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# Improved Hybrid Exponentially Weighted Moving Average Control Chart using Auxiliary Information

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Received 09 September 2019; Accepted 31 January 2020;  
Publication 15 October 2020

## Abstract

Statistical process control is an important tool for maintaining the quality of a production process. Several control charts are available to monitor changes in process parameters. In this study, a control chart for the process mean is proposed. For this purpose, an auxiliary variable is used in the form of a regression estimator under the configuration of the hybrid exponentially weighted moving average (HEWMA) control chart. The proposed chart is evaluated by conducting a simulation study. The results showed that the proposed chart is sensitive with respect to the HEWMA chart. A real-life application is also presented to demonstrate the performance of the proposed control chart.

**Keywords:** ARL, auxiliary variable, process control.

**Mathematics Subject Classification:** 62P30

*Journal of Reliability and Statistical Studies, Vol. 13, Issue 1 (2020), 113–126.*

doi: 10.13052/jrss0974-8024.1316

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## 1 Introduction

An important statistical tool for monitoring processes is the control chart. In the industry, product quality can be improved by removing the special cause of variation. A process is said to be statistically controlled if the cause of the change is due to natural or common causes. Montgomery & Runger (2007) showed that statistical process control (SPC) addresses specific causes of variability. Control charts are divided into two categories. One is a memory-less control chart like Shewhart control charts, and the other includes EWMA chart and CUSUM chart.

Control chart performance is evaluated based on average run length (ARL). The main goal when creating a new control chart is to detect changes as soon as possible. (Haq, 2013) proposed a new chart called the hybrid exponentially weighted moving average (HEWMA) chart. The author used two EWMA statistics to improve the sensitivity of the control chart. Auxiliary information can play an important role in improving the sensitivity of control charts. (Abbas, Riaz, & Does, 2014), Riaz, (2008) and Noor-ul-Amin, Khan, & Sanaullah (2019) used auxiliary information in control charts.

Several authors have proposed ratio and regression estimators that incorporate auxiliary information ((see Yasmeen, Noor-ul-Amin, & Hanif (2019) and Noor-ul-Amin and Hanif (2012)). The main purpose of this study is to improve the control chart by using HEWMA statistic and auxiliary variable. To do this, we propose a control chart called  $M_xHEWMA$  control chart using a regression estimator.

## 2 Proposed $M_xHEWMA$ Control Chart

We suppose that  $W_i$  is a supporting variable correlated with the variable of interest say  $X_i$  and that the correlation between  $X_i$  and  $W_i$  is denoted by  $\rho_{XW}$ . This is assumed that  $(X, W) \sim N_2(\mu_x, \mu_w, \sigma_x^2, \sigma_w^2, \rho_{XW})$  where  $N_2$  denotes the bivariate normal distribution. The auxiliary variable is incorporated in the form of regression estimator, *i.e.*

$$M_x = \bar{X} + b_{XW}(\mu_w - \bar{W}), \quad (1)$$

where,  $b_{xw} = \rho_{xw} \left( \frac{\sigma_x}{\sigma_w} \right)$ . For statistic  $M_x$  the mean and variance are given by

$$E(M_x) = \mu_x, V(M_x) = \sigma_M^2 = \frac{\sigma_x^2}{n} (1 - \rho_{XW}^2). \quad (2)$$

By using (1), the plotting statistic for the proposed control chart is defined as

$$M_x H E_t = \lambda_1 E_t + (1 - \lambda_1) H E_{t-1}, \tag{3}$$

here,

$$H E_0 = E_0 = \mu_x, \quad \text{and} \quad E_t = \lambda_2 M_x + (1 - \lambda_2) E_{t-1} \tag{4}$$

where two smoothing constants are  $\lambda_1$  and  $\lambda_2$  such that  $0 < \lambda_1, \lambda_2 \leq 1$ . The respective expected value, variance, and control limits of  $M_x H E W M A$  control chart are given by

$$E (M_x H E_t) = \mu_x ,$$

$$V(M_x H E_t) = \frac{(\lambda_2 \lambda_1)^2}{(\lambda_2 - \lambda_1)^2} \left[ \frac{(1-\lambda_1)^2(1-(1-\lambda_1)^{2t})}{1-(1-\lambda_1)^2} + \frac{(1-\lambda_2)^2(1-(1-\lambda_2)^{2t})}{1-(1-\lambda_2)^2} - \frac{2(1-\lambda_1)(1-\lambda_2)\{1-1-(1-\lambda_2)^t(1-\lambda_1)^t\}}{1-(1-\lambda_1)(1-\lambda_2)} \right] \sigma_M^2 \tag{5}$$

and

$$LCL_t = \mu_x - L \sqrt{\frac{(\lambda_2 \lambda_1)^2}{(\lambda_2 - \lambda_1)^2} \left[ \frac{(1-\lambda_1)^2(1-(1-\lambda_1)^{2t})}{1-(1-\lambda_1)^2} + \frac{(1-\lambda_2)^2(1-(1-\lambda_2)^{2t})}{1-(1-\lambda_2)^2} - \frac{2(1-\lambda_1)(1-\lambda_2)\{1-1-(1-\lambda_2)^t(1-\lambda_1)^t\}}{1-(1-\lambda_1)(1-\lambda_2)} \right]} \sigma_M \tag{6}$$

$$CL_t = \mu_x$$

$$UCL_t = \mu_x + L \sqrt{\frac{(\lambda_2 \lambda_1)^2}{(\lambda_2 - \lambda_1)^2} \left[ \frac{(1-\lambda_1)^2(1-(1-\lambda_1)^{2t})}{1-(1-\lambda_1)^2} + \frac{(1-\lambda_2)^2(1-(1-\lambda_2)^{2t})}{1-(1-\lambda_2)^2} - \frac{2(1-\lambda_1)(1-\lambda_2)\{1-1-(1-\lambda_2)^t(1-\lambda_1)^t\}}{1-(1-\lambda_1)(1-\lambda_2)} \right]} \sigma_M \tag{7}$$

The  $L$  is a control constant, which is determined that  $ARL_0$  of the  $M_x H E W M A$  control chart will reach the pre-described level. Here,  $ARL_0$  is fixed at 500. The  $H E W M A$  statistic using the auxiliary information is called  $M_x H E W M A$ . We define  $\delta$  as the shift in the process mean.

The ARLs are calculated through the simulations by using the following steps:

- i. A random sample of size one is selected from a bivariate normal distribution having the specified parameters, that is,  $(X, W) \sim N_2(0, 0, 1, 1, \rho_{xw})$  and  $M_x$  is computed.
- ii. Setting up the upper and lower limits for the different combinations of  $L$  and  $\lambda_2$ , where  $\lambda_1$  is fixed as 0.1 (for detail see Shabbir & Awan (2016)).
- iii. For comparison purposes with other control charts we fixed  $ARL_0 = 500$ , the control coefficient  $L$  has been determined to set in-control  $ARL$  by simulations, the decision of  $L$  relies upon  $n$  and  $\rho_{xw}$ .
- iv. We assume the value of the population correlation coefficient  $\rho_{xw}$  to be known in advance.
- v. In Tables A1–A5,  $\lambda_1 = 0.1$  is fixed.
- vi. Follow the steps (1)–(2), the number of subgroups before the first point plotted beyond the limits is recorded as run length.
- vii. The process is repeated 100,000 times to obtain the  $ARL$  and the standard deviation of the run length (SDRL).

The basic discoveries about  $M_xHEWMA$  control chart from Tables A1–A5 are discussed below:

- I Auxiliary information improves the sensitivity of the proposed control chart. As it can be seen from Tables A1–A5, a decrease in  $ARL$  is observed with an increase in the value of the correlation coefficient. We observed that the efficiency of the proposed chart is almost equal to the HEWMA chart for small value of  $\rho_{xw} = 0.05$ .
- II In all the tables, the proposed control chart  $ARL$  is fair (unbiased). In other words,  $ARL_0$  has never decreased from  $ARL_1$  for any value of  $\delta$ .
- III It is observed in Tables 1–2 and in Figures A1–A3 (listed in the Appendix) that the proposed control chart has gained the efficiency as compared to the HEWMA chart and  $M_xEt$  (exponentially weighted moving average using auxiliary variable) even for a small value of correlation coefficient.

### 3 Comparison

This section presents the comparisons between the proposed  $M_xHEWMA$  chart and the HEWMA control chart and the  $M_xEt$  control chart. The comparative results of the  $M_xHEWMA$  chart with the HEWMA are presented in Table 1 in the form of ARLs. Table 1 showed that the proposed chart is

**Table 1** Comparison of ARL of  $M_XHEWMA$  chart with Haq (2013) when  $\lambda_1 = 0.10$  and  $\rho_{xw} = 0.25, 0.5$

$\delta$	$\lambda_2$								
	0.1			0.25			0.5		
	Haq	$\rho_{xw}$		Haq	$\rho_{xw}$		Haq	$\rho_{xw}$	
		0.25	0.5		0.25	0.5		0.25	0.5
0	500.2	493.2	496.0	504.6	508.3	509.3	502.4	508.5	508.5
0.1	286.7	305.6	275.3	304.4	268.9	244.8	310.4	314.9	218.3
0.25	86.2	95.8	78.4	96.2	75.7	61.2	100.8	94.7	78.2
0.5	28.0	26.2	21.5	28.8	22.2	18.5	28.5	26.4	21.3
0.75	16.1	12.6	10.4	14.9	11.3	9.4	14.2	12.6	10.2
1	11.2	7.6	6.3	10.04	6.9	5.8	9.0	7.6	6.2
1.25	8.7	5.2	4.3	7.4	4.7	4.0	6.5	5.2	4.3
1.5	7.1	3.9	3.2	6.0	3.5	2.8	5.0	3.8	3.2
1.75	6.1	3.0	2.5	5.04	2.7	2.2	4.1	3.0	2.5
2	5.3	2.4	2.1	4.3	2.1	1.8	3.5	2.4	2.1

**Table 2** Comparison of ARL of  $M_XHEWMA$  chart with Riaz (2008)

$\delta$	$\lambda_2$								
	0.1			0.25			0.5		
	Riaz	$\rho_{xw}$		Riaz	$\rho_{xw}$		Riaz	$\rho_{xw}$	
		0.25	0.5		0.25	0.5		0.25	0.5
0	499.1	493.2	496.0	499.8	508.3	509.3	500.7	508.5	508.5
0.1	290.3	305.6	275.3	364.0	268.9	244.8	423.5	314.9	218.3
0.25	80.6	95.8	78.4	135.7	75.7	61.2	216.0	94.7	78.2
0.5	22.0	26.2	21.5	34.5	22.2	18.5	65.3	26.4	21.3
0.75	10.5	12.6	10.4	14.1	11.3	9.4	24.7	12.6	10.2
1	6.4	7.6	6.3	7.80	6.9	5.8	11.9	7.6	6.2
1.25	4.4	5.2	4.3	5.1	4.7	4.0	6.8	5.2	4.3
1.5	3.3	3.9	3.2	3.7	3.5	2.8	4.5	3.8	3.2
1.75	2.5	3.0	2.5	2.9	2.7	2.2	3.5	3.0	2.5
2	2.1	2.4	2.1	2.3	2.1	1.8	2.5	2.4	2.1

efficient than HEWMA chart to detect changes in process mean. From the results of the ARLs, it is clear that the proposed chart is more efficient for weak, moderate, and high correlation values. The same results are observed in Table 2 in comparison with  $M_xEt$  control chart.

### 4 Real-life Application

An example of the proposed control chart is presented. The proposed control chart is functional for the industry if they collect quantifiable data. The actual data from Shabbir and Awan (2016) are taken here. The qualitative characteristics are the net weight of the cooking oil, including the tin, indicated by  $X_t$ . The weight of the empty tin is considered an auxiliary variable ( $W_t$ ). The variables,  $X_t$ . and  $W_t$  are normally distributed so that  $\mu_x = 4.9, \mu_w = 22.6, \sigma_x^2 = 0.0003$ , and  $\sigma_w^2 = 0.005$ . The  $X_t$ . and  $W_t$  data and the corresponding  $M_xHEWMA$  statistics of each subgroup are shown in Table 3 using  $\lambda_1 = 0.1$  and,  $\lambda_2 = 0.25$ . The control limits of HEWMA and  $M_xHEWMA$  are presented in Table 3 of the data set indicated. The graphical representation is shown in Figures 1–2.

Table 3 and Figure 1 described that HEWMA is not able to efficiently detect changes in the process average as it appears in the  $M_xHEWMA$  control chart.

**Table 3** Results from cooking oil data for  $\delta = 0.1, \lambda_1 = 0.1, \lambda_2 = 0.25$  and  $ARL_0 = 500$ .

S.N	$X_t$	$W_t$	$M_x$	$M_x Et$	$M_xHEt$	Lcl- MxHEt	Ucl- MxHEt	Ucl- HEt	Lcl- HEt	HEt
1	4.89	22.6	0.000	0.000	0.000	-0.058	0.058	0.067	-0.067	0
2	4.889	22.5	0.610	0.305	0.031	-0.112	0.112	0.129	-0.129	-0.002
3	4.9	22.6	0.578	0.442	0.072	-0.163	0.163	0.189	-0.188	0.0248
4	4.899	22.6	0.520	0.481	0.113	-0.209	0.209	0.242	-0.242	0.062
5	4.888	22.6	-0.116	0.183	0.120	-0.250	0.250	0.289	-0.289	0.069
6	4.871	22.5	-0.430	-0.124	0.095	-0.285	0.285	0.330	-0.329	0.015
7	4.916	22.6	1.503	0.690	0.155	-0.314	0.314	0.364	-0.363	0.064
8	4.901	22.6	0.636	0.663	0.205	-0.339	0.339	0.393	-0.392	0.115
9	4.877	22.5	-0.084	0.290	0.214	-0.360	0.360	0.417	-0.416	0.095
10	4.904	22.7	0.141	0.215	0.214	-0.377	0.377	0.437	-0.436	0.121
11	4.899	22.5	1.188	0.702	0.263	-0.391	0.391	0.453	-0.453	0.153
12	4.894	22.5	0.899	0.800	0.317	-0.403	0.403	0.467	-0.467	0.171
13	4.888	22.5	0.552	0.676	0.353	-0.413	0.413	0.478	-0.478	0.165
14	4.886	22.6	-0.231	0.223	0.340	-0.421	0.421	0.488	-0.487	0.143
15	4.917	22.6	1.561	0.892	0.395	-0.427	0.427	0.495	-0.495	0.203
16	4.905	22.7	0.199	0.545	0.410	-0.433	0.433	0.501	-0.501	0.264
17	4.91	22.6	1.156	0.851	<b>0.454</b>	-0.437	0.437	0.506	-0.506	0.336
18	4.9	22.6	0.578	0.714	<b>0.480</b>	-0.441	0.441	0.511	-0.510	0.380
19	4.926	22.6	2.081	1.398	<b>0.572</b>	-0.444	0.444	0.514	-0.513	<b>0.525</b>
20	4.912	22.7	0.604	1.001	<b>0.615</b>	-0.446	0.446	0.517	-0.516	<b>0.572</b>

\* bold indicates out-of-control signal

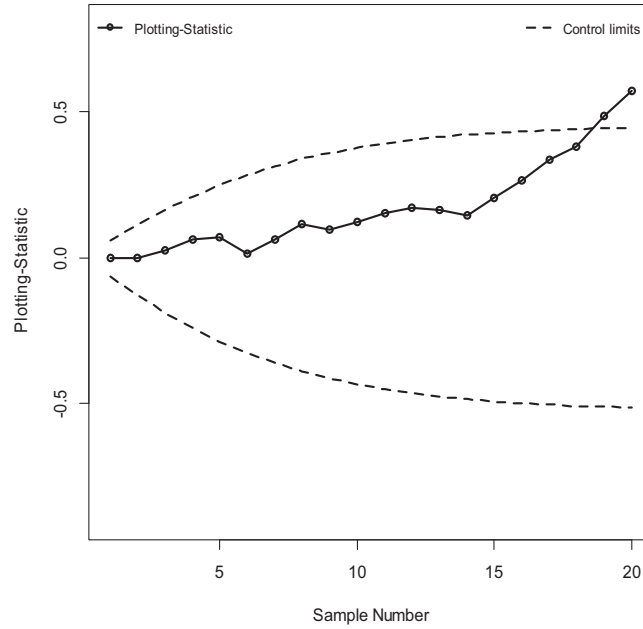


Figure 1 HEWMA control chart with  $ARL_0 = 500$

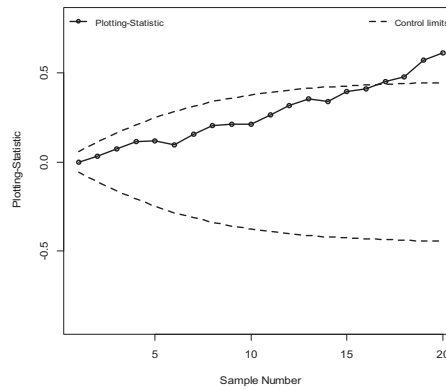


Figure 2  $M_x$ HEWMA control chart with  $ARL_0 = 500$

## 5 Conclusion

In this study,  $M_x$ HEWMA control chart is presented using auxiliary information in the form of a regression estimator to monitor the process mean. The proposed  $M_x$ HEWMA control chart is becoming more sensitive when the

correlation between the study variable and the auxiliary variable increases. It is observed from the results that the out-of-control ARLs and SDRLs are smaller for the proposed control chart as compared to the considered charts. The proposed control is superior in identifying minor/moderate variations in the process average. Therefore, professionals are advised to use the  $M_xHEWMA$  control chart. The study can be extended using different sampling methods, such as ranked set sampling. Furthermore, the effect of measurement error can also be studied.

### Appendix

The ARLs and SDRLs in Tables A1–A5 are calculated for three values of  $\lambda_2$ , i.e. 0.25, 0.5, 0.75 and taking  $\lambda_1 = 0.1$ . The values of control constant with different values of  $\lambda_2$  are given by

$\lambda_2$	0.25	0.5	0.75
L	2.548	2.675	2.76

**Table A1** ARLS for the proposed chart with  $\rho_{XW} = 0.05$  AND  $ARL_0 = 500$

$\delta$	ARLs	SDRLs	ARLs	SDRLs	ARLs	SDRLs
0	511.2	525.4	517.7	521.2	506.8	513.7
0.1	298.9	303.1	316.7	318.6	323.3	321.0
0.25	92.6	85.5	99.5	94.1	99.6	92.3
0.5	26.4	21.0	27.9	22.4	28.2	22.8
0.75	12.6	8.8	13.2	9.1	13.3	9.4
1	7.6	4.9	7.9	5.0	8.1	5.1
1.25	5.3	3.3	5.4	3.3	5.5	3.3
1.5	3.8	2.3	4.0	2.3	4.08	2.3
1.75	3.0	1.7	3.1	1.7	3.2	1.7
2	2.4	1.4	2.5	1.4	2.6	1.366

**Table A2** ARLS for the proposed chart with  $\rho_{XW} = 0.25$  AND  $ARL_0 = 500$

$\delta$	ARLs	SDRLs	ARLs	SDRLs	ARLs	SDRLs
0	508.3	532.0	508.5	507.4	505.4	508.6
0.1	268.9	274.6	314.9	311.0	320.5	307.9
0.25	75.7	68.9	94.7	89.7	98.4	92.2
0.5	22.2	16.8	26.4	20.6	26.8	21.3
0.75	11.3	8.0	12.6	8.6	12.7	8.9
1	6.9	4.7	7.6	4.8	7.67	4.8
1.25	4.7	3.1	5.2	3.1	5.26	3.1
1.5	3.5	2.2	3.8	2.2	3.9	2.2
1.75	2.7	1.7	3.0	1.6	3.0	1.6
2	2.1	1.3	2.4	1.3	2.5	1.3



**Table A3** ARLs for the proposed chart with  $\rho_{XW} = 0.5$  AND  $ARL_0 = 500$

$\delta$	ARLs	SDRLs	ARLs	SDRLs	ARLs	SDRLs
0	509.3	524.4	508.5	517.4	502.7	501.5
0.1	244.8	251.1	258.3	278.3	280.5	279.4
0.25	61.2	54.9	78.2	73.2	79.2	73.3
0.5	18.5	13.8	21.3	16.3	21.4	16.2
0.75	9.4	6.5	10.2	6.7	10.4	6.9
1	5.8	3.9	6.2	3.8	6.3	3.9
1.25	4.0	2.6	4.3	2.5	4.4	2.5
1.5	2.8	1.8	3.2	1.8	3.27	1.8
1.75	2.2	1.3	2.5	1.3	2.6	1.3
2	1.8	1.0	2.0	1.0	2.0	1.0

**Table A4** ARLs for the proposed chart with  $\rho_{XW} = 0.75$  AND  $ARL_0 = 500$

$\delta$	ARLs	SDRLs	ARLs	SDRLs	ARLs	SDRLs
0	509.6	519.0	506.3	514.1	499.2	501.1
0.1	174.5	172.6	209.3	201.9	213.0	211.9
0.25	38.5	31.9	47.2	41.7	48.2	42.0
0.5	11.7	8.2	13.0	9.0	13.3	9.2
0.75	5.9	4.0	6.4	4.0	6.5	4.0
1	3.6	2.4	3.9	2.2	4.0	2.3
1.25	2.5	1.5	2.8	1.5	2.84	1.5
1.5	1.8	1.0	2.1	1.1	2.16	1.0
1.75	1.5	0.7	1.7	0.8	1.7	0.8
2	1.3	0.5	1.4	0.6	1.4	0.6

**Table A5** ARLs for the proposed chart with  $\rho_{XW} = 0.95$  AND  $ARL_0 = 500$

$\delta$	ARLs	SDRLs	ARLs	SDRLs	ARLs	SDRLs
0	495.0	501.8	506.2	518.0	498.6	513.7
0.1	51.9	45.8	65.6	5.3	66.1	60.6
0.25	10.7	7.3	11.7	7.9	11.9	8.1
0.5	3.3	2.1	3.6	2.0	3.67	2.0
0.75	1.7	0.9	1.9	0.9	1.93	0.9
1	1.2	0.4	1.34	0.5	1.38	0.5
1.25	1.0	0.2	1.0	0.2	1.1	0.3
1.5	1.0	0.08	1.0	0.13	1.0	0.1
1.75	1.0	0.02	1.0	0.03	1.0	0.04
2	1.0	0	1.0	0.01	1.0	0.01

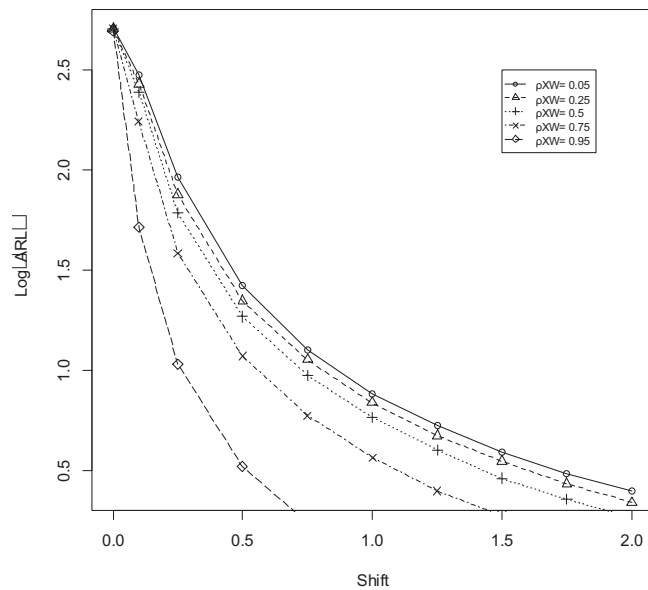


Figure A1 ARLs for various values of  $\rho_{XW}$  at  $\lambda_2 = 0.25$

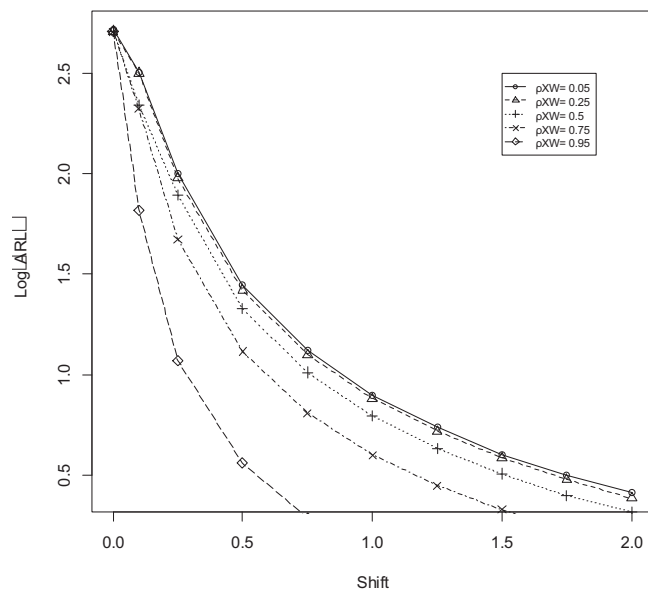
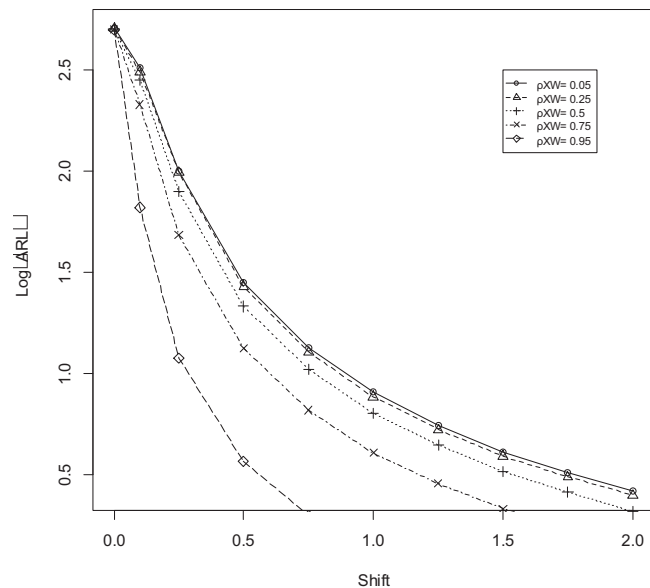


Figure A2 ARLs for various values of  $\rho_{XW}$  at  $\lambda_2 = 0.5$



**Figure A3** ARLs for various values of  $\rho_{XW}$  at  $\lambda_2 = 0.75$

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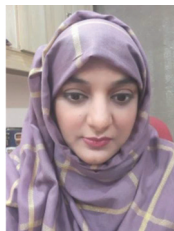
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