Inference on Positive Exponential Family of Distributions (PEFD) through Transformation Method

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Abstract

The estimation of $R(t)$ and $R = Pr(Y > X)$ for the Positive Exponential Family of Distribution (PEFD) is considered. The UMVUES, MLES and Confidence Interval are derived. These estimators are derived through the method of Transformation. The $\alpha = Pr(X > \gamma)$, which is termed as probability of disaster is also derived when random stress X follows PEFD and finite strength follows Power function distribution.

Keywords: Positive exponential family of distribution, uniformly minimum variance unbiased estimator, maximum likelihood estimator, confidence interval, probability of disaster, stress-strength reliability.

1 Introduction

Reliability measure $R = Pr(X > t)$, which defines the failure free operation of items / components until time 't' and the measure $R = Pr(Y > X)$

commonly represents the reliability of items / components, where the random variable X and Y are the random stress and random strength. Another measure of Reliability, $\alpha = Pr(X > \gamma)$ which represents the probability of disaster, where the random variable X represents the stress and γ is the maximum strength of the items / components.

In the literature of reliability, lot of work has been done since last few decades. For a brief review of literature, the most popular article on the related study are Pugh (1963) [12], Basu (1964) [3], Church and Harris (1970) [7], Enis and Geisser (1971) [9], Downton (1973) [8], Tong (1974) [19], Kelly et al. (1976) [10], Sinha and Kale (1980) [15], Sathe and Shah (1981) [14], Chao (1982) [4], Awad and Gharraf (1986) [2], Chaturvedi and Surinder (1999) [6], Kotz et al. (2003) [11], Rezaei and Mahmoodi (2010) [13], Chaturvedi and Pathak (2012) [5], Surinder and Mukesh (2015) [16] and Surinder and Mukesh (2016) [17]. In the present study, we have considered a positive exponential family of distribution, which covers various lifetime distributions as their specific cases.

2 Set Up of the Problem

Liang (2008)[18] proposed a positive exponential family of distributions, which covers gamma distribution as specific case. Let the random variable X has positive exponential family of distribution, then the pdf is given by

$$f(x; \Theta) = \frac{\rho x^{\rho\nu-1} exp\left(\frac{-x^\rho}{\theta}\right)}{\Gamma\nu\theta^\nu}; x > 0, \; \theta, \nu, \rho > 0 \tag{1}$$

where, $\Theta = (\rho, \nu, \theta)$ and θ is assumed to be unknown and ρ, ν are known constants. On assigning different values to ν and ρ, this family distribution covers following pdfs as

(1) For $\rho = \nu = 1$, we get one parameter exponential distribution.
(2) For $\rho = 1$, we get gamma distribution.
(3) For ν=1, we get Weibull distribution.
(4) For $\nu > 0$, $\rho = 1$, we get Erlang distribution.
(5) For $\nu > 1/2$, $\rho = 2$, we get half – normal distribution.
(6) For $\nu > m/2$, $\rho = 2$, we get Chi-distribution.
(7) For $\nu = 1$, $\rho = 2$, we get Rayleigh distribution.
(8) For $\nu = p + 1$, $\rho = 2$, we get Generalized Rayleigh distribution.

Let the random variable Y considered as strength follows Power function distribution whose cdf and pdf is

$$G(y; \mu, \gamma) = \left(\frac{y}{\gamma}\right)^{\mu} \tag{2}$$

and

$$g(y; \mu, \gamma) = \frac{\mu}{\gamma} \left(\frac{y}{\gamma}\right)^{\mu-1} ; 0 < y < \gamma, \ \mu > 0 \tag{3}$$

3 MLE and UMVUE of $R = Pr(Y > X)$ for PEFD

The MLE and UMVUE of $R = Pr(Y > X)$ for PEFD by using the transformation method are evaluated in the following theorems.

Theorem 3.1: The MLE of $R = Pr(Y > X)$ is given by

$$\tilde{R} = \left(\frac{\nu_1 \overline{\eta}}{\nu_2 \overline{\xi} + \nu_1 \overline{\eta}}\right)^{\nu_1} \frac{1}{B(\nu_1, \nu_2)} {}_2F_1\left(\nu_1, 1 - \nu_2; \nu_1; \frac{\nu_1 \overline{\eta}}{\nu_2 \overline{\xi} + \nu_1 \overline{\eta}}\right) \tag{4}$$

where, $\overline{\xi} = \frac{1}{n_1} \sum_{i=1}^{n_1} X_i^{\rho_1} = \overline{T}_X$ and $\overline{\eta} = \frac{1}{n_2} \sum_{j=1}^{n_2} Y_j^{\rho_2} = \overline{T}_Y$

Proof: In order to transform the given pdf (1), let us assume $x^{\rho_1} = \xi$, we get

$$f(\xi; \lambda_1, \nu_1) = \frac{\xi^{\nu_1 - 1} e^{-\left(\frac{\xi}{\lambda_1}\right)}}{\Gamma \nu_1 \lambda_1^{\nu_1}}; \ \xi, \lambda_1, \nu_1 > 0 \tag{5}$$

Similarly, for $\eta = y^{\rho_2}$

$$f(\eta; \lambda_2, \nu_2) = \frac{\xi^{\nu_1 - 2} e^{-\left(\frac{\eta}{\lambda_2}\right)}}{\Gamma \nu_2 \lambda_2^{\nu_2}}; \ \eta, \lambda_2, \nu_2 > 0 \tag{6}$$

where $\lambda_1 = \theta_1$ and $\lambda_2 = \theta_2$

Let ξ and η are two independent random variables with gamma pdfs given at (5) and (6). Thus,

$$R = Pr(\eta > \xi)$$

$$= Pr\left(\frac{\eta}{\xi} > 1\right)$$

$$= Pr\left(\frac{\eta/\lambda_2}{\xi/\lambda_1} > \frac{\lambda_1}{\lambda_2}\right)$$

$$= Pr\left(\frac{\eta/\lambda_2}{\xi/\lambda_1} + 1 > \frac{\lambda_1}{\lambda_2} + 1\right)$$

$$R = Pr\left(\frac{\xi/\lambda_1}{\xi/\lambda_1 + \eta/\lambda_2} > \frac{\lambda_2}{\lambda_2 + \lambda_1}\right)$$

Since, we know that, if ξ and η be two independent random variables which follow gamma distribution with parameters (λ_1, ν_1) and (λ_2, ν_2) then $z = \frac{\xi/\lambda_1}{\xi/\lambda_1 + \eta/\lambda_2}$ is a beta (ν_1, ν_2) random variable with the pdf

$$f(z, \nu_1, \nu_2) = [B(\nu_1, \nu_2)]^{-1} z^{\nu_1 - 1} (1 - z)^{\nu_2 - 1}$$

or,

$$R = I_{\left(\frac{\lambda_2}{\lambda_2 + \lambda_1}\right)}(\nu_1, \nu_2) \tag{7}$$

which is the incomplete beta function. Using the relation between the incomplete beta function and the hypergeometric series, we rewrite (7) as

$$R = \left(\frac{\lambda_2}{\lambda_2 + \lambda_1}\right)^{\nu_1} \frac{1}{B(\nu_1, \nu_2)} {}_2F_1\left(\nu_1, 1 - \nu_2; \nu_1; \frac{\lambda_2}{\lambda_2 + \lambda_1}\right) \tag{8}$$

The reliability $R = Pr(Y > X)$

$$R = \left(\frac{\theta_2}{\theta_2 + \theta_1}\right)^{\nu_1} \frac{1}{B(\nu_1, \nu_2)} {}_2F_1\left(\nu_1, 1 - \nu_2; \nu_1; \frac{\theta_2}{\theta_2 + \theta_1}\right)$$

Substituting the MLE's i.e. $\widetilde{\lambda_1} = \frac{\bar{\xi}}{\nu_1}$ and $\widetilde{\lambda_2} = \frac{\bar{\eta}}{\nu_2}$ in place of λ_1 and λ_2 in (8). The MLE of $R = Pr(\eta > \xi)$ is

$$\widetilde{R} = \left(\frac{\nu_1 \bar{\eta}}{\nu_2 \bar{\xi} + \nu_1 \bar{\eta}}\right)^{\nu_1} \frac{1}{B(\nu_1, \nu_2)} {}_2F_1\left(\nu_1, 1 - \nu_2; \nu_1; \frac{\nu_1 \bar{\eta}}{\nu_2 \bar{\xi} + \nu_1 \bar{\eta}}\right)$$

where, $\bar{\xi} = \frac{1}{n_1} \sum_{i=1}^{n_1} X_i^{\rho_1} = \overline{T}_X$ and $\bar{\eta} = \frac{1}{n_2} \sum_{j=1}^{n_2} Y_j^{\rho_2} = \overline{T}_Y$

Hence, the theorem follows.

Corollary 1.

1. MLE of $R = Pr(Y > X)$ for one parameter exponential distribution $(\rho = \nu = 1)$

$$\widetilde{R} = \frac{\overline{T}_Y}{\overline{T}_X + \overline{T}_Y}$$

where, $\overline{T}_Y = \frac{1}{n_2} \sum_{j=1}^{n_2} Y_j$ and $\overline{T}_X = \frac{1}{n_1} \sum_{i=1}^{n_1} X_i$

2. MLE of $R = Pr(Y > X)$ for gamma Distribution ($\rho = 1$)

$$\widetilde{R} = \left(\frac{\nu_1 \overline{Y}}{\nu_2 \overline{X} + \nu_1 \overline{Y}} \right)^{\nu_1} \frac{1}{B(\nu_1, \nu_2)} {}_2F_1 \left(\nu_1, 1 - \nu_2; \nu_1; \frac{\nu_1 \overline{Y}}{\nu_2 \overline{X} + \nu_1 \overline{Y}} \right)$$

where, $\overline{T}_Y = \frac{1}{n_2} \sum_{j=1}^{n_2} Y_j^{\rho_2}$ and $\overline{T}_X = \frac{1}{n_1} \sum_{i=1}^{n_1} X_i^{\rho_1}$

3. MLE of $R = Pr(Y > X)$ for Weibull Distribution ($\nu = 1$)

$$\widetilde{R} = \frac{\overline{T}_Y}{\overline{T}_X + \overline{T}_Y}$$

where, $\overline{T}_Y = \frac{1}{n_2} \sum_{j=1}^{n_2} Y_j^{\rho_2}$ and $\overline{T}_X = \frac{1}{n_1} \sum_{i=1}^{n_1} X_i^{\rho_1}$

4. MLE of $R = Pr(Y > X)$ for Erlang distribution ($\nu > 0, \rho = 1$)

$$\widetilde{R} = \left(\frac{\nu_1 \overline{Y}}{\nu_2 \overline{X} + \nu_1 \overline{Y}} \right)^{\nu_1} \frac{1}{B(\nu_1, \nu_2)} {}_2F_1 \left(\nu_1, 1 - \nu_2; \nu_1; \frac{\nu_1 \overline{Y}}{\nu_2 \overline{X} + \nu_1 \overline{Y}} \right)$$

where, $\overline{T}_Y = \frac{1}{n_2} \sum_{j=1}^{n_2} Y_j$ and $\overline{T}_X = \frac{1}{n_1} \sum_{i=1}^{n_1} X_i$

5. MLE of $R = Pr(Y > X)$ for half – normal distribution ($\nu = 1/2, \rho = 2$)

$$\widetilde{R} = \left(\frac{\overline{T}_Y}{\overline{T}_X + \overline{T}_Y} \right)^{1/2} \frac{1}{\pi^2} {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \frac{\overline{T}_Y}{\overline{T}_X + \overline{T}_Y} \right)$$

where, $\overline{T}_Y = \frac{1}{n_2} \sum_{j=1}^{n_2} Y_j^2$ and $\overline{T}_X = \frac{1}{n_1} \sum_{i=1}^{n_1} X_i^2$

6. MLE of $R = Pr(Y > X)$ for Chi-distribution ($\nu > m/2, \rho = 2$)

$$\widetilde{R} = \left(\frac{\overline{T}_Y}{\overline{T}_X + \overline{T}_Y} \right)^{m/2} \frac{1}{B\left(\frac{m}{2}, \frac{m}{2} \right)} {}_2F_1 \left(\frac{m}{2}, 1 - \frac{m}{2}; \frac{m}{2}; \frac{\overline{T}_Y}{\overline{T}_X + \overline{T}_Y} \right)$$

where, $\overline{T}_Y = \frac{1}{n_2} \sum_{j=1}^{n_2} Y_j^2$ and $\overline{T}_X = \frac{1}{n_1} \sum_{i=1}^{n_1} X_i^2$

7. MLE of $R = Pr(Y > X)$ for Rayleigh distribution ($\nu = 1, \rho = 2$)

$$\widetilde{R} = \frac{\overline{T}_Y}{\overline{T}_X + \overline{T}_Y}$$

where, $\overline{T}_Y = \frac{1}{n_2} \sum_{j=1}^{n_2} Y_j^2$ and $\overline{T}_X = \frac{1}{n_1} \sum_{i=1}^{n_1} X_i^2$

8. MLE of $R = Pr(Y > X)$ for Generalized Rayleigh distribution ($\nu = p + 1, \rho = 2$)

$$\widetilde{R} = \left(\frac{\overline{T}_Y}{\overline{T}_X + \overline{T}_Y} \right)^{p+1} \frac{1}{B(p+1, p+1)} {}_2F_1 \left(p + 1, -p; p + 1; \frac{\overline{T}_Y}{\overline{T}_X + \overline{T}_Y} \right)$$

where, $\overline{T}_Y = \frac{1}{n_2} \sum_{j=1}^{n_2} Y_j^2$ and $\overline{T}_X = \frac{1}{n_1} \sum_{i=1}^{n_1} X_i^2$

Theorem 3.2: The UMVUE of $R = Pr(Y > X)$ is given by

$$
\hat{R} = \begin{cases}
\begin{aligned}
&\frac{\rho_1\rho_2 B[(n_2-1)\nu_2+i+1,\nu_1+j]}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1\nu_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\nu_2+i} \\
&\binom{\nu_2-1}{i} \sum_{j=0}^{\infty}(-1)^j \binom{(n_1-1)\nu_1-1}{j} \left(\frac{T_Y}{T_X}\right)^{\nu_1+j} ; \quad T_Y < T_X \\
&\forall \quad 0 \le i \le \nu_1 - 1 < \infty \quad \& \quad 0 \le j \le (n_1-1)\nu_1 - 1 < \infty
\end{aligned} \\
\\
\begin{aligned}
&\frac{\rho_1\rho_2 B[(n_1-1)\nu_1,\nu_1+j]}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1\nu_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\nu_2+i} \\
&\binom{\nu_2-1}{i} \sum_{j=0}^{\infty}(-1)^j \binom{(n_2-1)\nu_1+i}{j} \left(\frac{T_X}{T_Y}\right)^{j} ; \quad T_Y > T_X \\
&\forall \quad 0 \le i \le \nu_2 - 1 < \infty \quad \& \quad 0 \le j \le (n_2-1)\nu_2 + i < \infty
\end{aligned}
\end{cases}
\tag{9}
$$

where, $T_X = \sum_{i=1}^{n_1} X_i{}^{\rho_1}$ and $T_Y = \sum_{i=1}^{n_2} Y_j{}^{\rho_2}$

Proof: Let ξ and η be two independent gamma distributions with pdfs (5) and (6). In order to obtain $Pr(\eta > \xi)$, we have to obtain the UMVUE of $f(\xi;\nu,\rho,\lambda)$ i.e. $\hat{f}(\xi;\nu,\rho,\lambda)$ and $f(\eta;\nu,\rho,\lambda)$ i.e. $\hat{f}(\eta;\nu,\rho,\lambda)$ which is given by

$$
\hat{f}(\xi;\nu_1,\rho_1,\lambda_1) = \frac{\rho_1}{B[\nu_1,(n_1-1)\nu_1]} \left\{ \frac{\xi^{\nu_1-1}}{(n_1\overline{\xi})^{\nu_1}} \right\} \left\{ 1 - \frac{\xi}{n_1\overline{\xi}} \right\}^{(n_1-1)\nu_1-1} ;
$$
$$
\text{if} \quad 0 < \xi < n_1\overline{\xi}
\tag{10}
$$

and

$$
\hat{f}(\eta;\nu_2,\rho_2,\lambda_2) = \frac{\rho_2}{B[\nu_2,(n_2-1)\nu_2]} \left\{ \frac{\eta^{\nu_2-1}}{(n_2\overline{\eta})^{\nu_2}} \right\} \left\{ 1 - \frac{\eta}{n_2\overline{\eta}} \right\}^{(n_2-1)\nu_2-1} ;
$$
$$
\text{if} \quad 0 < \eta < n_2\overline{\eta}
\tag{11}
$$

The Reliability is

$$
\hat{R} = Pr(\eta > \xi) = \int_0^{\infty} \int_{\xi}^{\infty} \hat{f}(\eta;\nu_2,\rho_2,\lambda_2)\hat{f}(\xi;\nu_1,\rho_1,\lambda_1)d\eta d\xi
$$
$$
= \frac{\rho_1\rho_2}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1)\nu_2]}
$$

$$\int_0^{n_1\overline{\xi}} \int_\xi^{n_2\overline{\eta}} \frac{\eta^{\nu_2-1}}{(n_2\overline{\eta})^{\nu_2}} \left\{1 - \frac{\eta}{n_2\overline{\eta}}\right\}^{(n_2-1)\nu_2-1}$$

$$\left\{\frac{\xi^{\nu_1-1}}{(n_1\overline{\xi})^{\nu_1}}\right\} \left\{1 - \frac{\xi}{n_1\overline{\xi}}\right\}^{(n_1-1)\nu_1-1} d\eta d\xi$$

Let $1 - \frac{\eta}{n_2\overline{\eta}} = w$

$$= \frac{\rho_1\rho_2}{B[\nu_1, (n_1-1)\nu_1]B[\nu_2, (n_2-1)\nu_2]}$$

$$\int_0^{n_1\overline{\xi}} \int_0^{1-\frac{\xi}{n_2\overline{\eta}}} (1-w)^{\nu_2-1} w^{(n_2-1)\nu_2-1}$$

$$\left\{\frac{\xi^{\nu_1-1}}{(n_1\overline{\xi})^{\nu_1}}\right\} \left\{1 - \frac{\xi}{n_1\overline{\xi}}\right\}^{(n_1-1)\nu_1-1} dw d\xi$$

$$= \frac{\rho_1\rho_2}{B[\nu_1, (n_1-1)\nu_1]B[\nu_2, (n_2-1)\nu_2]} \sum_{i=0}^\infty \frac{(-1)^i}{(n_2-1)\nu_2+i}$$

$$\binom{\nu_2-1}{i} \int_0^{min(n_1\overline{\xi}, n_2\overline{\eta})} \left\{\frac{\xi^{\nu_1-1}}{(n_1\overline{\xi})^{\nu_1}}\right\}$$

$$\left\{1 - \frac{\xi}{n_1\overline{\xi}}\right\}^{(n_1-1)\nu_1-1} dw d\xi$$

$$= \frac{\rho_1\rho_2}{B[\nu_1, (n_1-1)\nu_1]B[\nu_2, (n_2-1)\nu_2]} \sum_{i=0}^\infty \frac{(-1)^i}{(n_2-1)\nu_2+i}$$

$$\binom{\nu_2-1}{i} \int_0^{min(n_1\overline{\xi}, n_2\overline{\eta})} \left\{\frac{\xi^{\nu_1-1}}{(n_1\overline{\xi})^{\nu_1}}\right\}$$

$$\left\{1 - \frac{\xi}{n_1\overline{\xi}}\right\}^{(n_1-1)\nu_1-1} \left\{1 - \frac{\xi}{n_2\overline{\eta}}\right\}^{(n_2-1)\nu_2+i} d\xi$$

Now, we consider the case when $n_1\overline{\xi} > n_2\overline{\eta}$ and let $1 - \frac{\xi}{n_2\overline{\eta}} = w$

$$= \frac{\rho_1\rho_2}{B[\nu_1, (n_1-1)\nu_1]B[\nu_2, (n_2-1)\nu_2]} \sum_{i=0}^\infty \frac{(-1)^i}{(n_2-1)\nu_2+i}$$

$$\binom{\nu_2 - 1}{i} \int_0^1 z^{(n_2-1)\nu_2+i}$$

$$\left\{1 - \frac{n_2\overline{\nu_2}}{n_1\overline{\xi}}(1-z)\right\}^{(n_1-1)\nu_1-1} \left(\frac{n_2\overline{\eta}}{n_1\overline{\xi}}\right)^{\nu_1} (1-z)^{\nu_1-1} \quad dz$$

Using the Binomial expansion. we get,

$$= \frac{\rho_1\rho_2}{B[\nu_1, (n_1-1)\nu_1]B[\nu_2, (n_2-1)\nu_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\nu_2+i}$$

$$\binom{\nu_2 - 1}{i} \int_0^1 \sum_{j=0}^{\infty} (-1)^j \binom{(n_1-1)\nu_1 - 1}{j}$$

$$\left(\frac{n_2\overline{\eta}}{n_1\overline{\xi}}\right)^{\nu_1+j} (1-z)^{\nu_1+j-1} \quad z^{(n_2-1)\nu_2+i} \quad dz$$

$$= \frac{\rho_1\rho_2 B[(n_2-1)\nu_2+i+1, \nu_1+j]}{B[\nu_1, (n_1-1)\nu_1]B[\nu_2, (n_2-1\nu_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\nu_2+i}$$

$$\binom{\nu_2 - 1}{i} \sum_{j=0}^{\infty} (-1)^j \binom{(n_1-1)\nu_1 - 1}{j}$$

$$\left(\frac{n_2\overline{\eta}}{n_1\overline{\xi}}\right)^{\nu_1+j}; \quad if \quad n_2\overline{\eta} < n_1\overline{\xi}$$

Similarly, we can take the case $n_2\overline{\eta} > n_1\overline{\xi}$, we get

$$= \frac{\rho_1\rho_2 B[(n_1-1)\nu_1, \nu_1+j]}{B[\nu_1, (n_1-1)\nu_1]B[\nu_2, (n_2-1\nu_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\nu_2+i} \binom{\nu_2 - 1}{i}$$

$$\sum_{j=0}^{\infty} (-1)^j \binom{(n_2-1)\nu_++i}{j} \left(\frac{n_2\overline{\xi}}{n_1\overline{\eta}}\right)^j$$

For obtaining the value of UMVUE substituting $n_1\overline{\xi} = \sum_{i=1}^{n_1} X_i^{\rho_1} = T_X$ and $n_2\overline{\eta} = \sum_{j=1}^{n_2} Y_j^{\rho_2} = T_Y$. Hence, the theorem follows.

Corollary 2.
1. UMVUE of R for one parameter exponential distribution is

$$
\hat{R} = \begin{cases}
\frac{B[(n_2+i,1+j]}{B[1,(n_1-1)]B[1,(n_2-1)]} \sum_{j=0}^{\infty} (-1)^j \binom{n_1-2}{j} \left(\frac{T_Y}{T_X}\right)^{1+j} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad ; T_Y < T_X \\
\frac{B[(n_1-1,1+j]}{B[1,(n_1-1)]B[1,(n_2-1)]} \sum_{j=0}^{\infty} (-1)^j \binom{n_2-1+i}{j} \left(\frac{T_X}{T_Y}\right)^{j} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad ; T_Y > T_X
\end{cases}
$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j$ and $T_X = \sum_{i=1}^{n_1} X_i$
2. UMVUE of R for gamma distribution is

$$
\hat{R} = \begin{cases}
\frac{B[(n_2-1)\nu_2+i+1,\nu_1+j]}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1)\nu_2]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\nu_2+i} \binom{\nu_2-1}{i} \sum_{j=0}^{\infty} (-1)^j \\
\binom{(n_1-1)\nu_1+i}{j} \left(\frac{T_Y}{T_X}\right)^{\nu_1+j} ; \qquad T_Y < T_X \\
\frac{B[(n_1-1)\nu_1,\nu_1+j]}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1)\nu_1]} \sum_{i=0}^{\infty} \frac{(-1)^i}{(n_2-1)\nu_2+i} \binom{\nu_2-1}{i} \sum_{j=0}^{\infty} (-1)^j \\
\binom{(n_2-1)\nu_2+i}{j} \left(\frac{T_X}{T_Y}\right)^{j} ; \qquad T_Y > T_X
\end{cases}
$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j$ and $T_X = \sum_{i=1}^{n_1} X_i$
3. UMVUE of R for Weibull distribution is

$$
\hat{R} = \begin{cases}
\frac{B[(n_2+i,1+j]}{B[1,(n_1-1)]B[1,(n_2-1)]} \sum_{j=0}^{\infty} (-1)^j \binom{n_1-2}{j} \left(\frac{T_Y}{T_X}\right)^{1+j} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad ; T_Y < T_X \\
\frac{B[(n_1-1,1+j]}{B[1,(n_1-1)]B[1,(n_2-1)]} \sum_{j=0}^{\infty} (-1)^j \binom{n_2-1+i}{j} \left(\frac{T_X}{T_Y}\right)^{j} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad ; T_Y > T_X
\end{cases}
$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j$ and $T_X = \sum_{i=1}^{n_1} X_i$

4. UMVUE of R for Erlang distribution is

$$
\hat{R} = \begin{cases}
\dfrac{B[(n_2-1)\nu_2+i+1,\nu_1+j]}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1)\nu_2]} \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i}{(n_2-1)\nu_2+i} \binom{\nu_2-1}{i} \displaystyle\sum_{j=0}^{\infty}(-1)^j \\[2mm]
\binom{(n_1-1)\nu_1+i}{j} \left(\dfrac{T_Y}{T_X}\right)^{\nu_1+j}; \qquad T_Y < T_X \\[4mm]
\dfrac{B[(n_1-1)\nu_1,\nu_1+j]}{B[\nu_1,(n_1-1)\nu_1]B[\nu_2,(n_2-1)\nu_1]} \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i}{(n_2-1)\nu_2+i} \binom{\nu_2-1}{i} \displaystyle\sum_{j=0}^{\infty}(-1)^j \\[2mm]
\binom{(n_2-1)\nu_2+i}{j} \left(\dfrac{T_X}{T_Y}\right)^{j}; \qquad T_Y > T_X
\end{cases}
$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j$ and $T_X = \sum_{i=1}^{n_1} X_i$

5. UMVUE of R for half-normal distribution is

$$
\hat{R} = \begin{cases}
\dfrac{4B[(n_2-1)\frac{1}{2}+i+1,\frac{1}{2}+j]}{B[\frac{1}{2},(n_1-1)\frac{1}{2}]B[\frac{1}{2},(n_2-1)\frac{1}{2}]} \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i}{(n_2-1)\frac{1}{2}+i} \binom{\frac{1}{2}-1}{i} \displaystyle\sum_{j=0}^{\infty}(-1)^j \\[2mm]
\binom{(n_1-1)\frac{1}{2}-1}{j} \left(\dfrac{T_Y}{T_X}\right)^{\frac{1}{2}+j}; \qquad T_Y < T_X \\[4mm]
\dfrac{4B[(n_1-1)\frac{1}{2},\frac{1}{2}+j]}{B[\frac{1}{2},(n_1-1)\frac{1}{2}]B[\frac{1}{2},(n_2-1)\frac{1}{2}]} \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i}{(n_2-1)\frac{1}{2}+i} \binom{\frac{1}{2}-1}{i} \displaystyle\sum_{j=0}^{\infty}(-1)^j \\[2mm]
\binom{(n_2-1)\frac{1}{2}+i}{j} \left(\dfrac{T_X}{T_Y}\right)^{j}; \qquad T_Y > T_X
\end{cases}
$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j^2$ and $T_X = \sum_{i=1}^{n_1} X_i^2$

6. UMVUE of R for Chi-distribution is

$$
\hat{R} = \begin{cases}
\dfrac{4B[(n_2-1)\frac{m}{2}+i+1,\frac{m}{2}+j]}{B[\frac{m}{2},(n_1-1)\frac{m}{2}]B[\frac{m}{2},(n_2-1)\frac{m}{2}]} \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i}{(n_2-1)\frac{m}{2}+i} \binom{\frac{m}{2}-1}{i} \displaystyle\sum_{j=0}^{\infty}(-1)^j \\[2mm]
\binom{(n_1-1)\frac{m}{2}-1}{j} \left(\dfrac{T_Y}{T_X}\right)^{\frac{m}{2}+j}; \qquad T_Y < T_X \\[4mm]
\dfrac{4B[(n_1-1)\frac{m}{2},\frac{m}{2}+j]}{B[\frac{m}{2},(n_1-1)\frac{m}{2}]B[\frac{m}{2},(n_2-1)\frac{m}{2}]} \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i}{(n_2-1)\frac{m}{2}+i} \binom{\frac{m}{2}-1}{i} \displaystyle\sum_{j=0}^{\infty}(-1)^j \\[2mm]
\binom{(n_2-1)\frac{m}{2}+i}{j} \left(\dfrac{T_X}{T_Y}\right)^{j}; \qquad T_Y > T_X
\end{cases}
$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j^2$ and $T_X = \sum_{i=1}^{n_1} X_i^2$

7. UMVUE of R for Rayleigh distribution is

$$
\hat{R} =
\begin{cases}
\dfrac{B[(n_2+i,1+j)]}{B[1,(n_1-1)]B[1,(n_2-1)]} \displaystyle\sum_{j=0}^{\infty} (-1)^j \binom{n_1-2}{j} \left(\dfrac{T_Y}{T_X}\right)^{1+j} \\
\hspace{7cm} ; T_Y < T_X \\[4mm]
\dfrac{B[(n_1-1,1+j)]}{B[1,(n_1-1)]B[1,(n_2-1)]} \displaystyle\sum_{j=0}^{\infty} (-1)^j \binom{n_2-1+i}{j} \left(\dfrac{T_X}{T_Y}\right)^{j} \\
\hspace{7cm} ; T_Y > T_X
\end{cases}
$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j^2$ and $T_X = \sum_{i=1}^{n_1} X_i^2$

8. UMVUE of R for Generalized Rayleigh distribution is

$$
\hat{R} =
\begin{cases}
\dfrac{B[(n_2-1)(p+1)+i+1,(p+1)+j]}{B[(p+1),(n_1-1)(p+1)]B[(p+1),(n_2-1)(p+1)]} \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i}{(n_2-1)(p+1)+i} \\[4mm]
\binom{(p+1)-1}{i} \displaystyle\sum_{j=0}^{\infty} (-1)^j \binom{(n_1-1)(p+1)+i}{j} \left(\dfrac{T_Y}{T_X}\right)^{(p+1)+j} \\
\hspace{6cm} ; T_Y < T_X \\[4mm]
\dfrac{B[(n_1-1)(p+1),(p+1)+j]}{B[(p+1),(n_1-1)(p+1)]B[(p+1),(n_2-1)(p+1)]} \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i}{(n_2-1)(p+1)+i} \binom{(p+1)-1}{i} \\[4mm]
\displaystyle\sum_{j=0}^{\infty} (-1)^j \binom{(n_2-1)(p+1)+i}{j} \left(\dfrac{T_X}{T_Y}\right)^{j} ; \qquad T_Y > T_X
\end{cases}
$$

where, $T_Y = \sum_{j=1}^{n_2} Y_j^2$ and $T_X = \sum_{i=1}^{n_1} X_i^2$

4 Confidence Interval of $R = Pr(Y > X)$

Theorem 4.1: The confidence interval for $R = Pr(Y > X)$ is

$$
Pr\left(I_{\left(\frac{(\nu_1 \overline{T}_Y / \nu_2 \overline{T}_X) F_{1-\sigma_2}}{(\nu_1 \overline{T}_Y / \nu_2 \overline{T}_X) F_{1-\sigma_2}+1}\right)} (\nu_1, \nu_2) < R < I_{\left(\frac{(\nu_1 \overline{T}_Y / \nu_2 \overline{T}_X) F_{\sigma_1}}{(\nu_1 \overline{T}_Y / \nu_2 \overline{T}_X) F_{\sigma_1}+1}\right)} (\nu_1, \nu_2) \right)
$$

$$
= 1 - \sigma \tag{12}
$$

where, $\overline{T}_X = \frac{1}{n_1} \sum_{i=1}^{n_1} X_i^{\rho_1}$ and $\overline{T}_Y = \frac{1}{n_2} \sum_{j=1}^{n_2} Y_j^{\rho_2}$

Proof: It follows from the above theorems that ξ and η be two independent random variables with Gamma (ν_1, λ_1) and Gamma (ν_2, λ_2) respectively, then

$$\lambda = \frac{\lambda_2}{\lambda_1}; \widetilde{\lambda} = \frac{\nu_1 \overline{\eta}}{\nu_2 \overline{\xi}}$$

where, $\lambda_1 = \frac{\overline{\xi}}{\nu_1}$ and $\lambda_2 = \frac{\overline{\eta}}{\nu_2}$. As we know that,

$$\frac{2n_1 \overline{\xi}}{\lambda_1} \sim Gamma(n_1\nu_1, 2) \equiv \chi^2_{2n_1\nu_1}$$

Similarly,

$$\frac{2n_2 \overline{\eta}}{\lambda_1} \sim Gamma(n_2\nu_2, 2) \equiv \chi^2_{2n_2\nu_2}$$

where, χ^2_α is the pdf of chi-squared distribution with α degree of freedom. Hence,

$$\frac{\widetilde{\lambda}}{\lambda} = \frac{2n_2\overline{\eta}/n_2\nu_2\lambda_2}{2n_1\overline{\xi}/n_1\nu_1\lambda_1}$$

$$= \frac{\chi^2_{2n_2\nu_2}/2n_2\nu_2}{\chi^2_{2n_1\nu_1}/2n_1\nu_1} \sim F(2n_1\nu_1, 2n_2\nu_2) \tag{13}$$

where, $F(\varepsilon_1, \varepsilon_2)$ denotes Snedecor's F-distribution with ε_1 and ε_2 degree of freedom.

$$\frac{\widetilde{\lambda}}{\lambda} \sim F(2n_1\nu_1, 2n_2\nu_2)$$

For any δ denoted by $F_\delta = F_\delta(2n_1\nu_1, 2n_2\nu_2)$, then the relation to F_δ and $1 - \delta$ quantile of $F_\delta(2n_1\nu_1, 2n_2\nu_2)$ distribution is

$$F_\delta(2n_1\nu_1, 2n_2\nu_2) = [F_{1-\delta}(2n_1\nu_1, 2n_2\nu_2)]^{-1}$$

Let σ_1 and σ_2 be non-negative numbers such that $\sigma_1 + \sigma_2 = \sigma$. Then

$$Pr(\widetilde{\lambda}F_{1-\sigma_2} < \lambda < \widetilde{\lambda}F_{\sigma_1}) = 1 - \sigma \tag{14}$$

Since $R = I_{\left(\frac{\lambda}{\lambda+1}\right)}(\nu_1, \nu_2)$ and $I_Z(a, b)$ is an increasing function of z for any a, b. So, $I_{\left(\frac{\lambda}{\lambda+1}\right)}(\nu_1, \nu_2)$ as the function of λ . Hence (14) become

$$Pr\left(I_{\left(\frac{\widetilde{\lambda}F_{1-\sigma_2}}{\widetilde{\lambda}F_{1-\sigma_2}+1}\right)}(\nu_1, \nu_2) < R < I_{\left(\frac{\widetilde{\lambda}F_{\sigma_1}}{\widetilde{\lambda}F_{\sigma_1}+1}\right)}(\nu_1, \nu_2)\right) = 1 - \sigma \tag{15}$$

After the substituting i.e. $\widetilde{\lambda} = \frac{\nu_1 \overline{\eta}}{\nu_2 \overline{\xi}}$ and $\lambda_1 = \theta_1, \lambda_2 = \theta_2$

Then $R = I_{\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)}(\nu_1, \nu_2) = I_{\left(\frac{\theta_2}{\theta_1+\theta_2}\right)}(\nu_1, \nu_2)$. The confidence interval for R is

$$Pr\left(I_{\left(\frac{(\nu_1\overline{\eta}/\nu_2\overline{\xi})F_{1-\sigma_2}}{(\nu_1\overline{\eta}/\nu_2\overline{\xi})F_{1-\sigma_2}+1}\right)}(\nu_1, \nu_2) < R < I_{\left(\frac{(\nu_1\overline{\eta}/\nu_2\overline{\xi})F_{\sigma_1}}{(\nu_1\overline{\eta}/\nu_2\overline{\xi})F_{\sigma_1}+1}\right)}(\nu_1, \nu_2)\right) = 1-\sigma$$

where, $\overline{T}_X = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i^{\rho_1} = \overline{\xi}$ and $\overline{T}_Y = \frac{1}{n_2}\sum_{j=1}^{n_2} Y_j^{\rho_2} = \overline{\eta}$

Hence, the theorem follows.

Corollary 3.

1. Confidence interval for one parameter exponential distribution $(\rho = \nu = 1)$

$$Pr\left(I_{\left(\frac{\overline{\lambda}F_{1-\sigma_2}}{\overline{\lambda}F_{1-\sigma_2}+1}\right)} < R < I_{\left(\frac{\overline{\lambda}F_{\sigma_1}}{\overline{\lambda}F_{\sigma_1}+1}\right)}\right) = 1 - \sigma$$

where $\overline{\lambda} = \frac{\overline{Y}}{\overline{X}}$ and $R = I_{\left(\frac{\theta_2}{\theta_1+\theta_2}\right)}(1, 1)$

2. Confidence interval for gamma Distribution $(\rho = 1)$

$$Pr\left(I_{\left(\frac{\overline{\lambda}F_{1-\sigma_2}}{\overline{\lambda}F_{1-\sigma_2}+1}\right)}(\nu_1, \nu_2) < R < I_{\left(\frac{\overline{\lambda}F_{\sigma_1}}{\overline{\lambda}F_{\sigma_1}+1}\right)}(\nu_1, \nu_2)\right) = 1 - \sigma$$

where $\overline{\lambda} = \frac{\overline{Y}}{\overline{X}}$ and $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}(\nu_1, \nu_2)$

3. Confidence interval for Weibull Distribution $(\nu = 1)$

$$Pr\left(I_{\left(\frac{\overline{\lambda}F_{1-\sigma_2}}{\overline{\lambda}F_{1-\sigma_2}+1}\right)}(1, 1) < R < I_{\left(\frac{\overline{\lambda}F_{\sigma_1}}{\overline{\lambda}F_{\sigma_1}+1}\right)}(1, 1)\right) = 1 - \sigma$$

where $\overline{\lambda} = \frac{\overline{Y}}{\overline{X}}$ and $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}(1, 1)$

4. Confidence interval for Erlang Distribution $(\nu > 0, \rho = 1)$

$$Pr\left(I_{\left(\frac{\overline{\lambda}F_{1-\sigma_2}}{\overline{\lambda}F_{1-\sigma_2}+1}\right)}(\nu_1, \nu_2) < R < I_{\left(\frac{\overline{\lambda}F_{\sigma_1}}{\overline{\lambda}F_{\sigma_1}+1}\right)}(\nu_1, \nu_2)\right) = 1 - \sigma$$

where $\overline{\lambda} = \frac{\nu_1\overline{\eta}}{\nu_2\overline{\xi}}$ and $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}(\nu_1, \nu_2)$

5. Confidence interval for half-normal Distribution $(\nu > 1/2, \rho = 2)$

$$Pr\left(I_{\left(\frac{\overline{\lambda}F_{1-\sigma_2}}{\overline{\lambda}F_{1-\sigma_2}+1}\right)}\left(\frac{1}{2},\frac{1}{2}\right) < R < I_{\left(\frac{\overline{\lambda}F_{\sigma_1}}{\overline{\lambda}F_{\sigma_1}+1}\right)}\left(\frac{1}{2},\frac{1}{2}\right)\right) = 1-\sigma$$

where $\overline{\lambda} = \frac{\overline{Y}}{\overline{X}}$ and $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}\left(\frac{1}{2},\frac{1}{2}\right)$

6. Confidence interval for Chi-distribution $(\nu > m/2, \rho = 2)$

$$Pr\left(I_{\left(\frac{\overline{\lambda}F_{1-\sigma_2}}{\overline{\lambda}F_{1-\sigma_2}+1}\right)}\left(\frac{m}{2},\frac{m}{2}\right) < R < I_{\left(\frac{\overline{\lambda}F_{\sigma_1}}{\overline{\lambda}F_{\sigma_1}+1}\right)}\left(\frac{m}{2},\frac{m}{2}\right)\right) = 1-\sigma$$

where $\overline{\lambda} = \frac{\overline{Y}}{\overline{X}}$ and $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}\left(\frac{m}{2},\frac{m}{2}\right)$

7. Confidence interval for Rayleigh distribution $(\nu = 1, \rho = 2)$

$$Pr\left(I_{\left(\frac{\overline{\lambda}F_{1-\sigma_2}}{\overline{\lambda}F_{1-\sigma_2}+1}\right)}(1,1) < R < I_{\left(\frac{\overline{\lambda}F_{\sigma_1}}{\overline{\lambda}F_{\sigma_1}+1}\right)}(1,1)\right) = 1-\sigma$$

where $\overline{\lambda} = \frac{\overline{Y}}{\overline{X}}$ and $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}(1,1)$

8. Confidence interval for Generalized Rayleigh distribution $(\nu = p+1, \rho = 2)$

$$Pr\left(I_{\left(\frac{\overline{\lambda}F_{1-\sigma_2}}{\overline{\lambda}F_{1-\sigma_2}+1}\right)}(p+1,p+1) < R < I_{\left(\frac{\overline{\lambda}F_{\sigma_1}}{\overline{\lambda}F_{\sigma_1}+1}\right)}(p+1,p+1)\right) = 1-\sigma$$

where $\overline{\lambda} = \frac{\overline{Y}}{\overline{X}}$ and $R = I_{\frac{\theta_2}{\theta_1+\theta_2}}(p+1,p+1)$

5 Probability of Disaster $Pr(Y > \gamma)$

Theorem 5.1: If the stress and finite strength are denoted by the random variables X and Y which follows PEFD and Power function distribution,

that are shown in (1) and (2), respectively. Then probability of disaster $\alpha = Pr(Y > \gamma)$ is given by

$$\alpha = Pr(Y > \gamma) = \frac{1}{\Gamma\nu} \int_k^\infty u^{\nu-1} e^{-u} du \tag{16}$$

where, $k = \frac{\gamma^\rho}{\theta}$

Proof: From (1),

$$\alpha = Pr(Y > \gamma)$$

$$= \int_\gamma^\infty \frac{\rho x^{\rho\nu-1} e^{\frac{-x^\rho}{\theta}}}{\Gamma\nu\theta^\nu} dx$$

$$= \frac{\rho}{\Gamma\nu\theta^\nu} \int_\gamma^\infty x^{\rho\nu-1} e^{\frac{-x^\rho}{\theta}} dx$$

Let $\frac{x^\rho}{\theta} = u$

$$\alpha = \frac{1}{\Gamma\nu} \int_{\frac{\gamma^\rho}{\theta}}^\infty u^{\nu-1} e^{-u} du$$

$$= \Gamma(\nu, k)$$

which is the upper incomplete gamma function, where, $k = \frac{\gamma^\rho}{\theta}$.

6 Numerical Analysis

From (16) the probability of disaster $Pr(Y > \gamma)$ can be measured. The numerical values are obtained for different values of ν which is presented in Table 1. It can be easily interpreted from Table 1 that the probability of disaster decreases with an increase in the value of k. In order to overcome the problem of disaster (i.e. to attain the smallest value of $\alpha = Pr(X > \gamma)$), the values of $k = \frac{\gamma^\rho}{\theta}$, where ρ and θ is the parameter of PEF-distribution and γ is the scale parameter of the power function distribution, should be considered in such a manner that the value of α tends to zero.

Alternatively, we may also obtain the numerical values of k for fixed values ν from equation (16). These values are used to obtain the optimum cost for manufacturing of item at desired tolerance level.

Table 1   Numerical Values for the Probability of disaster $\alpha = Pr(X > \theta)$ and k for different values of ν

k	$\nu = 0.001$	$\nu = 0.01$	$\nu = 0.5$	$\nu = 1.05$	$\nu = 1.5$	$\nu = 2$
0.8	0.000311	0.003132	0.205903	0.472619	0.659390	**0.808792**
1.0	0.000219	0.002216	0.157299	0.389400	0.572407	**0.735759**
1.2	0.000158	0.001603	0.121335	0.320603	0.493635	**0.662627**
1.5	0.000064	0.001013	0.083264	0.239233	0.391625	**0.557825**
1.8	0.000032	0.000065	0.057779	0.178339	0.308022	**0.462837**
2.3	0.000016	0.000331	0.031972	0.109131	0.203542	**0.330854**
2.8	0	0.000172	0.017961	0.066687	0.132778	**0.231078**
3.4	0	0.000081	0.009116	0.036880	0.078555	**0.146842**
4.1	0	0.000034	0.004189	0.018454	0.042054	**0.084521**
5.1	0	0.000012	0.001404	0.006851	0.016940	**0.037190**
6.2	0	0	0.000429	0.002300	0.006131	**0.014612**
7.6	0	0	0.000087	0.000572	0.000165	**0.004304**
9.3	0	0	0.000016	0.000105	0.000331	**0.000942**
11.4	0	0	0	0.000013	0.000044	**0.000138**
13.9	0	0	0	0	0	**0.000014**
17.0	0	0	0	0	0	**0**

Table 2   Values of m at different tolerance level α for $\nu = 2$

α	0.05	0.02	0.01	0.001	0.0001	0.00001
k	2.996020	3.912310	4.605460	6.908040	9.210630	11.513200

7 Stress – Strength Reliability

Theorem 7.1: The Stress – Strength model $Pr(Y > X)$, where X follows PEFD and Y follows power function distribution, respectively is given as

$$Pr(Y > X) = \frac{1}{\Gamma \nu} \int_0^k u^{\nu-1} e^{-u} du - \frac{1}{\Gamma \nu k^{\mu/\alpha}} \int_0^k u^{\left(\frac{\mu}{\alpha}+\nu-1\right)} e^{-u} du \quad (17)$$

where, $\frac{x^\rho}{\theta} = u$

Proof: From (1) and (3), we have

$$Pr(Y > X) = \int_0^\gamma \int_x^\gamma f(x, \Theta) g(y, \mu) dy dx$$

where, $\Theta = (\rho, \nu, \theta)$

$$Pr(Y > X) = \int_0^\gamma \frac{\rho x^{\rho\nu-1} e^{\frac{-x^\rho}{\theta}}}{\Gamma\nu\theta^\nu} \left[\int_x^\gamma \frac{\mu}{\gamma} \left(\frac{y}{\gamma}\right)^{\mu-1} dy \right] dx$$

$$= \int_0^\gamma \frac{\rho x^{\rho\nu-1} e^{\frac{-x^\rho}{\theta}}}{\Gamma\nu\theta^\nu} \left[\frac{1}{\gamma^\mu} \{\gamma^\mu - x^\mu\} \right] dx$$

$$= \int_0^\gamma \frac{\rho x^{\rho\nu-1} e^{\frac{-x^\rho}{\theta}}}{\Gamma\nu\theta^\nu} dx - \int_0^\gamma \frac{\rho x^{\rho\nu+\mu-1} e^{\frac{-x^\rho}{\theta}}}{\Gamma\nu\theta^\nu\gamma^\mu} dx$$

Taking $\frac{x^\rho}{\theta} = u$ and solving the above integrals we finally get,

$$Pr(Y > X) = \frac{1}{\Gamma\nu} \int_0^k u^{\nu-1} e^{-u} du - \frac{1}{\Gamma\nu k^{\mu/\alpha}} \int_0^k u^{\left(\frac{\mu}{\alpha}+\nu-1\right)} e^{-u} du$$

where $k = \frac{\gamma^\rho}{\theta}$, Hence, the theorem follows.

Table 3   The Strength reliability of an item for $\nu = 2, \rho = 2$, and varying values of k and μ and varying values of k and μ

k	$\mu = 2$	$\mu = 5$	$\mu = 10$	$\mu = 15$	$\mu = 20$	$\mu = 30$
0.8	0.072651	0.116730	0.145622	0.158460	0.165680	0.173507
1.0	0.103638	0.164995	0.204307	0.221481	0.231046	0.241332
1.2	0.136518	0.215358	0.264713	0.285899	0.297582	0.310045
1.5	0.187304	0.291463	0.354377	0.380661	0.394936	0.409972
1.8	0.237853	0.365149	0.439257	0.469359	0.485451	0.502181
2.3	0.317875	0.477318	0.564419	0.598102	0.615615	0.633416
2.8	0.389960	0.573140	0.666799	0.701180	0.718538	0.735769
3.4	0.464769	0.666495	0.761480	0.794106	0.809972	0.825255
4.1	0.536852	0.749440	0.840064	0.868720	0.882015	0.894349
5.1	0.616331	0.831032	0.910101	0.932106	0.941591	0.949892
6.2	0.680103	0.887158	0.951925	0.967406	0.973505	0.978478
7.6	0.737474	0.928900	0.977654	0.987122	0.990404	0.992822
9.3	0.785057	0.956229	0.990580	0.995696	0.997186	0.998139
11.4	0.824575	0.973529	0.996360	0.998797	0.999357	0.999650
13.9	0.856116	0.983855	0.998620	0.999695	0.999878	0.999951
17.0	0.882353	0.990239	0.999493	0.999930	0.999981	0.999995

8 The Stress-Strength Reliability $R = Pr(Y > X)$ when both follows PEFD

Theorem 8.1: Let X and Y be two independent random variables from PEFD, where X and Y are the stress and the strength, respectively. Reliability $R = Pr(Y > X)$ is

$$R = \frac{\rho_1}{\theta_1^{\nu_1}\Gamma\nu_1} \int_{x=0}^{\infty} x^{\rho_1\nu_1-1}e^{(-x^{\rho_1}/\theta_1)}e^{(-x^{\rho_2}/\theta_2)} \sum_{k=0}^{\nu_2-1} \frac{1}{k!}\left(\frac{x^{\rho_2}}{\theta_2}\right)^k dx \quad (18)$$

Proof: Random variable X follows the PEF-distribution with pdf

$$f(x,\Theta) = \frac{\rho_1 x^{\rho_1\nu_1-1}e^{(-x^{\rho_1}/\theta_1)}}{\Gamma\nu_1\theta_1^{\nu_1}} \quad (19)$$

where, $\Theta = (\rho_1, \nu_1, \theta_1)$

Random variable Y follows the PEF-distribution with pdf

$$f(y,\Theta) = \frac{\rho_2 y^{\rho_2\nu_2-1}e^{(-y^{\rho_2}/\theta_2)}}{\Gamma\nu_2\theta_2^{\nu_2}} \quad (20)$$

where, $\Theta = (\rho_2, \nu_2, \theta_2)$

The Reliability $R = Pr(Y > X)$ is

$$R = \int_{x=0}^{\infty} \int_{y=x}^{\infty} f(x,\Theta)f(y,\Theta)dydx$$

$$= \int_{x=0}^{\infty} \left\{ \frac{\rho_1 x^{\rho_1\nu_1-1}e^{(-x^{\rho_1}/\theta_1)}}{\Gamma\nu_1\theta_1^{\nu_1}} \right\} \left[\int_{y=x}^{\infty} \frac{\rho_2 y^{\rho_2\nu_2-1}e^{(-y^{\rho_2}/\theta_2)}}{\Gamma\nu_2\theta_2^{\nu_2}} dy \right] dx$$

Let $\frac{y^{\rho_2}}{\theta_2} = t$

$$R = \int_0^{\infty} \left\{ \frac{\rho_1 x^{\rho_1\nu_1-1}e^{(-x^{\rho_1}/\theta_1)}}{\Gamma\nu_1\theta_1^{\nu_1}} \right\} \left[\int_{\frac{x^{\rho_2}}{\theta_2}}^{\infty} \frac{t^{\nu_2-1}}{\Gamma\nu_2}e^{-t}dt \right]$$

which is the upper incomplete gamma after solving, we get

$$R = \frac{\rho_1}{\theta_1^{\nu_1}\Gamma\nu_1} \int_{x=0}^{\infty} x^{\rho_1\nu_1-1}e^{(-x^{\rho_1}/\theta_1)}e^{(-x^{\rho_2}/\theta_2)} \sum_{k=0}^{\nu_2-1} \frac{1}{k!}\left(\frac{x^{\rho_2}}{\theta_2}\right)^k dx$$

where $r > 0$ is any positive number. Hence, the theorem follows.

9 Discussion

When an item/device is manufactured and if the strength of an item follows Power function distribution, it is expected that the maximum feasible values of γ may have an upper limit say γ_0 . For example, the maximum accelerating speed of a turbine must not be increased its permissible capacity. At a fixed tolerance level α, suppose γ_α is the desired value of γ. In case $\gamma_\alpha < \gamma_0$, we may obtain the required value of μ say μ_α, by using Table 3, so that the item is manufactured with the strength distribution having parameters $(\mu_\alpha, \gamma_\alpha)$ and consequently, the desired strength reliability can be achieved. However, if $\gamma_\alpha > \gamma_0$, we will have to either adjust α or look for an alternate item.

10 Study of the Cost with an Example

Let us assume that the maximum feasible value of k is 12. When $\alpha \leq 0.01$ the value of m must be greater or equal to 5.1 i.e. $m \geq 5.1$. As the value of m cannot exceed 12 , then one needs to fix the item / device in a way such that $5.1 \leq k \leq 12$ i.e. $2.7 \leq \gamma \leq 4.1$ and thus, the corresponding values of μ leads to a maximum of $Pr(Y > X)$. The cost factor of adjusting the parameters may be taken into consideration here as the cost of varying γ and μ may be different. Theoretically, the costs may be an increasing or decreasing function of γ and μ depending upon the nature of the parameters. Usually, Cost (Y) is an increasing function Y, if Y is the mean strength. In our study, $E(Y) = \mu\gamma/(\mu + 1)$, which implies that the mean strength increases by increasing either of the two parameters. Hence, we may assume the two costs to be an increasing function of the respective parameters. Assuming the costs to be directly proportional to the required values of the parameters, the problem may be further evaluated as follows:

Let C1 and C2 be the costs of adjusting one unit of γ and μ, respectively. Minimize $C = \gamma C_1 + \mu C_2$ subject to $2.7 \leq \gamma \leq 4.1$ and $Pr(Y > X) \geq 0.99$.

Analytically, the problem may be simplified as follows:

On using Table 3 for k = 5.1, 6.2, 7.6, 9.3 and 11.4 i.e. γ = 2.76, 3.04, 3.37, 3.73 and 4.1, respectively and obtain those values of μ for which $Pr(Y > X) \geq 0.99$, the cost function for each pair of (γ, μ) is evaluated:

Table 4 depicts that the minimum cost lies at $3.37C_1 + 20C_2$ depending upon the numerical values of C_1 and C_2.

Table 4   Table for optimum cost of manufactured items

γ	μ	$\gamma C_1 + \mu C_2$
3.37	20	$3.37C_1 + 20C_2$
3.37	30	$3.37C_1 + 30C_2$
3.7	10	$3.7C_1 + 10C_2$
3.7	15	$3.7C_1 + 15C_2$
3.7	20	$3.7C_1 + 20C_2$
3.7	30	$3.7C_1 + 30C_2$
4.1	10	$4.1C_1 + 10C_2$
4.1	15	$4.1C_1 + 15C_2$
4.1	20	$4.1C_1 + 20C_2$
4.1	30	$4.1C_1 + 30C_2$

Acknowledgments

References

[1] Alam and Roohi. 2003. On facing an exponential stress with strength having power function distribution. Aligarh J. Of Stat., 23:57-63.

[2] Awad, A. M. and Gharraf, M. K. 1986. Estimation of $Pr(Y < X)$ in the Burr case, A Comparative Study. Commun. Statist. – Simul., 15(2):389 – 403.

[3] Basu, D. 1964. Estimates of reliability for some distributions useful in life testing. Technometrics, 6:215-219.

[4] Chao, A. 1982. On comparing estimators of $Pr(X > Y)$ in the exponential case. IEEE transactions on reliability, 31:389-392.

[5] Chaturvedi, A. and Pathak. A. 2012. Estimation of the reliability functions for exponentiated Weibull distribution. J. Stat. Appl., 7:1–8.

[6] Chaturvedi, A. and Surinder, K. 1999. Further remarks on estimating the reliability function of exponential distribution under Type-I and Type-II censorings. Brazilian Journal of Probability and Statistics, 13:29-39.

[7] Church, J. D. and Harries, B. 1970. The estimation of reliability from stress-strength relationships. Technometrics, 12:49-54.

[8] Downton, F. 1973. The estimation of $P(Y < X)$ in the normal case. Technometrics, 15: 551-558.

[9] Enis, P. and Geisser. S. 1971. Estimation of the probability that $(Y > X)$. J. Amer. Statist. Asso., 66:162-168.

[10] Kelly, G. D., Kelly., J. A. and Schucany. W. R. 1976. Efficient estimation of $P(Y < X)$ in the exponential case. Technometrics, 18:359-360.

[11] Kotz, S., Lumelskii, Y. and Pensky, M. 2003. Stress-Strength Model and its Generalizations, World Scientic Publication Co. Pte. Ltd., Singapore.

[12] Pugh, E. L. 1963. The best estimate of reliability in the exponential case. Operations Research, 11:57-61

[13] Rezaei, S., Tahmasbi, R. and Mahmoodi, M. 2010. Estimation of $P(Y < X)$ for generalized Pareto distribution. J. Stat. Plan Inference, 140:480-494.

[14] Sathe, Y. S. and Shah. S. P. 1981. On estimating $P(X < Y)$ for the exponential distribution. Commun. Statist. Theor. Meth., A10:39-47.

[15] Sinha , S. K. And Kale, B.K. 1980: Life testing and Reliability Estimation. Wiley Eastern Ltd., New Delhi.

[16] Surinder, K. and Kumar. M. 2015. Study of the Stress-Strength Reliability among the Parameters of Generalized Inverse Weibull Distribution. Intern. Journal of Science, Technology and Management. 4: 751-757.

[17] Surinder, K. and Kumar. M. 2016. Point and Interval Estimation of $R = P(Y > X)$ for Generalized Inverse Weibull Distribution by Transformation Method. J. Stat. Appl. Pro. Lett., 3:1-6.

[18] T, Liang. 2008. Empirical Bayes estimation of reliability in a positive exponential family. Communications in Statistics -– Theory and Methods, 37(13):2052–2065.

[19] Tong, H. 1974. A note on the estimation of $P(Y < X)$ in the exponential case. Technometrics, 16:625.

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