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ESTIMATION OF PARAMETERS FOR THE LIFETIME DISTRIBUTIONS

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Abstract
This paper deals with various methods of estimation used for estimating the parameters of lifetime distributions. The distributions considered are exponential, Weibull, Rayleigh, lognormal and gamma and the method used are: method of moments, maximum likelihood, probability weighted moments, least squares and relative least squares. To compare the efficiency between the different methods of estimation, we used the total deviation, mean squared error and probability plot correlation coefficients. In order to study numerically, the execution of the different methods of estimation and goodness of fit analysis, their statistical properties have been simulated for different sample sizes. The graphs of bias designed for different methods of estimation have also been plotted against various sample sizes.

Key Words: Lifetime Distributions, Methods of Estimation, Goodness of Fit Analysis, Simulation.

1. Introduction
Numerous parametric models are used in the analysis of lifetime data and in the problems associated with the modeling of the ageing or failure process. A few specific distributions play a central role because they are useful in various situations. The exponential, Weibull, Rayleigh, gamma, and lognormal distributions are the primary distributions in this category. The exponential distribution is often used to model the time interval between successive random events. The Weibull distribution, perhaps the most widely used model of fatigue distribution, was derived in 1939 by W. Weibull, who used it in 1951 to experimentally observe variations in the fatigue resistance of steel, its elastic limits, etc. The Weibull distribution is the most widely used model for lifetime distribution. It is also widely used in biomedicine, e.g. in studies on the time until tumors occur in the human population or in laboratory animals and in various supplementary situations. The exponential distribution is a particular case of the Weibull distribution.

Epstein and Sobel (1953) introduced the maximum likelihood estimators of the scale parameter, the one-parameter exponential distribution in the case of censorship from the right. Lee and Thompson (1974) argued that within the class of the proportional hazard rate model, Weibull seemed to be the mainly suitable selection intended for describing lifetimes. Hosking and Wallis (1987) described ML as the most efficient method for estimating the parameters of the generalized Pareto distribution (GPD), algorithms for calculating ML estimates can cause convergence problems even with large sample sizes. Castillo and Hadi (1997) developed a method for estimating the parameters of the generalized Pareto distribution (GPD) that can be used for each of k. This method has the advantage of generating estimates of k' that always match the observable data (k is the shape parameter of GPD). Grimshaw (1993) performed ML estimations to make comparisons with the proposed Bayesian approach. Hirai (1998) described the L moments r; r=1,2.... of a real-valued random variable 'X' exist, if and only if "X' has a finite mean. A distribution whose mean exists is characterized by its L moments {: r=1, 2...}. Such a distribution specified by its L-moments, even if some of its conventional moments do not exist. Al-Fawzan (2000) presented both the graphical and the analytical methods for estimating the Weibull parameters, namely form and scale parameters. Afify (2003) conducted the study by estimating the parameters of the two parameters "Rayleigh distribution". He used the smallest square method (LS), the relatively smallest square method (RLS), the ridge regression and the robust ridge regression (RR, RRR), the moment estimator (ME) and the modified moment estimators (MME) to estimate the two Rayleigh distribution parameters. To compare different methods of estimation, he recommended the quality of fit analysis.

Bermudez and Tukrman (2003) proposed Bayesian approach to parameter estimation of the generalized Pareto distribution. Mahdi and Cenac (2006) presented results on parameter estimation of logistic and Rayleigh distributions. Three estimation methods are investigated, namely the MM, ML and PWM methods.

Inspired by the above mentioned study, the present study intends to fit a suitable distribution for survival data. Focusing on distribution belongs to exponential family. The structure of this article as follows: Exponential, Weibull, Rayleigh, Lognormal and gamma distributions are selected for study. Different methods of estimation for estimating the parameters of the exponential family of distributions are described in section **2**. Goodness of fit analysis is given in section **3**. In section **4**, the simulation study and the graphs of bias for different sample sizes is given. A real life data application is given in section **5**. This article concludes with a brief discussion in section 6.

2. Materials and Methods

Different methods of estimation used for estimating the parameters of the lifetime distributions are: method of moments (MM), maximum likelihood (ML), and probability weighted moments (PWM), least squares (LS) and relative least squares (RELS).

2.1 Exponential distribution

The pdf f (x) and cdf F (x) of the exponential distribution are given below:

$$f(x) = \lambda \exp(-\lambda x); \quad x > 0, \ \lambda > 0$$

$$F(x) = 1 - \exp(-\lambda x)$$

The method of moments and method of maximum likelihood estimates of λ is:

$$\hat{\lambda} = \frac{1}{\bar{x}}$$

While the probability weighted moment estimates of λ is:

$$\hat{\lambda} = \frac{1}{a_o}$$

Now the method of least squares and method of relative least squares estimates of λ is along these lines:

$$\hat{\lambda} = -\frac{\sum \left[\log e\{1 - F(x)\}\right]^2}{\sum x \log e\left[1 - F(x)\right]}$$

$$\hat{\lambda} = -\frac{\sum_{i=1}^{n} \left[\frac{\log e\left[1 - F(x_i)\right]}{x_i}\right]^2}{\sum_{i=1}^{n} \left[\frac{\log e\left[1 - F(x_i)\right]}{x_i}\right]}$$

2.2 Weibull distribution

The pdf f (x) and cdf F (x) of the weibull distribution are given by:

$$f(x) = \left(\frac{\alpha}{\lambda}\right)\left(\frac{x}{\lambda}\right)^{\alpha - 1} \exp\left(-\frac{x}{\lambda}\right)^{\alpha} ; x \geq 0 , \alpha,\lambda > 0$$

$$F(x) = 1 - \exp\left(-\frac{x}{\lambda}\right)^{\alpha}$$

The quantile function x (F) of the Weibull distribution is defined as:

$$x(F) = \left[\frac{1}{\lambda}\left[-\ln(1 - F)\right]\right]^{\frac{1}{\alpha}}$$

The Weibull distribution reduces to exponential distribution when $\alpha = 1$. The method of moments estimators of α & λ are obtained to solve these equations numerically.

$$m_1' = \hat{\lambda}\Gamma\left(1 + \frac{1}{\hat{\alpha}}\right)$$

$$m_2' = \hat{\lambda}^2\Gamma\left(1 + \frac{2}{\hat{\alpha}}\right)$$

While the methods of maximum likelihood estimating equations are:

$$\tilde{\alpha} = \frac{n}{\left(\frac{1}{\tilde{\lambda}}\right)^{\tilde{\alpha}} \sum x_i^{\tilde{\alpha}} \ln x_i - \sum \ln x_i}$$

$$\tilde{\lambda} = \left[\frac{1}{n}\sum x_i^{\tilde{\alpha}}\right]^{\frac{1}{\tilde{\alpha}}}$$

We solve these equations numerically to find the ML estimates of α and λ.
We have obtained the PWM estimates of α & λ

$$\hat{\alpha} = \frac{\ln 2}{\ln a_o - \ln a_1 - \ln 2}$$

$$\hat{\lambda} = \frac{a_0}{\Gamma\left(1 + \frac{1}{\hat{\alpha}}\right)}$$

by solving these above equations simultaneously for α & λ .

The least squares and relative least squares estimates for the shape and scale parameters are inclined away:

$$\hat{\alpha} = \frac{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i y_i}$$

$$\hat{\lambda} = \exp\left(\frac{\sum y_i}{n} - \frac{\sum x}{n\hat{\alpha}}\right)$$

while the relative least squares estimators are obtained by minimizing the relative sum of squares of residuals and are given as:

$$\hat{a} = \frac{\sum_{i=1}^{n} w_i z_i \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} w_i \sum_{i=1}^{n} z_i^2}{\left(\sum_{i=1}^{n} w_i z_i\right)^2 - \sum_{i=1}^{n} z_i^2 \sum_{i=1}^{n} w_i^2}$$

$$\hat{b} = \frac{\sum_{i=1}^{n} w_i z_i \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} w_i^2 \sum_{i=1}^{n} z_i}{\left(\sum_{i=1}^{n} w_i z_i\right)^2 - \sum_{i=1}^{n} z_i^2 \sum_{i=1}^{n} w_i^2}$$

Where $w_i = \frac{1}{y_i}, z_i = \frac{x_i}{y_i}$

2.3 Rayleigh distribution

The pdf f (x) and cdf F (x) of the Rayleigh distribution are given below

$$f(x) = \frac{2x}{\alpha^2} \exp\left(-\frac{x^2}{\alpha^2}\right); \quad \begin{matrix} x \geq 0 \\ \alpha > 0 \end{matrix}$$

$$F(x) = 1 - \exp\left(-\frac{x^2}{\alpha^2}\right)$$

The quantile function of the Rayleigh distribution is

$$x(F) = \alpha[\ln(F-1)]^{1/2}$$

The Rayleigh distribution, which is a special case of the Weibull distribution, is widely used, e.g. in service life testing. The MM and PWM estimate of the Rayleigh distribution is given by:

$$\hat{\alpha} = \frac{2\overline{x}}{\sqrt{\pi}}$$

While the ML estimate of the scale parameter of the rayleigh distribution is:

$$\tilde{\alpha} = \sqrt{\frac{\sum x^2}{n}}$$

The least squares and relative least squares estimates of the Rayleigh distribution defined as below:

$$\hat{\alpha} = \frac{\sum_{i=1}^{n} x_i [-\log e(1 - F(x_i))]^{1/2}}{\sum_{i=1}^{n} [-\log e(1 - F(x_i))]}$$

$$\hat{\alpha} = \frac{\sum_{i=1}^{n}\left[\frac{[-\log e(1-F(x_i))]^{\frac{1}{2}}}{x_i}\right]}{\sum_{i=1}^{n}\left[\frac{[-\log e(1-F(x_i))]^{1/2}}{x_i}\right]^2}$$

2.4 Lognormal distribution

The pdf f (x) and cdf F (x) of the lognormal distribution are given below:

$$\text{f(x)} = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln x-\mu)^2}{2\sigma^2}\right]; x \geq 0, \quad \mu,\sigma > 0$$

$$F(x) = \phi\left[\frac{(\ln x-\mu)}{\sigma}\right] \text{, Where } \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)dt$$

The moments estimate of $\mu \,\&\, \sigma^2$ are inclined away:

$$\hat{\mu} = 2\ln m_1' - \frac{1}{2}\ln m_2' = \ln\left(\frac{m_1'^2}{m_2'^{\frac{1}{2}}}\right)$$

$$\hat{\sigma}^2 = \ln m_2' - 2\ln m_1' = \ln\left(\frac{m_2'}{m_1'^2}\right)$$

While the maximum likelihood estimates of $\mu \,\&\, \sigma^2$ are defined as under:

$$\tilde{\mu} = \frac{\sum \ln x}{n}$$

$$\tilde{\sigma}^2 = \frac{\sum(\ln x - \tilde{\mu})^2}{n}$$

2.5 Gamma distribution

The pdf f (x) and cdf F (x) of the gamma distribution are given by:

$$f(x) = \frac{1}{\beta^{\alpha}\Gamma\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} \quad ; \quad 0 \leq x \leq \infty, \; \alpha, \beta > 0$$

$$F(x)=\int_0^x \frac{1}{\beta^{\alpha}\Gamma\alpha}x^{\alpha-1}e^{-\frac{x}{\beta}}dx$$

The method of moments estimators of α & β are given below:

$$\hat{\beta}=\frac{m'_2-m'^2_1}{m'_1}$$

$$\hat{\alpha}=\frac{m'^2_1}{m'_2-m'^2_1}$$

The maximum likelihood estimators of α & β are:

$$\tilde{\beta}=\frac{\overline{X}}{\tilde{\alpha}}$$

$$-n\ln\tilde{\beta}-n\psi(\tilde{\alpha})+\sum_{i=1}^{n}\ln x_i=0$$

Hosking (1986) showed that point estimates of the parameters of the Gamma distribution could be obtained by L-moment are described as:

$$\hat{\alpha}^{(K+1)}=\frac{\pi^{-\frac{1}{2}}\Gamma\left(\alpha^{k}+\frac{3}{2}\right)}{\tau\Gamma\left(\alpha^{k}+1\right)}-\frac{1}{2}\text{ With starting value }\alpha^{0}=\frac{1}{\pi\tau^{2}}$$

if $\tau<0.6$

$$=\frac{-\log_e\tau}{\log_e 4}\qquad\text{If }\tau\geq 0.6$$

$$\hat{\beta}=\frac{\overline{X}}{\hat{\alpha}}$$

3. Goodness of Fit Analysis

To compare the efficiency among different methods of estimation, we use the total deviation (TD), mean squared error (MES) and probability plot correlation coefficient (R^2).

3.1: Total Deviation (TD)

The TD will be calculated as under:

$$TD = \left|\frac{\hat{\gamma}-\gamma}{\gamma}\right| + \left|\frac{\hat{\delta}-\delta}{\delta}\right|$$

If γ & δ are the true values of the estimated parameters after each method, the best method yields the minimum total deviation.

3.2 Mean Square Error (MSE)

The mean square error can be calculated as given:

$$\text{MSE}=\frac{1}{n}\sum_{i=1}^{n}\left[\hat{F}(x_i)-F(x_i)\right]^2$$

3.3 Correlation Coefficient R^2

The adequacy of a fitted distribution can beassessed by the correlation coefficient, a value of R^2 close to1.0 indicatesthat the observations could have been drawn from the fitted distribution.

$$R^2 = \frac{\sum_{i=1}^{n}(Y_i-\bar{Y})(\hat{Y}_i-\bar{\hat{Y}})}{\left[\sum_{i=1}^{n}(Y_i-\bar{Y})^2\sum_{i=1}^{n}(\hat{Y}_i-\bar{\hat{Y}})^2\right]^{0.5}}$$

4. Simulation study

A simulation study for sample sizes of n=10, 20, 40 and 80 with different values of shape and scale parameters was performed to compare the performance of the proposed estimation methods for lifetime distributions. The results are based on 10,000 simulation runs. We have generated samples of different sizes using the Minitab statistics package. The results are presented in Tables 4.1 to 4.5.

Method	**Sample Size**	**True values**	**Estimated values**	**TD**	**MSE**	R^2	**Bias**
		λ	$\hat{\lambda}$				
MM,ML and PWM	10	1	0.9668	0.0332	0.0001	0.8789	-0.0332
		2	0.6131	0.6934	0.0164	0.9116	-1.3869
		3	0.5109	0.8297	0.0264	0.9903	-2.4891
	20	1	0.1755	0.2446	0.1621	0.5555	-0.8245
		2	0.6392	0.6804	0.0130	0.9490	-1.3608
		3	0.3382	0.8873	0.0543	0.9884	-2.6618
	40	1	0.8812	0.1188	0.0010	0.7935	-0.1188
		2	0.5387	0.7306	0.0293	0.9611	-1.4613

		3	0.2854	0.9049	0.0968	0.9945	-2.7146
	80	1	0.8506	0.1494	0.0018	0.7960	-0.1494
		2	0.5141	0.7430	0.1042	0.8193	-1.4859
		3	0.3138	0.8954	0.1949	0.7822	-2.6862
LS	10	1	1.1335	0.1335	0.0008	0.8457	0.1335
		2	0.9906	0.5047	0.0270	0.9693	-1.0094
		3	0.3732	0.8756	0.1239	0.9978	-2.6268
	20	1	0.9595	0.0405	0.0002	0.7449	-0.0405
		2	1.3813	0.3094	0.0098	0.9355	-0.6187
		3	0.4359	0.8547	0.2038	0.9912	-2.5641
	40	1	2.1814	1.1813	0.0423	0.6556	1.1814
		2	1.0177	0.4912	0.0263	0.9892	-0.9823
		3	0.5340	0.8220	0.1441	0.9883	-2.4660
	80	1	2.9284	1.9284	0.0702	0.6867	1.9284
		2	0.7088	0.6456	0.0607	0.8528	-1.2912
		3	0.7757	0.7414	0.0570	0.8130	-2.2243
RELS	10	1	0.1318	0.8682	0.2570	0.5786	-0.8682
		2	0.0723	0.9639	0.4010	0.9628	-1.9277
		3	0.1942	0.9353	0.2764	0.9732	-2.8058
	20	1	0.0877	0.9123	0.2315	0.5451	-0.9123
		2	0.1755	0.9123	0.2938	0.9923	-1.8245
		3	0.0161	0.9946	0.7789	0.9870	-2.9839
	40	1	0.0343	0.9657	0.3080	0.6751	-0.9657
		2	0.0374	0.9813	0.5064	0.9788	-1.9626
		3	0.1047	0.9651	0.4397	0.9925	-2.8953
	80	1	0.0119	0.9881	0.3237	0.5926	-0.9881
		2	0.2195	0.8902	0.2582	0.9164	-1.7805
		3	0.1418	0.9527	0.3986	0.8197	-2.8582

Table 4.1: Estimation of Parameter for Exponential Distribution

From Tables 4.1, it is observed that LS achieves the best estimate 8 times out of 12 which is approximately 67% of the time based on TD, MSE and R^2. The bias in MM, ML and PWM estimates remains constant with the increase in sample size for λ =2 and 3. But for λ=1, the bias is close to zero as the sample size increases. Fig. 4.2, shows that the bias in LS estimators for λ=1 increases with the increase in sample size, for λ=2 the bias in LS estimator decreases after λ=2. It can further be observed the bias in RELS estimator just about remains constant with the increase in sample size for different values of the scale parameter.

Method	Sample size	True values		Estimated values		TD	MSE	R^2	Bias	
		α	λ	$\hat{\alpha}$	$\hat{\lambda}$				$Bias(\hat{\alpha})$	$Bias(\hat{\lambda})$
MM	10	1	1	1.4064	0.8989	0.5075	0.1410	0.7662	0.4064	-0.1011
		2	2	2.6457	1.9205	0.3626	0.1241	0.7099	0.6457	-0.0795
		3	2	3.7986	1.8804	0.3260	0.2030	0.6948	0.7986	-0.1196
	20	1	1	0.8775	0.6621	0.4604	0.2066	0.6588	-0.1225	-0.3379
		2	2	1.8736	1.6103	0.2581	0.2576	0.9112	-0.1264	-0.3897
		3	2	2.7104	2.1410	0.1670	0.1204	0.7110	-0.2896	0.1410
	40	1	1	0.9997	1.0617	0.0620	0.1366	0.6329	-0.0003	0.0617
		2	2	2.1228	2.2655	0.1942	0.1363	0.8304	0.1228	0.2655
		3	2	4.5163	2.1345	0.5727	0.1622	0.9143	1.5163	0.1345
	80	1	1	1.0366	0.9176	0.1190	0.1369	0.7289	0.0366	-0.0824
		2	2	1.9758	1.9876	0.0183	0.1896	0.8977	-0.0242	-0.0124
		3	2	3.2815	1.9817	0.1030	0.1899	0.7165	0.2815	-0.0183
ML	10	1	1	2.3995	1.0905	1.4900	0.1146	0.8128	1.3995	0.0905
		2	2	1.6184	2.5225	0.4521	0.1046	0.9114	-0.3816	0.5225
		3	2	3.9490	1.6022	0.5152	0.3275	0.8875	0.9490	-0.3978
	20	1	1	1.2131	1.1419	0.3550	0.2076	0.8183	0.2131	0.1419
		2	2	1.9743	2.0215	0.0236	0.1646	0.9257	-0.0257	0.0215
		3	2	2.5043	2.3537	0.3421	0.2253	0.7843	0.4957	0.3537
	40	1	1	1.2180	0.9160	0.3020	0.1649	0.4551	0.2180	-0.0840
		2	2	2.0638	1.8486	0.1076	0.2005	0.8869	0.0638	-0.1514
		3	2	4.4655	1.6149	0.6811	0.1768	0.8552	-1.4655	-0.3851
	80	1	1	0.8880	0.9999	0.1121	0.1547	0.7833	-0.1120	-0.0001
		2	2	2.1028	1.7144	0.1942	0.1615	0.8782	0.1028	-0.2856
		3	2	3.3557	1.8877	0.1748	0.1327	0.7402	0.3557	-0.1123

Table 4.2: Estimation of Parameters for Weibull Distribution

For the Weibull distribution, obtained results are listed in Table 4.2. It is noticed that MM and ML method for estimates are found more complex and Newton-Raphson method is used to calculate the parameters. The bias in MM estimators decreases rapidly with the increase in sample size. It is observed that estimated value of the shape parameter $\hat{\alpha}$ by PWM, LS and RELS came out negative for samples of size less than 80. It might be due to large amount of bias for small samples. The bias in the shape parameter when estimated by using MM is large when $\lambda = 3$ and n=40. But when the sample size increases i.e. 80, then the bias for $\lambda = 1, 2 \ \& \ 3$ are almost similar. While the bias in scale parameter also decreases rapidly when n=80.The bias in the shape and scale parameters decreases when estimated by using the method of maximum likelihood.

Method	Sample size	True values α	Estimated values $\hat{\alpha}$	TD	MSE	R^2	Bias
MM and PWM	10	1	1.0986	0.0986	0.0018	0.4406	0.0986
		2	1.2062	0.3969	0.0724	0.4240	-0.7938
		3	2.7549	0.0817	0.0023	0.2030	-0.2451
	20	1	1.1663	0.1663	0.0079	0.1484	0.1663
		2	1.8960	0.0520	0.0008	0.0547	-0.1040
		3	2.7498	0.0830	0.0021	0.1230	-0.2502
	40	1	1.0528	0.0528	0.0009	0.2814	0.0528
		2	1.8422	0.0789	0.0019	0.3321	-0.1578
		3	2.9921	0.0026	0.0000	0.1341	-0.0079
	80	1	1.0450	0.0450	0.0005	0.4455	0.0450
		2	2.1067	0.0534	0.0007	0.8803	0.1067
		3	2.8992	0.0336	0.0004	0.9748	-0.1008
ML	10	1	0.9495	0.0505	0.0006	0.3573	-0.0505
		2	1.6675	0.1663	0.0093	0.3585	-0.3325
		3	2.2400	0.2533	0.0322	0.2761	-0.7600
	20	1	1.1502	0.1502	0.0062	0.0125	0.1502
		2	1.9040	0.0480	0.0008	0.0500	-0.0960
		3	3.4412	0.1471	0.0048	0.1125	0.4412
	40	1	0.9323	0.0667	0.0013	0.4332	-0.0677
		2	1.8471	0.0765	0.0018	0.2515	-0.1529
		3	3.0142	0.0047	0.0000	0.1200	0.0142
	80	1	1.0754	0.0754	0.0018	0.6564	0.0754
		2	2.1193	0.0597	0.0009	0.9009	0.1193
		3	2.9162	0.0279	0.0002	0.9715	-0.0838
LS	10	1	0.9980	0.0020	0.0000	0.7922	-0.0020
		2	1.3580	0.3210	0.0357	0.1797	-0.6420
		3	3.1590	0.0530	0.0007	0.1932	0.1590
	20	1	0.5930	0.4070	0.0885	0.2612	-0.4070
		2	2.0166	0.0083	0.0000	0.0898	0.0166
		3	2.8270	0.0577	0.0009	0.0612	-0.1730
	40	1	0.7899	0.2101	0.0144	0.3932	-0.2101
		2	1.5174	0.2413	0.0167	0.2955	-0.4826
		3	2.3687	0.2104	0.0159	0.1233	-0.6313
	80	1	0.7925	0.2075	0.0175	0.4198	-0.2075
		2	1.5600	0.2200	0.0188	0.9140	-0.4400
		3	2.4271	0.1910	0.0126	0.9802	-0.5729
	10	1	1.1374	0.1374	0.0044	0.6462	0.1374
		2	2.1104	0.0552	0.0008	0.4833	0.1104
		3	3.1454	0.0485	0.0006	0.4877	0.1454
	20	1	0.9271	0.0729	0.0014	0.6796	-0.0729
		2	2.6057	0.3028	0.0194	0.7260	0.6057

RELS		3	4.0599	0.3533	0.0202	0.4864	1.0599
	40	1	0.5725	0.4275	0.0983	0.3205	-0.4275
		2	2.5502	0.2751	0.0160	0.6452	0.5502
		3	5.4088	0.8029	0.0934	0.5478	2.4088
	80	1	0.1305	0.8695	0.3191	0.3066	-0.8695
		2	0.8004	0.5998	0.1511	0.8788	-1.1996
		3	1.0157	0.6614	0.1816	0.9706	-1.9843

Table 4.3: Estimation of Parameter for Rayleigh Distribution

For the Rayleigh distribution, the gained values are listed in Table 4.3. It can be observed that MM and PWM achieve the best estimates. In the table 4.3, the bias in the RELS estimator decreases rapidly with the increase in sample size, while the bias in ML and PWM estimator decreases rapidly for scale parameter is 3, with the increase in sample size.

Method	**Sample size**	**True values**		**Estimated values**		**TD**	**Bias**	
		μ	σ^2	$\hat{\mu}$	$\hat{\sigma}^2$		$Bias(\hat{\mu})$	$Bias(\hat{\sigma}^2)$
MM	10	1	1	1.2757	0.4491	0.8266	0.2757	-0.5509
		2	2	3.5384	0.8754	1.3315	1.5384	-1.1246
		3	2	3.5684	1.3290	0.5250	0.5684	-0.6710
	20	1	1	1.0601	0.4973	0.5628	0.0601	-0.5027
		2	2	2.2945	2.0548	0.1747	0.2945	0.0548
		3	2	3.5612	1.2303	0.5720	0.5612	-0.7697
	40	1	1	0.8554	0.4793	0.6653	-0.1446	-0.5207
		2	2	2.1449	1.7292	0.2079	0.1449	-0.2708
		3	1	3.9061	2.3275	0.4658	0.9061	0.3275
	80	1	2	1.0483	1.0543	0.1026	0.0483	0.0543
		2	2	2.7385	2.5039	0.6212	0.7385	0.5039
		3	2	3.9208	1.6055	0.5042	0.9208	-0.3945
ML	10	1	1	2.0250	0.8896	1.1354	1.0250	-0.1104
		2	2	1.3126	3.2025	0.9450	-0.6874	1.2025
		3	2	2.4563	2.2524	0.3074	-0.5437	0.2524
	20	1	1	0.9998	0.3973	0.6029	-0.0002	-0.6027
		2	2	1.4759	4.8743	1.6993	-0.5241	2.8743
		3	2	2.8942	3.5440	0.8073	-0.1058	1.5440
	40	1	1	1.3332	0.9016	0.4316	0.3332	-0.0984
		2	2	2.1351	3.7099	0.9226	0.1351	1.7099
		3	2	3.4759	2.9714	0.6443	0.4756	0.9714
	80	1	1	1.1679	0.9069	0.2611	0.1679	-0.0931
		2	2	2.1716	3.1256	0.6486	0.1716	1.1256
		3	2	3.2081	3.5614	0.8501	0.2081	1.5614

Table 4.4: Estimation of Parameters for Lognormal Distribution

For the lognormal distribution, the location parameter (μ) and the scale parameters (σ^2) were estimated by MM and ML. The results are given in Table 4.4. It can be observed that MM is 9 times best as compared to ML estimates, which is approximately 75% of the time on the basis of goodness of fit analysis. The bias in MM estimates decreases consistently with the increase in sample size.

Method	Sample size	True values		Estimated values		TD	Bias	
		α	β	$\hat{\alpha}$	$\hat{\beta}$		$Bias(\hat{\alpha})$	$Bias(\hat{\beta})$
MM	10	1	1	1.1318	1.3711	0.5029	0.1318	0.3711
		2	2	1.4461	3.2615	0.9077	-0.5539	1.2615
		3	2	1.7871	2.5746	0.6916	-1.2129	0.5746
	20	1	1	1.4323	1.0126	0.4539	0.4322	0.0126
		2	2	2.8663	1.4661	0.7001	0.8663	-0.5339
		3	2	3.8023	1.6883	0.4233	0.8023	0.3117
	40	1	1	0.9796	1.2157	0.2361	-0.0204	0.2157
		2	2	3.0653	1.4077	0.8289	1.0653	-0.5924
		3	1	2.5861	2.0902	0.1831	-0.4139	0.0902
	80	1	2	0.8362	1.3197	0.4835	0.1638	0.3197
		2	2	2.6806	1.7016	0.4895	0.6806	0.2984
		3	2	3.7878	1.8646	0.3303	0.7878	-0.1354
ML	10	1	1	0.8769	1.1202	0.2433	-0.1231	0.1202
		2	2	3.1494	1.0786	1.0354	1.1494	-0.9214
		3	2	2.9620	1.6846	0.1704	-0.0387	-0.3154
	20	1	1	1.5192	0.6996	0.8196	0.5192	-0.3004
		2	2	1.6238	2.5849	0.4806	-0.3762	0.5849
		3	2	3.5081	1.6152	0.3618	0.5081	-0.3848
	40	1	1	1.1676	0.9519	0.2157	0.1676	-0.0481
		2	2	1.4085	2.9548	0.7732	-0.5915	0.4774
		3	1	2.8139	1.9377	0.0932	-0.1861	-0.0623
	80	1	2	0.8746	1.0635	0.1889	-0.1254	0.0635
		2	2	2.2581	1.6849	0.2866	0.2581	-0.3151
		3	2	3.5204	1.7256	0.3107	0.5204	0.2744

Table 4.5: Estimation of Parameters for Gamma Distribution

For the gamma distribution, the scale parameter (β) and shape parameters (α) were estimated by MM and ML. The obtained results are listed in Table 4.5. The MM and ML method for estimated parameters of gamma distribution found more complex and Newton-Raphson method is used to calculate the parameters of gamma distribution. Fleeting look on Table 4.5shows that ML estimators are steadfast and unrelenting estimators for estimating the parameters of gamma distribution.

5. Data Analysis

In this section, we analyze real data set and discriminate between the exponential families of distributions. The following data set (Linhart and Zucchini; 1986, page 69) represents the failure times of the air conditioning system of an airplane:23,261,87,7,120,14,62,47,225,71,246,21,42,20,5,12,120,11,3,14,71,11,14,11,16,90,1,16,52,95.The study aim is to discriminate the lifetime distribution model. The output of the real life data is given in the following tables:

Distribution	Anderson-Darling (adj)
Weibull	0.910
Lognormal	0.754
Exponential	1.400
Rayleigh	1.350
Gamma	3.017

Table 5.1: Goodness of Fit Analysis by using the Real Life Data

Distribution	Percent	Percentile	Standard Error	Lower	Upper
Weibull	1	0.249	0.2132	0.047	1.3225
Lognormal	1	1.335	0.6192	0.538	3.3135
Exponential	1	0.599	0.1094	0.419	0.8567
Rayleigh	1	0.499	0.1000	0.400	0.7500
Gamma	1	-104.818	24.8407	-153.505	-56.1313
Weibull	5	1.683	1.0102	0.519	5.4575
Lognormal	5	3.281	1.2122	1.591	6.7688
Exponential	5	3.057	0.5581	2.137	4.3723
Rayleigh	5	3.000	0.4400	2.100	3.3000
Gamma	5	-56.653	19.7927	-95.445	17.8596

Table 5.2: Percentiles by using the Real Life Data

From the above tables (5.1& 5.2), it is noticed that results obtained through the real life data are compatible with the simulation results.

6. Conclusion

Estimation of parameters of the exponential family of distributions have been derived using the MM, ML, PWM, LS and RELS methods, for all the five distributions i.e. exponential, Weibull, Rayleigh, lognormal and gamma. It is found that PWM method is best suited for estimating the parameters of those distributions, whose CDF are expressible n the inverse form for small sample sizes. In case of large sample sizes, ML provides best estimates and the estimated value is close to the true value of the parameters. Intended for the exponential distribution, LS achieves the best estimate '8' times out of ‘12’, which is approximately 67% of the time based on the goodness of fit analysis.

For the Weibull distribution, it is noticed that MM and ML estimates are found more complex and Newton-Raphson method is used. The bias in MM estimators decreases rapidly by means of the increase in sample size. It is observed that estimated value of the shape parameter $\hat{\alpha}$ by PWM, LS and RELS came out negative for samples of size less than 80. It might be due to large amount of bias for small samples.

In case of Rayleigh distribution, MM and PWM provide the best estimates for scale parameter α . While the bias decreases rapidly with the increase in sample size and the estimated value is close to the true value of the parameter and the TD is minimum and ranges from 0.026 to 0.1663 and the value of the correlation coefficient is close to one.

For the lognormal distribution, MM provides the best estimates ‘9’ times out of ‘12’ times. The bias and MSE for the MM estimates decrease consistently.

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