
Families of Estimators for Estimating Mean Using Information of Auxiliary Variate Under Response and Non-Response

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Abstract

This research article is concerned with the efficiency improvement of estimators for finite population mean under complete and incomplete information arising as a result of non-response. Different families of estimators for estimating the mean of study variate via known population mean, proportion and rank of auxiliary variate under different situations are proposed along with their bias and mean square error (MSE). Optimum conditions are suggested to attain minimum mean square error of proposed families of estimators. Further the problem is extended for the situation of unknown parameters of auxiliary variate and two phase sampling families of estimators are suggested along with their properties under fixed cost and precision. Employing real data sets, theoretical and empirical comparisons are executed to explain the efficiency of the proposed families of estimators.

Keywords: Non-response, bias, mean square error, rank, auxiliary variate.

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1 Introduction

It is a well-known fact that the efficiency of the estimators often increases by utilising the available information of the auxiliary variate [see Cochran (1940), Tripathi et al. (1994), Khare (2003)], but there are many causes when the required information may not be available specifically or completely, on different variates. For instance, sometimes respondents are reluctant to answer the questionnaire or do not provide complete information, this circumstance is called non-response which causes the decrease in the efficiency. To resolve this problem of non-response, Hansen and Hurwitz (1946) introduced a methodology of sub-sampling from non-responding group of population to suggest an unbiased estimator for estimating the population mean. Several authors Rao (1986, 90), Khare and Srivastava (1993, 95, 97, 2000), Khare and Sinha (2002, 2009), Singh and Kumar (2009), Sinha and Kumar (2011, 13, 14) suggested various type of estimators/classes of estimators for estimating the population mean using the methodology of sub-sampling the non-respondents with known and unknown population mean of auxiliary variate(s).

Let us suppose that $\mathbb{S} = (\mathbb{S}_1, \mathbb{S}_2 \dots \mathbb{S}_N)$ is a finite population consist with \mathcal{N} units. A sample \mathbb{S}_n of size n is drawn from \mathbb{S} using simple random sampling without replacement (SRSWOR) method and value of study and auxiliary variates for the i^{th} unit ($i = 1, 2, \dots n$) of the population is denoted by y_i and x_i respectively. Let the population is dichotomy in two non-overlapping strata of responding (\mathcal{N}_1 units) and non-responding (\mathcal{N}_2 units) groups, i.e. $\mathbb{S} = \mathbb{S}_{\mathcal{N}_1} + \mathbb{S}_{\mathcal{N}_2}$. Further, in a sample (\mathbb{S}_n) of n units, n_1 responding (\mathbb{S}_{n_1}) and n_2 non-responding units (\mathbb{S}_{n_2}) are supposed to come from $\mathbb{S}_{\mathcal{N}_1}$ responding and $\mathbb{S}_{\mathcal{N}_2}$ non-responding group of population. Let a subsample of size $r = \frac{n_2}{h}$, ($h > 1$) units is drawn arbitrarily from \mathbb{S}_{n_2} , where h denotes the sub-sampling fraction at second phase sample. If $w_1 = \frac{n_1}{n}$ and $w_2 = \frac{n_2}{n}$ are the estimates of unknown proportions $W_1 = \frac{\mathcal{N}_1}{\mathcal{N}}$ and $W_2 = \frac{\mathcal{N}_2}{\mathcal{N}}$ respectively, therefore on the basis of available information on $(n_1 + r)$ units, Hansen and Hurwitz (1946) suggested an unbiased estimator for estimating the population mean as

$$t_h = \bar{y}^h = w_1 \bar{y}_1 + w_2 \bar{y}_{2(r)} \quad (1)$$

whose variance is given by

$$\mathcal{V}(t_h) = \bar{Y}^2 [\lambda_n C_0^2 + \lambda_h C_{0(2)}^2] \quad (2)$$

where $\lambda_n = \frac{1}{n} - \frac{1}{N}$, $\lambda_h = w_2 \frac{(h-1)}{n}$, $C_0^2 (= \frac{S_y^2}{\bar{Y}^2})$ and $C_{0(2)}^2 (= \frac{S_{y(2)}^2}{\bar{Y}^{(2)2}})$ are the coefficients of variation of complete and non-responding group of the study variate while \bar{y}_1 and $\bar{y}_{2(r)}$ are sample means of the study variate depending upon n_1 and r units respectively.

Following the strategies of Rao (1986, 90), Khare and Srivastava (1993), and Khare and Srivastava (2000), the conventional ratio, product, regression, generalized and class of estimators for estimating the population mean with known mean and proportion of auxiliary variate under unit non-response on study as well as auxiliary variates may be defined as

$$\begin{aligned} t_{r1}^{(1)} &= \bar{y}^h \frac{\bar{X}}{\bar{x}^h}, & t_{r1}^{(2)} &= \bar{y}^h \frac{\bar{P}_x}{\bar{p}_x^h}, \\ t_{p1}^{(1)} &= \bar{y}^h \frac{\bar{x}^h}{\bar{X}}, & t_{p1}^{(2)} &= \bar{y}^h \frac{\bar{p}_x^h}{\bar{P}_x}, \\ t_{reg1}^{(1)} &= \bar{y}^h + b_1^h (\bar{X} - \bar{x}^h), & t_{reg1}^{(2)} &= \bar{y}^h + b_2^h (\bar{P}_x - \bar{p}_x^h), \\ t_{g1}^{(1)} &= \bar{y}^h \left(\frac{\bar{x}^h}{\bar{X}} \right)^{\alpha_1}, & t_{g1}^{(2)} &= \bar{y}^h \left(\frac{\bar{p}_x^h}{\bar{P}_x} \right)^{\alpha_2}, \\ t_{c1}^{(1)} &= f_1(\bar{y}^h, \theta_1) \text{ where } \theta_1 = \frac{\bar{x}^h}{\bar{X}}, & t_{c1}^{(2)} &= f_2(\bar{y}^h, \theta_2) \text{ where } \theta_2 = \frac{\bar{p}_x^h}{\bar{P}_x}, \end{aligned}$$

The *Bias* and *MSE* of all the above estimators up to the order of n^{-1} are given as:

$$\mathcal{B}(t_{r1}^{(1)}) = \bar{Y} [\lambda_n (C_1^2 - 2\rho_{01} C_0 C_1) + \lambda_h (C_{1(2)}^2 - 2\rho_{01(2)} C_{0(2)} C_{1(2)})] \quad (3)$$

$$\begin{aligned} \mathcal{M}(t_{r1}^{(1)}) &= \bar{Y}^2 [\lambda_n (C_0^2 + C_1^2 - 2\rho_{01} C_0 C_1) + \lambda_h (C_{0(2)}^2 \\ &\quad + C_{1(2)}^2 - 2\rho_{01(2)} C_{0(2)} C_{1(2)})] \end{aligned} \quad (4)$$

$$\mathcal{B}(t_{p1}^{(1)}) = \bar{Y} [\lambda_n C_0 C_1 + \lambda_h \rho_{01(2)} C_{0(2)} C_{1(2)}] \quad (5)$$

$$\begin{aligned} \mathcal{M}(t_{p1}^{(1)}) &= \bar{Y}^2 [\lambda_n (C_0^2 + C_1^2 + 2\rho_{01} C_0 C_1) + \lambda_h (C_{0(2)}^2 + C_{1(2)}^2 \\ &\quad + 2\rho_{01(2)} C_{0(2)} C_{1(2)})] \end{aligned} \quad (6)$$

$$\mathcal{B}(t_{reg1}^{(1)}) = \beta \lambda_n \left(\frac{\mu_{30}}{S_x^2} - \frac{\mu_{21}}{S_{yx}} \right) \quad (7)$$

where $\beta = \frac{s_{yx}}{S_x^2}$, $\mu_{30} = E(\bar{x}^h - \bar{X})^3(\bar{y}^h - \bar{Y})^0$ and $\mu_{21} = E(\bar{x}^h - \bar{X})^2(\bar{y}^h - \bar{Y})$

$$\mathcal{M}(t_{reg1}^{(1)}) = \{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\} - \frac{\{\lambda_n \rho_{01} S_y S_x + \lambda_h \rho_{01(2)} S_{y(2)} S_{x(2)}\}^2}{\{\lambda_n S_x^2 + \lambda_h S_{x(2)}^2\}} \quad (8)$$

at

$$(b_1^h)_{opt} = \frac{\{\lambda_n \rho_{01} S_y S_x + \lambda_h \rho_{01(2)} S_{y(2)} S_{x(2)}\}}{\{\lambda_n S_x^2 + \lambda_h S_{x(2)}^2\}},$$

$$\mathcal{B}(t_{g1}^{(1)}) = \bar{Y} \left[\frac{\alpha_1(\alpha_1 - 1)}{2} \{\lambda_n C_1^2 + \lambda_h C_{1(2)}^2\} + \alpha_1 \{\lambda_n \rho_{01} C_0 C_1 + \lambda_h \rho_{01(2)} C_{0(2)} C_{1(2)}\} \right] \quad (9)$$

$$[\mathcal{M}(t_{g1}^{(1)})]_{min} = \{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\} - \frac{\{\lambda_n \rho_{01} S_y S_x + \lambda_h \rho_{01(2)} S_{y(2)} S_{x(2)}\}^2}{\{\lambda_n S_x^2 + \lambda_h S_{x(2)}^2\}} \quad (10)$$

at

$$(\alpha_1)_{opt} = - \frac{\{\lambda_n \rho_{01} C_0 C_1 + \lambda_h \rho_{01(2)} C_{0(2)} C_{1(2)}\}}{\{\lambda_n C_1^2 + \lambda_h C_{1(2)}^2\}},$$

$$\mathcal{B}(t_{c1}^{(1)}) = \frac{\{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\}}{2} \frac{\partial^2 f_1}{\partial \bar{y}^h{}^2} \Big|_{(\bar{y}^{h*}, \theta_1^*)} + \frac{\{\lambda_n C_1^2 + \lambda_h C_{1(2)}^2\}}{2} \frac{\partial^2 f_1}{\partial \theta_1^2} \Big|_{(\bar{y}^{h*}, \theta_1^*)} + \bar{Y} \{\lambda_n \rho_{01} C_0 C_1 + \lambda_h \rho_{01(2)} C_{0(2)} C_{1(2)}\} \frac{\partial^2 f_1}{\partial \bar{y}^h \partial \theta_1} \Big|_{(\bar{y}^{h*}, \theta_1^*)} \quad (11)$$

where

$$\bar{y}^{h*} = \bar{Y} + \psi_0(\bar{y}^h - \bar{Y}), \quad \theta_1^* = 1 + \psi_1(\theta_1 - 1), \quad 0 < \psi_0, \psi_1 < 1,$$

$$[\mathcal{M}(t_{c1}^{(1)})]_{min} = \{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\} - \frac{\{\lambda_n \rho_{01} S_y S_x + \lambda_h \rho_{01(2)} S_{y(2)} S_{x(2)}\}^2}{\{\lambda_n S_x^2 + \lambda_h S_{x(2)}^2\}} \quad (12)$$

at

$$f_1^{(2)}(\bar{Y}, 1) = \left. \frac{\partial f_1}{\partial \theta_1} \right|_{(\bar{Y}, 1)} = \frac{-\bar{X}\{\lambda_n \rho_{01} S_y S_x + \lambda_h \rho_{01(2)} S_{y(2)} S_{x(2)}\}}{\{\lambda_n S_x^2 + \lambda_h S_{x(2)}^2\}},$$

$$\mathcal{B}(t_{r1}^{(2)}) = \bar{Y}[\{\lambda_n C_2^2 + \lambda_h C_{2(2)}^2\} - \{\lambda_n \rho_{02} C_0 C_2 + \lambda_h \rho_{02(2)} C_{0(2)} C_{2(2)}\}] \quad (13)$$

$$\mathcal{M}(t_{r1}^{(2)}) = \bar{Y}^2[\lambda_n(C_0^2 + C_2^2 - 2\rho_{02} C_0 C_2) + \lambda_h(C_{0(2)}^2 + C_{2(2)}^2 - 2\rho_{02(2)} C_{0(2)} C_{2(2)})] \quad (14)$$

$$\mathcal{B}(t_{p1}^{(2)}) = \bar{Y}[\lambda_n \rho_{02} C_0 C_2 + \lambda_h \rho_{02(2)} C_{0(2)} C_{2(2)}] \quad (15)$$

$$\mathcal{M}(t_{p1}^{(2)}) = \bar{Y}^2[\lambda_n(C_0^2 + C_2^2 + 2\rho_{02} C_0 C_2) + \lambda_h(C_{0(2)}^2 + C_{2(2)}^2 + 2\rho_{02(2)} C_{0(2)} C_{2(2)})] \quad (16)$$

$$\mathcal{B}(t_{reg1}^{(2)}) = \beta' \lambda_n \left(\frac{\partial_{30}}{S_p^2} - \frac{\partial_{21}}{S_{yp}} \right) \quad (17)$$

where $\beta' = \frac{s_{yp}}{S_p^2}$, $\delta_{30} = E(\bar{p}_x^h - \bar{P}_x)^3 (\bar{y}^h - \bar{Y})^0$ and $\delta_{21} = E(\bar{p}_x^h - \bar{P}_x)^2 (\bar{y}^h - \bar{Y})$,

$$\mathcal{M}(t_{reg1}^{(2)}) = \{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\} - \frac{\{\lambda_n \rho_{02} S_y S_p + \lambda_h \rho_{02(2)} S_{y(2)} S_{p(2)}\}^2}{\{\lambda_n S_p^2 + \lambda_h S_{p(2)}^2\}} \quad (18)$$

at

$$(b_2^h)_{opt} = \frac{\{\lambda_n \rho_{02} S_y S_p + \lambda_h \rho_{02(2)} S_{y(2)} S_{p(2)}\}}{\{\lambda_n S_p^2 + \lambda_h S_{p(2)}^2\}},$$

$$\mathcal{B}(t_{g1}^{(2)}) = \bar{Y} \left[\frac{\alpha_2(\alpha_2 - 1)}{2} \{\lambda_n C_2^2 + \lambda_h C_{2(2)}^2\} + \alpha_2 \{\lambda_n \rho_{02} C_0 C_2 + \lambda_h \rho_{02(2)} C_{0(2)} C_{2(2)}\} \right] \quad (19)$$

$$[\mathcal{M}(t_{g1}^{(2)})]_{min} = \{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\} - \frac{\{\lambda_n \rho_{02} S_y S_p + \lambda_h \rho_{02(2)} S_{y(2)} S_{p(2)}\}^2}{\{\lambda_n S_p^2 + \lambda_h S_{p(2)}^2\}} \quad (20)$$

at

$$\begin{aligned}
 (\alpha_2)_{opt} &= -\frac{\{\lambda_n \rho_{02} C_0 C_2 + \lambda_h \rho_{02(2)} C_{0(2)} C_{2(2)}\}}{\{\lambda_n C_2^2 + \lambda_h C_{2(2)}^2\}}, \\
 \mathcal{B}(t_{c1}^{(2)}) &= \frac{\{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\}}{2} \frac{\partial^2 f_2}{\partial \bar{y}^{h^2}} \Big|_{(\bar{y}^{h^*}, \theta_2^*)} \\
 &\quad + \frac{\{\lambda_n C_2^2 + \lambda_h C_{2(2)}^2\}}{2} \frac{\partial^2 f_2}{\partial \theta_2^2} \Big|_{(\bar{y}^{h^*}, \theta_2^*)} \\
 &\quad + \bar{Y} \{\lambda_n \rho_{02} C_0 C_2 + \lambda_h \rho_{02(2)} C_{0(2)} C_{2(2)}\} \frac{\partial^2 f_2}{\partial \bar{y}^h \partial \theta_2} \Big|_{(\bar{y}^{h^*}, \theta_2^*)}
 \end{aligned} \tag{21}$$

where

$$\bar{y}^{h^*} = \bar{Y} + \psi_0(\bar{y}^h - \bar{Y}), \quad \theta_2^* = 1 + \psi_2(\theta_2 - 1), \quad 0 < \psi_0, \psi_2 < 1,$$

and

$$\mathcal{M}(t_{c1}^{(2)})_{min} = \{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\} - \frac{\{\lambda_n \rho_{02} S_y S_p + \lambda_h \rho_{02(2)} S_{y(2)} S_{p(2)}\}^2}{\{\lambda_n S_p^2 + \lambda_h S_{p(2)}^2\}} \tag{22}$$

at

$$f_2^{(2)}(\bar{Y}, 1) = \frac{\partial f_2}{\partial \theta_2} \Big|_{(\bar{Y}, 1)} = -\frac{\bar{P}_x \{\lambda_n \rho_{02} S_y S_p + \lambda_h \rho_{02(2)} S_{y(2)} S_{p(2)}\}}{\{\lambda_n S_p^2 + \lambda_h S_{p(2)}^2\}}.$$

However, it has been observed in some cases when complete information is available on auxiliary variate but not on study variate [see Rao (1990)], so in this circumstance the alternative ratio, product, regression, generalized and class of estimators using the known population mean and proportion of the auxiliary variate may be defined as

$$\begin{aligned}
 t_{r2}^{(1)} &= \bar{y}^h \frac{\bar{X}}{\bar{x}}, & t_{r2}^{(2)} &= \bar{y}^h \frac{\bar{P}_x}{\bar{p}_x}, \\
 t_{p2}^{(1)} &= \bar{y}^h \frac{\bar{x}}{\bar{X}}, & t_{p2}^{(2)} &= \bar{y}^h \frac{\bar{p}_x}{\bar{P}_x}, \\
 t_{reg2}^{(1)} &= \bar{y}^h + b_3^h (\bar{X} - \bar{x}), & t_{reg2}^{(2)} &= \bar{y}^h + b_4^h (\bar{P}_x - \bar{p}_x), \\
 t_{g2}^{(1)} &= \bar{y}^h \left(\frac{\bar{x}}{\bar{X}} \right)^{a_3}, & t_{g2}^{(2)} &= \bar{y}^h \left(\frac{\bar{p}_x}{\bar{P}_x} \right)^{a_4}, \\
 t_{c2}^{(1)} &= f_3(\bar{y}^h, \theta_3) \text{ where } \theta_3 = \frac{\bar{x}}{\bar{X}}, & t_{c2}^{(2)} &= f_4(\bar{y}^h, \theta_4) \text{ where } \theta_4 = \frac{\bar{p}_x}{\bar{P}_x},
 \end{aligned}$$

The *Bias* and *MSE* of all the above estimators up to the order of n^{-1} are given as follows:

$$\mathcal{B}(t_{r2}^{(1)}) = \lambda_n \bar{Y} (C_1^2 - 2\rho_{01} C_0 C_1) \quad (23)$$

$$\mathcal{M}(t_{r2}^{(1)}) = \bar{Y}^2 \{ \lambda_n C_1^2 + (\lambda_n C_0^2 + \lambda_h C_{0(2)}^2) - 2\lambda_n \rho_{01} C_0 C_1 \} \quad (24)$$

$$\mathcal{B}(t_{p2}^{(1)}) = \lambda_n \bar{Y} \rho_{01} C_0 C_1 \quad (25)$$

$$\mathcal{M}(t_{p2}^{(1)}) = \bar{Y}^2 \{ \lambda_n C_1^2 + (\lambda_n C_0^2 + \lambda_h C_{0(2)}^2) + 2\lambda_n \rho_{01} C_0 C_1 \} \quad (26)$$

$$\mathcal{B}(t_{reg2}^{(1)}) = \beta \lambda_n \left(\frac{\mu'_{30}}{S_x^2} - \frac{\mu'_{21}}{S_{yx}} \right) \quad (27)$$

where $\beta = \frac{s_{yx}}{S_x^2}$, $\mu'_{30} = E(\bar{x} - \bar{X})^3 (\bar{y} - \bar{Y})^0$ and $\mu'_{21} = E(\bar{x} - \bar{X})^2 (\bar{y} - \bar{Y})$

$$\mathcal{M}(t_{reg2}^{(1)}) = \{ \lambda_n S_y^2 + \lambda_h S_{y(2)}^2 \} - \lambda_n \rho_{01}^2 S_y^2 \text{ at } (b_3^h)_{opt} = \rho_{01} \frac{S_y}{S_x}, \quad (28)$$

$$\mathcal{B}(t_{g2}^{(1)}) = \bar{Y} \left[\frac{\alpha_3(\alpha_3 - 1)}{2} \lambda_n C_1^2 + \alpha_3 \lambda_n \rho_{01} C_0 C_1 \right] \quad (29)$$

$$[\mathcal{M}(t_{g2}^{(1)})]_{min} = \{ \lambda_n S_y^2 + \lambda_h S_{y(2)}^2 \} - \lambda_n \rho_{01}^2 S_y^2 \text{ at } (\alpha_3)_{opt} = -\rho_{01} \frac{C_0}{C_1} \quad (30)$$

$$\begin{aligned} \mathcal{B}(t_{c2}^{(1)}) &= \frac{\{ \lambda_n S_y^2 + \lambda_h S_{y(2)}^2 \}}{2} \frac{\partial^2 f_3}{\partial \bar{y}^{h^2}} \Big|_{(\bar{y}^{h^*}, \theta_3^*)} \\ &+ \frac{\{ \lambda_n C_1^2 \}}{2} \frac{\partial^2 f_3}{\partial \theta_3^2} \Big|_{(\bar{y}^{h^*}, \theta_3^*)} \\ &+ \bar{Y} \{ \lambda_n \rho_{01} C_0 C_1 \} \frac{\partial^2 f_3}{\partial \bar{y}^h \partial \theta_3} \Big|_{(\bar{y}^{h^*}, \theta_3^*)} \end{aligned} \quad (31)$$

where $\bar{y}^{h^*} = \bar{Y} + \psi_0(\bar{y}^h - \bar{Y})$, $\theta_3^* = 1 + \psi_3(\theta_3 - 1)$, $0 < \psi_0, \psi_3 < 1$,

$$\begin{aligned} [\mathcal{M}(t_{c2}^{(1)})]_{min} &= \{ \lambda_n S_y^2 + \lambda_h S_{y(2)}^2 \} - \lambda_n \rho_{01}^2 S_y^2 \text{ at } f_3^{(2)}(\bar{Y}, 1) \\ &= \frac{\partial f_3}{\partial \theta_3} \Big|_{(\bar{Y}, 1)} = -\bar{X} \rho_{01} \frac{S_y}{S_x} \end{aligned} \quad (32)$$

$$\mathcal{B}(t_{r2}^{(2)}) = \lambda_n \bar{Y} (C_2^2 - \rho_{02} C_0 C_2) \quad (33)$$

$$\mathcal{M}(t_{r2}^{(2)}) = \bar{Y}^2 \{ \lambda_n C_2^2 + (\lambda_n C_0^2 + \lambda_h C_{0(2)}^2) - 2\lambda_n \rho_{02} C_0 C_2 \} \quad (34)$$

$$\mathcal{B}(t_{p2}^{(2)}) = \lambda_n \bar{Y} \rho_{02} C_0 C_2 \quad (35)$$

$$\mathcal{M}(t_{p2}^{(2)}) = \bar{Y}^2 \{ \lambda_n C_2^2 + (\lambda_n C_0^2 + \lambda_h C_{0(2)}^2) + 2\lambda_n \rho_{02} C_0 C_2 \} \quad (36)$$

$$\mathcal{B}(t_{reg2}^{(2)}) = \beta' \lambda_n \left(\frac{\delta'_{30}}{S_p^2} - \frac{\delta'_{21}}{S_{yp}} \right) \quad (37)$$

where $\beta' = \frac{s_{yp}}{S_p^2}$, $\delta'_{30} = E(\bar{p}_x - \bar{P}_x)^3 (\bar{y} - \bar{Y})^0$ and $\delta'_{21} = E(\bar{p}_x - \bar{P}_x)^2 (\bar{y} - \bar{Y})$

$$\mathcal{M}(t_{g2}^{(2)}) = \{ \lambda_n S_y^2 + \lambda_h S_{y(2)}^2 \} - \lambda_n \rho_{02}^2 S_y^2 \text{ at } (b_4^h)_{opt} = \rho_{02} \frac{S_y}{S_p} \quad (38)$$

$$\mathcal{B}(t_{g2}^{(2)}) = \bar{Y} \left[\frac{\alpha_4 (\alpha_4 - 1)}{2} \lambda_n C_2^2 + \alpha_4 \lambda_n \rho_{02} C_0 C_2 \right] \quad (39)$$

$$[\mathcal{M}(t_{g2}^{(2)})]_{min} = \{ \lambda_n S_y^2 + \lambda_h S_{y(2)}^2 \} - \lambda_n \rho_{02}^2 S_y^2 \text{ at } (\alpha_4)_{opt} = -\rho_{02} \frac{C_0}{C_2} \quad (40)$$

$$\begin{aligned} \mathcal{B}(t_{c2}^{(2)}) &= \frac{\{ \lambda_n S_y^2 + \lambda_h S_{y(2)}^2 \}}{2} \frac{\partial^2 f_4}{\partial \bar{y}^h} \Big|_{(\bar{y}^{h*}, \theta_4^*)} \\ &+ \frac{\{ \lambda_n C_2^2 \}}{2} \frac{\partial^2 f_4}{\partial \theta_4^2} \Big|_{(\bar{y}^{h*}, \theta_4^*)} \\ &+ \bar{Y} \{ \lambda_n \rho_{02} C_0 C_2 \} \frac{\partial^2 f_4}{\partial \bar{y}^h \partial \theta_4} \Big|_{(\bar{y}^{h*}, \theta_4^*)} \end{aligned} \quad (41)$$

where

$$\bar{y}^{h*} = \bar{Y} + \psi_0 (\bar{y}^h - \bar{Y}), \theta_4^* = 1 + \psi_4 (\theta_4 - 1), 0 < \psi_0, \psi_4 < 1,$$

and

$$[\mathcal{M}(t_{c2}^{(2)})]_{min} = \{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\} - \lambda_n \rho_{02}^2 S_y^2 \text{ at } f_4^{(2)}(\bar{Y}, 1)$$

$$= \left. \frac{\partial f_4}{\partial \theta_4} \right|_{(\bar{Y}, 1)} = -\bar{P}_x \rho_{02} \frac{S_y}{S_p} \quad (42)$$

Further, Sinha and Kumar (2014) extended this problem to propose the classes of estimators using population mean and proportion of auxiliary variate when non-response occurs only on study variate and studied their properties.

2 Proposed Estimators

Following the approach of Sinha and Kumar (2014) and using the information of known mean, proportion and rank of auxiliary variable, two situations are considered for wider families of estimators to estimate the population mean of study variate.

Situation I: The families of estimators proposed for the situation when non-response takes place on both study and auxiliary variates

$$\mathcal{T}_1 = \bar{y}^h g_1(\theta_1, \eta_1) \quad \text{where } \theta_1 = \frac{\bar{x}^h}{\bar{X}}, \eta_1 = \frac{\bar{r}_x^h}{\bar{R}_x} \quad (43)$$

and

$$\mathcal{T}_2 = \bar{y}^h g_2(\theta_2, \eta_1) \quad \text{where } \theta_2 = \frac{\bar{p}_x^h}{\bar{P}_x}, \eta_1 = \frac{\bar{r}_x^h}{\bar{R}_x} \quad (44)$$

such that $g_i(1, 1) = 1 \forall i = 1, 2$.

Situation II: When non-response takes place only on study variate but complete response accessible on auxiliary variate, the families of estimators proposed for this situation are

$$\mathcal{T}_3 = \bar{y}^h g_3(\theta_3, \eta_2) \quad \text{where } \theta_3 = \frac{\bar{x}}{\bar{X}}, \eta_2 = \frac{\bar{r}_x}{\bar{R}_x} \quad (45)$$

and

$$\mathcal{T}_4 = \bar{y}^h g_4(\theta_4, \eta_2) \quad \text{where } \theta_4 = \frac{\bar{p}_x}{\bar{P}_x}, \eta_2 = \frac{\bar{r}_x}{\bar{R}_x} \quad (46)$$

such that $g_j(1, 1) = 1 \forall j = 3, 4$.

The families of estimators $[\mathcal{T}_1, \mathcal{T}_2]$ and $[\mathcal{T}_3, \mathcal{T}_4]$ may be combine to study their properties and they may be defined as

$$\mathcal{T}_i = \bar{y}^h g_i(\theta_i, \eta_1); i = 1, 2 \quad (47)$$

and

$$\mathcal{T}_j = \bar{y}^h g_j(\theta_j, \eta_2); j = 3, 4 \quad (48)$$

such that $g_i(1, 1) = g_j(1, 1) = 1 \forall i = 1, 2; j = 3, 4$.

It is to be noted here that the continuous functions $g_i(\theta_i, \eta_1)$ and $g_j(\theta_j, \eta_2)$ assume positive values in a bounded subsets \mathcal{D}_i and \mathcal{D}_j containing the point $(1, 1)$ on the real line. The first and second order partial derivatives of functions $g_i(\theta_i, \eta_1)$ and $g_j(\theta_j, \eta_2)$ with respect to (θ_i, η_1) and (θ_j, η_2) are assumed to be continuous and bounded in \mathcal{D}_i and \mathcal{D}_j respectively.

To obtain *Bias* and *MSE* of \mathcal{T}_i and \mathcal{T}_j , we assume for ease that the population size is large enough as compared to the sample size so that the finite population correction (*f.p.c*) terms may be ignored. Therefore, we define the following terms under the large sample approximation as

$$\begin{aligned} \mathcal{E}_0 &= \frac{\bar{y}^h - \bar{Y}}{\bar{Y}}, \mathcal{E}_1 = \frac{\bar{x}^h - \bar{X}}{\bar{X}}, \mathcal{E}_2 = \frac{\bar{p}_x^h - \bar{P}_x}{\bar{P}_x}, \mathcal{E}_3 = \frac{\bar{r}_x^h - \bar{R}_x}{\bar{R}_x}, \\ \mathcal{E}_4 &= \frac{\bar{x} - \bar{X}}{\bar{X}}, \mathcal{E}_5 = \frac{\bar{p}_x - \bar{P}_x}{\bar{P}_x}, \quad \text{and} \quad \mathcal{E}_6 = \frac{\bar{r}_x - \bar{R}_x}{\bar{R}_x} \end{aligned}$$

such that $E(\mathcal{E}_i) = 0, \forall i = 0, 1, 2, \dots, 6$.

and the following terms are obtained up to the first order of approximation

$$\begin{aligned} E(\varepsilon_0^2) &= \lambda_n C_0^2 + \lambda_h C_{0(2)}^2, & E(\varepsilon_1^2) &= \lambda_n C_1^2 + \lambda_h C_{1(2)}^2, \\ E(\varepsilon_2^2) &= \lambda_n C_2^2 + \lambda_h C_{2(2)}^2, & E(\varepsilon_3^2) &= \lambda_n C_3^2 + \lambda_h C_{3(2)}^2, \\ E(\varepsilon_4^2) &= \lambda_n C_1^2, & E(\varepsilon_5^2) &= \lambda_n C_2^2, \\ E(\varepsilon_6^2) &= \lambda_n C_3^2, & E(\mathcal{E}_0 \mathcal{E}_1) &= \lambda_n C_{01} + \lambda_h C_{01(2)}, \\ E(\mathcal{E}_0 \mathcal{E}_2) &= \lambda_n C_{02} + \lambda_h C_{02(2)}, & E(\mathcal{E}_0 \mathcal{E}_3) &= \lambda_n C_{03} + \lambda_h C_{03(2)}, \\ E(\mathcal{E}_0 \mathcal{E}_4) &= \lambda_n C_{01}, & E(\mathcal{E}_0 \mathcal{E}_5) &= \lambda_n C_{02}, \\ E(\mathcal{E}_0 \mathcal{E}_6) &= \lambda_n C_{03}, & E(\mathcal{E}_1 \mathcal{E}_2) &= \lambda_n C_{12} + \lambda_h C_{12(2)}, \\ E(\mathcal{E}_1 \mathcal{E}_3) &= \lambda_n C_{13} + \lambda_h C_{13(2)}, & E(\mathcal{E}_2 \mathcal{E}_3) &= \lambda_n C_{23} + \lambda_h C_{23(2)}, \\ E(\mathcal{E}_4 \mathcal{E}_5) &= \lambda_n C_{12}, & E(\mathcal{E}_4 \mathcal{E}_6) &= \lambda_n C_{13}, \\ E(\mathcal{E}_5 \mathcal{E}_6) &= \lambda_n C_{23}. \end{aligned}$$

where

$$\begin{aligned}
 C_0^2 &= \frac{S_y^2}{Y^2}, & C_1^2 &= \frac{S_x^2}{X^2}, & C_2^2 &= \frac{S_p^2}{P_x^2}, & C_3^2 &= \frac{S_r^2}{R_x^2}, & C_{01} &= \frac{S_{yx}}{YX}, \\
 C_{02} &= \frac{S_{yp}}{YP_x}, & C_{03} &= \frac{S_{yr}}{YR_x}, & C_{12} &= \frac{S_{xp}}{XP_x}, & C_{13} &= \frac{S_{xr}}{XR_x}, \\
 C_{23} &= \frac{S_{pr}}{P_x R_x}, & C_{0(2)}^2 &= \frac{S_{y(2)}^2}{Y^2}, & C_{1(2)}^2 &= \frac{S_{x(2)}^2}{X^2}, & C_{2(2)}^2 &= \frac{S_{p(2)}^2}{P_x^2}, \\
 C_{3(2)}^2 &= \frac{S_{r(2)}^2}{R_x^2}, & C_{01(2)} &= \frac{S_{yx(2)}}{YX}, & C_{02(2)} &= \frac{S_{yp(2)}}{YP_x}, \\
 C_{03(2)} &= \frac{S_{yr(2)}}{YR_x}, & C_{12(2)} &= \frac{S_{xp(2)}}{XP_x}, & C_{13(2)} &= \frac{S_{xr(2)}}{XR_x},
 \end{aligned}$$

$$\begin{aligned}
 C_{23(2)} &= \frac{S_{pr(2)}}{P_x R_x}, & S_{yx} &= \rho_{01} S_y S_x, & S_{yp} &= \rho_{02} S_y S_p, \\
 S_{yr} &= \rho_{03} S_y S_r, & S_{xp} &= \rho_{12} S_x S_p, & S_{xr} &= \rho_{13} S_x S_r, \\
 S_{pr} &= \rho_{23} S_p S_r, & S_{yx(2)} &= \rho_{01(2)} S_{y(2)} S_{x(2)}, \\
 S_{yp(2)} &= \rho_{02(2)} S_{y(2)} S_{p(2)}, & S_{yr(2)} &= \rho_{03(2)} S_{y(2)} S_{r(2)}, \\
 S_{xp(2)} &= \rho_{12(2)} S_{x(2)} S_{p(2)}, & S_{xr(2)} &= \rho_{13(2)} S_{x(2)} S_{r(2)}, \\
 S_{pr(2)} &= \rho_{23(2)} S_{p(2)} S_{r(2)}.
 \end{aligned}$$

Here $(\rho_{01}, \rho_{02}, \rho_{03}, \rho_{12}, \rho_{13}, \rho_{23})$ and $(\rho_{01(2)}, \rho_{02(2)}, \rho_{03(2)}, \rho_{12(2)}, \rho_{13(2)}, \rho_{23(2)})$ denote the coefficient of correlation between $(y, x), (y, p), (y, r), (x, p), (x, r), (p, r)$ respectively for complete and non-responding group of population while $(\rho_{02}, \rho_{12}, \rho_{23})$ and $(\rho_{02(2)}, \rho_{12(2)}, \rho_{23(2)})$ denote the coefficient of bi-serial correlation between $(y, p), (x, p), (p, r)$ for complete and non-responding group of population.

Now, expanding the functions $g_i(\theta_i, \eta_1); i = 1, 2$ and $g_j(\theta_j, \eta_2); j = 3, 4$ about the point $(\theta_i, \eta_1) = (\theta_j, \eta_2) = (1, 1)$ by Taylor's series up to the partial derivatives of second order and applying the condition

$g_i(1, 1) = g_j(1, 1) = 1$, we have

$$\begin{aligned} \mathcal{T}_1 = & \bar{Y}[1 + \mathcal{E}_1 g_1^{(1)} + \mathcal{E}_3 g_1^{(2)} + \frac{\varepsilon_1^2}{2} g_1^{(11)}(\theta_1^*, \eta_1^*) + \frac{\varepsilon_3^2}{2} g_1^{(22)}(\theta_1^*, \eta_1^*) \\ & + \mathcal{E}_1 \mathcal{E}_3 g_1^{(12)}(\theta_1^*, \eta_1^*) + \mathcal{E}_0 + \mathcal{E}_0 \mathcal{E}_1 g_1^{(1)} + \mathcal{E}_0 \mathcal{E}_3 g_1^{(2)}] \end{aligned} \quad (49)$$

$$\begin{aligned} \mathcal{T}_2 = & \bar{Y}[1 + \mathcal{E}_2 g_2^{(1)} + \mathcal{E}_3 g_2^{(2)} + \frac{\varepsilon_2^2}{2} g_2^{(11)}(\theta_2^*, \eta_1^*) + \frac{\varepsilon_3^2}{2} g_2^{(22)}(\theta_2^*, \eta_1^*) \\ & + \mathcal{E}_2 \mathcal{E}_3 g_2^{(12)}(\theta_2^*, \eta_1^*) + \mathcal{E}_0 + \mathcal{E}_0 \mathcal{E}_2 g_2^{(1)} + \mathcal{E}_0 \mathcal{E}_3 g_2^{(2)}] \end{aligned} \quad (50)$$

$$\begin{aligned} \mathcal{T}_3 = & \bar{Y}[1 + \mathcal{E}_4 g_3^{(1)} + \mathcal{E}_6 g_3^{(2)} + \frac{\varepsilon_4^2}{2} g_3^{(11)}(\theta_3^*, \eta_2^*) + \frac{\varepsilon_6^2}{2} g_3^{(22)}(\theta_3^*, \eta_2^*) \\ & + \mathcal{E}_4 \mathcal{E}_6 g_3^{(12)}(\theta_3^*, \eta_2^*) + \mathcal{E}_0 + \mathcal{E}_0 \mathcal{E}_4 g_3^{(1)} + \mathcal{E}_0 \mathcal{E}_6 g_3^{(2)}] \end{aligned} \quad (51)$$

$$\begin{aligned} \mathcal{T}_4 = & \bar{Y}[1 + \mathcal{E}_5 g_4^{(1)} + \mathcal{E}_6 g_4^{(2)} + \frac{\varepsilon_5^2}{2} g_4^{(11)}(\theta_4^*, \eta_2^*) + \frac{\varepsilon_6^2}{2} g_4^{(22)}(\theta_4^*, \eta_2^*) \\ & + \mathcal{E}_5 \mathcal{E}_6 g_4^{(12)}(\theta_4^*, \eta_2^*) + \mathcal{E}_0 + \mathcal{E}_0 \mathcal{E}_5 g_4^{(1)} + \mathcal{E}_0 \mathcal{E}_6 g_4^{(2)}] \end{aligned} \quad (52)$$

where

$$\begin{aligned} g_i^{(1)} &= \left. \frac{\partial g_i}{\partial \theta_i} \right|_{(1,1)}, \quad g_i^{(2)} = \left. \frac{\partial g_i}{\partial \eta_1} \right|_{(1,1)}, \quad g_i^{(11)}(\theta_1^*, \eta_1^*) = \left. \frac{\partial^2 g_i}{\partial \theta_i^2} \right|_{(\theta_1^*, \eta_1^*)}, \\ g_i^{(22)}(\theta_i^*, \eta_1^*) &= \left. \frac{\partial^2 g_i}{\partial \eta_1^2} \right|_{(\theta_i^*, \eta_1^*)}, \quad g_i^{(12)}(\theta_i^*, \eta_1^*) = \left. \frac{\partial^2 g_i}{\partial \theta_i \partial \eta_1} \right|_{(\theta_i^*, \eta_1^*)} \\ \theta_i^* &= 1 + \psi_i(\theta_i - 1), \quad \eta_1^* = 1 + \phi_1(\eta_1 - 1), \\ &0 < \psi_i, \phi_1 < 1, \quad \forall i = 1, 2. \\ g_j^{(1)} &= \left. \frac{\partial g_j}{\partial \theta_j} \right|_{(1,1)}, \quad g_j^{(2)} = \left. \frac{\partial g_j}{\partial \eta_2} \right|_{(1,1)}, \\ g_j^{(11)}(\theta_j^*, \eta_2^*) &= \left. \frac{\partial^2 g_j}{\partial \theta_j^2} \right|_{(\theta_j^*, \eta_2^*)}, \\ g_j^{(22)}(\theta_j^*, \eta_2^*) &= \left. \frac{\partial^2 g_j}{\partial \eta_2^2} \right|_{(\theta_j^*, \eta_2^*)}, \quad g_j^{(12)}(\theta_j^*, \eta_2^*) = \left. \frac{\partial^2 g_j}{\partial \theta_j \partial \eta_2} \right|_{(\theta_j^*, \eta_2^*)} \end{aligned}$$

$$\begin{aligned}\theta_j^* &= 1 + \psi_j(\theta_j - 1), \quad \eta_2^* = 1 + \phi_2(\eta_2 - 1), \\ 0 &< \psi_j, \phi_2 < 1, \quad \forall j = 3, 4.\end{aligned}$$

Since, it is assumed that the sample size is so large to justify the first degree of approximation and under the regularity conditions imposed on $\mathcal{T}_i; i = 1, 2, \dots, 4$, their *Bias* and *MSE* will always exist. Therefore, the bias and *MSE* of $\mathcal{T}_i; i = 1, 2, \dots, 4$ up to the first order of approximation [$O(n^{-1})$] are as follows

$$\begin{aligned}\mathcal{B}(\mathcal{T}_1) &= \bar{Y} \left[\left(\frac{\lambda_n C_1^2 + \lambda_h C_{1(2)}^2}{2} \right) g_1^{(11)}(\theta_1^*, \eta_1^*) \right. \\ &\quad + \left(\frac{\lambda_n C_3^2 + \lambda_h C_{3(2)}^2}{2} \right) g_1^{(22)}(\theta_1^*, \eta_1^*) \\ &\quad + \{ \lambda_n \rho_{13} C_1 C_3 + \lambda_h \rho_{13(2)} C_{1(2)} C_{3(2)} \} g_1^{(12)}(\theta_1^*, \eta_1^*) \\ &\quad + \{ \lambda_n \rho_{01} C_0 C_1 + \lambda_h \rho_{01(2)} C_{0(2)} C_{1(2)} \} g_1^{(1)} \\ &\quad \left. + \{ \lambda_n \rho_{03} C_0 C_3 + \lambda_h \rho_{03(2)} C_{0(2)} C_{3(2)} \} g_1^{(2)} \right] \quad (53)\end{aligned}$$

$$\begin{aligned}\mathcal{M}(\mathcal{T}_1) &= \bar{Y}^2 \left[\left(\frac{\lambda_n C_1^2 + \lambda_h C_{1(2)}^2}{2} \right) \{ g_1^{(1)} \}^2 \right. \\ &\quad + \left(\frac{\lambda_n C_3^2 + \lambda_h C_{3(2)}^2}{2} \right) \{ g_1^{(2)} \}^2 + \left(\frac{\lambda_n C_0^2 + \lambda_h C_{0(2)}^2}{2} \right) \\ &\quad + 2 \{ \lambda_n \rho_{13} C_1 C_3 + \lambda_h \rho_{13(2)} C_{1(2)} C_{3(2)} \} g_1^{(1)} g_1^{(2)} \\ &\quad + 2 \{ \lambda_n \rho_{01} C_0 C_1 + \lambda_h \rho_{01(2)} C_{0(2)} C_{1(2)} \} g_1^{(1)} \\ &\quad \left. + 2 \{ \lambda_n \rho_{03} C_0 C_3 + \lambda_h \rho_{03(2)} C_{0(2)} C_{3(2)} \} g_1^{(2)} \right] \quad (54)\end{aligned}$$

$$\mathcal{B}(\mathcal{T}_2) = \bar{Y} \left[\left(\frac{\lambda_n C_2^2 + \lambda_h C_{2(2)}^2}{2} \right) g_2^{(11)}(\theta_2^*, \eta_1^*) \right]$$

$$\begin{aligned}
& + \left(\frac{\lambda_n C_3^2 + \lambda_h C_{3(2)}^2}{2} \right) g_2^{(22)}(\theta_2^*, \eta_1^*) \\
& + \{ \lambda_n \rho_{23} C_2 C_3 + \lambda_h \rho_{23(2)} C_{2(2)} C_{3(2)} \} g_2^{(12)}(\theta_2^*, \eta_1^*) \\
& + \{ \lambda_n \rho_{02} C_0 C_2 + \lambda_h \rho_{02(2)} C_{0(2)} C_{2(2)} \} g_2^{(1)} \\
& + \{ \lambda_n \rho_{03} C_0 C_3 + \lambda_h \rho_{03(2)} C_{0(2)} C_{3(2)} \} g_2^{(2)} \Big] \quad (55)
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}(\mathcal{T}_2) = \bar{Y}^2 & \left[\left(\frac{\lambda_n C_2^2 + \lambda_h C_{2(2)}^2}{2} \right) \{g_2^{(1)}\}^2 \right. \\
& + \left(\frac{\lambda_n C_3^2 + \lambda_h C_{3(2)}^2}{2} \right) \{g_2^{(2)}\}^2 + \left(\frac{\lambda_n C_0^2 + \lambda_h C_{0(2)}^2}{2} \right) \\
& + 2\{ \lambda_n \rho_{23} C_2 C_3 + \lambda_h \rho_{23(2)} C_{2(2)} C_{3(2)} \} g_2^{(1)} g_2^{(2)} \\
& + 2\{ \lambda_n \rho_{02} C_0 C_2 + \lambda_h \rho_{02(2)} C_{0(2)} C_{2(2)} \} g_2^{(1)} \\
& \left. + 2\{ \lambda_n \rho_{03} C_0 C_3 + \lambda_h \rho_{03(2)} C_{0(2)} C_{3(2)} \} g_2^{(2)} \right] \quad (56)
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}(\mathcal{T}_3) = \bar{Y} & \left[\left(\frac{\lambda_n C_1^2}{2} \right) g_3^{(11)}(\theta_3^*, \eta_2^*) + \left(\frac{\lambda_n C_3^2}{2} \right) g_3^{(22)}(\theta_3^*, \eta_2^*) \right. \\
& + \{ \lambda_n \rho_{13} C_1 C_3 \} g_3^{(12)}(\theta_3^*, \eta_2^*) \\
& \left. + \{ \lambda_n \rho_{01} C_0 C_1 \} g_3^{(1)} + \{ \lambda_n \rho_{03} C_0 C_3 \} g_3^{(2)} \right] \quad (57)
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}(\mathcal{T}_3) = \bar{Y}^2 & \left[\left(\frac{\lambda_n C_1^2}{2} \right) \{g_3^{(1)}\}^2 + \left(\frac{\lambda_n C_3^2}{2} \right) \{g_3^{(2)}\}^2 \right. \\
& + \left(\frac{\lambda_n C_0^2 + \lambda_h C_{0(2)}^2}{2} \right) + 2\{ \lambda_n \rho_{13} C_1 C_3 \} g_3^{(1)} g_3^{(2)} \\
& \left. + 2\{ \lambda_n \rho_{01} C_0 C_1 \} g_3^{(1)} + 2\{ \lambda_n \rho_{03} C_0 C_3 \} g_3^{(2)} \right] \quad (58)
\end{aligned}$$

$$\begin{aligned} \mathcal{B}(\mathcal{T}_4) = \bar{Y} \left[\left(\frac{\lambda_n C_2^2}{2} \right) g_4^{(11)}(\theta_4^*, \eta_2^*) + \left(\frac{\lambda_n C_3^2}{2} \right) g_4^{(22)}(\theta_4^*, \eta_2^*) \right. \\ \left. + \{\lambda_n \rho_{23} C_2 C_3\} g_4^{(12)}(\theta_4^*, \eta_2^*) + \{\lambda_n \rho_{02} C_0 C_2\} g_4^{(1)} \right. \\ \left. + \{\lambda_n \rho_{03} C_0 C_3\} g_4^{(2)} \right] \end{aligned} \quad (59)$$

$$\begin{aligned} \mathcal{M}(\mathcal{T}_4) = \bar{Y}^2 \left[\left(\frac{\lambda_n C_2^2}{2} \right) \{g_4^{(1)}\}^2 + \left(\frac{\lambda_n C_3^2}{2} \right) \{g_4^{(2)}\}^2 \right. \\ \left. + \left(\frac{\lambda_n C_0^2 + \lambda_h C_{0(2)}^2}{2} \right) + 2\{\lambda_n \rho_{23} C_2 C_3\} g_4^{(1)} g_4^{(2)} \right. \\ \left. + 2\{\lambda_n \rho_{02} C_0 C_2\} g_4^{(1)} + 2\{\lambda_n \rho_{03} C_0 C_3\} g_4^{(2)} \right] \end{aligned} \quad (60)$$

Using the principle of maxima and minima, partially differentiating the MSE 's of $\mathcal{T}_i; i = 1, 2, \dots, 4$ with respect to the corresponding $g_i^{(1)}$ and $g_i^{(2)}$, the optimum values of the functions can be obtained. Hence, the minimum MSE 's of $\mathcal{T}_i; i = 1, 2, \dots, 4$ along with the optimum conditions are given by

$$\begin{aligned} [\mathcal{M}(\mathcal{T}_1)]_{min} = \{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\} - \frac{\{\lambda_n \rho_{01} S_y S_x + \lambda_h \rho_{01(2)} S_{y(2)} S_{x(2)}\}^2}{\{\lambda_n S_x^2 + \lambda_h S_{x(2)}^2\}} \\ - \frac{\{[\lambda_n S_x^2 + \lambda_h S_{x(2)}^2]\{\lambda_n \rho_{03} S_y S_r + \lambda_h \rho_{03(2)} S_{y(2)} S_{r(2)}\} \\ - \{\lambda_n \rho_{01} S_y S_x + \lambda_h \rho_{01(2)} S_{y(2)} S_{x(2)}\} \\ \{\lambda_n \rho_{13} S_x S_r + \lambda_h \rho_{13(2)} S_{x(2)} S_{r(2)}\}^2}{\{\lambda_n S_x^2 + \lambda_h S_{x(2)}^2\}[\{\lambda_n S_x^2 + \lambda_h S_{x(2)}^2\}\{\lambda_n S_r^2 + \lambda_h S_{r(2)}^2\} \\ - \{\lambda_n \rho_{13} S_x S_r + \lambda_h \rho_{13(2)} S_{x(2)} S_{r(2)}\}^2]} \end{aligned} \quad (61)$$

if

$$g_1^{(1)} = \frac{\{[\lambda_n \rho_{03} C_0 C_3 + \lambda_h \rho_{03(2)} C_{0(2)} C_{3(2)}] \\ \{\lambda_n \rho_{13} C_1 C_3 + \lambda_h \rho_{13(2)} C_{1(2)} C_{3(2)}\} - \{\lambda_n C_3^2 + \lambda_h C_{3(2)}^2\}}{\{\lambda_n C_1^2 + \lambda_h C_{1(2)}^2\}\{\lambda_n C_3^2 + \lambda_h C_{3(2)}^2\} \\ - \{\lambda_n \rho_{13} C_1 C_3 + \lambda_h \rho_{13(2)} C_{1(2)} C_{3(2)}\}^2}$$

and

$$g_1^{(2)} = \frac{[\{\lambda_n \rho_{01} C_0 C_1 + \lambda_h \rho_{01(2)} C_{0(2)} C_{1(2)}\} \{\lambda_n \rho_{13} C_1 C_3 + \lambda_h \rho_{13(2)} C_{1(2)} C_{3(2)}\} - \{\lambda_n C_1^2 + \lambda_h C_{1(2)}^2\} \{\lambda_n \rho_{03} C_0 C_3 + \lambda_h \rho_{03(2)} C_{0(2)} C_{3(2)}\}]}{\{\lambda_n C_1^2 + \lambda_h C_{1(2)}^2\} \{\lambda_n C_3^2 + \lambda_h C_{3(2)}^2\} - \{\lambda_n \rho_{13} C_1 C_3 + \lambda_h \rho_{13(2)} C_{1(2)} C_{3(2)}\}^2}$$

$$[\mathcal{M}(\mathcal{T}_2)]_{min} = \{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\} - \frac{\{\lambda_n \rho_{02} S_y S_p + \lambda_h \rho_{02(2)} S_{y(2)} S_{p(2)}\}^2}{\{\lambda_n S_p^2 + \lambda_h S_{p(2)}^2\}}$$

$$- \frac{[\{\lambda_n S_p^2 + \lambda_h S_{p(2)}^2\} \{\lambda_n \rho_{03} S_y S_r + \lambda_h \rho_{03(2)} S_{y(2)} S_{r(2)}\} - \{\lambda_n \rho_{02} S_y S_p + \lambda_h \rho_{02(2)} S_{y(2)} S_{p(2)}\} \{\lambda_n \rho_{23} S_p S_r + \lambda_h \rho_{23(2)} S_{p(2)} S_{r(2)}\}]^2}{\{\lambda_n S_p^2 + \lambda_h S_{p(2)}^2\} \{\lambda_n S_r^2 + \lambda_h S_{r(2)}^2\} - \{\lambda_n \rho_{23} S_p S_r + \lambda_h \rho_{23(2)} S_{p(2)} S_{r(2)}\}^2}$$
(62)

if

$$g_2^{(1)} = \frac{[\{\lambda_n \rho_{03} C_0 C_3 + \lambda_h \rho_{03(2)} C_{0(2)} C_{3(2)}\} \{\lambda_n \rho_{23} C_2 C_3 + \lambda_h \rho_{23(2)} C_{2(2)} C_{3(2)}\} - \{\lambda_n C_3^2 + \lambda_h C_{3(2)}^2\} \{\lambda_n \rho_{02} S_y S_p + \lambda_h \rho_{02(2)} S_{y(2)} S_{p(2)}\}]}{[\{\lambda_n C_2^2 + \lambda_h C_{2(2)}^2\} \{\lambda_n C_3^2 + \lambda_h C_{3(2)}^2\} - \{\lambda_n \rho_{23} C_2 C_3 + \lambda_h \rho_{23(2)} C_{2(2)} C_{3(2)}\}^2]}$$

and

$$g_2^{(2)} = \frac{[\{\lambda_n \rho_{02} C_0 C_2 + \lambda_h \rho_{02(2)} C_{0(2)} C_{2(2)}\} \{\lambda_n \rho_{23} C_2 C_3 + \lambda_h \rho_{23(2)} C_{2(2)} C_{3(2)}\} - \{\lambda_n C_2^2 + \lambda_h C_{2(2)}^2\} \{\lambda_n \rho_{03} C_0 C_3 + \lambda_h \rho_{03(2)} C_{0(2)} C_{3(2)}\}]}{[\{\lambda_n C_2^2 + \lambda_h C_{2(2)}^2\} \{\lambda_n C_3^2 + \lambda_h C_{3(2)}^2\} - \{\lambda_n \rho_{23} C_2 C_3 + \lambda_h \rho_{23(2)} C_{2(2)} C_{3(2)}\}^2]}$$

$$[\mathcal{M}(\mathcal{T}_3)]_{min} = \{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\} - \lambda_n \rho_{01}^2 S_y^2 - \frac{\lambda_n S_y^2 (\rho_{03} - \rho_{01} \rho_{13})^2}{1 - \rho_{13}^2}$$
(63)

if

$$g_3^{(1)} = \frac{\bar{X}S_y}{\bar{Y}S_x} \left(\frac{\rho_{03}\rho_{13} - \rho_{01}}{1 - \rho_{13}^2} \right)$$

and

$$g_3^{(2)} = \frac{-\bar{R}_xS_y}{\bar{Y}S_x} \left(\frac{\rho_{03} - \rho_{13}\rho_{01}}{1 - \rho_{13}^2} \right)$$

$$[\mathcal{M}(\mathcal{T}_4)]_{min} = \{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\} - \lambda_n \rho_{03}^2 S_y^2 - \frac{\lambda_n S_y^2 (\rho_{03}\rho_{23} - \rho_{02})^2}{1 - \rho_{23}^2} \quad (64)$$

if

$$g_4^{(1)} = \frac{\bar{P}_xS_y}{\bar{Y}S_p} \left(\frac{\rho_{03}\rho_{23} - \rho_{02}}{1 - \rho_{23}^2} \right)$$

and

$$g_4^{(2)} = \frac{-\bar{R}_xS_y}{\bar{Y}S_p} \left(\frac{\rho_{02} - \rho_{23}\rho_{02}}{1 - \rho_{23}^2} \right).$$

Since the values of $g_i^{(1)}$ and $g_i^{(2)}$, $i = 1, 2, \dots, 4$ involve the unknown parameters, therefore their values can be obtained by using the prior data or replacing with their consistent estimate, Reddy (1978), Srivastava and Jhaji (1983) and Koyuncu and Kadilar (2009) have shown that these estimates do not affect the minimum mean square error of the estimators up to the order n^{-1} .

3 Extension of This Problem to the Two-Phase Sampling

In this segment, the problem is extended to the two-phase sampling for estimating the population mean \bar{Y} when non-response observes only on study variate (y) while mean \bar{X} and rank \bar{R}_x of auxiliary variate are unknown. In this situation, a larger sample of size n' at first phase is chosen from the population of size N using SRSWOR to estimate unknown population mean \bar{X} and rank \bar{R}_x . Let the estimates are \bar{x}' and \bar{r}_x' based on complete information available on n' units of auxiliary variate thereafter a sample of second phase is drawn from the sample of first phase, which is used to obtain

the obligatory information on study variate (y). Now, the three different families of estimators under these situations are as follows

$$\mathcal{T}_{c1} = f(\bar{y}^h, \theta_5) \quad \text{where } \theta_5 = \frac{\bar{x}}{\bar{x}'} \quad [\text{Khare and Sinha (2002)}] \quad (65)$$

$$\mathcal{T}_{c2} = g(\bar{y}^h, \theta_6) \quad \text{where } \theta_6 = \frac{\bar{p}_x}{\bar{p}_x'} \quad [\text{Sinha and Kumar (2014)}] \quad (66)$$

and

$$\mathcal{T}_{c3} = \bar{y}^h F(\theta_5, \theta_6) \quad \text{where } \theta_5 = \frac{\bar{x}}{\bar{x}'} \quad \text{and} \quad \theta_6 = \frac{\bar{p}_x}{\bar{p}_x'} \\ [\text{Sinha and Kumar (2014)}] \quad (67)$$

Following Khare and Sinha (2002) and Sinha and Kumar (2014), the proposed family of two-phase sampling estimators for estimating the population mean \bar{Y} using the estimates \bar{x}' and \bar{r}_x' is given by

$$\mathcal{T}_{c4} = \bar{y}^h G(\theta_5, \eta_3) \quad \text{where } \theta_5 = \frac{\bar{x}}{\bar{x}'} \quad \text{and} \quad \eta_3 = \frac{\bar{r}_x}{\bar{r}_x'} \quad (68)$$

such that $G(1, 1) = 1$ and the function \mathcal{T}_{c4} satisfy some regularity conditions required for the its expansion by Taylor's series.

Proceeding from the previous section, some large approximations under SRSWOR are defined to obtain bias and MSE 's of \mathcal{T}_{ci} , $i = 1, 2, \dots, 4$ as

$$\bar{x}' = \bar{X}(1 + \mathcal{E}_1'), \quad \bar{p}_x' = \bar{P}_x(1 + \mathcal{E}_2') \quad \text{and} \quad \bar{r}_x' = \bar{R}_x(1 + \mathcal{E}_3')$$

such that $E(\varepsilon_i') = 0$; $i = 1, 2, 3$,

$$\begin{aligned} E(\mathcal{E}_1'^2) &= \lambda' C_1^2, & E(\mathcal{E}_2'^2) &= \lambda' C_2^2, & E(\mathcal{E}_3'^2) &= \lambda' C_3^2, \\ E(\mathcal{E}_0 \mathcal{E}_1') &= \lambda' C_{01}, & E(\mathcal{E}_0 \mathcal{E}_2') &= \lambda' C_{02}, & E(\mathcal{E}_0 \mathcal{E}_3') &= \lambda' C_{03}, \\ E(\mathcal{E}_1 \mathcal{E}_1') &= \lambda' C_1^2, & E(\mathcal{E}_1 \mathcal{E}_2') &= \lambda' C_{12}, & E(\mathcal{E}_1 \mathcal{E}_3') &= \lambda' C_{13}, \\ E(\mathcal{E}_1' \mathcal{E}_2) &= \lambda' C_{12}, & E(\mathcal{E}_1' \mathcal{E}_2') &= \lambda' C_{12}, & E(\mathcal{E}_1' \mathcal{E}_3) &= \lambda' C_{13}, \\ E(\mathcal{E}_1' \mathcal{E}_3') &= \lambda' C_{13}, & E(\mathcal{E}_2 \mathcal{E}_2') &= \lambda' C_2^2, & E(\mathcal{E}_2 \mathcal{E}_3') &= \lambda' C_{23}, \\ E(\mathcal{E}_2' \mathcal{E}_3) &= \lambda' C_{23}, & E(\mathcal{E}_2' \mathcal{E}_3') &= \lambda' C_{23}, & E(\mathcal{E}_3 \mathcal{E}_3') &= \lambda' C_3^2 \end{aligned}$$

where $\lambda' = \frac{1}{n'} - \frac{1}{N}$.

Now, expanding the function $G(\theta_5, \eta_3)$ by Taylor's series about the point $(\theta_5, \eta_3) = (1, 1)$ and using the condition $G(1, 1) = 1$, we have

$$\begin{aligned} \mathcal{T}_{c4} = & \bar{Y}[G(1, 1) + (\theta_5 - 1)G^{(1)} + (\eta_3 - 1)G^{(2)} + \frac{1}{2}(\theta_5 - 1)^2G^{(11)}(\theta_5^*, \eta_3^*) \\ & + \frac{1}{2}(\eta_3 - 1)^2G^{(22)}(\theta_5^*, \eta_3^*) + (\theta_5 - 1)(\eta_3 - 1) \\ & G^{(11)}(\theta_5^*, \eta_3^*)G^{(12)}(\theta_5^*, \eta_3^*)] \end{aligned} \quad (69)$$

where

$$\begin{aligned} G^{(1)} = \frac{\partial G}{\partial \theta_5} \Big|_{(1,1)}, \quad G^{(2)} = \frac{\partial G}{\partial \eta_3} \Big|_{(1,1)}, \quad G^{(11)}(\theta_5^*, \eta_3^*) = \frac{\partial^2 G}{\partial \theta_5^2} \Big|_{(\theta_5^*, \eta_3^*)}, \\ G^{(22)}(\theta_5^*, \eta_3^*) = \frac{\partial^2 G}{\partial \eta_3^2} \Big|_{(\theta_5^*, \eta_3^*)}, \quad G^{(12)}(\theta_5^*, \eta_3^*) = \frac{\partial^2 G}{\partial \theta_5 \partial \eta_3} \Big|_{(\theta_5^*, \eta_3^*)} \end{aligned}$$

and

$$\eta_3^* = \eta_3 + \psi_3(\eta_3 - 1), \quad 0 < \psi_3 < 1.$$

The *Bias* and *MSE* of proposed family of estimators \mathcal{T}_{c4} up to order n^{-1} are given as

$$\begin{aligned} \mathcal{B}(\mathcal{T}_{c4}) = & \bar{Y}(\lambda_n - \lambda')[\rho_{01}C_0C_1G^{(1)} + \rho_{03}C_0C_3G^{(2)} + \frac{C_1^2}{2}G^{(11)}(\theta_5^*, \eta_3^*) \\ & + \rho_{13}C_1C_3G^{(11)}(\theta_5^*, \eta_3^*)G^{(12)}(\theta_5^*, \eta_3^*) + \frac{C_3^2}{2}G^{(22)}(\theta_5^*, \eta_3^*)] \end{aligned} \quad (70)$$

$$\begin{aligned} \mathcal{M}(\mathcal{T}_{c4}) = & \{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\} + \bar{Y}^2(\lambda_n - \lambda')[C_1^2G^{(1)2} + C_3^2G^{(2)2} \\ & + 2\rho_{01}C_0C_1G^{(1)} + 2\rho_{03}C_0C_3G^{(2)} + 2\rho_{13}C_1C_3G^{(1)}G^{(2)}] \end{aligned} \quad (71)$$

The *MSE* of proposed family of estimators \mathcal{T}_{c4} will attain its minimum value when

$$G^{(1)} = \frac{\bar{X}S_y(\rho_{03}\rho_{13} - \rho_{01})}{\bar{Y}S_x(1 - \rho_{13}^2)} \quad \text{and} \quad G^{(2)} = \frac{\bar{R}_xS_y(\rho_{01}\rho_{13} - \rho_{03})}{\bar{Y}S_r(1 - \rho_{13}^2)}$$

and the minimum MSE of \mathcal{T}_{c4} is given by

$$[\mathcal{M}(\mathcal{T}_{c4})]_{min} = \{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\} - (\lambda_n - \lambda') \rho_{03}^2 S_y^2 - \frac{(\lambda_n - \lambda') S_y^2 (\rho_{03} \rho_{13} - \rho_{01})^2}{1 - \rho_{13}^2}. \quad (72)$$

4 Calculation of n' , n and h Under Fixed Cost $C \leq C^{(0)}$

Let the total fixed cost apart from the overhead charge of the survey is $C^{(0)}$. The cost function is

$$\mathcal{C}(\mathcal{T}_{ci}) = \mathcal{C}^{(1)'} n' + n \left(\mathcal{C}^{(1)} + \mathcal{C}^{(2)} W_1 + \mathcal{C}^{(3)} \frac{W_2}{h} \right) \quad (73)$$

Here $\mathcal{C}^{(1)'}$ -Cost for observing and identifying an auxiliary variate.

$\mathcal{C}^{(1)}$ -Cost of sending a questionnaire or visiting the unit at second phase.

$\mathcal{C}^{(2)}$ -Cost for processing and collecting information on a unit of study variate y obtained from n_1 responding units and $\mathcal{C}^{(3)}$ -Cost for processing and collecting information on a sub-sampled unit of study variate y by interview basis.

The $MSE(\mathcal{T}_{ci})$, $i = 1, 2, 3, 4$ can be expressed as given below:

$$\mathcal{M}(\mathcal{T}_{ci}) = \frac{1}{n} A_i + \frac{1}{n'} B_i + \frac{h}{n} D_i + \text{terms not containing } n, n' \text{ and } h. \quad (74)$$

where A_i = coefficient of $\frac{1}{n}$ terms, B_i = coefficient of $\frac{1}{n'}$ terms, D_i = coefficient of $\frac{h}{n}$ terms.

In order to optimize MSE of the estimators and n' , n and h for the fixed cost $C \leq C^{(0)}$, a function can be defined as

$$\begin{aligned} \chi &= \frac{1}{n} A_i + \frac{1}{n'} B_i + \frac{h}{n} D_i \\ &+ \delta_i \left[\mathcal{C}^{(1)'} n' + n \left(\mathcal{C}^{(1)} + \mathcal{C}^{(2)} W_1 + \mathcal{C}^{(3)} \frac{W_2}{h} \right) - C^{(0)} \right], \\ i &= 1, 2, 3, 4 \end{aligned} \quad (75)$$

where δ_i is a Lagrange's multipliers.

Differentiate the function χ with respect to n' , n and h and equating them to zero, we have

$$n' = \sqrt{\frac{B_i}{\delta_i \mathcal{C}^{(1)'}}} \quad (76)$$

$$n = \sqrt{\frac{A_i + hD_i}{\delta_i(\mathcal{C}^{(1)} + \mathcal{C}^{(2)}W_1 + \mathcal{C}^{(3)}\frac{W_2}{h})}} \quad (77)$$

and

$$\frac{n}{h} = \sqrt{\frac{D_i}{\delta_i \mathcal{C}^{(3)}W_2}} \quad (78)$$

Solving (77) and (78), we get

$$h_{opt} = \sqrt{\frac{A_i \mathcal{C}^{(3)}W_2}{(\mathcal{C}^{(1)} + \mathcal{C}^{(2)}W_1)D_i}} \quad (79)$$

Substituting the obtained values of n' , n and h in (73), we get

$$\sqrt{\delta_i} = \frac{\sqrt{B_i \mathcal{C}^{(1)'}} + \sqrt{A_i(\mathcal{C}^{(1)} + \mathcal{C}^{(2)}W_1)} + \sqrt{\mathcal{C}^{(3)}W_2 D_i}}{C_0} \quad (80)$$

Finally, the optimum value of $MSE(\mathcal{T}_{ci})$ is given by:

$$[\mathcal{M}(\mathcal{T}_{ci})]_{opt} = \frac{(\sqrt{B_i \mathcal{C}^{(1)'}} + \sqrt{A_i(\mathcal{C}^{(1)} + \mathcal{C}^{(2)}W_1)} + \sqrt{\mathcal{C}^{(3)}W_2 D_i})^2}{\mathcal{C}^{(0)}} - \frac{S_y^2}{\mathcal{N}} \quad (81)$$

For t_h , the expected total cost is considered as

$$C = n \left(\mathcal{C}^{(1)} + \mathcal{C}^{(2)}W_1 + \mathcal{C}^{(3)}\frac{W_2}{h} \right) \quad (82)$$

and

$$[\mathcal{M}(\bar{y}^h)]_{opt} = \frac{\left(\sqrt{A_0(\mathcal{C}^{(1)} + \mathcal{C}^{(2)}W_1)} + \sqrt{\mathcal{C}^{(3)}W_2 D_0} \right)^2}{\mathcal{C}^{(0)}} - \frac{S_y^2}{\mathcal{N}} \quad (83)$$

5 Calculation of n , n' and h Under Specified Variance

Suppose \mathcal{V}_0'' is the fixed variance for the estimator $\mathcal{M}(\mathcal{T}_{ci})$, $i = 1, 2, 3, 4$ and let

$$\mathcal{V}_0'' = \frac{1}{n}A_i + \frac{1}{n'}B_i + \frac{h}{n}D_i - \frac{S_y^2}{\mathcal{N}} \quad (84)$$

Now consider a function to optimize the average total cost $\mathcal{C}(\mathcal{T}_{ci})$ for the fixed variance of the estimator \mathcal{T}_{ci}

$$\chi' = \left\{ \mathcal{C}^{(1)'}n' + n \left(\mathcal{C}^{(1)} + \mathcal{C}^{(2)}W_1 + \mathcal{C}^{(3)}\frac{W_2}{h} \right) \right\} + \mu_i \mathcal{M}(\mathcal{T}_{ci}) - \mathcal{V}_0'' \quad (85)$$

where $\mu_i (i = 1, 2, 3, 4)$ is a Lagrange's multiplier.

In order to optimize the cost function, differentiate χ' w.r.t. n' , n and h and equating them to zero, we have

$$n' = \sqrt{\frac{\mu_i B_i}{\mathcal{C}^{(1)'}}}, \quad (86)$$

$$n = \sqrt{\frac{\mu_i (A_i + hD_i)}{(\mathcal{C}^{(1)} + \mathcal{C}^{(2)}W_1 + \mathcal{C}^{(3)}\frac{W_2}{h})}}, \quad (87)$$

and

$$\frac{n}{h} = \sqrt{\frac{\mu_i D_i}{\mathcal{C}^{(3)}W_2}}. \quad (88)$$

Solving (87) and (88) we get,

$$h_{opt} = \sqrt{\frac{A_i \mathcal{C}^{(3)} W_2}{(\mathcal{C}^{(1)} + \mathcal{C}^{(2)} W_1) D_i}} \quad (89)$$

Putting the values of n , n' and h in (84), we have

$$\sqrt{\mu_i} = \frac{\sqrt{B_i \mathcal{C}^{(1)'}} + \sqrt{A_i (\mathcal{C}^{(1)} + \mathcal{C}^{(2)} W_1)} + \sqrt{\mathcal{C}^{(3)} W_2 D_i}}{\mathcal{V}_0'' + \frac{S_y^2}{\mathcal{N}}} \quad (90)$$

The optimum expected total cost incurred in attaining the fixed variance \mathcal{V}_0'' for the families of estimators \mathcal{T}_{ci} is given by:

$$[\mathcal{C}(\mathcal{T}_{ci})]_{opt} = \frac{\left(\sqrt{B_i \mathcal{C}^{(1)'}} + \sqrt{A_i (\mathcal{C}^{(1)} + \mathcal{C}^{(2)} W_1)} + \sqrt{\mathcal{C}^{(3)} W_2 D_i} \right)^2}{\mathcal{V}_0'' + \frac{S_y^2}{\mathcal{N}}}; \quad (91)$$

$i = 1, 2, 3, 4$

For t_h , the optimum cost for fixed variance is given by

$$[C(t_h)]_{opt} = \frac{\left(\sqrt{A_0(C^{(1)} + C^{(2)}W_1)} + \sqrt{C^{(3)}W_2D_0}\right)^2}{\mathcal{V}_0'' + \frac{S_y^2}{N}} \quad (92)$$

6 Efficiency Comparisons

(i) From Equations (61) and (2), we get

$$\begin{aligned} & \mathcal{M}(t_h) - [\mathcal{M}(\mathcal{T}_1)]_{min} \\ &= \frac{\{Cov(\bar{y}^h, \bar{x}^h)\}^2}{\{V(\bar{x}^h)\}} \\ & \quad + \frac{[\{V(\bar{x}^h)\}\{Cov(\bar{y}^h, \bar{r}_x^h)\} - \{Cov(\bar{y}^h, \bar{x}^h)\}\{Cov(\bar{x}^h, \bar{r}_x^h)\}]^2}{\{V(\bar{x}^h)\}[\{V(\bar{x}^h)\}\{V(\bar{r}_x^h)\} - \{Cov(\bar{x}^h, \bar{r}_x^h)\}^2]} > 0 \end{aligned}$$

(ii) From Equations (61) and (12), we get

$$\begin{aligned} & [\mathcal{M}(t_{c1}^{(1)})]_{min} - [\mathcal{M}(\mathcal{T}_1)]_{min} \\ &= \frac{[\{V(\bar{x}^h)\}\{Cov(\bar{y}^h, \bar{r}_x^h)\} - \{Cov(\bar{y}^h, \bar{x}^h)\}\{Cov(\bar{x}^h, \bar{r}_x^h)\}]^2}{\{V(\bar{x}^h)\}[\{V(\bar{x}^h)\}\{V(\bar{r}_x^h)\} - \{Cov(\bar{x}^h, \bar{r}_x^h)\}^2]} > 0 \end{aligned}$$

(iii) From Equations (61) and (22), we get

$$[\mathcal{M}(t_{c1}^{(2)})]_{min} - [\mathcal{M}(\mathcal{T}_1)]_{min} > 0$$

if

$$\begin{aligned} & \frac{\{Cov(\bar{y}^h, \bar{x}^h)\}^2}{\{V(\bar{x}^h)\}} \\ & \quad + \frac{[\{V(\bar{x}^h)\}\{Cov(\bar{y}^h, \bar{r}_x^h)\} - \{Cov(\bar{y}^h, \bar{x}^h)\}\{Cov(\bar{x}^h, \bar{r}_x^h)\}]^2}{\{V(\bar{x}^h)\}[\{V(\bar{x}^h)\}\{V(\bar{r}_x^h)\} - \{Cov(\bar{x}^h, \bar{r}_x^h)\}^2]} \\ & > \frac{\{Cov(\bar{y}^h, \bar{p}_x^h)\}^2}{\{V(\bar{p}_x^h)\}} \end{aligned}$$

(iv) From Equations (62) and (2), we get

$$\begin{aligned} & \mathcal{M}(t_h) - [\mathcal{M}(\mathcal{T}_2)]_{min} \\ &= \frac{\{Cov(\bar{y}^h, \bar{p}_x^h)\}^2}{\{V(\bar{p}_x^h)\}} + \frac{[\{V(\bar{p}_x^h)\}\{Cov(\bar{y}^h, \bar{r}_x^h)\} - \{Cov(\bar{y}^h, \bar{p}_x^h)\}\{Cov(\bar{p}_x^h, \bar{r}_x^h)\}]^2}{\{V(\bar{p}_x^h)\}[\{V(\bar{p}_x^h)\}\{V(\bar{r}_x^h)\} - \{Cov(\bar{p}_x^h, \bar{r}_x^h)\}^2]} > 0 \end{aligned}$$

(v) From Equations (62) and (12), we get

$$\begin{aligned} & [\mathcal{M}(t_{c1}^{(1)})]_{min} - [\mathcal{M}(\mathcal{T}_2)]_{min} > 0 \quad \text{if} \\ & \frac{\{Cov(\bar{y}^h, \bar{p}_x^h)\}^2}{\{V(\bar{p}_x^h)\}} \\ & + \frac{[\{V(\bar{p}_x^h)\}\{Cov(\bar{y}^h, \bar{r}_x^h)\} - \{Cov(\bar{y}^h, \bar{p}_x^h)\}\{Cov(\bar{p}_x^h, \bar{r}_x^h)\}]^2}{\{V(\bar{p}_x^h)\}[\{V(\bar{p}_x^h)\}\{V(\bar{r}_x^h)\} - \{Cov(\bar{p}_x^h, \bar{r}_x^h)\}^2]} \\ & > \frac{\{Cov(\bar{y}^h, \bar{x}^h)\}^2}{\{V(\bar{x}^h)\}} \end{aligned}$$

(vi) From Equations (62) and (22), we get

$$\begin{aligned} & [\mathcal{M}(t_{c1}^{(2)})]_{min} - [\mathcal{M}(\mathcal{T}_2)]_{min} \\ &= \frac{[\{V(\bar{p}_x^h)\}\{Cov(\bar{y}^h, \bar{r}_x^h)\} - \{Cov(\bar{y}^h, \bar{p}_x^h)\}\{Cov(\bar{p}_x^h, \bar{r}_x^h)\}]^2}{\{V(\bar{p}_x^h)\}[\{V(\bar{p}_x^h)\}\{V(\bar{r}_x^h)\} - \{Cov(\bar{p}_x^h, \bar{r}_x^h)\}^2]} > 0 \end{aligned}$$

(vii) From Equations (63) and (2), we get

$$\mathcal{M}(t_h) - [\mathcal{M}(\mathcal{T}_3)]_{min} = \lambda_n \rho_{01}^2 S_y^2 + \frac{\lambda_n S_y^2 (\rho_{03} - \rho_{01} \rho_{13})^2}{1 - \rho_{13}^2} > 0$$

(viii) From Equations (63) and (12), we get

$$[\mathcal{M}(t_{c2}^{(1)})]_{min} - [\mathcal{M}(\mathcal{T}_3)]_{min} = \frac{\lambda_n S_y^2 (\rho_{03} - \rho_{01} \rho_{13})^2}{1 - \rho_{13}^2} > 0$$

(ix) From Equations (63) and (22), we get

$$[\mathcal{M}(t_{c2}^{(2)})]_{min} - [\mathcal{M}(\mathcal{T}_3)]_{min} > 0$$

if

$$\lambda_n \rho_{01}^2 S_y^2 + \frac{\lambda_n S_y^2 (\rho_{03} - \rho_{01} \rho_{13})^2}{1 - \rho_{13}^2} - \lambda_n \rho_{02}^2 S_y^2 > 0$$

(x) From Equations (72) and (2), we get

$$\mathcal{M}(t_h) - [\mathcal{M}(\mathcal{T}_4)]_{min} = \lambda_n \rho_{02}^2 S_y^2 + \frac{\lambda_n S_y^2 (\rho_{03} - \rho_{02} \rho_{23})^2}{1 - \rho_{23}^2} > 0$$

(xi) From Equations (72) and (12), we get

$$[\mathcal{M}(t_{c2}^{(1)})]_{min} - [\mathcal{M}(\mathcal{T}_4)]_{min} > 0$$

if

$$\lambda_n \rho_{02}^2 S_y^2 + \frac{\lambda_n S_y^2 (\rho_{03} - \rho_{02} \rho_{23})^2}{1 - \rho_{23}^2} - \lambda_n \rho_{01}^2 S_y^2 > 0$$

(xii) From Equations (72) and (22), we get

$$[\mathcal{M}(t_{c2}^{(2)})]_{min} - [\mathcal{M}(\mathcal{T}_4)]_{min} = \frac{\lambda_n S_y^2 (\rho_{03} - \rho_{02} \rho_{23})^2}{1 - \rho_{23}^2} > 0$$

where

$$\begin{aligned} V(\bar{x}^h) &= \{\lambda_n S_x^2 + \lambda_h S_{x(2)}^2\}, \{V(\bar{r}_x^h)\} = \{\lambda_n S_r^2 + \lambda_h S_{r(2)}^2\}, \\ \{V(\bar{p}_x^h)\} &= \{\lambda_n S_p^2 + \lambda_h S_{p(2)}^2\}, \\ \{Cov(\bar{y}^h, \bar{x}^h)\} &= \{\lambda_n \rho_{01} S_y S_x + \lambda_h \rho_{01(2)} S_{y(2)} S_{x(2)}\}, \\ \{Cov(\bar{y}^h, \bar{r}_x^h)\} &= \{\lambda_n \rho_{03} S_y S_r + \lambda_h \rho_{03(2)} S_{y(2)} S_{r(2)}\}, \\ \{Cov(\bar{y}^h, \bar{p}_x^h)\} &= \{\lambda_n \rho_{02} S_y S_p + \lambda_h \rho_{02(2)} S_{y(2)} S_{p(2)}\}, \\ \{Cov(\bar{x}^h, \bar{r}_x^h)\} &= \{\lambda_n \rho_{13} S_x S_r + \lambda_h \rho_{13(2)} S_{x(2)} S_{r(2)}\}, \\ \{Cov(\bar{p}_x^h, \bar{r}_x^h)\} &= \{\lambda_n \rho_{23} S_p S_r + \lambda_h \rho_{23(2)} S_{p(2)} S_{r(2)}\}. \end{aligned}$$

For the efficiency comparisons of \mathcal{T}_{c4} with respect to the relevant estimators, the minimum MSE of the estimators \mathcal{T}_{ci} ; $i = 1, 2, 3$ can be defined as follows

$$[\mathcal{M}(\mathcal{T}_{c1})]_{min} = \{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\} - (\lambda_n - \lambda') \rho_{01}^2 S_y^2$$

at

$$f^{(2)}(\bar{Y}, 1) = \left. \frac{\partial f}{\partial \theta_5} \right|_{(\bar{Y}, 1)} = -\bar{X} \rho_{01} \frac{S_y}{S_x}, \quad (93)$$

$$[\mathcal{M}(\mathcal{T}_{c2})]_{min} = \{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\} - (\lambda_n - \lambda') \rho_{02}^2 S_y^2 \quad (94)$$

at

$$g^{(2)}(\bar{Y}, 1) = \left. \frac{\partial g}{\partial \theta_6} \right|_{(\bar{Y}, 1)} = -\bar{P}_x \rho_{02} \frac{S_y}{S_p}, \quad (95)$$

$$\begin{aligned} [\mathcal{M}(\mathcal{T}_{c3})]_{min} &= \{\lambda_n S_y^2 + \lambda_h S_{y(2)}^2\} - (\lambda_n - \lambda') \rho_{02}^2 S_y^2 \\ &\quad - \frac{(\lambda_n - \lambda') S_y^2 (\rho_{02} \rho_{12} - \rho_{01})^2}{1 - \rho_{12}^2} \end{aligned} \quad (96)$$

at

$$F^{(1)} = \left. \frac{\partial F}{\partial \theta_5} \right|_{(1,1)} = \frac{\bar{X} S_y (\rho_{02} \rho_{12} - \rho_{01})}{\bar{Y} S_x (1 - \rho_{12}^2)}$$

and

$$F^{(2)} = \left. \frac{\partial F}{\partial \theta_6} \right|_{(1,1)} = \frac{\bar{P}_x S_y (\rho_{01} \rho_{12} - \rho_{02})}{\bar{Y} S_p (1 - \rho_{12}^2)}$$

(xiii) From Equations (72) and (2), we get

$$\begin{aligned} &[\mathcal{M}(\mathfrak{t}_h)]_{min} - [\mathcal{M}(\mathcal{T}_{c4})]_{min} \\ &= (\lambda_n - \lambda') \rho_{03}^2 S_y^2 + \frac{(\lambda_n - \lambda') S_y^2 (\rho_{03} \rho_{13} - \rho_{01})^2}{1 - \rho_{13}^2} > 0 \end{aligned}$$

(xiv) From Equations (72) and (93), we get

$$[\mathcal{M}(\mathcal{T}_{c1})]_{min} - [\mathcal{M}(\mathcal{T}_{c4})]_{min} > 0$$

if

$$\rho_{03}^2 + \frac{(\rho_{03} \rho_{13} - \rho_{01})^2}{1 - \rho_{13}^2} - \rho_{01}^2 > 0$$

(xv) From Equations (72) and (94), we get

$$[\mathcal{M}(\mathcal{T}_{c2})]_{min} - [\mathcal{M}(\mathcal{T}_{c4})]_{min} > 0$$

if

$$\rho_{03}^2 + \frac{(\rho_{03}\rho_{13} - \rho_{01})^2}{1 - \rho_{13}^2} - \rho_{02}^2 > 0$$

(xvi) From Equations (72) and (96), we get

$$[\mathcal{M}(\mathcal{T}_{c3})]_{min} - [\mathcal{M}(\mathcal{T}_{c4})]_{min} > 0$$

if

$$\rho_{03}^2 + \frac{(\rho_{03}\rho_{13} - \rho_{01})^2}{1 - \rho_{13}^2} - \rho_{02}^2 - \frac{(\rho_{02}\rho_{12} - \rho_{01})^2}{1 - \rho_{12}^2} > 0$$

7 Empirical Study

An empirical study is carried out using real data sets of 109 village wise population of Baria (Urban) Police station Champua Tahsil, District-Orissa, India taken from Census Handbook of Orissa, 1981 published by Government of India. 25% villages (i.e. 27 villages) from upper part are considered to constitute non-respondents of the population to show the efficiency of suggested families of estimators.

Data 1: The study and auxiliary variates are as follows:

y: Agricultural labours, **x:** Occupied houses
p_x: Occupied houses more than 70, **r_x:** Rank of *x*

The parameters for data 1 are:

$$\begin{array}{lll} N = 109 & \bar{Y} = 41.2385 & \bar{X} = 88.8624 \\ \bar{P}_x = 0.5229 & \bar{R}_x = 54.6789 & \\ n = 30 & S_y = 46.64779 & S_x = 58.9933 \\ S_p = 0.50178 & S_r = 31.49570 & \\ \lambda_n = 0.02416 & \bar{Y}_{(2)} = 51.7037 & \bar{X}_{(2)} = 108.56 \\ \bar{P}_{x(2)} = 0.7037 & \bar{R}_{x(2)} = 13.8148 & \\ W_2 = 0.2477 & S_{y(2)} = 38.42857 & S_{x(2)} = 68.07029 \end{array}$$

$$\begin{aligned}
S_{p(2)} &= 0.46532 & S_{r(2)} &= 7.93259 \\
\rho_{01} &= 0.456 & \rho_{02} &= 0.426 & \rho_{12} &= 0.706 \\
\rho_{03} &= -0.507 & \rho_{13} &= -0.899 \\
\rho_{01(2)} &= 0.071 & \rho_{02(2)} &= 0.227 & \rho_{12(2)} &= 0.535 \\
\rho_{03(2)} &= -0.273 & \rho_{13(2)} &= -0.867 \\
\rho_{23} &= -0.866 & \rho_{23(2)} &= -0.797
\end{aligned}$$

Data 2: The study and auxiliary variates are as follows:

y: Agricultural labours, **x:** Total population
p_x: Population greater than 500, **r_x:** Rank of *x*

The parameters for data 2 are:

$$\begin{aligned}
N &= 109 & \bar{Y} &= 41.2385 & \bar{X} &= 485.92 \\
\bar{P}_x &= 0.3761 & \bar{R}_x &= 54.9633 \\
n &= 30 & S_y &= 46.64779 & S_x &= 320.20928 \\
S_p &= 0.48666 & S_r &= 31.60372 \\
\lambda_n &= 0.02416 & \bar{Y}_{(2)} &= 51.7037 & \bar{X}_{(2)} &= 595.15 \\
\bar{P}_{x(2)} &= 0.4815 & \bar{R}_{x(2)} &= 14.0000 \\
W_2 &= 0.2477 & S_{y(2)} &= 38.42857 & S_{x(2)} &= 363.23119 \\
S_{p(2)} &= 0.50918 & S_{r(2)} &= 7.93725 \\
\rho_{01} &= 0.451 & \rho_{02} &= 0.451 & \rho_{12} &= 0.786 \\
\rho_{03} &= -0.497 & \rho_{13} &= -0.897 \\
\rho_{01(2)} &= 0.126 & \rho_{02(2)} &= 0.365 & \rho_{12(2)} &= 0.751 \\
\rho_{03(2)} &= -0.269 & \rho_{13(2)} &= -0.886 \\
\rho_{23} &= -0.839 & \rho_{23(2)} &= -0.866
\end{aligned}$$

Two different data sets are considered to demonstrate the efficiency of the suggested families of estimators, their minimum mean square errors are calculated along with relevant estimators at various levels of sub-sampling fractions. The percentage relative efficiency (*PRE*) of \mathcal{T}_i ; $i = 1, 2, \dots, 4$ with respect to corresponding relevant estimators is calculated by

$$PRE = \frac{\mathcal{M}(\cdot)}{\mathcal{M}(\mathcal{T})|_{min.}} \times 100.$$

The minimum mean square errors and *PRE* of $(\mathcal{T}_1, \mathcal{T}_2)$, $(\mathcal{T}_3, \mathcal{T}_4)$ and \mathcal{T}_{ci} , $i = 1, 2, 3, 4$ with respect to t_h for data 1 and 2 are respectively given in Table 1–3 while the analysis of cost functions are given in Table 4.

Table 1 Mean square errors and *FRE* (shown in parenthesis) of T_1 and T_2 with other estimators for data 1 and 2

Estimators	For Data 1		
	$h = 4$	$h = 3$	$h = 2$
$t_h = w_1 \bar{y}_1 + w_2 \bar{y}_{2(r)}$	89.1518 (100%)*	76.9587 (100%)	64.7656 (100%)
$t_{r-1}^{(1)} = \bar{y}^h \frac{\bar{X}}{\bar{x}^h}$	99.5688 (89.5%)	80.5597 (95.5%)	61.5506 (105.2%)
$t_{p-1}^{(1)} = \bar{y}^h \frac{\bar{x}^h}{\bar{X}}$	164.3870 (54.2%)	142.5310 (53.9%)	120.6750 (53.7%)
$t_{reg1}^{(1)} = \bar{y}^h + b_1^h (\bar{X} - \bar{x}^h)$	83.0203 (107.4%)	70.0188 (109.9%)	56.4732 (114.7%)
$t_{g1}^{(1)} = \bar{y}^h \left(\frac{\bar{x}^h}{\bar{X}} \right)^{\alpha_1}$	83.0203 (107.4%)	70.0188 (109.9%)	56.4732 (114.7%)
$t_{c1}^{(1)} = f_1(\bar{y}^h, \theta_1); \theta_1 = \frac{\bar{x}^h}{\bar{X}}$	$(\alpha_1)_{opt} = -0.3784$	$(\alpha_1)_{opt} = -0.4479$	$(\alpha_1)_{opt} = -0.561013$
	83.0203 (107.4%)	70.0188 (109.9%)	56.4732 (114.7%)
	$(f_1^{(2)})_{opt} = -15.6038$	$(f_1^{(2)})_{opt} = -18.4725$	$(f_1^{(2)})_{opt} = -23.1353$
$t_{r-1}^{(2)} = \bar{y}^h \frac{\bar{P}_x}{\bar{p}_x^h}$	106.4870 (83.7%)	88.4612 (86.9%)	70.4351 (92%)
$t_{p-1}^{(2)} = \bar{y}^h \frac{\bar{p}_x^h}{\bar{P}_x}$	214.2020 (41.6%)	185.6030 (41.5%)	157.0040 (41.3%)
$t_{reg1}^{(2)} = \bar{y}^h + b_2^h (\bar{P}_x - \bar{p}_x^h)$	78.9660 (112.9%)	67.1410 (114.6%)	55.1977 (117.3%)
$t_{g1}^{(2)} = \bar{y}^h \left(\frac{\bar{p}_x^h}{\bar{P}_x} \right)^{\alpha_2}$	78.9660 (112.9%)	67.1410 (114.6%)	55.1977 (117.3%)
	$(\alpha_2)_{opt} = -0.3782$	$(\alpha_2)_{opt} = -0.40426$	$(\alpha_2)_{opt} = -0.442094$
$t_{c1}^{(2)} = f_2(\bar{y}^h, \theta_2); \theta_2 = \frac{\bar{p}_x^h}{\bar{P}_x}$	78.9660 (112.9%)	67.1410 (114.6%)	55.1977 (117.3%)
	$(f_2^{(2)})_{opt} = -15.5985$	$(f_2^{(2)})_{opt} = -16.6712$	$(f_2^{(2)})_{opt} = -18.2313$
	$(f_5^{(2)})_{opt} = -0.3177$	$(f_5^{(2)})_{opt} = -0.3153$	$(f_5^{(2)})_{opt} = -0.3021$

(Continued)

Table 1 Continued

Estimators	For Data 1	
	$h = 4$	$h = 3$
$T_1 = \bar{y}^h \mathbf{g}_1(\theta_1, \eta_1); \theta_1 = \frac{\bar{y}^h}{\bar{X}}, \theta_1 = \frac{\bar{y}^h}{R_x}$	73.0153 (121.5%)	61.7334 (124.6%)
	$(g_1^{(1)})_{opt} = 0.1367$	$(g_1^{(1)})_{opt} = 0.1223$
	$(g_1^{(2)})_{opt} = 1.2131$	$(g_1^{(2)})_{opt} = 1.1719$
$T_2 = \bar{y}^h \mathbf{g}_2(\theta_2, \eta_1); \theta_2 = \frac{\bar{y}^h}{P_x}, \eta_1 = \frac{\bar{y}^h}{R_x}$	73.273 0(121.7%)	61.9016 (124.3%)
	$(g_2^{(1)})_{opt} = -0.0609$	$(g_2^{(1)})_{opt} = -0.0459$
	$(g_2^{(2)})_{opt} = 0.9412$	$(g_2^{(2)})_{opt} = 0.9543$
		$(g_1^{(1)})_{opt} = -0.0167$
		$(g_2^{(2)})_{opt} = 0.9865$
For Data 2		
Estimators	$h = 4$	$h = 3$
$t_h = w_1 \bar{y}_1 + w_2 \bar{y}_2(r)$	89.1518 (100%)	76.9587 (100%)
$t_{r_1}^{(1)} = \bar{y}^h \frac{\bar{X}}{\bar{y}^h}$	95.5120 (93.3%)	77.9377 (98.7%)
$t_{p_1}^{(1)} = \bar{y}^h \frac{\bar{y}^h}{\bar{X}}$	165.5510 (53.9%)	143.0470 (53.8%)
$t_{reg1}^{(1)} = \bar{y}^h + b_1^h(\bar{X} - \bar{x}^h)$	81.7425 (109.1%)	69.0576 (111.4%)
$t_{g_1}^{(1)} = \bar{y}^h \left(\frac{\bar{y}^h}{\bar{X}} \right)^{\alpha_1}$	81.7425 (109.1%)	69.0576 (111.4%)
	$(\alpha_1)_{opt} = -0.4231$	$(\alpha_1)_{opt} = -0.4854$
$t_{e1}^{(1)} = f_1(\bar{y}^h, \theta_1); \theta_1 = \frac{\bar{y}^h}{\bar{X}}$	81.7425 (109.1%)	69.0576 (111.4%)
	$(f_1^{(2)})_{opt} = -17.45$	$(f_1^{(2)})_{opt} = -20.0173$
		$(f_1^{(2)})_{opt} = -24.1528$
		$(\alpha_1)_{opt} = -0.5857$
		55.9540 (115.7%)
		55.9540 (115.7%)
		64.7656 (110%)
		60.3634 (107.3%)
		120.5440 (53.7%)
		55.9540 (115.7%)
		55.9540 (115.7%)

$t_{r_1}^{(2)} = \bar{y}^h \frac{\bar{P}_x^h}{\bar{P}_x^h}$	159.7890 (55.8%)	129.9020 (59.2%)	100.0150 (64.8%)
$t_{p_1}^{(2)} = \bar{y}^h \frac{\bar{p}_x^h}{\bar{P}_x^h}$	310.5190 (28.7%)	264.5480 (29.1%)	218.5760 (29.6%)
$t_{reg_1}^{(2)} = \bar{y}^h + b_2^h(\bar{P}_x^h - \bar{p}_x^h)$	79.4260 (112.2%)	67.5372 (114%)	55.4718 (116.8%)
$t_{g_1}^{(2)} = \bar{y}^h \left(\frac{\bar{p}_x^h}{\bar{P}_x^h} \right)^{\alpha_2}$	79.4260 (112.2%)	67.5372 (114%)	55.4718 (116.8%)
	$(a_2)_{opt} = -0.2581$	$(a_2)_{opt} = -0.2799$	$(a_2)_{opt} = -0.3136$
$t_{\theta_1}^{(2)} = f_2(\bar{y}^h, \theta_2); \theta_2 = \frac{\bar{p}_x^h}{\bar{P}_x^h}$	79.4260 (112.2%)	67.5372 (114%)	55.4718 (116.8%)
	$(f_2^{(2)})_{opt} = -10.6435$	$(f_2^{(2)})_{opt} = -11.5423$	$(f_2^{(2)})_{opt} = -12.9305$
	$(f_5^{(2)})_{opt} = -0.1922$	$(f_5^{(2)})_{opt} = -0.1964$	$(f_5^{(2)})_{opt} = -0.1988$
$T_1 = \bar{y}^h \mathbf{g}_1(\theta_1, \eta_1); \theta_1 = \frac{\bar{x}^h}{\bar{X}}, \eta_1 = \frac{\bar{p}_x^h}{\bar{R}_x}$	73.9494 (120.6%)	62.5112 (123.1%)	51.0590 (126.8%)
	$(g_1^{(1)})_{opt} = 0.0480$	$(g_1^{(1)})_{opt} = 0.0375$	$(g_1^{(1)})_{opt} = 0.0189$
	$(g_1^{(2)})_{opt} = 1.0835$	$(g_1^{(2)})_{opt} = 1.0531$	$(g_1^{(2)})_{opt} = 1.0146$
$T_2 = \bar{y}^h \mathbf{g}_2(\theta_2, \eta_1); \theta_2 = \frac{\bar{p}_x^h}{\bar{P}_x^h}, \eta_1 = \frac{\bar{r}_x^h}{\bar{R}_x}$	73.7819 (120.8%)	62.4101 (123.3%)	51.0242 (126.9%)
	$(g_2^{(1)})_{opt} = -0.0547$	$(g_2^{(1)})_{opt} = -0.0477$	$(g_2^{(1)})_{opt} = -0.0328$
	$(g_2^{(2)})_{opt} = 0.8988$	$(g_2^{(2)})_{opt} = 0.9052$	$(g_2^{(2)})_{opt} = 0.9260$

(Continued)

Table 2 Mean square errors and *PRE* (shown in parenthesis) of T_3 and T_4 with other estimators for data 1 and 2

Estimators	For Data 1		
	$h = 4$	$h = 3$	$h = 2$
$t_h = w_1 \bar{y}_1 + w_2 \bar{y}_{2(r)}$	89.1518 (100%)*	76.9587 (100%)	64.7656 (100%)
$t_{r2}^{(1)} = \bar{y}^h \frac{\bar{X}}{\bar{x}}$	79.1207 (112.7%)	66.9277 (115%)	54.7346 (118.2%)
$t_{p2}^{(1)} = \bar{y}^h \frac{\bar{X}}{\bar{x}}$	135.3990 (65.8%)	123.2060 (62.5%)	111.0130 (58.3%)
$t_{reg2}^{(1)} = \bar{y}^h + b_3^h (\bar{X} - \bar{x})$	78.2200 (114%)	66.0270 (116.6%)	53.8339 (120.3%)
$t_{g2}^{(1)} = \bar{y}^h \left(\frac{\bar{x}}{\bar{X}} \right)^{\alpha_3}$	78.2200 (114%)	66.0270 (116.6%)	53.8339 (120.3%)
	$(\alpha_3)_{opt} = -0.7770$	$(\alpha_3)_{opt} = -0.7770$	$(\alpha_3)_{opt} = -0.7770$
$t_{e2}^{(1)} = f_3(\bar{y}^h, \theta_3); \theta_3 = \frac{\bar{x}}{\bar{X}}$	78.2200 (114%)	66.0270 (116.6%)	53.8339 (120.3%)
	$(f_3^{(2)})_{opt} = -32.0414$	$(f_3^{(2)})_{opt} = -32.0414$	$(f_3^{(2)})_{opt} = -32.0414$
$t_{r2}^{(2)} = \bar{y}^h \frac{\bar{P}_x}{\bar{p}_x}$	88.9882 (100.2%)	76.7952 (100.2%)	64.6021 (100.2%)
$t_{p2}^{(2)} = \bar{y}^h \frac{\bar{P}_x}{\bar{p}_x}$	164.9850 (54%)	152.7920 (50.4%)	140.5990 (46.1%)
$t_{reg2}^{(2)} = \bar{y}^h + b_4^h (\bar{P}_x - \bar{p}_x)$	79.6111 (112%)	67.4180 (114.2%)	55.2250 (117.2%)
$t_{g2}^{(2)} = \bar{y}^h \left(\frac{\bar{p}_x}{\bar{P}_x} \right)^{\alpha_4}$	79.6111 (112%)	67.4180 (114.2%)	55.2250 (117.2%)
	$(\alpha_4)_{opt} = -0.5022$	$(\alpha_4)_{opt} = -0.5022$	$(\alpha_4)_{opt} = -0.5022$

$t_{e2}^{(2)} = f_4(\bar{y}^h, \theta_4); \theta_4 = \frac{\bar{P}_x}{P_x}$	79.6111 (112%)	67.4180 (114.2%)	55.2250 (117.2%)
	$(f_4^{(2)})_{opt} = -20.7084$	$(f_4^{(2)})_{opt} = -20.7084$	$(f_4^{(2)})_{opt} = -20.7084$
$T_3 = \bar{y}^h \mathbf{g}_3(\theta_3, \eta_2); \theta_3 = \frac{\bar{P}_x}{\bar{X}}, \eta_2 = \frac{\bar{P}_x}{R_x}$	75.6380 (117.9%)	63.4450 (121.3%)	51.2519 (126.4%)
	$(g_3^{(1)})_{opt} = -0.0018$	$(g_3^{(1)})_{opt} = -0.0018$	$(g_3^{(1)})_{opt} = -0.0018$
	$(g_3^{(2)})_{opt} = 0.9937$	$(g_3^{(2)})_{opt} = 0.9937$	$(g_3^{(2)})_{opt} = 0.9937$
$T_4 = \bar{y}^h \mathbf{g}_4(\theta_4, \eta_2); \theta_4 = \frac{\bar{P}_x}{\bar{P}}, \eta_2 = \frac{\bar{P}_x}{R_x}$	75.6022 (117.9%)	63.4091 (121.4%)	51.2160 (126.4%)
	$(g_4^{(1)})_{opt} = 0.0616$	$(g_4^{(1)})_{opt} = 0.0616$	$(g_4^{(1)})_{opt} = 0.0616$
	$(g_4^{(2)})_{opt} = 1.0845$	$(g_4^{(2)})_{opt} = 1.0845$	$(g_4^{(2)})_{opt} = 1.0845$

For Data 2

Estimators	h = 4	
	h = 4	h = 4
$t_h = w_1 \bar{y}_1 + w_2 \bar{y}_2(r)$	89.1518 (100%)	89.1518 (100%)
$t_{r2}^{(1)} = \bar{y}^h \frac{\bar{X}}{\bar{x}}$	79.3684 (112.3%)	79.3684 (112.3%)
$t_{p2}^{(1)} = \bar{y}^h \frac{\bar{X}}{\bar{X}}$	134.6190 (66.2%)	134.6190 (66.2%)
$t_{re g_2}^{(1)} = \bar{y}^h + b_3^h (\bar{X} - \bar{x})$	78.4585 (113.7%)	78.4585 (113.7%)
$t_{g_2}^{(1)} = \bar{y}^h \left(\frac{\bar{x}}{\bar{X}} \right)^{\alpha_3}$	78.4585 (113.7%)	78.4585 (113.7%)
	$(\alpha_3)_{opt} = -0.7742$	$(\alpha_3)_{opt} = -0.7742$

(Continued)

Table 2 Continued

Estimators	For Data 2	
	$h = 4$	$h = 4$
$t_{e2}^{(1)} = f_3(\bar{y}^h, \theta_3); \theta_3 = \frac{\bar{P}_x}{\bar{X}}$	78.4585 (113.7%)	78.4585 (113.7%)
	$(f_3^{(2)})_{opt} = -31.9256$	$(f_3^{(2)})_{opt} = -31.9256$
$t_{r2}^{(2)} = \bar{y}^h \frac{\bar{P}_x}{\bar{P}_x}$	103.7000 (86%)	103.7000 (86%)
$t_{p2}^{(2)} = \bar{y}^h \frac{\bar{P}_x}{\bar{P}_x}$	212.1900 (42%)	212.1900 (42%)
$t_{reg2}^{(2)} = \bar{y}^h + b_4^h(\bar{P}_x - \bar{P}_x)$	78.4585 (113.6%)	78.4585 (113.6%)
$t_{g2}^{(2)} = \bar{y}^h \left(\frac{\bar{P}_x}{\bar{P}_x}\right)^{\alpha_4}$	78.4585 (113.6%)	78.4585 (113.6%)
	$(\alpha_4)_{opt} = -0.3943$	$(\alpha_4)_{opt} = -0.3943$
$t_{e2}^{(2)} = f_4(\bar{y}^h, \theta_4); \theta_4 = \frac{\bar{P}_x}{\bar{P}_x}$	78.4585 (113.6%)	78.4585 (113.6%)
	$(f_4^{(2)})_{opt} = -16.2587$	$(f_4^{(2)})_{opt} = -16.2587$
$T_3 = \bar{y}^h \mathbf{g}_3(\theta_3, \eta_2); \theta_3 = \frac{\bar{P}_x}{\bar{X}}, \eta_2 = \frac{\bar{P}_x}{\bar{R}_x}$	76.1586 (117.1%)	76.1586 (117.1%)
	$(g_3^{(1)})_{opt} = -0.0456$	$(g_3^{(1)})_{opt} = -0.0456$
	$(g_3^{(2)})_{opt} = 0.9308$	$(g_3^{(2)})_{opt} = 0.9308$
$T_4 = \bar{y}^h \mathbf{g}_4(\theta_4, \eta_2); \theta_4 = \frac{\bar{P}_x}{\bar{P}_x}, \eta_2 = \frac{\bar{P}_x}{\bar{R}_x}$	75.9604 (117.4%)	75.9604 (117.4%)
	$(g_4^{(1)})_{opt} = -0.1004$	$(g_4^{(1)})_{opt} = -0.1004$
	$(g_4^{(2)})_{opt} = 0.7881$	$(g_4^{(2)})_{opt} = 0.7881$

Table 3 Mean square errors and PRE (shown in parentheses) of \mathcal{T}_{ei} ; $i = 1, 2, 3, 4$ with t_h for data 1 and 2

Estimators	For Data 1		
	$h = 4$	$h = 3$	$h = 2$
$t_h = w_1 \bar{y}_1 + w_2 \bar{y}_2(r)$	89.1518 (100%)*	76.9587 (100%)	64.7656 (100%)
$\mathcal{T}_{e1} = f(\bar{y}^h, \theta_5), \theta_5 = \frac{\bar{x}}{\bar{y}}$	79.7253 (112.5%) $f^{(2)} = -32.0414$	67.5322 (114%) $f^{(2)} = -32.0414$	55.3391 (117%) $f^{(2)} = -32.0414$
$\mathcal{T}_{e2} = g(\bar{y}^h, \theta_6)$ where $\theta_6 = \frac{\bar{p}_x}{\bar{p}_y}$	80.9248 (110.2%) $g^{(2)} = -20.7084$	68.7317 (112%) $g^{(2)} = -20.7084$	56.5387 (115%) $g^{(2)} = -20.7084$
$\mathcal{T}_{e3} = \bar{y}^h F(\theta_5, \theta_6); \theta_5 = \frac{\bar{x}}{\bar{y}}$ and $\theta_6 = \frac{\bar{p}_x}{\bar{p}_y}$	78.7465 (113%) $F^{(1)} = -0.5274$ $F^{(2)} = -0.2446$	66.5534 (115.6%) $F^{(1)} = -0.5274$ $F^{(2)} = -0.2446$	54.3603 (119.1%) $F^{(1)} = -0.5274$ $F^{(2)} = -0.2446$
$\mathcal{T}_{e4} = \bar{y}^h G(\theta_5, \eta_3); \theta_5 = \frac{\bar{x}}{\bar{y}}$ and $\eta_3 = \frac{\bar{r}_x}{\bar{r}_y}$	77.4988 (115%) $G^{(1)} = -0.0018$ $G^{(2)} = 0.99374$	65.3057 (117.8%) $G^{(1)} = -0.0018$ $G^{(2)} = 0.99374$	53.1126 (121.9%) $G^{(1)} = -0.0018$ $G^{(2)} = 0.99374$

(Continued)

Table 3 Continued

Estimators	For Data 2		
	$h = 4$	$h = 3$	$h = 2$
$t_h = w_1 \bar{y}_1 + w_2 \bar{y}_2(r)$	89.1518 (100%)	76.9587 (100%)	64.7656 (100%)
$\mathcal{T}_{c1} = f(\bar{y}^h, \theta_5), \theta_5 = \frac{\bar{y}}{\bar{x}}$	79.9309 (111.5%) $f^{(2)} = -31.9256$	67.7378 (113.6%) $f^{(2)} = -31.9256$	55.5447 (116.6%) $f^{(2)} = -31.9256$
$\mathcal{T}_{c2} = g(\bar{y}^h, \theta_6)$ where $\theta_6 = \frac{\bar{p}_x}{\bar{r}_x}$	80.9248 (110.2%) $g^{(2)} = -15.3574$	68.7317 (120%) $g^{(2)} = -15.3574$	56.5387 (114.6%) $g^{(2)} = -15.3574$
$\mathcal{T}_{c3} = \bar{y}^h F(\theta_5, \theta_6); \theta_5 = \frac{\bar{y}}{\bar{x}}$ and $\theta_6 = \frac{\bar{p}_x}{\bar{r}_x}$	78.8845 (113%) $F^{(1)} = -0.5142$ $F^{(2)} = -0.1875$	66.6915 (115.4%) $F^{(1)} = -0.5142$ $F^{(2)} = -0.1875$	54.4984 (118.8%) $F^{(1)} = -0.5142$ $F^{(2)} = -0.1875$
$\mathcal{T}_{c4} = \bar{y}^h G(\theta_5, \eta_3); \theta_5 = \frac{\bar{y}}{\bar{x}}$ and $\eta_3 = \frac{\bar{r}_x}{\bar{r}_x'}$	77.9477 (114.4%) $G^{(1)} = -0.0456$ $G^{(2)} = 0.9308$	65.7546 (117%) $G^{(1)} = -0.0456$ $G^{(2)} = 0.9308$	53.5616 (120.9%) $G^{(1)} = -0.0456$ $G^{(2)} = 0.9308$

Table 4 PRE and expected cost of \mathcal{T}_c , $i = 1, 2, 3, 4$ with respect to t_h for data 1 and 2
 $\mathcal{C}^{(1)'} = 2.9, \mathcal{C}^{(1)} = 25, \mathcal{C}^{(2)} = 100, \mathcal{C}^{(3)} = 250$
 $\mathcal{C}^{(0)} = Rs.3700$ (fixed)

		$n_{opt.}$	$h_{opt.}$	$n'_{opt.}$	MSE	PRE
Estimators		(Approx.)	(Approx.)	(Approx.)	(Approx.)	
For Data 1	t_h	27	1.75	–	69.8493	100%
	\mathcal{T}_{c1}	25	1.51	83	63.4754	110%
	\mathcal{T}_{c2}	25	1.55	77	65.0922	107.3%
	\mathcal{T}_{c3}	24	1.49	88	62.0982	112.5%
	\mathcal{T}_{c4}	24	1.45	94	60.2736	115.9%
For Data 2	t_h	27	1.75	–	69.8493	100%
	\mathcal{T}_{c1}	25	1.51	82	63.7582	110%
	\mathcal{T}_{c2}	25	1.52	82	63.7582	110%
	\mathcal{T}_{c3}	24	1.49	88	62.2119	112.3%
	\mathcal{T}_{c4}	24	1.47	92	60.9387	114.6%

		$n_{opt.}$	$h_{opt.}$	$n'_{opt.}$	Expected cost in Rs.
Estimators		(Approx.)	(Approx.)	(Approx.)	
For Data 1	t_h	27	1.75	–	3613.47
	\mathcal{T}_{c1}	22	1.36	75	3357.02
	\mathcal{T}_{c2}	23	1.49	71	3422.08
	\mathcal{T}_{c3}	22	1.27	79	3301.62
	\mathcal{T}_{c4}	21	1.18	82	3228.21
For Data 2	t_h	27	1.75	–	3613.47
	\mathcal{T}_{c1}	22	1.38	75	3368.40
	\mathcal{T}_{c2}	22	1.38	75	3368.40
	\mathcal{T}_{c3}	22	1.28	78	3306.19
	\mathcal{T}_{c4}	21	1.21	81	3254.97

8 Conclusion

Tables 1 and 2 exhibit that the suggested families of estimators $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ and \mathcal{T}_4 are more efficient than the corresponding estimators at all the sub-sampling fractions for both the data 1 and 2. The mean square errors of the suggested families of estimators are decreasing as the sub-sampling fraction increases for both the data 1 and 2. Similarly, in the case of two-phase sampling estimation, \mathcal{T}_{c4} shows efficient results compared to the existing estimators $t_h, \mathcal{T}_{c1}, \mathcal{T}_{c2}, \mathcal{T}_{c3}$. From Table 4, it has been observed that the estimator \mathcal{T}_{c4} is more efficient than the existing estimators $t_h, \mathcal{T}_{c1}, \mathcal{T}_{c2}, \mathcal{T}_{c3}$ for

fixed cost while expected cost incurred for \mathcal{T}_{c4} is less compared to expected cost incurred for existing estimators $t_h, \mathcal{T}_{c1}, \mathcal{T}_{c2}, \mathcal{T}_{c3}$. Therefore, the suggested families of estimators can be recommended on the account of theoretical and empirical studies discussed in the text.

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