Approximate Bayes Estimators of the Parameters of the Inverse Gaussian Distribution Under Different Loss Functions

Ilhan Usta[1,*] and Merve Akdede[2]

[1] *Eskisehir Technical University, Department of Statistics, Eskisehir, Turkey*
[2] *Usak University Department of Statistics, Usak, Turkey*
E-mail: iusta@eskisehir.edu.tr; merve.akdede@usak.edu.tr
[*] *Corresponding Author*

Abstract

Inverse Gaussian is a popular distribution especially in the reliability and life time modelling, and thus the estimation of its unknown parameters has received considerable interest. This paper aims to obtain the Bayes estimators for the two parameters of the inverse Gaussian distribution under varied loss functions (squared error, general entropy and linear exponential). In Bayesian procedure, we consider commonly used non-informative priors such as the vague and Jeffrey's priors, and also propose using the extension of Jeffrey's prior. In the case where the two parameters are unknown, the Bayes estimators cannot be obtained in the closed-form. Hence, we employ two approximation methods, namely Lindley and Tierney Kadane (TK) approximations, to attain the Bayes estimates of the parameters. In this paper. the effects of considered loss functions, priors and approximation methods on Bayesian parameter estimation are also presented. The performance of Bayes estimates is compared with the corresponding classical estimates in terms of the bias and the relative efficiency throughout an extensive simulation study. The results of the comparison show that Bayes estimators obtained by TK method under linear exponential loss function using the proposed

prior outperform the other estimators for estimating the parameters of inverse Gaussian distribution most of the time. Finally, a real data set is provided to illustrate the results.

Keywords: Inverse Gaussian distribution, Bayes estimator, extension of Jeffrey's prior, Lindley approximation, Tierney Kadane approximation.

1 Introduction

The Inverse Gaussian (IG) distribution arises as the distribution of the passage time in Brownian motion with positive drift. Initial studies on the statistical properties of IG distribution was conducted by Tweedie (1957). The theory of IG distribution and its statistical properties were also studied by Chhikara and Folks (1975). For more details on the IG distribution, the readers can referred to Chhikara and Folks (1989), Johnson et al. (1994) and Seshadri (1999).

The probability density function (pdf) of two parameter IG distribution is

$$f(x; \mu, \lambda) = \left(\frac{\lambda}{2\pi x^3} \right)^{1/2} \exp \left(-\frac{\lambda(x - \mu)^2}{2\mu^2 x} \right), \quad x > 0 \qquad (1)$$

with $\mu > 0, \lambda > 0$. We denote the IG distribution by $\text{IG}(\mu, \lambda)$. The expected value and variance of the IG distribution are given as μ and μ^2/λ, respectively. The cumulative distribution function (cdf) is given as follows

$$F(x; \mu, \lambda) = \Phi \left[\sqrt{\frac{\lambda}{x}} \left(\frac{x}{\mu} - 1 \right) \right] + \exp \left(\frac{2\lambda}{\mu} \right) \Phi \left[-\sqrt{\frac{\lambda}{x}} \left(\frac{x}{\mu} + 1 \right) \right] \qquad (2)$$

where Φ is the cdf for the standard normal distribution.

Iyengar and Patwardhan (1988) suggested the IG distribution as an alternative to log-normal, gamma, Weibull and generalized gamma distributions where the initial failure rate is high. Lemeshko et al. (2010) proposed the IG distribution as a competitor of the families of generalized Weibull, log-normal and log-logistic distributions since its hazard function has $\cap$ shape. Therefore, many authors have shown interest in the classical and Bayesian estimation of parameters of the IG distribution. For instance, the maximum likelihood estimators (MLEs) of μ and λ were obtained by Tweedie (1957). Chhikara and Folks (1989) derived the minimum variance unbiased estimators (MVUEs) of parameters. Banarjee and Bhattacharyya (1979) obtained the Bayes estimators of the parameters and reliability function of $\text{IG}(1/\mu, \lambda)$ under the vague

prior, $\pi(\mu|\lambda) \propto constant$ and $\pi(\lambda) = 1/\lambda$. Sinha (1986) used Lindley approximation to obtain Bayes estimates of the unknown parameters as well as the reliability function using under the squared error loss function (SELF) with a vague prior. Ahmad and Jaheen (1995) used Lindley and Tierney and Kadane (TK) approximations to compute the Bayes estimators for the parameters and reliability function of $\text{IG}(\mu, \lambda)$ considering the conjugate prior under SELF. Recently, Singh et al. (2008) obtained Bayes estimators using Lindley's approximation under SELF and the general entropy loss function (GELF). Pandey and Bandyopadhyay (2012) used Markov Chain Monte Carlo method and Lindley approximation to compute Bayes estimators using gamma prior under SELF. In addition, many authors have studied the Bayesian estimation of the parameters of IG distribution using censored data, see for example, Basak and Balakrishnan (2012), Jia et al. (2017), Rostamian and Nematollahi (2019).

The purpose of this study is to derive Bayes estimators for the unknown parameters of $\text{IG}(\mu, \lambda)$ based on a class of non-informative priors: the vague prior, Jeffrey's prior, and the proposed extension of Jeffrey's prior, under symmetric (SELF) and asymmetric (linear exponential and GELF) loss functions. It is noticed that the Bayes estimators of the parameters cannot be expressed in explicit forms. Thus, we use two approximation methods, Lindley and TK, to compute the Bayes estimates. A comprehensive simulation study is conducted to evaluate the performance of the proposed Bayes estimators, MLEs and MVUEs in terms of bias and the mean squared error.

The rest of this study is set as follows: In Section 2, the MLEs and MVUEs of the two-parameter IG distribution are given. Section 3 presents the non-informative priors, the posterior distributions and the considered loss functions. Bayes estimation using Lindley and TK approximation methods is also outlined in this section. By using the approximation methods, the approximate Bayes estimates under the proposed extension of Jeffrey's prior are derived in Section 4. The results of simulation study are provided in Section 5. An application to real data set is given in Section 6. We conclude the study in Section 7.

2 Classical Estimation Methods

Let $X = (X_1, X_2, \cdots, X_n)$ be a random sample of size n from the $\text{IG}(\mu, \lambda)$ distribution, then the likelihood and log-likelihood functions are respectively

as follows:

$$
L(\mu, \lambda|x) = \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \prod_{i=1}^{n} x_i^{-3/2} \exp\left(-\lambda\left(2\mu^2\right)^{-1}\sum_{i=1}^{n} z_i(\mu)\right), \tag{3}
$$

$$
\ell(\mu, \lambda|x) = n/2(ln\lambda - ln2\pi) - 3/2\sum_{i=1}^{n}\ln x_i - \lambda(2\mu^2)^{-1}\sum_{i=1}^{n} z_i(\mu). \tag{4}
$$

where $z_i(\mu) = \frac{(x_i - \mu)^2}{x_i}$. By differentiating (5) with respect to the parameters and setting to zero, we have

$$
\frac{\partial \ell}{\partial \mu} = \sum_{i=1}^{n}\left(\frac{x_i}{\mu} - 1\right)\frac{\lambda}{\mu^2} = 0, \tag{5}
$$

$$
\frac{\partial \ell}{\partial \lambda} = \frac{n}{2\lambda} - \frac{1}{2\mu^2}\sum_{i=1}^{n} z_i(\mu) = 0. \tag{6}
$$

From (6) and (7), the MLEs of μ and λ are obtained as follows, respectively:

$$
\widehat{\mu}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{n} = \overline{X} \quad \text{and} \quad \widehat{\lambda}_{MLE} = \frac{n}{\sum_{i=1}^{n}\left(\frac{1}{x_i} - \frac{1}{\overline{X}}\right)}. \tag{7}
$$

According to Chhikara and Folks [5], the MVUEs of μ and λ are given as

$$
\widehat{\mu}_{MVUE} = \overline{X} \quad \text{and} \quad \widehat{\lambda}_{MVUE} = \frac{n-3}{\sum_{i=1}^{n}\left(\frac{1}{x_i} - \frac{1}{\overline{X}}\right)} \tag{8}
$$

3 Bayesian Estimation

The property that distinguishes the Bayesian estimation approach from classical estimation is that the model parameter is considered as a random variable. It has a certain model named as the prior distribution. The prior represents any knowledge or beliefs about the parameter. In case that the prior knowledge about the parameters is available, informative priors are used for Bayesian estimation. However, in many real cases, it is very difficult to know in advance information about the parameters.

Therefore, using non-informative priors is the more suitable alternative to informative priors. For this reason, in this study, we assume the absence of the prior knowledge about the parameters, and obtain Bayes estimators under

three different non-informative priors: the vague prior, Jeffrey's prior, and the proposed extension of Jeffrey's prior.

3.1 Posterior Distribution Under the Vague Prior

First, we consider the vague prior for μ and λ given as follows

$$\pi(\mu, \lambda) \propto 1/\lambda, \tag{10}$$

which also used by Banarjee and Bhattacharyya (1979), and Sinha (1986).

Thus, the joint posterior distribution of (μ, λ) is derived as follows

$$\pi(\mu, \lambda \mid x) = \frac{L(\mu, \lambda \mid x)\pi(\mu, \lambda)}{\int_0^\infty \int_0^\infty L(\mu, \lambda \mid x)\pi(\mu, \lambda)d\mu d\lambda}, \tag{11}$$

Substituting the vague prior (10) in (11), the posterior distribution becomes as the following

$$\pi(\mu, \lambda | x) = \frac{\frac{1}{\lambda}\left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}}\prod_{i=1}^n x_i^{-3/2}\exp\left(-\lambda\left(2\mu^2\right)^{-1}\sum_{i=1}^n z_i\left(\mu\right)\right)}{\int_0^\infty \int_0^\infty \frac{1}{\lambda}\left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}}\prod_{i=1}^n x_i^{-3/2}\exp\left(-\lambda\left(2\mu^2\right)^{-1}\sum_{i=1}^n z_i\left(\mu\right)\right)d\mu d\lambda}. \tag{12}$$

or

$$\pi(\mu, \lambda | x) = A_1 \lambda^{n/2-1}\exp\left(-\lambda\left(2\mu^2\right)^{-1}\sum_{i=1}^n z_i\left(\mu\right)\right), \tag{13}$$

where $A_1^{-1} = \int_0^\infty \int_0^\infty \lambda^{n/2-1}\exp\left(-\lambda\left(2\mu^2\right)^{-1}\sum_{i=1}^n z_i\left(\mu\right)\right)d\mu d\lambda$ is the normalizing constant.

3.2 Posterior Distribution Under Jeffrey's Prior

One of the commonly used non-informative priors was suggested by Jeffrey (1961) as $\pi(\theta) \propto \sqrt{|I(\theta)|}$, where $I(\theta) = -E\left(\frac{\partial^2 \log L(\theta)}{\partial \theta^2}\right)$ is the Fisher Information matrix. For two parameter IG distribution, the joint Jeffrey's prior of (μ, λ) is given as

$$\pi(\mu, \lambda) \propto \frac{1}{\sqrt{\mu^3 \lambda}}. \tag{14}$$

Substituting the Jeffrey's prior (14) in (11), we obtain

$$\pi(\mu, \lambda | x) = A_2 \mu^{-3/2}\lambda^{(n-1)/2}\exp\left(-\lambda\left(2\mu^2\right)^{-1}\sum_{i=1}^n z_i\left(\mu\right)\right), \tag{15}$$

where $A_2^{-1} = \int_0^\infty \int_0^\infty \mu^{-3/2}\lambda^{(n-1)/2}\exp\left(-\lambda\left(2\mu^2\right)^{-1}\sum_{i=1}^n z_i\left(\mu\right)\right) d\mu d\lambda$ is the normalizing constant.

3.3 Posterior Distribution Under Extended Jeffrey's Prior

Al-Kutubi and Ibrahim (2009) suggested an extension of Jeffrey's prior for exponential distribution as $\pi(\theta) \propto \left(\sqrt{|I(\theta)|}\right)^c, c \in \mathbb{R}^+$. Following the similar path, we propose to use the extended Jeffrey's prior for two parameters of IG distribution which can be express as the following form

$$\pi(\mu, \lambda) \propto \left(1/\sqrt{\mu^3\lambda}\right)^c, \quad c > 0. \tag{16}$$

This is also a generalisation of the non-informative priors in this study. For instance, the extended Jeffrey's prior in (16) yields Jeffrey's prior in (16) when $c = 1$. If $c = 2$ and μ is constant, (18) becomes the vague prior in (12).

The posterior distribution of (μ, λ) is derived by using the proposed prior in (16) as follows

$$\pi(\mu, \lambda|x) = A_3\mu^{-3c/2}\lambda^{(n-c)/2}\exp\left(-\lambda\left(2\mu^2\right)^{-1}\sum_{i=1}^n z_i\left(\mu\right)\right), \tag{17}$$

where $A_3^{-1} = \int_0^\infty \int_0^\infty \mu^{-3c/2}\lambda^{(n-c)/2}\exp\left(-\frac{\lambda}{2\mu^2}\sum_{i=1}^n z_i\left(\mu\right)\right) d\mu d\lambda$ is the normalizing constant.

3.4 Loss Functions

In this subsection, we briefly present various loss functions used to drive Bayes estimators of the IG parameters in this study.

3.4.1 Squared error loss function

One of the widely used loss functions is SELF, which has the symmetry property. SELF is defined as follows

$$L(\widehat{\theta}; \theta) = (\widehat{\theta} - \theta)^2. \tag{18}$$

Let $u(\theta)$ be an arbitrary function of θ. The Bayes estimator of $u(\theta)$ under SELF, denoted by $\hat{u}_{SELF}$, that equals to the posterior mean is given as

$$\hat{u}_{SELF} = E\left[u(\theta)|X\right] = \frac{\int_\Theta u(\theta)L(\theta|x)\pi(\theta)d\theta}{\int_\Theta L(\theta|x)\pi(\theta)d\theta}. \tag{19}$$

SELF is widely used due to the simplicity of its algebraic calculations. However, as SELF is symmetric, it may not be appropriate for many estimation problems especially where overestimation or underestimation of a parameter is important. Therefore, asymmetric loss functions are suggested as more appropriate alternatives.

3.4.2 Linear exponential loss function

Varian (1975) defined an asymmetric loss function, called linear exponential (LINEX) loss function, as an alternative to the symmetric loss functions where overestimation is more serious than underestimation. LINEX loss function is defined as follows

$$L(\widehat{\theta}; \theta) = \exp(k(\widehat{\theta} - \theta)) - k(\widehat{\theta} - \theta) - 1, \quad k \neq 0, \tag{20}$$

where k is the loss parameter. The sign and the magnitude of k represent the direction and the degree of symmetry, respectively.

The Bayes estimator of $u(\theta)$ under (20), denoted by $\hat{u}_{LINEX}$, was obtained by Zellner (1986) as following

$$\hat{u}_{LINEX} = -\frac{1}{k} \ln E\left[\exp(-ku(\theta))|\, X\right]. \tag{21}$$

3.4.3 General entropy loss function

General entropy loss function (GELF) introduced by Calabria and Pulcini (1994) was defined as the following

$$L(\widehat{\theta}; \theta) = \left(\frac{\widehat{\theta}}{\theta}\right)^k - k \ln\left(\frac{\widehat{\theta}}{\theta}\right) - 1, \quad k \neq 0, \tag{22}$$

where k is the loss parameter and its magnitude shows the degree of symmetry.

The Bayes estimator of $u(\theta)$ under GELF, denoted by $\hat{u}_{GELF}$, is given by

$$\hat{u}_{GELF} = \left(E\left[(u(\theta))^{-k}\middle|\, X\right]\right)^{-1/k}. \tag{23}$$

Note that if $k = -1$ in (25), the Bayes estimator under GELF, $\hat{u}_{GELF}$, coincides with $\hat{u}_{SELF}$.

We would like to emphasize that Bayes estimators under the considered loss functions contain the ratio of the integrals which cannot be simplified in close forms. Therefore, Lindley and TK approximations are employed to obtain the approximate Bayes estimates of IG(μ,λ).

3.5 Lindley Approximation

Lindley (1980) suggested an approximation for the integrals of the form

$$\frac{\int_{\Theta} g\left(\theta\right) h(\theta) \exp\left(\ell(\theta)\right) d\theta}{\int_{\Theta} h(\theta) \exp\left(\ell(\theta)\right) d\theta}, \tag{24}$$

where $g(\theta)$ and $h(\theta)$ are any functions of θ and $\ell(\theta)$ is the log-likelihood function. If $h(\theta)$ is assumed to be the prior distribution of θ, then (24) becomes the expectation of $g(\theta)$ as in the following form

$$E\left[g(\theta)|X\right] = \frac{\int_{\Theta} g(\theta) \exp\left(\tau\left(\theta\right)\right) d\theta}{\int_{\Theta} \exp\left(\tau\left(\theta\right) + \ell(\theta)\right) d\theta}, \tag{25}$$

where τ is the logarithm of the prior.

Applying Lindley's approximation method for two parameter case, say $\theta = (\theta_1, \theta_2)$, $E\left[g(\theta)|X\right]$ can be asymptotically obtained as

$$E(g(\theta)|X) = u + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} (u_{ij} + 2u_i \tau_j)\sigma_{ij}$$

$$+ \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} \sum_{l=1}^{2} L_{ijk}\sigma_{ij}\sigma_{kl}u_l, \tag{26}$$

where all terms are evaluated at the MLEs of $\theta = (\theta_1, \theta_2)$, $u = g$, $\sigma_{ij} = [-L_{ij}]^{-1}$, $u_i = \frac{\partial g}{\partial \theta_i}$, $u_{ij} = \frac{\partial^2 g}{\partial \theta_i \partial \theta_j}$, $L_{ijk} = \frac{\partial^3 \ell}{\partial \theta_i \partial \theta_j \partial \theta_k}$ and $\tau_j = \frac{\partial \tau}{\partial \theta_j}$.

3.6 Tierney-Kadane Approximation

Lindley approximation method is often used to evaluate the integrals in (25); however, this method requires the third derivatives of the log-likelihood function. An alternative approximation method was derived by Tierney and Kadane (1986). By defining the functions $\mathcal{L} = \frac{1}{n}\left(\ell(\theta) + \tau(\theta)\right)$ and $\mathcal{L}^* = \frac{1}{n}\left(\ln u + \ell(\theta) + \tau(\theta)\right)$, respectively, the expectation of the $g(\theta)$ can be expressed as

$$E\left[g(\theta)|\,X\right] = \frac{\int_{\Theta} e^{n\mathcal{L}^*} d\theta}{\int_{\Theta} e^{n\mathcal{L}} d\theta}. \tag{27}$$

Then applying TK method, the approximate posterior expectation of the $g(\theta)$ in (27) is obtained as following

$$E\left[g(\theta)|\,X\right] \approx \left[\frac{\det\Sigma^*}{\det\Sigma}\right]^{1/2} \exp\left\{n\left[\mathcal{L}^*(\widehat{\theta}_{\mathcal{L}^*}) - \mathcal{L}(\widehat{\theta}_{\mathcal{L}})\right]\right\}, \qquad (28)$$

where $\mathcal{L}^*$ and $\mathcal{L}$ attain their maximum value at $\widehat{\theta}_{\mathcal{L}^*}$ and $\widehat{\theta}_{\mathcal{L}}$, respectively. Σ^* and Σ are the inverses of the negatives Hessian matrices of $\mathcal{L}^*$ and $\mathcal{L}$ at $\widehat{\theta}_{\mathcal{L}^*}$ and $\widehat{\theta}_{\mathcal{L}}$, respectively.

4 Bayes Estimators Under Extended Jeffrey's Prior

In this section, the extended Jeffrey's prior is taken into the consideration as the prior distribution and Bayes estimators of the parameters of the IG distribution are approximately computed by using Lindley and TK methods under the considered loss functions.

4.1 Bayes Estimators Using Lindley Approximation

For our estimation problem with $\theta = (\mu, \lambda)$, we derive the following expressions by using the log-likelihood function in (5) and the extension of Jeffrey's prior in (18) as

$$L_{111} = 6n\widehat{\lambda}/\widehat{\mu}^4,\ L_{112} = -n/\widehat{\mu}^3,\ L_{121} = L_{122} = L_{211} = L_{221} = 0,\ L_{222} = n/\widehat{\lambda}^3$$

$$\sigma_{11} = \widehat{\mu}^3/n\widehat{\lambda},\ \sigma_{12} = \sigma_{21} = 0,\ \sigma_{22} = 2\widehat{\lambda}^2/n, \tau_1 = -3c/2\widehat{\mu}, \quad \tau_2 = -c/2\widehat{\lambda},$$

where $\widehat{\mu} = \widehat{\mu}_{MLE}$ and $\widehat{\lambda} = \widehat{\lambda}_{MLE}$ given in (8).

Substituting the above expressions into (28), the approximate posterior expectation of $g(\mu, \lambda)$ can be expressed as in the following form

$$E\left[g(\mu, \lambda)|X\right] = \hat{u} + \frac{1}{2}\left(\hat{u}_{11} + 2\hat{u}_1\tau_1\right)\sigma_{11} + \frac{1}{2}(\hat{u}_{22} + 2\hat{u}_2\tau_2)\sigma_{22}$$
$$+ \frac{1}{2}(L_{111}\sigma_{11}{}^2\hat{u}_1 + L_{112}\sigma_{11}\sigma_{22}\hat{u}_2 + L_{222}\sigma_{22}{}^2\hat{u}_2). \qquad (29)$$

4.1.1 Bayes estimators under LINEX loss function

- Let $g(\mu, \lambda) = e^{-k\mu}$, $(u_1 = -ke^{-k\mu}, u_{11} = k^2e^{-k\mu}, u_2 = u_{12} = u_{21} = u_{22} = 0)$ then the approximate posterior expectation of $g(\mu, \lambda)$ is

obtained as

$$E\left[e^{-k\mu}|X\right] = e^{-k\widehat{\mu}} + (k\widehat{\mu} + 3c - 6)\frac{ke^{-k\widehat{\mu}}\widehat{\mu}^2}{2n\widehat{\lambda}}.$$

The approximate Bayes estimator of μ under LINEX loss function, $\widehat{\mu}_{L-LINEX}$, is

$$\widehat{\mu}_{L-LINEX} = -\frac{1}{k}\ln\left[E(e^{-k\mu}|X)\right]. \tag{30}$$

- Let $g(\mu, \lambda) = e^{-k\lambda}$, $(u_2 = -ke^{-k\lambda}, u_{22} = k^2 e^{-k\lambda}, u_1 = u_{11} = u_{12} = u_{21} = 0)$ then the approximate posterior expectation of $u(\mu, \lambda)$ is obtained as

$$E(e^{-k\lambda}|X) = e^{-k\widehat{\lambda}} + (k\widehat{\lambda} + c - 1)\frac{ke^{-k\widehat{\lambda}}\widehat{\lambda}}{n}.$$

The approximate Bayes estimator of λ under LINEX loss function, $\widehat{\lambda}_{L-LINEX}$, is

$$\widehat{\lambda}_{L-LINEX} = -\frac{1}{k}\ln\left[E(e^{-k\lambda}|X)\right]. \tag{31}$$

4.1.2 Bayes estimators under general entropy loss function

- Let $g(\mu, \lambda) = \mu^{-k}$, $(u_1 = -k\mu^{-k-1}, u_{11} = (k^2 + k)\mu^{-k-2}, u_2 = u_{12} = u_{21} = u_{22} = 0)$, then

$$E(\mu^{-k}|X) = \widehat{\mu}^{-k} + (k + 3c - 5)\frac{k\widehat{\mu}^{1-k}}{2n\widehat{\lambda}}.$$

The approximate Bayes estimator of μ under GELF, $\widehat{\mu}_{L-GELF}$, is

$$\widehat{\mu}_{L-GELF} = \left[E(\mu^{-k}|X)\right]^{-1/k}. \tag{32}$$

- Let $g(\mu, \lambda) = \lambda^{-k}$, $(u_2 = -k\lambda^{-k-1}, u_{22} = (k^2 + k)\lambda^{-k-2}, u_1 = u_{11} = u_{12} = u_{21} = 0)$, then

$$E(\lambda^{-k}|X) = \widehat{\lambda}^{-k} + \frac{(k^2 + ck)\widehat{\lambda}^{-k}}{n}.$$

The approximate Bayes estimator of λ, under GELF, $\widehat{\lambda}_{L-GELF}$, is

$$\widehat{\lambda}_{L-GELF} = \left[E(\lambda^{-k}|X)\right]^{-1/k}. \tag{33}$$

We note that the approximate Bayes estimators of μ and λ under SELF, denoted by $\widehat{\mu}_{L-SELF}$ and $\widehat{\lambda}_{L-SELF}$, respectively, are obtained from $\widehat{\mu}_{L-GELF}$ and $\widehat{\lambda}_{L-GELF}$ with $k = -1$.

4.2 Bayes Estimators Using TK Approximation

For our case with $\theta = (\mu, \lambda)$, we observe the following expression by using the log-likelihood function in (5) and the extended Jeffrey's prior in (18) as

$$
\mathcal{L}(\mu, \lambda) = \frac{1}{n} \left[\frac{n}{2} \ln \left(\frac{\lambda}{2\pi} \right) - \frac{3}{2} \sum_{i=1}^{n} \ln x_i - \frac{\lambda}{2\mu^2} \sum_{i=1}^{n} z_i(\mu) \right.
$$

$$
\left. - \frac{3c}{2} \ln \mu - \frac{c}{2} \ln \lambda \right].
\tag{34}
$$

Then, $(\widehat{\mu}_{\mathcal{L}}, \widehat{\lambda}_{\mathcal{L}})$ are obtained by solving the following equations:

$$
\frac{\partial \mathcal{L}}{\partial \mu} = \frac{\lambda}{n\mu^3} \sum_{i=1}^{n} x_i - \frac{\lambda}{\mu^2} - \frac{3c}{2n\mu} = 0
$$

$$
\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{n - c}{2n\lambda} - \frac{1}{2n\mu^2} \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{x_i} = 0.
\tag{35}
$$

Further, we obtain the second order derivatives of $\mathcal{L}(\mu, \lambda)$ with respect to μ and λ as

$$
\frac{\partial^2 \mathcal{L}}{\partial \mu^2} = \frac{3c}{2n\mu^2} + \frac{2\lambda}{\mu^3} - \frac{3\lambda}{n\mu^4} \sum_{i=1}^{n} x_i,
$$

$$
\frac{\partial^2 \mathcal{L}}{\partial \lambda^2} = \frac{c - n}{2n\lambda^2},
\tag{36}
$$

$$
\frac{\partial^2 \mathcal{L}}{\partial \mu \partial \lambda} = \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \mu} = \frac{1}{n\mu^3} \sum_{i=1}^{n} x_i - \frac{1}{\mu^2}.
$$

Next, using the expressions in (36), we compute the inverse of negatives hessian matrix of $\mathcal{L}(\mu, \lambda)$ at $(\widehat{\mu}_{\mathcal{L}}, \widehat{\lambda}_{\mathcal{L}})$ as follows

$$
\Sigma = \begin{bmatrix} -\frac{\partial^2 \mathcal{L}}{\partial \mu^2} & -\frac{\partial^2 \mathcal{L}}{\partial \mu \partial \lambda} \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \mu} & -\frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{bmatrix}^{-1}_{\mu = \widehat{\mu}_{\mathcal{L}}, \lambda = \widehat{\lambda}_{\mathcal{L}}}.
\tag{37}
$$

4.2.1 Bayes estimators under LINEX loss function

Let $g(\mu, \lambda) = e^{-k\mu}$, then $\mathcal{L}_\mu{}^*$ function is defined

$$\mathcal{L}_\mu{}^*(\mu, \lambda) = -\frac{k\mu}{n} + \mathcal{L}(\mu, \lambda), \tag{38}$$

and $(\widehat{\mu}_{\mathcal{L}^*}, \widehat{\lambda}_{\mathcal{L}^*})$ are obtained by solving the following equations:

$$\frac{\partial \mathcal{L}_\mu{}^*}{\partial \mu} = \frac{\partial \mathcal{L}}{\partial \mu} - \frac{k}{n} = 0,$$

$$\frac{\partial \mathcal{L}_\mu{}^*}{\partial \lambda} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0. \tag{39}$$

Then, we obtain the inverse of negatives hessian matrix of $\mathcal{L}_\mu^*(\mu, \lambda)$ at $(\widehat{\mu}_{\mathcal{L}^*}, \widehat{\lambda}_{\mathcal{L}^*})$ as

$$\Sigma_\mu^* = \begin{bmatrix} -\frac{\partial^2 \mathcal{L}_\mu{}^*}{\partial \mu^2} & -\frac{\partial^2 \mathcal{L}_\mu{}^*}{\partial \mu \partial \lambda} \\ -\frac{\partial^2 \mathcal{L}_\mu{}^*}{\partial \lambda \partial \mu} & -\frac{\partial^2 \mathcal{L}_\mu{}^*}{\partial \lambda^2} \end{bmatrix}^{-1}_{\mu = \widehat{\mu}_{\mathcal{L}^*}, \lambda = \widehat{\lambda}_{\mathcal{L}^*}}. \tag{40}$$

Thus, the approximate Bayes estimator of μ, denoted by $\widehat{\mu}_{TK-LINEX}$, is derived as

$$\widehat{\mu}_{TK-LINEX} = -\frac{1}{k} \ln \left(\left[\frac{\det \Sigma_\mu^*}{\det \Sigma} \right]^{\frac{1}{2}} \right.$$

$$\left. \exp \left\{ n \left[\mathcal{L}_\mu{}^* \left(\widehat{\mu}_{\mathcal{L}^*}, \widehat{\lambda}_{\mathcal{L}^*} \right) - \mathcal{L} \left(\widehat{\mu}_{\mathcal{L}}, \widehat{\lambda}_{\mathcal{L}} \right) \right] \right\} \right). \tag{41}$$

We obtain the approximate Bayes estimator of λ under LINEX loss function likewise.

4.2.2 Bayes estimators under GELF

Let $g(\mu, \lambda) = \mu^{-k}$, then $\mathcal{L}_\mu{}^*$ function is defined

$$\mathcal{L}_\mu{}^*(\mu, \lambda) = -\frac{k}{n} \ln \mu + \mathcal{L}(\mu, \lambda) \tag{42}$$

We obtain $(\widehat{\mu}_{\mathcal{L}^*}, \widehat{\lambda}_{\mathcal{L}^*})$ by solving the following equations:

$$\frac{\partial \mathcal{L}_\mu{}^*}{\partial \mu} = \frac{\partial \mathcal{L}}{\partial \mu} - \frac{k}{n\mu} = 0,$$

$$\frac{\partial \mathcal{L}_\mu{}^*}{\partial \lambda} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0. \tag{43}$$

Taking the second order derivatives of $\mathcal{L}_\mu{}^*(\mu, \lambda)$, we compute the inverse of negatives hessian matrix of $\mathcal{L}_\mu^*(\mu, \lambda)$ at $(\widehat{\mu}_{\mathcal{L}^*}, \widehat{\lambda}_{\mathcal{L}^*})$ as

$$\Sigma_\mu^* == \begin{bmatrix} -\dfrac{\partial^2 \mathcal{L}_\mu{}^*}{\partial \mu^2} & -\dfrac{\partial^2 \mathcal{L}_\mu{}^*}{\partial \mu \partial \lambda} \\ -\dfrac{\partial^2 \mathcal{L}_\mu{}^*}{\partial \lambda \partial \mu} & -\dfrac{\partial^2 \mathcal{L}_\mu{}^*}{\partial \lambda^2} \end{bmatrix}^{-1}_{\mu = \widehat{\mu}_{\mathcal{L}^*}, \lambda = \widehat{\lambda}_{\mathcal{L}^*}} . \tag{44}$$

Hence, the approximate Bayes estimator of μ, denoted by $\widehat{\mu}_{TK-GELF}$, is obtained as follows

$$\widehat{\mu}_{TK-GELF} = \left(\left[\frac{\det \Sigma_\mu^*}{\det \Sigma} \right]^{\frac{1}{2}} \exp \left\{ n \left[\mathcal{L}_\mu{}^* \left(\widehat{\mu}_{\mathcal{L}^*}, \widehat{\lambda}_{\mathcal{L}^*} \right) \right. \right. \right.$$

$$\left. \left. \left. - \mathcal{L} \left(\widehat{\mu}_{\mathcal{L}}, \widehat{\lambda}_{\mathcal{L}} \right) \right] \right\} \right)^{-1/k} . \tag{45}$$

Similarly, we derive the approximate Bayes estimator of λ under GELF.

It is noted that the approximate Bayes estimators of μ and λ under SELF, $\widehat{\mu}_{TK-SELF}$ and $\widehat{\lambda}_{TK-SELF}$, respectively, are derived from $\widehat{\mu}_{TK-GELF}$ and $\widehat{\lambda}_{TK-GELF}$ with $k = -1$. Besides, the same procedure in Section 4 is repeated for the vague prior and Jeffrey's prior.

5 Simulation Study

We conduct a comprehensive simulation study to investigate the performances of the classical and Bayesian estimators of the parameters in terms of bias and relative efficiency (RE). In the simulation study, we consider a sample of sizes $n = 25, 50, 100$ to illustrate the effect of small, moderate and large samples on the estimators. The true values of (μ, λ) are taken as $(1, 2)$, $(1, 4), (3, 2)$ and $(3, 4)$. The constant values for the extended Jeffrey's prior are considered as $c = (2, 3)$. The loss parameter for LINEX loss function and GELF is taken as $k = \pm 0.75$.

For 5000 repetitions, the performance of the classical and Bayesian estimators is measured with different criteria such as bias and RE (see Usta (2013)), given as follows:

$$Bias = \frac{1}{5000} \sum_{i=1}^{5000} (\widehat{\theta}_i - \theta),$$

Table 1   Bias values of estimates for μ

	c	$\widehat{\mu}_{MLE}$	$\widehat{\mu}_{MVUE}$	Lindley					Tierney-Kadane				
				$\widehat{\mu}_{BS}$	$\widehat{\mu}_{BL}$	$\widehat{\mu}_{BL}$	$\widehat{\mu}_{BG}$	$\widehat{\mu}_{BG}$	$\widehat{\mu}_{BS}$	$\widehat{\mu}_{BL}$	$\widehat{\mu}_{BL}$	$\widehat{\mu}_{BG}$	$\widehat{\mu}_{BG}$
					k					K			
					−.75	.75	−.75	.75		−.75	.75	−.75	.75
n=25													
$\mu=4,\ \lambda=1$	−	.007	.007	.067	.073	.061	.065	.052	.078	.086	.060	.074	.053
	1	.007	.007	.037	.044	.030	.035	.020	.035	.042	.022	.032	.015
	2	.007	.007	.007	.015	−.001	.005	−.010	.001	.007	−.010	−.001	−.017
	3	.007	.007	−.023	−.015	−.030	−.025	−.038	−.031	−.026	−.040	−.033	−.048
$\mu=2,\ \lambda=1$	−	.004	.004	.034	.037	.030	.033	.026	.038	.042	.031	.036	.027
	1	.004	.004	.019	.023	.015	.018	.010	.019	.022	.013	.017	.009
	2	.004	.004	.004	.008	.000	.003	−.005	.002	.005	−.003	.001	−.008
	3	.004	.004	−.011	−.007	−.015	−.012	−.019	−.015	−.012	−.020	−.017	−.024
$\mu=4,\ \lambda=3$	−	−.002	−.002	.539	.571	.369	.528	.438	.792	.382	.216	.714	.403
	1	−.002	−.002	.268	.399	.043	.249	.115	.215	.319	−.064	.184	.026
	2	−.002	−.002	−.002	.198	−.203	−.025	−.152	−.088	.099	−.276	−.109	−.223
	3	−.002	−.002	−.273	−.047	−.404	−.291	−.377	−.318	−.169	−.458	−.334	−.425
$\mu=2,\ \lambda=3$	−	.014	.014	.281	.336	.186	.273	.216	.345	.502	.148	.324	.216
	1	.014	.014	.147	.231	.041	.137	.071	.134	.283	−.001	.120	.043
	2	.014	.014	.014	.115	−.088	.003	−.062	−.018	.098	−.127	−.030	−.095
	3	.014	.014	−.120	−.013	−.203	−.130	−.184	−.153	−.061	−.245	−.163	−.222
n=50													
$\mu=4,\ \lambda=1$	−	.005	.005	.035	.038	.031	.034	.026	.037	.041	.031	.036	.027
	1	.005	.005	.020	.024	.016	.019	.011	.019	.023	.014	.018	.010
	2	.005	.005	.005	.009	.001	.004	−.004	.003	.006	−.002	.002	−.006
	3	.005	.005	−.010	−.007	−.014	−.012	−.019	−.013	−.009	−.017	−.014	−.021
$\mu=2,\ \lambda=1$	−	.000	.000	.015	.017	.013	.014	.011	.016	.018	.013	.015	.011
	1	.000	.000	.007	.009	.006	.007	.003	.007	.009	.005	.007	.003
	2	.000	.000	.000	.002	−.002	−.001	−.004	−.001	.001	−.003	−.001	−.005
	3	.000	.000	−.007	−.006	−.009	−.008	−.012	−.008	−.007	−.011	−.009	−.013
$\mu=4,\ \lambda=3$	−	−.013	−.013	.257	.313	.163	.249	.191	.303	.474	.121	.285	.184
	1	−.013	−.013	.122	.206	.016	.112	.044	.108	.251	−.024	.094	.018
	2	−.013	−.013	−.013	.089	−.114	−.024	−.090	−.041	.069	−.146	−.052	−.116
	3	−.013	−.013	−.148	−.042	−.232	−.158	−.213	−.168	−.082	−.256	−.178	−.233
$\mu=2,\ \lambda=3$	−	−.008	−.008	.125	.164	.077	.121	.089	.139	.211	.067	.132	.090
	1	−.008	−.008	.059	.105	.008	.053	.020	.056	.115	−.005	.049	.013
	2	−.008	−.008	−.008	.043	−.058	−.013	−.046	−.017	.037	−.071	−.022	−.056
	3	−.008	−.008	−.074	−.023	−.120	−.080	−.110	−.085	−.037	−.134	−.090	−.121

(*Continued*)

Table 1 (*Continued*)

	c	$\widehat{\mu}_{MLE}$	$\widehat{\mu}_{MVUE}$	Lindley $\widehat{\mu}_{BS}$	$\widehat{\mu}_{BL}$	$\widehat{\mu}_{BL}$	$\widehat{\mu}_{BG}$	$\widehat{\mu}_{BG}$	Tierney-Kadane $\widehat{\mu}_{BS}$	$\widehat{\mu}_{BL}$	$\widehat{\mu}_{BL}$	$\widehat{\mu}_{BG}$	$\widehat{\mu}_{BG}$
					k					K			
n=100					−.75	.75	−.75	.75		−.75	.75	−.75	.75
$\lambda=1,\ \mu=4$	−	−.001	−.001	.014	.016	.012	.014	.010	.015	.017	.012	.014	.010
	1	−.001	−.001	.007	.009	.005	.006	.002	.007	.008	.004	.006	.002
	2	−.001	−.001	−.001	.001	−.003	−.002	−.005	−.001	.001	−.004	−.002	−.006
	3	−.001	−.001	−.008	−.007	−.010	−.009	−.013	−.009	−.007	−.011	−.010	−.013
$\lambda=2,\ \mu=4$	−	−.001	−.001	.007	.007	.006	.006	.004	.007	.008	.006	.006	.004
	1	−.001	−.001	.003	.004	.002	.002	.001	.003	.004	.002	.002	.001
	2	−.001	−.001	−.001	.000	−.002	−.001	−.003	−.001	.000	−.002	−.002	−.003
	3	−.001	−.001	−.005	−.004	−.006	−.005	−.007	−.005	−.004	−.006	−.005	−.007
$\lambda=3,\ \mu=2$	−	−.006	−.006	.131	.170	.081	.126	.094	.141	.212	.070	.134	.092
	1	−.006	−.006	.062	.109	.010	.057	.023	.059	.119	−.003	.052	.016
	2	−.006	−.006	−.006	.046	−.057	−.012	−.045	−.014	.039	−.069	−.020	−.053
	3	−.006	−.006	−.074	−.022	−.121	−.080	−.110	−.081	−.034	−.131	−.086	−.118
$\lambda=3,\ \mu=3$	−	.015	.015	.083	.106	.058	.081	.064	.087	.117	.055	.083	.064
	1	.015	.015	.049	.074	.023	.046	.029	.048	.076	.019	.045	.027
	2	.015	.015	.015	.041	−.011	.012	−.005	.012	.039	−.015	.009	−.008
	3	.015	.015	−.019	.007	−.044	−.022	−.039	−.022	.003	−.049	−.025	−.042

$$MSE = \frac{1}{5000}\sum_{i=1}^{5000}\left(\widehat{\theta}_i - \theta\right)^2,$$

$$RE = \frac{MSE(\widehat{\theta})}{MSE(\widehat{\theta}_{MVUE})},$$

where $\widehat{\theta}_i(=\widehat{\mu}_i,\widehat{\lambda}_i)$ is the estimate of $\theta(=\mu,\lambda)$ for the ith simulated sample. All programs and random number generation are accomplished in MATLAB (2019a).

Simulation results are given in Tables 1-4 for all combinations of n,μ,λ,c and k. Tables 1-2 show the biases for the estimates of μ and λ. Tables 3-4 present the REs of the estimates. It is to be noted that, in these tables, $c=$ " $-$ " and $c=1$ stand for the vague and Jeffrey's priors, respectively.

Based on the results in Table 1, MLE and MVUE of μ have the smallest bias values in most of the considered cases as expected, since the classical estimators are unbiased for μ, while Bayes estimators perform well for the large sample size according to bias. Among Bayes estimates of μ, ones

Table 2 Bias values of estimates for λ

	c	$\widehat{\lambda}_{MLE}$	$\widehat{\lambda}_{MVUE}$	$\widehat{\lambda}_{BS}$	$\widehat{\lambda}_{BL}$	$\widehat{\lambda}_{BL}$	$\widehat{\lambda}_{BG}$	$\widehat{\lambda}_{BG}$	$\widehat{\lambda}_{BS}$	$\widehat{\lambda}_{BL}$	$\widehat{\lambda}_{BL}$	$\widehat{\lambda}_{BG}$	$\widehat{\lambda}_{BG}$
						Lindley					**Tierney-Kadane**		
						k					**k**		
					−.75	.75	−.75	.75		−.75	.75	−.75	.75
n=25													
$\mu=1$ $\lambda=2$	−	.255	−.016	.164	.327	.024	.143	.028	.164	.345	.019	.141	.006
	1	.255	−.016	.255	.409	.101	.232	.106	.261	.451	.111	.239	.104
	2	.255	−.016	.164	.327	.024	.143	.028	.170	.353	.027	.148	.013
	3	.255	−.016	.074	.240	−.048	.054	−.044	.072	.247	−.066	.050	−.086
$\mu=1$ $\lambda=4$	−	.477	−.060	.298	.862	−.144	.255	.028	.297	1.135	−.228	.252	−.016
	1	.477	−.060	.477	.985	−.031	.432	.182	.484	1.358	−.063	.439	.171
	2	.477	−.060	.298	.862	−.144	.255	.028	.304	1.144	−.220	.260	−.008
	3	.477	−.060	.119	.726	−.248	.079	−.116	.117	.922	−.385	.073	−.196
$\mu=3$ $\lambda=2$	−	.296	.020	.204	.372	.059	.182	.065	.203	.626	.053	.181	.043
	1	.296	.020	.296	.455	.136	.273	.144	.313	.553	.158	.291	.154
	2	.296	.020	.204	.372	.059	.182	.065	.216	.416	.067	.194	.056
	3	.296	.020	.112	.284	−.014	.091	−.009	.102	.285	−.041	.079	−.060
$\mu=3$ $\lambda=4$	−	.595	.044	.412	1.000	−.048	.367	.134	.410	1.431	−.140	.365	.089
	1	.595	.044	.595	1.125	.066	.549	.292	.615	1.590	.043	.570	.295
	2	.595	.044	.412	1.000	−.048	.367	.134	.428	1.353	−.121	.383	.108
	3	.595	.044	.228	.862	−.153	.186	−.014	.220	1.107	−.306	.175	−.102
n=50													
$\mu=1$ $\lambda=2$	−	.123	−.004	.081	.151	.015	.070	.012	.081	.153	.015	.070	.006
	1	.123	−.004	.123	.192	.055	.113	.051	.125	.199	.058	.114	.051
	2	.123	−.004	.081	.151	.015	.070	.012	.083	.155	.017	.072	.008
	3	.123	−.004	.038	.109	−.023	.028	−.026	.038	.110	−.026	.028	−.036
$\mu=1$ $\lambda=4$	−	.259	.003	.173	.440	−.063	.152	.035	.173	.479	−.081	.152	.024
	1	.259	.003	.259	.512	.005	.237	.114	.260	.573	.001	.239	.111
	2	.259	.003	.173	.440	−.063	.152	.035	.175	.481	−.079	.154	.026
	3	.259	.003	.088	.364	−.127	.068	−.041	.088	.388	−.160	.067	−.061
$\mu=3$ $\lambda=2$	−	.127	−.001	.084	.155	.018	.074	.015	.084	.168	.018	.073	.010
	1	.127	−.001	.127	.195	.058	.116	.054	.132	.208	.064	.121	.058
	2	.127	−.001	.084	.155	.018	.074	.015	.089	.162	.022	.078	.014
	3	.127	−.001	.042	.112	−.020	.031	−.023	.040	.112	−.024	.030	−.034
$\mu=3$ $\lambda=4$	−	.278	.022	.193	.462	−.046	.172	.054	.192	.503	−.065	.171	.043
	1	.278	.022	.278	.534	.022	.257	.133	.284	.601	.022	.262	.134
	2	.278	.022	.193	.462	−.046	.172	.054	.198	.509	−.059	.177	.048
	3	.278	.022	.107	.387	−.110	.087	−.023	.107	.412	−.145	.086	−.043

(*Continued*)

Table 2   (*Continued*)

	c	$\hat{\lambda}_{MLE}$	$\hat{\lambda}_{MVUE}$	$\hat{\lambda}_{BS}$	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{BG}$	$\hat{\lambda}_{BG}$	$\hat{\lambda}_{BS}$	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{BG}$	$\hat{\lambda}_{BG}$
				Lindley					**Tierney-Kadane**				
						k					**k**		
n=100				−.75	.75	−.75	.75		−.75	.75	−.75	.75	
$\lambda=2$ $\mu=1$	−	.052	−.010	.031	.063	.000	.026	−.004	.031	.064	.000	.026	−.005
	1	.052	−.010	.052	.083	.020	.046	.016	.052	.085	.020	.047	.016
	2	.052	−.010	.031	.063	.000	.026	−.004	.032	.064	.000	.026	−.004
	3	.052	−.010	.011	.043	−.020	.006	−.023	.011	.043	−.020	.005	−.025
$\lambda=4$ $\mu=1$	−	.083	−.039	.042	.167	−.075	.032	−.026	.042	.175	−.079	.032	−.029
	1	.083	−.039	.083	.205	−.038	.073	.013	.084	.217	−.039	.074	.012
	2	.083	−.039	.042	.167	−.075	.032	−.026	.043	.175	−.078	.033	−.029
	3	.083	−.039	.002	.128	−.111	−.008	−.065	.002	.133	−.118	−.009	−.070
$\lambda=2$ $\mu=3$	−	.043	−.019	.022	.054	−.009	.017	−.012	.022	.054	−.009	.017	−.014
	1	.043	−.019	.043	.074	.011	.037	.007	.044	.077	.013	.039	.008
	2	.043	−.019	.022	.054	−.009	.017	−.012	.023	.056	−.008	.018	−.012
	3	.043	−.019	.002	.034	−.028	−.003	−.032	.002	.034	−.029	−.004	−.034
$\lambda=4$ $\mu=3$	−	.097	−.026	.056	.182	−.062	.046	−.013	.056	.189	−.066	.046	−.016
	1	.097	−.026	.097	.220	−.025	.087	.027	.099	.233	−.025	.089	.027
	2	.097	−.026	.056	.182	−.062	.046	−.013	.058	.191	−.064	.048	−.014
	3	.097	−.026	.015	.143	−.098	.005	−.052	.016	.147	−.105	.005	−.056

obtained under the proposed prior with $c = 2$ perform better than the other Bayes estimates in terms of bias. Furthermore, the Bayes estimators under asymmetric loss functions show a good performance to estimate μ in most cases especially when the loss parameter k is 0.75. We also observed that, in general, Bayes estimates of μ obtained using Lindley method have smaller bias as compared to TK method.

The results from Table 2 show that the MVUEs of the shape parameter λ outperforms the MLEs and Bayes estimators in terms of bias for $n = 25, 50$. However, for $n = 100$, the Bayes estimators have the smallest biases in most of the cases. On the other hand, once the loss parameter $k = 0.75$, indicating overestimation, the Bayes estimators of λ under LINEX loss function and GELF again using the proposed prior with $c = 2$ provide smaller biases than the other Bayes estimators. Moreover, the approximate Bayes estimators obtained using Lindley method is preferable to the estimators based on TK method.

We observe from Table 3 that as the sample size n increases, the MSEs of all estimates of μ decrease. The results for μ also show that all Bayes

Table 3 The relative efficiency of estimators for μ

				Lindley					Tierney-Kadane				
	c	$\hat\mu_{MVUE}$	$\hat\mu_{MLE}$	$\hat\mu_{BS}$	$\hat\mu_{BL}$	$\hat\mu_{BL}$	$\hat\mu_{BG}$	$\hat\mu_{BG}$	$\hat\mu_{BS}$	$\hat\mu_{BL}$	$\hat\mu_{BL}$	$\hat\mu_{BG}$	$\hat\mu_{BG}$
					k					k			
n=25					−.75	.75	−.75	.75		−.75	.75	−.75	.75
$\lambda=2$, $\lambda=1$, $\mu=$	−	1.00	1.00	1.483	1.562	1.394	1.463	1.320	1.635	1.773	1.374	1.581	1.330
	1	1.00	1.00	1.187	1.266	1.113	1.172	1.069	1.167	1.241	1.054	1.143	1.034
	2	1.00	1.00	1.000	1.054	0.951	0.990	0.936	0.970	1.005	0.916	0.961	0.921
	3	1.00	1.00	0.911	0.941	0.897	0.911	0.911	0.906	0.921	0.901	0.906	0.926
$\lambda=4$, $\lambda=1$, $\mu=$	−	1.00	1.00	1.245	1.294	1.196	1.235	1.157	1.294	1.363	1.206	1.275	1.167
	1	1.00	1.00	1.098	1.137	1.069	1.088	1.039	1.098	1.137	1.049	1.088	1.029
	2	1.00	1.00	1.000	1.029	0.980	1.000	0.971	0.990	1.020	0.961	0.990	0.961
	3	1.00	1.00	0.961	0.971	0.941	0.951	0.951	0.951	0.961	0.941	0.951	0.961
$\lambda=2$, $\lambda=3$, $\mu=$	−	1.00	1.00	2.446	2.425	1.677	2.421	2.146	4.167	1.507	1.159	3.547	1.941
	1	1.00	1.00	1.542	1.935	0.902	1.493	1.194	1.366	1.533	0.802	1.292	1.007
	2	1.00	1.00	1.000	1.484	0.749	0.971	0.860	0.887	1.327	0.767	0.872	0.832
	3	1.00	1.00	0.822	1.116	0.847	0.824	0.866	0.846	1.006	0.897	0.852	0.912
$\lambda=4$, $\lambda=3$, $\mu=$	−	1.00	1.00	1.691	1.893	1.301	1.664	1.478	2.000	2.953	1.153	1.899	1.465
	1	1.00	1.00	1.269	1.548	0.966	1.243	1.096	1.224	1.960	0.902	1.189	1.032
	2	1.00	1.00	1.000	1.256	0.843	0.984	0.916	0.949	1.268	0.833	0.937	0.892
	3	1.00	1.00	0.884	1.041	0.859	0.882	0.888	0.883	0.989	0.888	0.885	0.910
n=50													
$\lambda=2$, $\lambda=1$, $\mu=$	−	1.00	1.00	1.240	1.279	1.192	1.231	1.154	1.269	1.327	1.192	1.250	1.154
	1	1.00	1.00	1.096	1.135	1.058	1.087	1.038	1.096	1.125	1.048	1.077	1.029
	2	1.00	1.00	1.000	1.029	0.981	1.000	0.971	0.990	1.019	0.962	0.990	0.962
	3	1.00	1.00	0.952	0.971	0.942	0.952	0.952	0.952	0.962	0.942	0.952	0.952
$\lambda=4$, $\lambda=1$, $\mu=$	−	1.00	1.00	1.106	1.128	1.085	1.106	1.064	1.128	1.149	1.085	1.106	1.064
	1	1.00	1.00	1.043	1.064	1.021	1.043	1.000	1.043	1.064	1.021	1.043	1.000
	2	1.00	1.00	1.000	1.000	0.979	1.000	0.979	1.000	1.000	0.979	1.000	0.979
	3	1.00	1.00	0.979	0.979	0.979	0.979	0.979	0.979	0.979	0.979	0.979	0.979
$\lambda=2$, $\lambda=3$, $\mu=$	−	1.00	1.00	1.686	1.895	1.288	1.658	1.468	1.918	2.950	1.136	1.828	1.430
	1	1.00	1.00	1.258	1.538	0.961	1.232	1.089	1.210	1.877	0.905	1.177	1.030
	2	1.00	1.00	1.000	1.247	0.861	0.986	0.931	0.957	1.259	0.862	0.947	0.914
	3	1.00	1.00	0.913	1.048	0.915	0.913	0.938	0.917	1.004	0.942	0.921	0.957
$\lambda=4$, $\lambda=3$, $\mu=$	−	1.00	1.00	1.302	1.455	1.124	1.287	1.195	1.355	1.753	1.086	1.328	1.194
	1	1.00	1.00	1.117	1.264	0.985	1.105	1.041	1.106	1.330	0.963	1.093	1.025
	2	1.00	1.00	1.000	1.116	0.927	0.994	0.965	0.987	1.112	0.921	0.980	0.957
	3	1.00	1.00	0.953	1.020	0.937	0.952	0.958	0.952	1.006	0.947	0.952	0.964

(*Continued*)

Table 3 (*Continued*)

	c	$\hat{\mu}_{MVUE}$	$\hat{\mu}_{MLE}$	Lindley $\hat{\mu}_{BS}$	$\hat{\mu}_{BL}$	$\hat{\mu}_{BL}$	$\hat{\mu}_{BG}$	$\hat{\mu}_{BG}$	Tierney-Kadane $\hat{\mu}_{BS}$	$\hat{\mu}_{BL}$	$\hat{\mu}_{BL}$	$\hat{\mu}_{BG}$	$\hat{\mu}_{BG}$
						k					k		
n=100					−.75	.75	−.75	.75		−.75	.75	−.75	.75
$\lambda=1, \mu=1$	−	1.00	1.00	1.104	1.125	1.083	1.104	1.063	1.104	1.146	1.083	1.104	1.063
	1	1.00	1.00	1.042	1.063	1.021	1.042	1.021	1.042	1.063	1.021	1.042	1.021
	2	1.00	1.00	1.000	1.021	1.000	1.000	0.979	1.000	1.000	0.979	1.000	0.979
	3	1.00	1.00	0.979	0.979	0.979	0.979	0.979	0.979	0.979	0.979	0.979	0.979
$\lambda=1, \mu=4$	−	1.00	1.00	1.042	1.083	1.042	1.042	1.042	1.042	1.083	1.042	1.042	1.042
	1	1.00	1.00	1.042	1.042	1.000	1.000	1.000	1.042	1.042	1.000	1.000	1.000
	2	1.00	1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	3	1.00	1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\lambda=3, \mu=2$	−	1.00	1.00	1.328	1.495	1.137	1.311	1.210	1.372	1.768	1.096	1.344	1.203
	1	1.00	1.00	1.124	1.283	0.986	1.112	1.043	1.114	1.345	0.964	1.098	1.026
	2	1.00	1.00	1.000	1.121	0.926	0.993	0.963	0.986	1.114	0.921	0.980	0.956
	3	1.00	1.00	0.952	1.017	0.942	0.952	0.961	0.951	1.003	0.949	0.951	0.964
$\lambda=3, \mu=4$	−	1.00	1.00	1.202	1.317	1.091	1.190	1.128	1.216	1.389	1.078	1.203	1.128
	1	1.00	1.00	1.081	1.179	0.997	1.073	1.029	1.078	1.195	0.989	1.068	1.023
	2	1.00	1.00	1.000	1.075	0.948	0.995	0.971	0.993	1.072	0.943	0.989	0.966
	3	1.00	1.00	0.958	1.003	0.938	0.956	0.953	0.956	0.997	0.940	0.954	0.953

estimators under the considered loss functions, except LINEX loss function with $k = -0.75$, using the proposed prior outperform the others for all the sample sizes. In particular, Bayes estimators under LINEX loss function with $k = 0.75$ using the proposed prior with $c = 3$ are generally the best in terms of RE. This is followed by the proposed prior with $c = 2$. Furthermore, Bayes estimates obtained through TK method compete quite well with those obtained through Lindley method.

The results from Table 4 show that for each estimator, the MSEs decrease when the sample size increases. It is observed that MVUE performs better than MLE in terms of RE. However, considering the REs, Bayes estimators under LINEX loss function with $k = 0.75$ present the best result for all the sample sizes. Among Bayes estimators of λ, the estimates under the proposed prior, especially with $c = 3$, have generally lower RE. We further observed that Bayes estimates computed by TK approximation method, in general, provides smaller REs than Lindley's approximation for estimating the shape parameter.

Table 4   The relative efficiency of estimators for λ

	C	$\hat{\lambda}_{MVUE}$	$\hat{\lambda}_{MLE}$	$\hat{\lambda}_{BS}$	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{BG}$	$\hat{\lambda}_{BG}$	$\hat{\lambda}_{BS}$	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{BG}$	$\hat{\lambda}_{BG}$
						Lindley					**Tierney-Kadane**		
						k					**k**		
n=25					$-.75$	$.75$	$-.75$	$.75$		$-.75$	$.75$	$-.75$	$.75$
$\mu=1\lambda=2$	−	1.00	1.460	1.260	1.862	0.919	1.218	1.047	1.259	2.013	0.889	1.217	1.022
	1	1.00	1.460	1.460	2.096	0.996	1.406	1.155	1.470	2.379	0.997	1.415	1.145
	2	1.00	1.460	1.260	1.862	0.919	1.218	1.047	1.266	2.029	0.891	1.223	1.024
	3	1.00	1.460	1.106	1.647	0.878	1.079	0.976	1.106	1.724	0.828	1.076	0.953
$\mu=1\lambda=4$	−	1.00	1.430	1.243	2.253	0.852	1.204	1.043	1.241	3.639	0.716	1.202	1.020
	1	1.00	1.430	1.430	2.417	0.854	1.379	1.145	1.433	4.225	0.744	1.383	1.132
	2	1.00	1.430	1.243	2.253	0.852	1.204	1.043	1.245	3.652	0.714	1.205	1.021
	3	1.00	1.430	1.099	2.090	0.866	1.073	0.978	1.099	3.135	0.721	1.070	0.955
$\mu=3\lambda=2$	−	1.00	1.501	1.289	1.937	0.917	1.245	1.054	1.288	2.960	0.876	1.243	1.026
	1	1.00	1.501	1.501	2.174	0.999	1.444	1.176	1.532	2.669	1.000	1.473	1.177
	2	1.00	1.501	1.289	1.937	0.917	1.245	1.054	1.309	2.216	0.887	1.262	1.037
	3	1.00	1.501	1.122	1.716	0.868	1.091	0.971	1.120	1.841	0.806	1.086	0.943
$\mu=3\lambda=4$	−	1.00	1.503	1.290	2.393	0.845	1.246	1.055	1.289	6.087	0.680	1.244	1.026
	1	1.00	1.503	1.503	2.573	0.858	1.446	1.177	1.518	6.678	0.728	1.460	1.169
	2	1.00	1.503	1.290	2.393	0.845	1.246	1.055	1.301	5.853	0.680	1.255	1.031
	3	1.00	1.503	1.123	2.214	0.847	1.091	0.971	1.122	5.101	0.672	1.088	0.940
n=50													
$\mu=1\lambda=2$	−	1.00	1.219	1.124	1.371	0.958	1.104	1.016	1.124	1.388	0.954	1.104	1.010
	1	1.00	1.219	1.219	1.493	0.997	1.193	1.071	1.222	1.534	0.995	1.195	1.068
	2	1.00	1.219	1.124	1.371	0.958	1.104	1.016	1.126	1.392	0.954	1.106	1.011
	3	1.00	1.219	1.051	1.265	0.928	1.037	0.981	1.051	1.272	0.918	1.036	0.976
$\mu=1\lambda=4$	−	1.00	1.238	1.135	1.669	0.887	1.113	1.018	1.134	1.818	0.856	1.112	1.012
	1	1.00	1.238	1.238	1.805	0.901	1.210	1.077	1.239	2.034	0.881	1.211	1.074
	2	1.00	1.238	1.135	1.669	0.887	1.113	1.018	1.135	1.821	0.855	1.113	1.012
	3	1.00	1.238	1.055	1.543	0.889	1.040	0.981	1.055	1.634	0.852	1.039	0.974
$\mu=3\lambda=2$	−	1.00	1.222	1.127	1.374	0.960	1.107	1.017	1.127	1.400	0.955	1.106	1.011
	1	1.00	1.222	1.222	1.496	0.999	1.196	1.073	1.230	1.544	0.994	1.204	1.074
	2	1.00	1.222	1.127	1.374	0.960	1.107	1.017	1.133	1.401	0.958	1.112	1.013
	3	1.00	1.222	1.053	1.267	0.929	1.038	0.981	1.053	1.274	0.920	1.039	0.976
$\mu=3\lambda=4$	−	1.00	1.239	1.138	1.666	0.880	1.116	1.019	1.138	1.824	0.845	1.116	1.012
	1	1.00	1.239	1.239	1.792	0.897	1.212	1.080	1.244	2.036	0.874	1.216	1.079
	2	1.00	1.239	1.138	1.666	0.880	1.116	1.019	1.142	1.833	0.844	1.119	1.014
	3	1.00	1.239	1.058	1.546	0.877	1.043	0.978	1.059	1.648	0.835	1.042	0.971

(*Continued*)

Table 4   (*Continued*)

	C	$\hat{\lambda}_{MVUE}$	$\hat{\lambda}_{MLE}$	$\hat{\lambda}_{BS}$	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{BG}$	$\hat{\lambda}_{BG}$	$\hat{\lambda}_{BS}$	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{BL}$	$\hat{\lambda}_{BG}$	$\hat{\lambda}_{BG}$
						Lindley					**Tierney-Kadane**		
						k					**k**		
n=100					−.75	.75	−.75	.75		−.75	.75	−.75	.75
λ=1, μ=2	−	1.00	1.093	1.053	1.158	0.977	1.044	1.006	1.053	1.160	0.976	1.043	1.005
	1	1.00	1.093	1.093	1.214	1.000	1.082	1.029	1.094	1.221	1.000	1.082	1.029
	2	1.00	1.093	1.053	1.158	0.977	1.044	1.006	1.053	1.161	0.976	1.044	1.005
	3	1.00	1.093	1.022	1.111	0.964	1.016	0.993	1.022	1.112	0.962	1.016	0.992
λ=1, μ=4	−	1.00	1.078	1.042	1.249	0.941	1.035	1.004	1.042	1.273	0.934	1.035	1.003
	1	1.00	1.000	1.000	1.212	0.875	0.990	0.948	1.000	1.249	0.871	0.991	0.947
	2	1.00	1.000	0.966	1.158	0.873	0.959	0.931	0.966	1.181	0.866	0.959	0.930
	3	1.00	1.000	0.942	1.110	0.879	0.938	0.923	0.942	1.124	0.870	0.937	0.922
λ=3, μ=2	−	1.00	1.080	1.043	1.139	0.975	1.036	1.004	1.043	1.142	0.975	1.036	1.004
	1	1.00	1.080	1.080	1.190	0.994	1.069	1.023	1.081	1.199	0.995	1.071	1.023
	2	1.00	1.080	1.043	1.139	0.975	1.036	1.004	1.044	1.144	0.975	1.036	1.002
	3	1.00	1.080	1.017	1.098	0.967	1.012	0.995	1.017	1.099	0.964	1.012	0.994
λ=3, μ=4	−	1.00	1.088	1.049	1.266	0.938	1.041	1.005	1.049	1.291	0.931	1.041	1.004
	1	1.00	1.088	1.088	1.327	0.943	1.077	1.026	1.089	1.369	0.938	1.078	1.026
	2	1.00	1.088	1.049	1.266	0.938	1.041	1.005	1.049	1.293	0.930	1.041	1.004
	3	1.00	1.088	1.019	1.212	0.941	1.014	0.993	1.019	1.227	0.932	1.014	0.992

6 An Application

In this section, a real data set is analyzed to verify how the considered estimators perform in a real-life context. The data set is given in Table 6 taken from Chhikara and Folks (1989). It was also studied by Sinha (1986), and Pandey and Bandyopadhyay (2012).

The estimates of the parameters and obtained by using the MVUE, MLE and Bayes estimators under the vague prior and the proposed extension of Jeffrey's prior with c=3 are reported in Table 6. The results of the Kolmogorov–Smirnov (K-S) test are also given in Table 6.

We observe from Table 6 that results obtained from the real data set are compatible with the simulation results.

Table 5   Active repair times for an airborne communication

0.2, 0.3. 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

Table 6   Estimation of parameters and K-S statistics for the real data set

		k	$\widehat{\mu}$	$\widehat{\lambda}$	$K-S$	p
	MVUE		3.6065	1,5507	0,0743	0,9451
	MLE		3.065	1,6589	0,0682	0,9731
Vague Prior	SELF		4.2423	1,6226	0,0804	0,9045
	LINEX	−0,75	4.3174	1,6683	0,1115	0,8777
	LINEX	0,75	3.8057	1,5804	0,0686	0,9717
	GELF	−0,75	4.1984	1,6137	0,0782	0,9206
	GELF	0,75	3.9784	1,5596	0,0690	0,9700
Ext. Jeffrey Prior c=3	SELF		3.3151	1,5839	0,0776	0,9242
	LINEX	−0,75	3.1538	1,5428	0,0898	0,8205
	LINEX	0,75	3.4877	1,6288	0,0658	0,9809
	GELF	−0,75	3.2060	1,5208	0,0875	0,8427
	GELF	0,75	3.2986	1,5749	0,0792	0,9130

7 Conclusions

In this paper, we consider Bayesian estimation of the two-parameter inverse Gaussian distribution under symmetric (squared error) and asymmetric (linear exponential and general entropy) loss functions. In Bayesian approach, we consider commonly used non-informative priors: the vague and Jeffrey's priors, as well as propose using the extended Jeffrey's prior.

However, Bayes estimators cannot be expressed analytically, because the ratio of the integrals in the posterior expectations is not in closed forms. For this reason, we employ Lindley and TK approximations to compute the approximate Bayes estimators of parameters μ and λ. The comparison between Bayesian estimates and the corresponding classical estimates (MLE and MVUE) is carried out based on biases and REs with a simulation study.

The results of the simulation study show that the Bayes estimators of parameters under LINEX with $k = 0.75$ using the proposed prior generally outperform others according to REs. while the IG parameters are best estimated by Bayes estimators under LINEX with $k = 0.75$ using the proposed extension of Jeffrey's prior according to REs. We also observed that Bayes estimates using the proposed prior with $c = 3$ are superior to the other considered priors in terms of RE. On the other hand, TK method competes quite well with Lindley's method to obtain the approximate Bayes estimates of parameters. Based on all we recommend to use Bayes estimators obtained by TK method under LINEX loss function ($k = 0.75$) using the proposed extension of Jeffrey's prior ($c = 3$) for estimating the parameters of IG distribution.

Acknowledgements

References

Sandhya, K. and Umamaheswari, T. S. (2013). Reliability of a multi-component stress strength model with standby system using mixture of two Exponential distributions, Journal of Reliability and Statistical Studies, 6(2), pp. 105–113.

Ahmad, K. E. and Jaheen, Z. F. (1995). Approximate Bayes estimators applied to the inverse Gaussian lifetime model, Computers and Mathematics with Applications, 29(12), pp. 39–47.

Al-Kutubi, H. S. and Ibrahim, N. A. (2009). Bayes estimator for exponential distribution with extension of Jeffery prior information, Malaysian Journal of Mathematical Sciences, 3(2), pp. 297–313.

Banerjee, A. K. and Bhattacharyya, G. K. (1979). Bayesian results for the inverse Gaussian distribution with an application, Technometrics, 21(2), pp. 247–251.

Basak, P. and Balakrishnan, N. (2012). Estimation for the three-parameter inverse Gaussian distribution under progressive Type-II censoring, Journal of the Statistics Computation and Simulation, 82 (7), pp. 1055–1072.

Calabria, R. and Pulcini, G. (1994). Bayes 2-sample prediction for the inverse Weibull distribution, Communication in Statistics Theory and Methods, 23(6), pp. 1811–1824.

Chhikara, R. S. and Folks, J. L. (1975). Statistical distributions related to the inverse Gaussian, Communication in Statistics, 4(12), pp. 1081–1091.

Chhikara, R. S. and Folks, J. L. (1989). The Inverse Gaussian Distribution. New York, Marcel Dekker.

Jia, J., Yan, Z. and Peng X. (2017). Estimation for inverse Gaussian distribution under first-failure progressive hybrid censored samples, Filomat, 31(18), 5743–5752.

Iyengar, S. and Patwardhan, G. (1988). Recent developments in the inverse Gaussian distribution, (In Handbook of Statistics, Vol. 7 edited by Krishnaiah P. R. and Rao C. R.), 479–490.

Jeffreys H. (1961). Theory of Probability. 3 Ed., Oxford, Clarendon Press.

Johnson, N. L., Kotz, S. and Balakrishnan, N. (1994). Continuous Univariate Distributions, Vol. II, John Wiley & Sons.

Lemeshko, B. Y., Lemeshko, S. B. and Akushkina, K. A. (2010). Inverse Gaussian Model and Its Applications in Reliability and Survival Analysis, (In. Mathematical and Statistical Models and Methods in Reliability Statistics for Industry and Technology edited by Rykov, V. V., Balakrishnan, N. and Nikulin, M. S.), Basel, Birkhäuser.

Lindley, D. V. (1980). Approximate Bayes method, Trabajos de Estadistica Y de Investigacion Operativa, 31(1), pp. 223–237.

Pandey, B. N. and Bandyopadhyay P. (2012). Bayesian estimation of inverse Gaussian distribution, Located at: arXiv:1210.4524v1.

Rostamian, S. and Nematollahi N. (2019). Estimation of stress–strength reliability in the inverse Gaussian distribution under progressively type II censored data, Mathematical Sciences, 13, pp. 175–191.

Seshadri, V. (1999). The Inverse Gaussian Distribution: Statistical Theory and Applications, Berlin, Springer - Verlag.

Singh, P. K., Singh, S. K. and Singh, U. (2008). Bayes estimator of inverse Gaussian parameters under general entropy loss function using Lindley's approximation, Communications in Statistics - Simulation and Computation, 37(9), pp. 1750–1762.

Sinha, S. K. (1986). Bayesian estimation of the reliability function of the inverse Gaussian distribution, Statistics & Probability Letters, 4(6), pp. 319–323.

Tierney, L. and Kadane, J. B. (1986). Accurate approximations for posterior moments and marginal densities, Journal of the American Statistical Association. 81(393), pp. 82–86.

Tweedie, M. C. K. (1957). Statistical properties of inverse Gaussian distribution, The Annals of Mathematical Statistics, 28, pp. 362–377.

Usta, I. (2013). Different estimation methods for the parameters of the extended Burr XII distribution, Journal of Applied Statistics, 40(2), pp. 397–414.

Varian, H. R. (1975). A Bayesian Approach to Real Estate Assessment (In Savage LJ. Feinberg SE. Zellner A. editors. Studies in Bayesian Econometrics and Statistics: In Honor of L. J. Savage. Amsterdam: North-Holland Pub. Co), pp. 195–208.

Zellner, A. (1986). Bayesian estimation and prediction using asymmetric loss functions, Journal of the American Statistical Association, 81(394), pp. 446–451.

Biographies

Ilhan Usta received his B.Sc. and B.Sc. (double major) degrees in Statistics and Mathematics from Anadolu University, Turkey in 2003 and 2004, respectively; M.Sc. and Ph.D. degrees in Statistics from Anadolu University, Institute of Science, Turkey in 2006 and 2009, respectively. He still works as a Professor at Eskisehir Technical University, Department of Statistics. His major areas of interest are theory of statistics, censored data and parameter estimation, Bayesian estimation, wind energy, entropy optimization distribution and portfolio theory.

Merve Akdede received her B.Sc. degree in Statistics from Dokuz Eylul University, Turkey in 2008; M.Sc. degree in Statistics from Texas A&M University, USA in 2013. Although she has been a Ph.D. student at Eskisehir Technical University since 2015, she died suddenly at an early age in 2019. We will continue to keep her memory alive with this study and do good by her until the day we meet again.