for data sets IV, V, VI)

Data IV			Data V			Data VI		
α_1	α_2	*PRE*	α_1	α_2	*PRE*	α_1	α_2	*PRE*
−5.5	−5.6	**18686.57**	−13.8	−13.8	**146480.3**	−11.3	−11.3	56544.35
−5.5	−5.5	9127.82	−13.5	−13.5	16447.9	−11	−11	7475.86
−5	−5	1011.91	−13	−13	6808.17	−10	−10	2084.82
−4	−4	443.93	−12	−12	3275.97	−8	−8	986.33
−3	−3	323.39	−10	−10	1749.71	−5	−5	659.03
−2	−2	273.50	−8	−8	1285.83	−4	−4	616.64
−1	−1	248.82	−5	−5	1012.38	−3	−3	588.91
1	1	235.59	−4	−4	967.68	−2	−2	572.19
2	2	244.97	−3	−3	936.87	−1	−1	564.63
3	3	274.84	−2	−2	917.52	1	1	575.12
4	4	372.70	−1	−1	908.24	2	2	594.85
5	5	2084.42	1	1	918.17	3	3	627.61
5.1	5.1	8806.16	2	2	938.25	4	4	679.08
−5.5	−5.4	6136.23	3	3	970.32	5	5	760.58
−5.5	−5.2	3812.23	4	4	1017.20	6	6	897.22
−5.5	−5.1	3242.17	5	5	1083.64	7	7	1156.67
−5.5	−5	2838.5	6	6	1177.62	8	8	1802.22
−5.5	−4.5	1851.49	7	7	1313.39	9	9	5816.34
−5.5	−4	1470.75	8	8	1518.41	9.3	9.3	24933.39
−5.5	−3	1210.17	10	10	2479.08	9	9.4	**100422**
−5.5	−2	1286.14	12	12	13846.51	*	*	*
−5.5	−1	2344.68	12.1	12.1	18724.25	*	*	*
−5.4	−5.6	5179.15	12.2	12.2	29101.79	*	*	*
−5.3	−5.6	3067.52	12.3	12.3	66293.65	*	*	*
−5.2	−5.6	2208.87	*	*	*	*	*	*
−5.1	−5.6	1743.17	*	*	*	*	*	*

We measured Percent Relative Efficiencies (*PRE*s) of various estimators along with our proposed generalized class of estimators *t* with respect to$\bar{y}$. It is observed that from the entries of the Tables 1, 2, 3(a) and 3(b) that the suggested generalized class of estimators *t* gives the largest *PRE* (17654.39%, 772384.00%, 9201.75%, 18686.57%, 146480.30%, and 100422.00%) for data set I to IV respectively. Using the proposed generalized class of estimators *t* over other existing estimators, there is considerable gain in efficiency. Thus there is ample room to pick up the scalars (α_1, α_2) in order to obtain estimators better than the existing estimators. Finally our recommendation is in favor of the proposed generalized class of estimators *t* for its use in practice.

6 Estimation of Population Mean Under Stratified Random Sampling

We consider a finite population $\Omega = \{\Omega_1, \Omega_2, \ldots, \Omega_N\}$ of *N* units divided into *L* strata with the hth stratum $(h = 1, 2, \ldots, L)$ having N_h units such that $\sum_{i=1}^{L} N_h = N$. Let y_{hi} and $(x_{hi}, z_{hi})(i = 1, 2, \ldots, N_h)$ respectively be the observations of study variable *y* and auxiliary variables (*x*, *z*) for the ith population unit in the hth stratum. A simple random sample of size n_h is drawn without replacement from the hth stratum such that $\sum_{i=1}^{L} n_h = n$.

Let $\bar{y}_{(st)} = \sum_{h=1}^{L} W_h \bar{y}_h, \bar{x}_{(st)} = \sum_{h=1}^{L} W_h \bar{x}_h$ and $\bar{z}_{(st)} = \sum_{h=1}^{L} W_h \bar{z}_h$ be the sample means corresponding to the population means $\bar{Y} = \sum_{h=1}^{L} W_h \bar{Y}_h$, $\bar{X} = \sum_{h=1}^{L} W_h \bar{X}_h$ and $\bar{Z} = \sum_{h=1}^{L} W_h \bar{Z}_h$ of the variables *y*, *x* and *z* respectively, where $\bar{y}_h = \frac{1}{n_h} \sum_{i=1}^{n_h} y_{hi}$, $\bar{x}_h = \frac{1}{n_h} \sum_{i=1}^{n_h} x_{hi}$ and $\bar{z}_h = \frac{1}{n_h} \sum_{i=1}^{n_h} z_{hi}$ be the sample means corresponding to the population means $\bar{Y}_h = \sum_{i=1}^{N_h} \frac{y_{hi}}{N_h}$, $\bar{X}_h = \sum_{i=1}^{N_h} \frac{x_{hi}}{N_h}$ and $\bar{Z}_h = \sum_{i=1}^{N_h} \frac{z_{hi}}{N_h}$ in the hth stratum respectively with known stratum weight $W_h = \frac{N_h}{N}$.

Further we denote

$$C_{yh} = \frac{S_{yh}}{\bar{Y}_h}, \quad C_{xh} = \frac{S_{xh}}{\bar{X}_h}, \quad C_{zh} = \frac{S_{zh}}{\bar{Z}_h}, \quad \rho_{yxh} = \frac{S_{yxh}}{S_{yh} S_{xh}},$$

$$\rho_{yzh} = \frac{S_{yzh}}{S_{yh} S_{zh}}, \quad \rho_{xzh} = \frac{S_{xzh}}{S_{xh} S_{zh}},$$

$$S_{yh}^2=\frac{1}{N_h-1}\sum_{i=1}^{N_h}\left(y_{hi}-\bar{Y}_h\right)^2,\quad S_{xh}^2=\frac{1}{N_h-1}\sum_{i=1}^{N_h}\left(x_{hi}-\bar{X}_h\right)^2,$$

$$S_{zh}^2=\frac{1}{N_h-1}\sum_{i=1}^{N_h}\left(z_{ih}-\bar{Z}_h\right)^2,$$

$$S_{yxh}=\frac{1}{N_h-1}\sum_{i=1}^{N_h}\left(y_{hi}-\bar{Y}_h\right)\left(x_{hi}-\bar{X}_h\right),$$

$$S_{yzh}=\frac{1}{N_h-1}\sum_{i=1}^{N_h}\left(y_{hi}-\bar{Y}_h\right)\left(z_{hi}-\bar{Z}_h\right),$$

$$S_{xzh}=\frac{1}{N_h-1}\sum_{i=1}^{N_h}\left(x_{hi}-\bar{X}_h\right)\left(z_{hi}-\bar{Z}_h\right),\quad V_{200}=\sum_{h=1}^{L}\gamma_h W_h^2 S_{yh}^2,$$

$$V_{020}=\sum_{h=1}^{L}\gamma_h W_h^2 S_{xh}^2,\quad V_{002}=\sum_{h=1}^{L}\gamma_h W_h^2 S_{zh}^2,$$

$$V_{110}=\sum_{h=1}^{L}\gamma_h W_h^2 S_{yxh},\quad V_{101}=\sum_{h=1}^{L}\gamma_h W_h^2 S_{yzh},$$

$$V_{011}=\sum_{h=1}^{L}\gamma_h W_h^2 S_{xzh},\quad \gamma_h=\left(\frac{1-f_h}{n_h}\right).$$

In the following section we have presented review of some existing estimators with their properties.

7 Reviewing Some Existing Estimators in Stratified Random Sampling

The conventional stratified sample mean estimator for population mean $\bar{Y}$ of y is defined by

$$\hat{\bar{Y}}_{0(st)}=\bar{y}_{(st)}=\sum_{h=1}^{L}W_h\bar{y}_h \tag{54}$$

whose variance/*MSE* is given by

$$Var\left(\hat{\bar{Y}}_{0(st)}\right) = MSE\left(\hat{\bar{Y}}_{0(st)}\right) = V_{200} = \sum_{h=1}^{L} \gamma_h W_h^2 S_{yh}^2 \tag{55}$$

The combined ratio estimator for $\bar{Y}$ is given by

$$\hat{\bar{Y}}_{R(st)} = \bar{y}_{(st)} \left(\frac{\bar{X}}{\bar{x}_{(st)}}\right) \tag{56}$$

To the *fda*, the *MSE* of $\hat{\bar{Y}}_{R(st)}$ is given by

$$MSE\left(\hat{\bar{Y}}_{R(st)}\right) = \left(V_{200} + R_1^2 V_{020} - 2R_1 V_{110}\right) \tag{57}$$

The combined product estimator for $\bar{Y}$ is defined by

$$\hat{\bar{Y}}_{P(st)} = \bar{y}_{(st)} \left(\frac{\bar{x}_{(st)}}{\bar{X}}\right) \tag{58}$$

The *MSE* of $\hat{\bar{Y}}_{P(st)}$ to the *fda* is given by

$$MSE\left(\hat{\bar{Y}}_{P(st)}\right) = \left(V_{200} + R_1^2 V_{020} + 2R_1 V_{110}\right) \tag{59}$$

Following the approach adopted by Srivastava (1967), we define a class of estimators for population mean $\bar{Y}$ as

$$\hat{\bar{Y}}_{S(st)} = \bar{y}_{(st)} \left(\frac{\bar{x}_{(st)}}{\bar{X}}\right)^{\alpha_1} \tag{60}$$

We mention that for $\alpha_1 = -1$, $\hat{\bar{Y}}_{S(st)}$ reduces to $\hat{\bar{Y}}_{R(st)}$ while for $\alpha_1 = 1$ it reduces to the product estimator $\hat{\bar{Y}}_{P(st)}$. If we set $\alpha_1 = 0$, then $\hat{\bar{Y}}_{S(st)}$ reduces to usual unbiased estimator $\bar{y}_{(st)}$.

The *MSE* of $\hat{\bar{Y}}_{S(st)}$ to the *fda* is given by

$$MSE\left(\hat{\bar{Y}}_{S(st)}\right) = \left(V_{200} + \alpha_1^2 R_1^2 V_{020} + 2\alpha_1 R_1 V_{110}\right) \tag{61}$$

which is minimum when

$$\alpha_1 = -\frac{V_{110}}{R_1 V_{020}} = \alpha_{1(opt)}, \text{ say} \tag{62}$$

Thus the corresponding minimum *MSE* of $\hat{\hat{Y}}_{S(st)}$ is given by

$$MSE_{\min}\left(\hat{\hat{Y}}_{S(st))}\right) = \left(V_{200} - \frac{V_{110}^2}{V_{020}}\right) \tag{63}$$

which is same as the minimum *MSE* of the difference estimator

$$\hat{\hat{Y}}_{D1(st)} = \bar{y}_{(st)} + d\left(\bar{X} - \bar{x}_{(st)}\right) \tag{64}$$

i.e.

$$MSE_{\min}\left(\hat{\hat{Y}}_{S(st)}\right) = MSE_{\min}\left(\hat{\hat{Y}}_{D1(st)}\right) = \left(V_{200} - \frac{V_{110}^2}{V_{020}}\right) \tag{65}$$

The stratified version of Upadhyaya et al. (1985) estimator is given by

$$\hat{\hat{Y}}_{USV(st)} = w_0\bar{y}_{(st)} + w_1\bar{y}_{(st)}\left(\frac{\bar{x}_{(st)}}{\bar{X}}\right)^{\alpha_1} \tag{66}$$

The *MSE* of $\hat{\hat{Y}}_{USV(st)}$ to the *fda* is given by

$$\begin{aligned} MSE\left(\hat{\hat{Y}}_{USV(st)}\right) = [&\bar{Y}^2 + w_0^2 A_{0(st)} + w_1^2 A_{1(st)} \\ &+ 2w_0w_1A_{3(st)} - 2w_0\bar{Y}^2 - 2w_1A_{6(st)}] \end{aligned} \tag{67}$$

where

$$\begin{aligned} A_{0(st)} &= \left(\bar{Y}^2 + V_{200}\right) \\ A_{1(st)} &= \left[\bar{Y}^2 + V_{200} + 4\alpha_1R_1V_{110} + \alpha_1(2\alpha_1 - 1)R_1^2V_{020}\right] \\ A_{3(st)} &= \left[\bar{Y}^2 + V_{200} + 2\alpha_1R_1V_{110} + \alpha_1\frac{(\alpha_1 - 1)}{2}R_1^2V_{020}\right] \\ A_{6(st)} &= \left[\bar{Y}^2 + \alpha_1R_1V_{110} + \alpha_1\frac{(\alpha_1 - 1)}{2}R_1^2V_{020}\right] \end{aligned}$$

The $MSE(\hat{\hat{Y}}_{USV(st)})$ is minimized for

$$w_0 = \frac{\Delta_{0(st)}^*}{\Delta_{(st)}^*}, w_1 = \frac{\Delta_{1(st)}^*}{\Delta_{(st)}^*} \tag{68}$$

Thus the corresponding minimum *MSE* of $\hat{\hat{Y}}_{USV(st)}$ is given by

$$MSE_{\min}\left(\hat{\hat{Y}}_{USV(st)}\right)$$

$$= \left[\bar{Y}^2 - \frac{\left\{A_{1(st)}\bar{Y}^4 - 2A_{3(st)}A_{6(st)}\bar{Y}^2 + A_{0(st)}A_{6(st)}^2\right\}}{\left(A_{0(st)}A_{1(st)} - A_{3(st)}^2\right)}\right] \tag{69}$$

where

$$\Delta_{(st)}^* = \left(A_{0(st)}A_{1(st)} - A_{3(st)}^2\right)$$

$$\Delta_{0(st)}^* = \left(A_{1(st)} - A_{3(st)}A_{6(st)}\right)$$

$$\Delta_{1(st)}^* = \left(A_{0(st)}A_{6(st)} - A_{3(st)}\right)$$

For $w_0 + w_1 = 1 \Rightarrow w_1 = (1 - w_0)$ in (66), the class of estimators $\hat{\hat{Y}}_{USV(st)}$ reduces to the estimator

$$\hat{\hat{Y}}_{USV(st)}^* = w_0\bar{y}_{st} + (1 - w_0)\bar{y}_{st}\left(\frac{\bar{x}_{st}}{\bar{X}}\right)^{\alpha_1} \tag{70}$$

To the *fda*, the *MSE* of $\hat{\hat{Y}}_{USV(st)}^*$ is given by

$$MSE\left(\hat{\hat{Y}}_{USV(st)}^*\right) = \begin{bmatrix} V_{100} + A_{1(st)} - 2A_{6(st)} \\ \quad +w_0^2\left(A_{0(st)} + A_{1(st)} - 2A_{3(st)}\right) \\ \quad -2w_0\left(\bar{Y}^2 + A_{1(st)} - A_{3(st)} - A_{6(st)}\right) \end{bmatrix} \tag{71}$$

which is minimized for

$$w_{0(opt)} = \frac{\left(\bar{Y}^2 + A_{1(st)} - A_{3(st)} - A_{6(st)}\right)}{\left(A_{0(st)} + A_{1(st)} - 2A_{3(st)}\right)} = \frac{(V_{110} + \alpha_1 R_1 V_{020})}{\alpha_1 R_1 V_{020}} \tag{72}$$

Thus the corresponding minimum *MSE* of $\hat{\hat{Y}}_{USV(st)}^*$ is given by

$$MSE_{\min}(\hat{\hat{Y}}_{USV(st)}^*)$$

$$= \left[\bar{Y}^2 + A_{1(st)} - 2A_{6(st)} - \frac{\left(\bar{Y}^2 + A_{1(st)} - A_{3(st)} - A_{6(st)}\right)^2}{\left(A_{0(st)} + A_{1(st)} - 2A_{3(st)}\right)}\right]$$

$$= \left(V_{200} - \frac{V_{110}^2}{V_{020}}\right) = MSE_{\min}\left(\hat{\hat{Y}}_{D1(st)}^*\right) \tag{73}$$

Stratified version of Rao (1991) estimator for population mean $\bar{Y}$ is given by

$$\hat{\bar{Y}}_{Rao(st)} = \alpha_1 \bar{y}_{st} + \alpha_2 (\bar{X} - \bar{x}_{st}) \tag{74}$$

where (α_1, α_2) are suitably chosen constants such that *MSE*$(\hat{\bar{Y}}_{Rao(st)})$ is minimum.

The optimum values of (α_1, α_2) along with minimum *MSE* of $\hat{\bar{Y}}_{Rao(st)}$ are respectively given by

$$\left.\begin{aligned} \alpha_{1(opt)} &= \left\{ \frac{\bar{Y}^2 V_{020}}{V_{020}\left(\bar{Y}^2 + V_{200}\right) - V_{110}^2} \right\} \\ \alpha_{2(opt)} &= -\left\{ \frac{\bar{Y}^2 V_{110}}{V_{020}\left(\bar{Y}^2 + V_{200}\right) - V_{110}^2} \right\} \end{aligned}\right\} \tag{75}$$

and

$$MSE_{\min}\left(\hat{\bar{Y}}_{Rao(st)}\right) = \frac{\bar{Y}^2 \left\{V_{020} V_{200} - V_{110}^2\right\}}{\left\{V_{020}\left(\bar{Y}^2 + V_{200}\right) - V_{110}^2\right\}} \tag{76}$$

Gupta and Shabbir (2008) suggested the following estimator for $\bar{Y}$ as,

$$\hat{\bar{Y}}_{GS(st)} = \left\{\alpha_3 \bar{y}_{st} + \alpha_4 \left(\bar{X} - \bar{x}_{st}\right)\right\} \left(\frac{\bar{X}}{\bar{x}_{st}}\right) \tag{77}$$

The *MSE* of $\hat{\bar{Y}}_{GS(st)}$ to the *fda* is given by

$$\begin{aligned} MSE(\hat{\bar{Y}}_{GS(st)}) = [&\bar{Y}^2 + \alpha_3^2 a_{1(st)} + \alpha_4^2 a_{2(st)} + 2\alpha_3 \alpha_4 a_{3(st)} \\ &- 2\alpha_3 a_{4(st)} - 2\alpha_4 a_{5(st)}] \end{aligned} \tag{78}$$

where

$$\begin{aligned} a_{1(st)} &= \left[\bar{Y}^2 + V_{200} + 3R_1^2 V_{020} - 4R_1 V_{110}\right] \\ a_{2(st)} &= V_{020} \\ a_{3(st)} &= (2R_1 V_{020} - V_{110}) \\ a_{4(st)} &= \left(R_1^2 V_{020} - R_1 V_{110} + \bar{Y}^2\right) \\ a_{5(st)} &= R_1 V_{020} \end{aligned}$$

The *MSE* of $\hat{\bar{Y}}_{GS(st)}$ is minimum when

$$\left.\begin{aligned}\alpha_{3(opt)} &= \frac{(a_{2(st)}a_{4(st)} - a_{3(st)}a_{5(st)})}{(a_{1(st)}a_{2(st)} - a_{3(st)}^2)} \\ \alpha_{4(opt)} &= \frac{(a_{1(st)}a_{5(st)} - a_{3(st)}a_{4(st)})}{(a_{1(st)}a_{2(st)} - a_{3(st)}^2)}\end{aligned}\right\} \tag{79}$$

Thus the minimum *MSE* of $\hat{\bar{Y}}_{GS(st)}$ is given by

$$MSE_{\min}\left(\hat{\bar{Y}}_{GS(st)}\right) = \left[\bar{Y}^2 - \frac{\left(a_{2(st)}a_{4(st)}^2 - 2a_{3(st)}a_{4(st)}a_{5(st)} + a_{4(st)}a_{5(st)}^2\right)}{\left(a_{1(st)}a_{2(st)} - a_{3(st)}^2\right)}\right] \tag{80}$$

The usual difference estimator using two auxiliary variables in stratified random sampling is defined by

$$\hat{\bar{Y}}_{D2(st)} = \bar{y}_{st} + k_1\left(\bar{X} - \bar{x}_{st}\right) + k_2\left(\bar{Z} - \bar{z}_{st}\right) \tag{81}$$

where k_1 and k_2 are constants whose values are to be obtained.

The *MSE* of $\hat{\bar{Y}}_{D2(st)}$ is given by

$$MSE\left(\hat{\bar{Y}}_{D2(st)}\right) = \left[\begin{array}{l} V_{200} + k_1^2 V_{020} + k_2^2 V_{002} \\ \quad +2k_1k_2V_{011} - 2k_1V_{110} - 2k_2V_{101}\end{array}\right] \tag{82}$$

which is minimized for

$$\left.\begin{aligned}k_{1(opt)} &= \frac{(V_{002}V_{110} - V_{011}V_{101})}{(V_{020}V_{002} - V_{011}^2)} \\ k_{2(opt)} &= \frac{(V_{020}V_{101} - V_{011}V_{110})}{(V_{020}V_{002} - V_{011}^2)}\end{aligned}\right\} \tag{83}$$

Thus the corresponding minimum *MSE* of $\hat{\bar{Y}}_{D2(st)}$ is given by

$$MSE_{\min}\left(\hat{\bar{Y}}_{D2(st)}\right) = \left[V_{200} - \frac{\left(V_{110}^2V_{002} - 2V_{011}V_{101}V_{110} + V_{020}V_{101}^2\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}\right] \tag{84}$$

8 Suggested Class of Estimators for Population Mean in Stratified Random Sampling

Motivated by Upadhyaya et al. (1985), we propose a generalized class of estimators based on two auxiliary variables (x, z) for population mean $\bar{Y}$in stratified random sampling as

$$t_{(st)} = w_0 \bar{y}_{st} + w_1 \bar{y}_{st} \left(\frac{\bar{x}_{st}}{\bar{X}} \right)^{\alpha_1} + w_2 \bar{y}_{st} \left(\frac{\bar{z}_{st}}{\bar{Z}} \right)^{\alpha_2} \tag{85}$$

where (w_0, w_1, w_2) are appropriately elected weights whose sum need not be unity and (α_1, α_2) are design parameters. The constants (α_1, α_2) may take positive $(+,+)$ or negative $(-,-)$ or positive-negative $(+,-)$ or negative-positive $(-,+)$ values to form product-type or ratio-type or product-cum-ratio-type or ratio-cum-product-type estimator.

To obtain the bias and *MSE* of the proposed estimator $t_{(st)}$, we write, $\bar{y}_{st} = \bar{Y}(1 + e_{0(st)}), \bar{x}_{st} = \bar{X}(1 + e_{1(st)})$ and $\bar{z}_{st} = \bar{Z}(1 + e_{2(st)})$ such that $E(e_{0(st)}) = E(e_{1(st)}) = E(e_{2(st)}) = 0$,

$$E(e_{0(st)}^2) = \frac{1}{\bar{Y}^2} \sum_{h=1}^{L} W_h^2 \gamma_h S_{yh}^2, \quad E\left(e_{1(st)}^2\right) = \frac{1}{\bar{X}^2} \sum_{h=1}^{L} W_h^2 \gamma_h S_{xh}^2,$$

$$E\left(e_{2(st)}^2\right) = \frac{1}{\bar{Z}^2} \sum_{h=1}^{L} W_h^2 \gamma_h S_{zh}^2,$$

$$E(e_{0(st)} e_{1(st)}) = \frac{1}{\bar{Y}\bar{X}} \sum_{h=1}^{L} W_h^2 \gamma_h S_{yxh},$$

$$E(e_{0(st)} e_{2(st)}) = \frac{1}{\bar{Y}\bar{Z}} \sum_{h=1}^{L} W_h^2 \gamma_h S_{yzh} \quad \text{and}$$

$$E(e_{1(st)} e_{2(st)}) = \frac{1}{\bar{X}\bar{Z}} \sum_{h=1}^{L} W_h^2 \gamma_h S_{xzh}.$$

Expressing (85) in terms of *e*'s we have

$$t_{(st)} = \bar{Y} \left[\begin{array}{l} w_0 \left(1 + e_{0(st)}\right) + w_1 \left(1 + e_{0(st)}\right) \left(1 + e_{1(st)}\right)^{\alpha_1} \\ \quad + w_2 \left(1 + e_{0(st)}\right) \left(1 + e_{1(st)}\right)^{\alpha_2} \end{array} \right] \tag{86}$$

We assume that $|e_{i(st)}| \ll 1, i = 1, 2$ so that $(1 + e_{i(st)})^{\alpha_i}, i = 1, 2$ are expandable. Expanding the right hand side of (85), multiplying out and ignoring terms of *e*'s having power greater than two, we have

$$t_{(st)} \cong \bar{Y} \left[\begin{array}{l} w_0 \left(1 + e_{0(st)}\right) \\ \quad +w_1 \left\{ 1 + e_{0(st)} + \alpha_1 e_{1(st)} + \alpha_1 e_{0(st)} e_{1(st)} \right. \\ \quad \left. + \frac{\alpha_1(\alpha_1 - 1)}{2} e_{1(st)}^2 \right\} \\ \quad +w_2 \left\{ 1 + e_{0(st)} + \alpha_2 e_{2(st)} + \alpha_2 e_{0(st)} e_{2(st)} \right. \\ \quad \left. + \frac{\alpha_2(\alpha_2 - 1)}{2} e_{2(st)}^2 \right\} \end{array} \right]$$

or

$$\left(t_{(st)} - \bar{Y}\right) \cong \bar{Y} \left[\begin{array}{l} w_0 \left(1 + e_{0(st)}\right) \\ \quad +w_1 \left\{ 1 + e_{0(st)} + \alpha_1 e_{1(st)} + \alpha_1 e_{0(st)} e_{1(st)} \right. \\ \quad \left. + \frac{\alpha_1 \left(\alpha_1 - 1\right)}{2} e_{1(st)}^2 \right\} \\ \quad +w_2 \left\{ 1 + e_{0(st)} + \alpha_2 e_{2(st)} + \alpha_2 e_{0(st)} e_{2(st)} \right. \\ \quad \left. + \frac{\alpha_2 \left(\alpha_2 - 1\right)}{2} e_{2(st)}^2 \right\} - 1 \end{array} \right] \tag{87}$$

Taking expectation of both sides of (87), we get the bias of $t_{(st)}$ to the *fda* as

$$\begin{aligned} B\left(t_{(st)}\right) = & \left[\bar{Y}(w_0 - 1) + w_1 \left\{ \bar{Y} + \frac{\alpha_1 \left(\alpha_1 - 1\right)}{2} R_1 \frac{V_{020}}{\bar{X}} + \alpha_1 \frac{V_{110}}{\bar{X}} \right\} \right. \\ & \left. + w_2 \left\{ \bar{Y} + \frac{\alpha_2 \left(\alpha_2 - 1\right)}{2} R_2 \frac{V_{002}}{\bar{Z}} + \alpha_2 \frac{V_{101}}{\bar{Z}} \right\} \right] \end{aligned} \tag{88}$$

Squaring both sides of (87), ignoring terms of *e*'s having power greater than two and then taking expectation of both sides we get the *MSE* of $t_{(st)}$ to the *fda* as

$$MSE\left(t_{(st)}\right) = \left[\begin{array}{l} \bar{Y}^2 + w_0^2 A_{0(st)} + w_1^2 A_{1(st)} + w_2^2 A_{2(st)} + 2w_0 w_1 A_{3(st)} \\ \quad +2w_0 w_2 A_{4(st)} + 2w_1 w_2 A_{5(st)} - 2w_0 \bar{Y}^2 \\ \quad -2w_1 A_{6(st)} - 2w_2 A_{7(st)} \end{array}\right] \tag{89}$$

where

$$A_{2(st)} = \left[\bar{Y}^2 + V_{200} + 4\alpha_2 R_2 V_{101} + \alpha_2 \left(2\alpha_2 - 1\right) R_2^2 V_{002}\right]$$

$$A_{4(st)} = \left[\bar{Y}^2 + V_{200} + 2\alpha_2 R_2 V_{101} + \alpha_2 \frac{(\alpha_2 - 1)}{2} R_2^2 V_{002}\right]$$

$$A_{5(st)} = \Bigg[\bar{Y}^2 + V_{200} + 2\alpha_1 R_1 V_{110} + 2\alpha_2 R_2 V_{101} + \alpha_1 \alpha_2 R_1 R_2 V_{011} + \alpha_1 \frac{(\alpha_1 - 1)}{2} R_1^2 V_{020} + \alpha_2 \frac{(\alpha_2 - 1)}{2} R_2^2 V_{002}\Bigg]$$

$$A_{7(st)} = \left[\bar{Y}^2 + \alpha_2 R_2 V_{101} + \alpha_2 \frac{(\alpha_2 - 1)}{2} R_2^2 V_{002}\right]$$

$(A_{0(st)}, A_{1(st)}, A_{3(st)}$ and $A_{6(st)})$ are same as defined earlier.

Minimization of $MSE(t_{(st)})$ at (89) with respect to (w_0, w_1, w_2) yields

$$\begin{bmatrix} A_{0(st)} & A_{3(st)} & A_{4(st)} \\ A_{3(st)} & A_{1(st)} & A_{5(st)} \\ A_{4(st)} & A_{5(st)} & A_{2(st)} \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \bar{Y}^2 \\ A_{6(st)} \\ A_{7(st)} \end{bmatrix} \tag{90}$$

Solving (90), we get the optimum values of (w_0, w_1, w_2) respectively as

$$w_{00} = \frac{\Delta_{0(st)}}{\Delta_{(st)}}, \quad w_{10} = \frac{\Delta_{1(st)}}{\Delta_{(st)}}, \quad w_{20} = \frac{\Delta_{2(st)}}{\Delta_{(st)}}. \tag{91}$$

where

$$\Delta_{(st)} = \begin{vmatrix} A_{0(st)} & A_{3(st)} & A_{4(st)} \\ A_{3(st)} & A_{1(st)} & A_{5(st)} \\ A_{4(st)} & A_{5(st)} & A_{2(st)} \end{vmatrix}$$

$$= A_{0(st)} \left(A_{1(st)} A_{2(st)} - A_{5(st)}^2\right)$$

$$
\begin{aligned}
&- A_{3(st)}\left(A_{2(st)}A_{3(st)} - A_{4(st)}A_{5(st)}\right)\\
&+ A_{4(st)}\left(A_{3(st)}A_{5(st)} - A_{1(st)}A_{4(st)}\right)
\end{aligned}
$$

$$
\Delta_{0(st)} = \begin{vmatrix} \bar{Y}^2 & A_{3(st)} & A_{4(st)} \\ A_{6(st)} & A_{1(st)} & A_{5(st)} \\ A_{7(st)} & A_{5(st)} & A_{2(st)} \end{vmatrix}
$$

$$
\begin{aligned}
&= \bar{Y}^2\left(A_{1(st)}A_{2(st)} - A_{5(st)}^2\right)\\
&- A_{3(st)}\left(A_{2(st)}A_{6(st)} - A_{5(st)}A_{7(st)}\right)\\
&+ A_{4(st)}\left(A_{5(st)}A_{6(st)} - A_{1(st)}A_{7(st)}\right)
\end{aligned}
$$

$$
\Delta_{1(st)} = \begin{vmatrix} A_{0(st)} & \bar{Y}^2 & A_{4(st)} \\ A_{3(st)} & A_{6(st)} & A_{5(st)} \\ A_{4(st)} & A_{7(st)} & A_{2(st)} \end{vmatrix}
$$

$$
\begin{aligned}
&= A_{0(st)}\left(A_{2(st)}A_{6(st)} - A_{5(st)}A_{7(st)}\right)\\
&- \bar{Y}^2\left(A_{2(st)}A_{3(st)} - A_{4(st)}A_{5(st)}\right)\\
&+ A_{4(st)}\left(A_{3(st)}A_{7(st)} - A_{4(st)}A_{6(st)}\right)
\end{aligned}
$$

$$
\Delta_{2(st)} = \begin{vmatrix} A_{0(st)} & A_{3(st)} & \bar{Y}^2 \\ A_{3(st)} & A_{1(st)} & A_{6(st)} \\ A_{4(st)} & A_{5(st)} & A_{7(st)} \end{vmatrix}
$$

$$
\begin{aligned}
&= A_{0(st)}\left(A_{1(st)}A_{7(st)} - A_{5(st)}A_{6(st)}\right)\\
&- A_{3(st)}\left(A_{3(st)}A_{7(st)} - A_{4(st)}A_{6(st)}\right)\\
&+ \bar{Y}^2\left(A_{3(st)}A_{5(st)} - A_{1(st)}A_{4(st)}\right)
\end{aligned}
$$

Thus the corresponding minimum *MSE* of $t_{(st)}$ is given by

$$
MSE_{\min}(t_{(st)}) = \left[\bar{Y}^2 - \frac{\Delta_{0(st)}\bar{Y}^2}{\Delta_{(st)}} - \frac{A_{6(st)}\Delta_{1(st)}}{\Delta_{(st)}} - \frac{A_{7(st)}\Delta_{2(st)}}{\Delta_{(st)}}\right] \tag{92}
$$

Thus we state the following theorem.

Theorem-8.1 – The *MSE* of t_{st} is always greater than equal to the minimum *MSE* of $t_{(st)}$ i.e.

$$MSE(t_{(st)}) \geq MSE_{\min}(t_{(st)}$$
$$= \left[\bar{Y}^2 - \frac{\Delta_{0(st)}\bar{Y}^2}{\Delta_{(st)}} - \frac{A_{6(st)}\Delta_{1(st)}}{\Delta_{(st)}} - \frac{A_{7(st)}\Delta_{2(st)}}{\Delta_{(st)}}\right]$$

with equality holding if

$$w_{0i} = \frac{\Delta_i}{\Delta}, \quad i = 0, 1, 2.$$

***A remark similar to Remark 3.1 follows here.**

9 Comparison of the Proposed Class of Estimator with Some Existing Estimators in Stratified Random Sampling

From (55), (57), (59) and (65), we have

$$MSE\left(\bar{y}_{(st)}\right) - \left[MSE_{\min}\left(\hat{\bar{Y}}_{D1(st)}\right) = MSE_{\min}\left(\hat{\bar{Y}}_{S(st)}\right)\right]$$
$$= \frac{V_{110}^2}{V_{020}} \geq 0 \tag{93}$$

$$MSE\left(\bar{y}_{R(st)}\right) - \left[MSE_{\min}\left(\hat{\bar{Y}}_{D1(st)}\right) = MSE_{\min}\left(\hat{\bar{Y}}_{S(st)}\right)\right]$$
$$= \frac{\left(R_1V_{020} - V_{110}\right)^2}{V_{020}} \geq 0 \tag{94}$$

$$MSE\left(\bar{y}_{P(st)}\right) - \left[MSE_{\min}\left(\hat{\bar{Y}}_{D1(st)}\right) = MSE_{\min}\left(\hat{\bar{Y}}_{S(st)}\right)\right]$$
$$= \frac{\left(R_1V_{020} + V_{110}\right)^2}{V_{020}} \geq 0 \tag{95}$$

Expressions (93), (94) and (95) clearly indicates that the usual difference estimator $\hat{\bar{Y}}_{D1(st)}$ and Srivastava (1967) estimator $\hat{\bar{Y}}_{S(st)}$ are better than the estimators $\bar{y}_{st}, \hat{\bar{Y}}_{R(st)}$ and $\hat{\bar{Y}}_{P(st)}$.

From (65) and (76), we have

$$\left[MSE_{\min}\left(\hat{\bar{Y}}_{D1(st)}\right) = MSE_{\min}\left(\hat{\bar{Y}}_{S(st)}\right)\right] - MSE_{\min}\left(\hat{\bar{Y}}_{Rao(st)}\right)$$
$$= \frac{\left(V_{200}V_{020} - V_{110}^2\right)^2}{\left(\bar{Y}^2 V_{020} + V_{020}V_{200} - V_{110}^2\right)} \geq 0 \tag{96}$$

From (93)–(96), we have the following inequalities:

$$MSE_{\min}\left(\hat{\bar{Y}}_{Rao(st)}\right) \leq \left[MSE_{\min}\left(\hat{\bar{Y}}_{D1(st)}\right) = MSE_{\min}\left(\hat{\bar{Y}}_{S(st)}\right)\right] \leq MSE\left(\bar{y}_{st}\right) \tag{97}$$

$$MSE_{\min}\left(\hat{\bar{Y}}_{Rao(st)}\right) \leq \left[MSE_{\min}\left(\hat{\bar{Y}}_{D1(st)}\right) = MSE_{\min}\left(\hat{\bar{Y}}_{S(st)}\right)\right] \leq MSE\left(\hat{\bar{Y}}_{R(st)}\right) \tag{98}$$

$$MSE_{\min}\left(\hat{\bar{Y}}_{Rao(st)}\right) \leq \left[MSE_{\min}\left(\hat{\bar{Y}}_{D1(st)}\right) = MSE_{\min}\left(\hat{\bar{Y}}_{S(st)}\right)\right] \leq MSE\left(\hat{\bar{Y}}_{P(st)}\right) \tag{99}$$

It follows from (97), (98) and (99) that the Rao (1991) estimator $\hat{\bar{Y}}_{Rao(st)}$ is more precise than $\bar{y}_{st}, \hat{\bar{Y}}_{R(st)}$, $\hat{\bar{Y}}_{P(st)}$, $\hat{\bar{Y}}_{D1(st)}$ and Srivastava's (1967) estimator $\hat{\bar{Y}}_{S(st)}$.

From (65) and (69), we have

$$\left[MSE_{\min}\left(\hat{\bar{Y}}_{D1(st)}\right) = MSE_{\min}\left(\hat{\bar{Y}}_{S(st)}\right) = MSE_{\min}\left(\hat{\bar{Y}}^*_{USV(st)}\right)\right] - MSE_{\min}\left(\hat{\bar{Y}}_{USV(st)}\right)$$
$$= \frac{\bar{Y}^2\left[\begin{array}{c} A_{1(st)}\left(\bar{Y}^2 - A_{0(st)}\right) + A_{3(st)}\left(A_{3(st)} - \bar{Y}^2\right) \\ +A_{6(st)}\left(A_{0(st)} - A_{3(st)}\right)\end{array}\right]^2}{\left(A_{0(st)}A_{1(st)} - A_{3(st)}^2\right)\left(A_{0(st)} + A_{1(st)} - 2A_{3(st)}\right)} \geq 0 \tag{100}$$

It follows that the Upadhyaya et al. (1985) estimator $\hat{\bar{Y}}_{USV(st)}$ is more efficient than $\hat{\bar{Y}}_{D1(st)}, \hat{\bar{Y}}_{S(st)}$ and $\hat{\bar{Y}}^*_{USV(st)}$.

From (69) and (92), we have

$$\begin{aligned} & MSE_{\min}\left(\hat{\hat{Y}}_{USV(st)}\right) - MSE_{\min}\left(t_{(st)}\right) \\ & \quad = \frac{\bar{Y}^2}{\Delta_{(st)}\Delta_{1(st)}} \left[\begin{array}{l} A_{7(st)}\left(A_{0(st)}A_{1(st)} - A^2_{3(st)}\right) \\ \quad + \left(A_{3(st)}A_{5(st)} - A_{1(st)}A_{4(st)}\right) \\ \quad + A_{6(st)}\left(A_{3(st)}A_{4(st)} - A_{0(st)}A_{5(st)}\right) \end{array} \right]^2 \\ & \quad \geq 0 \end{aligned} \tag{101}$$

which shows that the proposed generalized class of estimators $t_{(st)}$ is more efficient than Upadhyaya et al. (1985) estimator $\hat{\hat{Y}}_{USV(st)}$. Hence the estimator $t_{(st)}$ is more precise than the estimators $\bar{y}_{st}, \hat{\hat{Y}}_{R(st)}, \hat{\hat{Y}}_{P(st)}, \hat{\hat{Y}}_{D1(st)}, \hat{\hat{Y}}_{S(st)}$ and $\hat{\hat{Y}}^*_{USV(st)}$.

From (73) and (84) we have

$$\begin{aligned} & \left[MSE_{\min}\left(\hat{\hat{Y}}_{D1(st)}\right) = MSE_{\min}\left(\hat{\hat{Y}}_{S(st)}\right) = MSE_{\min}\left(\hat{\hat{Y}}^*_{USV(st)}\right)\right] \\ & \quad - MSE_{\min}\left(\hat{\hat{Y}}_{D2(st)}\right) = \frac{\left(V_{020}V_{101} - V_{110}V_{011}\right)}{V_{020}\left(V_{020}V_{002} - V^2_{011}\right)} \geq 0 \end{aligned} \tag{102}$$

which shows that the difference estimator $\hat{\hat{Y}}_{D2(st)}$ is more efficient than the estimator $\hat{\hat{Y}}_{D1(st)}$.

From (76), (80), (84) and (92),we have

- $MSE_{\min}(t_{(st)}) < MSE_{\min}(\hat{\hat{Y}}_{Rao(st)})$ if

$$\begin{aligned} & \frac{\bar{Y}^4 V_{020}}{\left[V_{020}\left(\bar{Y}^2 + V_{200}\right) - V^2_{110}\right]} \\ & \quad < \left[\frac{\left(\Delta_{0(st)}\bar{Y}^2 + A_{6(st)}\Delta_{1(st)} + A_{7(st)}\Delta_{2(st)}\right)}{\Delta_{(st)}}\right] \end{aligned} \tag{103}$$

- $MSE_{\min}(t_{(st)}) < MSE_{\min}(\hat{\hat{Y}}_{GS(st)})$ if

$$\begin{aligned} & \left[\frac{\left(a_{2(st)}a^2_{4(st)} - 2a_{3(st)}a_{4(st)}a_{5(st)} + a_{4(st)}a^2_{5(st)}\right)}{\left(a_{1(st)}a_{2(st)} - a^2_{3(st)}\right)}\right] \\ & \quad < \left[\frac{\left(\Delta_{0(st)}\bar{Y}^2 + A_{6(st)}\Delta_{1(st)} + A_{7(st)}\Delta_{2(st)}\right)}{\Delta_{(st)}}\right] \end{aligned} \tag{104}$$

- $MSE_{\min}(t_{(st)}) < MSE_{\min}(\hat{\bar{Y}}_{D2(st)})$ if

$$\left[\bar{Y}^2 + \frac{\left(V_{110}^2 V_{002} - 2V_{011}V_{101}V_{110} + V_{020}V_{101}^2\right)}{\left(V_{020}V_{002} - V_{011}^2\right)}\right] < \left[V_{200} + \frac{\left(\Delta_{0(st)}\bar{Y}^2 + A_{6(st)}\Delta_{1(st)} + A_{7(st)}\Delta_{2(st)}\right)}{\Delta_{(st)}}\right] \quad (105)$$

It is observed from (103), (104) and (105) that the proposed generalized class of estimators $t_{(st)}$ is more efficient than the estimators $\hat{\bar{Y}}_{Rao(st)}$, $\hat{\bar{Y}}_{GS(st)}$ and $\hat{\bar{Y}}_{D2(st)}$ as long as the conditions (103), (104) and (105) are satisfied respectively.

10 Numerical Illustration

To examine the performance of the proposed generalized class of estimators $t_{(st)}$ over existing estimators, we use the data sets given below

Data I: Source: [Murthy (1967), P. 228]

N = 80, *n* = 22

$N_1 = 19$	$N_2 = 32$	$N_3 = 29$	$n_1 = 5$	$n_2 = 9$	$n_3 = 8$
$\bar{Y}_1 = 2967.95$	$\bar{Y}_2 = 4657.63$	$\bar{Y}_3 = 7212.97$	$\bar{X}_1 = 65.16$	$\bar{X}_2 = 139.97$	$\bar{X}_3 = 589.41$
$\bar{Z}_1 = 349.68$	$\bar{Z}_2 = 706.59$	$\bar{Z}_3 = 2098.69$	$C_{y1} = 0.25509$	$C_{y2} = 0.14366$	$C_{y3} = 0.11848$
$C_{x1} = 0.17158$	$C_{x2} = 0.31693$	$C_{x3} = 0.38415$	$C_{z1} = 0.3130$	$C_{z2} = 0.15457$	$C_{z3} = 0.30386$
$\rho_{yx1} = 0.81$	$\rho_{yx2} = 0.89$	$\rho_{yx3} = 0.98$	$\rho_{yz1} = 0.94$	$\rho_{yz2} = 0.93$	$\rho_{yz3} = 0.98$
$\rho_{xz1} = 0.90$	$\rho_{xz2} = 0.85$	$\rho_{xz3} = 0.97$			

Data II: [Source: Koyuncu and Kadilar (2009)]

N = 923, n =180

$N_1 = 127$	$N_2 = 117$	$N_3 = 103$	$N_4 = 170$	$N_5 = 205$	$N_6 = 201$
$n_1 = 31$	$n_2 = 21$	$n_3 = 29$	$n_4 = 38$	$n_5 = 22$	$n_6 = 39$
$\bar{Y}_1 = 703.74$	$\bar{Y}_2 = 413.0$	$\bar{Y}_3 = 513.17$	$\bar{Y}_4 = 424.66$	$\bar{Y}_5 = 267.03$	$\bar{Y}_6 = 393.84$
$\bar{X}_1 = 20804.59$	$\bar{X}_2 = 9211.79$	$\bar{X}_3 = 14309.30$	$\bar{X}_4 = 9478.85$	$\bar{X}_5 = 5569.95$	$\bar{X}_6 = 12997.59$
$\bar{Z}_1 = 498.28$	$\bar{Z}_2 = 318.33$	$\bar{Z}_3 = 413.36$	$\bar{Z}_4 = 311.32$	$\bar{Z}_5 = 227.20$	$\bar{Z}_6 = 313.71$
$S_{y1} = 883.84$	$S_{y2} = 644.92$	$S_{y3} = 1033.46$	$S_{y4} = 810.58$	$S_{y5} = 403.65$	$S_{y6} = 711.72$
$S_{x1} = 30486.7$	$S_{x2} = 15180.77$	$S_{x3} = 27549.78$	$S_{x4} = 18218.93$	$S_{x5} = 8497.77$	$S_{x6} = 2394.14$
$S_{z1} = 555.58$	$S_{z2} = 365.46$	$S_{z3} = 612.95$	$S_{z4} = 458.03$	$S_{z5} = 260.85$	$S_{z6} = 397.05$
$\rho_{yx1} = 0.936$	$\rho_{yx2} = 0.996$	$\rho_{yx3} = 0.994$	$\rho_{yx4} = 0.983$	$\rho_{yx5} = 0.989$	$\rho_{yx6} = 0.965$
$\rho_{yz1} = 0.979$	$\rho_{yz2} = 0.976$	$\rho_{yz3} = 0.984$	$\rho_{yz4} = 0.983$	$\rho_{yz5} = 0.964$	$\rho_{yz6} = 0.983$
$\rho_{xz1} = 0.9396$	$\rho_{xz2} = 0.9696$	$\rho_{xz3} = 0.977$	$\rho_{xz4} = 0.964$	$\rho_{xz5} = 0.9676$	$\rho_{xz6} = 0.996$

Table 4 *PRE*'s of different estimators of population mean $\bar{Y}$ with respect to $\bar{y}_{st}$

Estimator	Data I	Data II
$\hat{\hat{Y}}_{R(st)}$	14.42	1025.10
$\hat{\hat{Y}}_{P(st)}$	5.89	24.22
$\hat{\hat{Y}}_{D1(st)}$	235.83	1141.85
$\hat{\hat{Y}}_{Rao(st)}$	235.91	1143.02
$\hat{\hat{Y}}_{GS(st)}$	183.44	1109.11
$\hat{\hat{Y}}_{D2(st)}$	273.99	2621.61

Table 5 *PRE*'s of the estimator $\hat{\hat{Y}}_{USV(st)}$ with respect to $\bar{y}_{st}$

Data I		Data II	
α_1	PRE	α_1	PRE
−1	238.07	−1	1146.96
1	235.84	1	1260.08

Table 4 presents the *PRE*'s of $\hat{\hat{Y}}_{R(st)}, \hat{\hat{Y}}_{P(st)}, \hat{\hat{Y}}_{D1(st)}, \hat{\hat{Y}}_{Rao(st)}, \hat{\hat{Y}}_{GS(st)}$ and $\hat{\hat{Y}}_{D2(st)}$ estimators with respect to $\bar{y}_{(st)}$ for two data sets respectively.

Table 5 shows the *PRE* of $\hat{\hat{Y}}_{USV(st)}$ with respect to $\bar{y}_{(st)}$ for $\alpha_1 = -1, 1$, for two data sets.

Table 6 depicts the *PRE* of proposed estimator $t_{(st)}$ *wrt* $\bar{y}_{(st)}$ at different values of α_1 and α_2, for two data sets.

Table 6 *PRE*'s of the proposed estimator $t_{(st)}$ with respect to $\bar{y}_{st}$ for different values of (α_1, α_2)

Data I			Data II		
α_1	α_2	PRE	α_1	α_2	PRE
−1	−1	275.29	−1	−1	2626.83
−2	−2	277.33	−2	−2	2720.91
−3	−3	280.47	−3	−3	3216.10
−4	−4	284.92	−4	−4	4826.13
−5	−5	291.07	−5	−5	20279.5
−8	−8	327.57	−5.2	−5.2	**96957.81**
−10	−10	391.89	1	1	3590.02
−12	−12	642.36	2	2	6380.74
−13	−13	1645.27	2.1	2.1	7060.90
−13.1	−13.1	2078.85	2.2	2.2	7938.96
−13.2	−13.2	2887.43	2.5	2.5	13252.63
−13.3	−13.3	4928.01	2.8	2.8	51458.14
−13.4	−13.4	20017.07	−1	1	3067.82

(*Continued*)

Table 6 Continued

Data I			Data II		
α_1	α_2	PRE	α_1	α_2	PRE
1	1	274.03	2	-1	2923.05
2	2	274.74	-1	2	1002.942
3	3	276.39	3	-1	2734.047
4	4	279.07	4	-1	2871.735
5	5	282.99	*	*	*
8	8	306.25	*	*	*
10	10	342.15	*	*	*
12	12	434.69	*	*	*
14	14	1047.46	*	*	*
14.1	14.1	1180.86	*	*	*
14.3	14.3	1634.81	*	*	*
14.5	14.5	2885.63	*	*	*
14.6	14.6	4977.54	*	*	*
14.7	14.7	**22036.36**	*	*	*
-1	1	274.16	*	*	*
-1	2	275.62	*	*	*
3	-1	277.21	*	*	*
4	-1	279.03	*	*	*
-5	5	259.76	*	*	*

It is observed from Tables 4, 5 and 6 that for various values of (α_1, α_2) the proposed generalized class of estimators $t_{(st)}$ is more efficient than the estimators $\bar{y}_{(st)}\hat{\bar{Y}}_{R(st)}, \hat{\bar{Y}}_{P(st)}$, $\hat{\bar{Y}}_{D1(st)}, \hat{\bar{Y}}_{Rao(st)}$, $\hat{\bar{Y}}_{GS(st)}$, $\hat{\bar{Y}}_{D2(st)}$ and $\hat{\bar{Y}}_{USV(st)}$, with considerable gain in efficiency. The proposed generalized class of estimators $t_{(st)}$ yields the largest percent relative efficiency 22036.60% for data set I while it is 96957.81% for data set II. It is further observed from Table 6 that there is enough scope of selecting the scalars (α_1, α_2) in acquiring efficient estimators (from the suggested generalized class of estimators $t_{(st)}$) than the existing estimators. Thus we conclude that the proposed generalized class of estimators $t_{(st)}$ can be used in practice just by selecting the appropriate values of (α_1, α_2).

11 Discussion and Conclusion

This article considers the problem of estimating the population mean $\bar{Y}$ of the study variable *y* using information on two auxiliary variables *x* and *z*. We have proposed a generalized class of estimators for the population mean $\bar{Y}$ using information on two supplementary variables x and z. Expressions of bias and mean square error up to the *fda* have been obtained in *SRSWOR* as well as in stratified random sampling. It is interesting to mention that the envisaged class of estimators includes several existing estimators. Thus the properties of the proposed generalized class of estimators unify results at one place. We have proved theoretically that the proposed generalized class of estimators is more efficient than the several existing estimators in both sampling designs *SRSWOR* and stratified random sampling.

Empirical studies are carried out to throw light on the merits of the envisaged generalized class of estimators over some existing competitors. Larger gain in efficiency is observed by using the proposed generalized class of estimators over some existing estimators in both the sampling designs: *SRSWOR* and stratified random sampling. Results incorporated in this study are very sound and quite illuminating. Thus it is recommended that the proposed study is useful in practice.

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