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ON THE BAYESIAN ANALYSIS OF EXTENDED WEIBULL-GEOMETRIC DISTRIBUTION

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Abstract

The paper deals with the Bayes estimation of Extended Weibull-Geometric (EWG) distribution. In particular, we discuss Bayes estimators and their posterior risks using the noninformative and informative priors under different loss functions. Since the posterior summaries cannot be obtained analytically, we adopt Markov Chain Monte Carlo (MCMC) technique to assess the performance of Bayes estimates for different sample sizes. A real life example is also part of this study.

Key Words: Extended Weibull Distribution, Mcmc, Bayes Estimator, Posterior Risk, Loss Function, Precautionary Loss Function.

1. Introduction

The Weibull distribution is one of the most commonly used lifetime probability distribution in statistics and reliability. The history of the Weibull distribution dated back to Frechet (1927), Rosin and Rammler (1933). More than ten years later, Weibull (1951) was the first to describe the distribution in detail for practical problems. Recently, there is also a debate about its correct name (Stoyan, 2013). The Weibull distribution has a flexible hazard rate, i.e., increasing, decreasing, constant and bathtub. Many researchers have proposed different extensions of the Weibull distribution, for example, to assess the reliability of commercial vehicle engines Keller et al., (1985) proposed the inverse Weibull while Mudholkar et al. (1995) studied exponentiated Weibull distribution for bus motor failure data. Later, the Weibull distribution was also used by Mudholkar and Hutson (1996) to model flood data. For a comprehensive overview on Weibull distributions different extensions and applications, we refer to Rinne (2008).

In recent years, many new generalizations of the Weibull have been proposed and studied. For example, Adamidis and Loukas (1998) discussed exponential geometric distribution, whereas Kus (2007) proposed a two-parameter exponential Poisson distribution to accommodate decreasing hazard rate. Barreto-Souza et al. (2011) studied Weibull-geometric distribution, which has the following special cases:

extended exponential geometric distribution, the exponential geometric distribution and Weibull distribution. Wang and Elbatal (2015) also proposed a modified Weibull geometric distribution by compounding modified Weibull with geometric distributions. Recently, Azeem et al. (2017) introduced an extended Weibull-geometric (EWG) distribution to analyze lifetime data. The EWG has increasing or decreasing hazard function, depending on the values of parameters.

The extended Weibull geometric distribution is defined as

$$f(x)=\frac{\alpha\beta(1-p)x^{\beta-1}(1+\gamma\alpha x^{\beta})^{-(1+\frac{1}{\gamma})}}{(1-p(1+\gamma\alpha x^{\beta})^{-\frac{1}{\gamma}})^{2}} \tag{1}$$

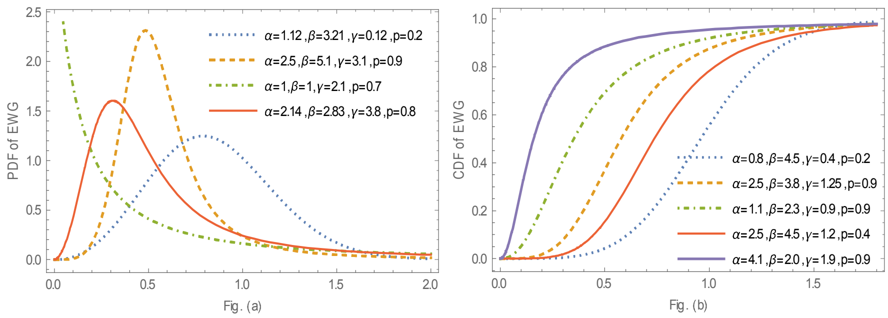
with $x>0$, $\alpha,\beta,\gamma>0$ and $p\in(0,1)$; where α is the scale, $(\beta,\gamma)>0$ are the shape parameters. We abbreviate this distribution as EWG(α,β,γ,p).
The CDF of EWG distribution is given as

$$F(x)=\frac{1-(1+\alpha\gamma x^{\beta})^{-\frac{1}{\gamma}}}{1-p(1+\alpha\gamma x^{\beta})^{-\frac{1}{\gamma}}}; \qquad x>0, \alpha>0, \beta>0, \gamma>0, p\in(0,1) \tag{2}$$

whereas the survival and hazard functions of EWG distribution are given as

$$S(x)=\frac{(1+\alpha\gamma x^{\beta})^{-\frac{1}{\gamma}}(1-p)}{1-p(1+\alpha\gamma x^{\beta})^{-\frac{1}{\gamma}}} \tag{3}$$

$$\mathrm{h}(x)=\frac{\alpha\beta x^{\beta-1}}{(1+\alpha\gamma x^{\beta})\left[1-p(1+\alpha\gamma x^{\beta})^{-\frac{1}{\gamma}}\right]} \tag{4}$$



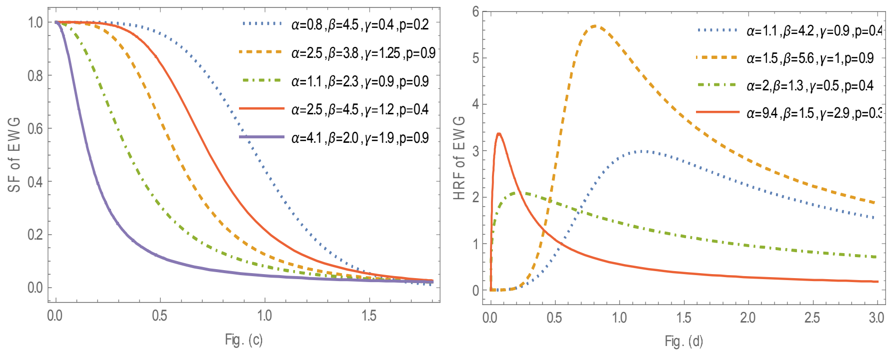


Figure 1: (a) PDF of the EWG, (b) CDF of the EWG, (c) Survival function of EWG, (d) Hazard function of EWG

Figure 1 presents the probability density, cumulative density, survival function and hazard function plots of the extended Weibull geometric distribution. From panel (a) of the probability density plot one can observe that EWG shows different shapes assuming different parameter values. To be more specific, the EWG distribution is skewed as well as symmetric. The hazard rate function of the EWG first increases and then decreases, which show its flexibility.

In the literature, Xiong (2008) proposed a Bayesian approach to estimate the reliability of geometric distribution under entropy and square loss functions assuming the power distribution as the prior. Banerjee and Kundu (2008) presented the statistical inferences on Weibull parameters for Type-II hybrid censored data and discussed the ML estimators and the approximate ML estimators. The authors used Markov Chain Monte Carlo (MMC) technique to obtain the Bayes estimates and the highest posterior density (HPD) credible intervals of the unknown parameters. The method of obtaining the optimum censoring scheme based on the maximum information measure was also developed in the article. Ramos (2014) discussed different non informative prior distributions such as Jeffreys and maximal data information for extended exponential-geometric distribution.

Singh (2014) presented the Bayesian analysis of Poisson-exponential distribution under symmetric and asymmetric loss functions using MCMC. Saqib and Dar (2016) assessed the performance of ML and Bayes estimates using noninformative priors for the Weibull distribution. The authors concluded that the performance of both methods was identical for large sample sizes. Thus, motivated by the previous studies, the aim of this article is to conduct the Bayesian analysis of EWG distribution. In particular, we derive the posterior distribution assuming the noninformative and informative priors. We also assume different loss functions to discuss the posterior summaries. To obtain posterior summaries, we use MCMC approach. The rest of the paper is organized as follows: Section 2 presents posterior distributions assuming different priors. Different type of loss functions to obtain the summaries are also discussed in the same section. In Section 3, we present an MCMC algorithm to obtain the numerical summaries under different sample sizes. Section 4 has an illustrative example, while some final remarks are presented in Section 5.

2. Bayesian Estimation

This section presents the derivation of the posterior distribution of α, β, γ and p of the EWG distribution. The posterior distribution is the basic element of the Bayesian inference and is obtained by combining the prior information with the likelihood function. The prior information is a subjective assessment of an expert and plays a crucial role in Bayesian analysis. For further information, we refer to Dey et al. (2015,2016,2017) and references cited therein. Here, we use the uniform prior and different informative priors to derive the posterior distribution.

2.1 The likelihood function

Suppose that $x_1, x_2 \cdots x_n$ is an independent and identically random sample of size n from the EWG distribution. The likelihood function takes the form

$$L(\psi \mid \mathbf{x}) = \prod_{i=1}^{n} f(\mathbf{x}; \alpha, \beta, \gamma, p) = \left[\alpha\beta(1-p)\right]^n \prod_{i=1}^{n} \left[x_i^{\beta-1} (1+\gamma\alpha x_i^{\beta})^{-(1+\frac{1}{\gamma})} \right]$$

$$\times \prod_{i=1}^{n} \left[1 - p(1+\gamma\alpha x_i^{\beta})^{-\frac{1}{\gamma}} \right]^{-2}$$

$$= \left[\alpha\beta(1-p)\right]^n \mathrm{e}^{\beta\sum_{i=1}^{n}\ln x_i} \mathrm{e}^{-\sum_{i=1}^{n}\ln x_i} e^{-(1+\frac{1}{\gamma})\sum_{i=1}^{n}\ln(1+\gamma\alpha x_i^{\beta})} e^{-2\sum_{i=1}^{n}\ln\left[1-p\left(1+\gamma\alpha x_i^{\beta}\right)^{-\left(\frac{1}{\gamma}\right)}\right]}$$

2.2 The posterior distribution assuming the Uniform Prior (UP)

The uniform prior is used as a noninformative prior when no or little prior information is available. In fact, it was initially used by Bayes (1763) and Laplace (1820) and later by Geisser (1984). We assume the following joint uniform prior for the parameters

$$g_1(\psi) \propto 1; \quad \alpha, \beta, \gamma > 0, \ 0 < p < 1$$

It is worth mentioning that we assumed the UP for α, β, γ over the intervals $(0, \infty)$ and $p \in (0,1)$. Thus, the joint posterior distribution of parameters α, β, γ and p given data $\mathbf{x}$, using the UP is:

$$\pi_1(\psi \mid \mathrm{x}) \propto \left[\alpha^n \beta^n e^{\beta\sum_{i=1}^{n}\ln x_i} (1-p)^n e^{-2\sum_{i=1}^{n}\ln[1-p(1+\alpha\gamma x^{\beta}_i)^{-\frac{1}{\gamma}}] - \left(1+\frac{1}{\gamma}\right)\sum_{i=1}^{n}\ln(1+\alpha\gamma x^{\beta}_i)} \right] \tag{5}$$

2.3 The posterior distribution assuming Informative Priors

An informative prior contains fairly precise, specific, definite and scientific information about the quantity of interest. In this section, we derive the posterior distributions assuming gamma and inverse Levy priors as the informative priors.

2.3.1 The posterior distribution assuming gamma prior

We take gamma prior for the α,β,γ while beta distribution for the parameter p to derive the posterior distribution, i.e.,

$$g_2(\alpha;a_1,b_1)=\frac{b_1^{a_1}}{\Gamma(a_1)}\alpha^{(a_1-1)}e^{-(b_1\alpha)},\qquad \alpha>0\,,a_1>0\,,b_1>0$$

$$g_3(\beta;a_1,b_1)=\frac{b_2^{a_2}}{\Gamma(a_2)}\beta^{(a_2-1)}e^{-(b_2\beta)},\qquad \beta>0\,,a_2>0\,,b_2>0$$

$$g_4(\gamma;a_3,b_3)=\frac{b_3^{a_3}}{\Gamma(a_3)}\gamma^{(a_3-1)}e^{-(b_3\gamma)},\qquad \gamma>0\,,a_3>0\,,b_3>0$$

$$g_5(p;a_4,b_4)=\frac{1}{B(a_4,b_4)}p^{(a_4-1)}(1-p)^{(b_4-1)},\quad 0<p<1\,,a_4>0\,,b_4>0$$

The joint prior distribution of parameters α,β,γ and p would be

$$g_6(\psi)\propto\alpha^{(a_1-1)}e^{-(b_1\alpha)}\beta^{(a_2-1)}e^{-(b_2\beta)}\gamma^{(a_3-1)}e^{-(b_3\gamma)}p^{(a_4-1)}(1-p)^{(b_4-1)}\tag{6}$$

The joint posterior distribution of parameters α,β,γ and p given the data **x**:

$$\pi_2(\psi\,|\,\mathbf{x})\propto L(\psi\,|\,\mathrm{x})\times g_6(\psi)$$

$$\pi_2(\psi\,|\,\mathbf{x})\propto\left[\begin{array}{l}\alpha^{n+a_1-1}e^{-\alpha b_1}\beta^{n+a_2-1}e^{-\beta(b_2-\sum_{i=1}^{n}\ln x_i)}p^{a_4-1}(1-p)^{n+b_4-1}\gamma^{a_3-1}.\\ e^{-\gamma b_3-2\sum_{i=1}^{n}\ln[1-\mathrm{p}(1+\gamma\alpha x_i^{\beta})^{-\frac{1}{\gamma}}]-\left(\frac{1}{\gamma}+1\right)\sum_{i=1}^{n}\ln(1+\gamma\alpha x_i^{\beta})}\end{array}\right]\tag{7}$$

The marginal distribution for each parameter can be derived by integrating the other parameters. To this end, we have the following marginal posterior distributions.

$$f(\alpha\,|\,\mathbf{x})=Gamma(\mathrm{n}+a_1,b_1)$$

$$f(\beta\,|\,\mathbf{x})=Gamma(\mathrm{n}+a_2,b_2-\sum_{i=1}^{n}\ln x_i),\qquad f(p\,|\,\mathbf{x})=Beta(a_4,\mathrm{n}+b_4)$$

$$f(\gamma\,|\,\alpha,\beta,p,\mathbf{x})\propto\gamma^{a_3-1}e^{-\gamma b_3-2\sum_{i=1}^{n}\ln[1-\mathrm{p}(1+\gamma\alpha x_i^{\beta})^{-\frac{1}{\gamma}}]-\left(\frac{1}{\gamma}+1\right)\sum_{i=1}^{n}\ln(1+\gamma\alpha x_i^{\beta})}$$

It is easy to show that $f(\gamma\,|\,\alpha,\beta,p,\mathbf{x})$ is log-concave density and thus, the idea of Gilks and Wild (1992) of Acceptance and Rejection sampling can be used to generate $f(\gamma\,|\,\alpha,\beta,p,\mathbf{x})$. Next, we derive the posterior distribution assuming the inverse Levy prior (ILP).

2.3.2 The posterior distribution using the ILP

We assume that the prior distribution of α, β, γ is the inverse Levy (IL) while the beta distribution for parameter p with the following densities:

$$h_1(\alpha;\mathrm{c}_1)=\sqrt{\frac{c_1}{2\pi}}\alpha^{-\frac{1}{2}}e^{-\frac{\alpha c_1}{2}}\ ;\ \alpha>0, c_1>0$$

$$h_2(\beta;\mathrm{c}_2)=\sqrt{\frac{c_2}{2\pi}}\beta^{-\frac{1}{2}}e^{-\frac{\beta c_2}{2}}\ ;\ \beta>0, c_2>0$$

$$h_3(\gamma;\mathrm{c}_3)=\sqrt{\frac{c_3}{2\pi}}\gamma^{-\frac{1}{2}}e^{-\frac{\gamma c_3}{2}}\ ;\quad \gamma>0, c_3>0$$

$$h_4(p;\mathrm{c}_4,\mathrm{d})=\frac{\Gamma c_4+d}{\Gamma c_4\Gamma d}p^{c_4-1}(1-p)^{d-1};\ 0<p<1, c_4>0, d>0$$

The joint prior distribution of parameters α, β, γ and p using the ILP is

$$h_5(\psi)\propto \alpha^{-\frac{1}{2}}e^{-\frac{\alpha c_1}{2}}\beta^{-\frac{1}{2}}e^{-\frac{\beta c_2}{2}}\gamma^{-\frac{1}{2}}e^{-\frac{\gamma c_3}{2}}p^{c_4-1}(1-p)^{d-1} \tag{8}$$

and the joint posterior distribution is:

$$\pi_3(\psi\,|\,\mathbf{x})\propto L(\psi\,|\,\mathrm{x})\times h_6(\psi)$$

$$\pi_3(\psi\,|\,\mathrm{x})\propto\left[\begin{array}{l}\alpha^{\left(n+\frac{1}{2}\right)-1}e^{-\frac{\alpha c_1}{2}}\beta^{\left(n+\frac{1}{2}\right)-1}e^{-\beta\left(\frac{c_2}{2}-\sum_{i=1}^{n}\ln x_i\right)}p^{a_4-1}(1-p)^{n+b_4-1}\gamma^{-\frac{1}{2}}.\\ e^{-\frac{\gamma c_3}{2}-2\sum_{i=1}^{n}\ln\left[1-p\left(1+\gamma\alpha x_i^{\beta}\right)^{-\left(\frac{1}{\gamma}\right)}\right]-\left(1+\frac{1}{\gamma}\right)\sum_{i=1}^{n}\ln(1+\gamma\alpha x_i^{\beta})}\end{array}\right] \tag{9}$$

The marginal distribution for each parameter can be written as:

$$f(\alpha\,|\,\mathrm{x})=Gamma(n+\frac{1}{2},\frac{\mathrm{c}_1}{2})$$

$$f(\beta\,|\,\mathrm{x})=Gamma(n+\frac{1}{2},\frac{\mathrm{c}_2}{2}-\sum_{i=1}^{n}\ln x_i)\ ,\quad f(p\,|\,\mathrm{x})=Beta(a_4,n+b_4)$$

$$f(\gamma\,|\,\alpha,\beta,p,\mathrm{x})\propto\gamma^{-\frac{1}{2}}e^{-\frac{\gamma c_3}{2}-2\sum_{i=1}^{n}\ln[1-\mathrm{p}(1+\gamma\alpha x_i^{\beta})^{-\frac{1}{\gamma}}]-\left(\frac{1}{\gamma}+1\right)\sum_{i=1}^{n}\ln(1+\gamma\alpha x_i^{\beta})}$$

Again, it is easy to show that $f(\gamma\,|\,\alpha,\beta,p,\mathrm{x})$ is log-concave density and the idea of Gilks and Wild (1992) of Acceptance and Rejection sampling can be used to generate γ from $f(\gamma\,|\,\alpha,\beta,p,\mathrm{x})$.

2.4 Bayes estimators and posterior risks under different loss functions

Here, we present the derivation of the Bayes estimator (BE) and posterior risks (PR) assuming different loss functions. To be more specific, we use the squared error loss function (SELF), precautionary loss function (PLF), weighted squared error loss function (WSELF), squared-logarithmic error loss function (SLELF), and entropy loss function (ELF). The Bayes estimators and the respective posterior risks under the mentioned loss functions are given in Table 1.

Loss Function $L(\psi,\widehat{\psi})$	**Bayes Estimator**	**Posterior Risk**
SELF $\left(\psi-\widehat{\psi}\right)^2$	$E(\psi\mid \mathrm{x})$	$\mathrm{var}(\psi\mid \mathrm{x})$
PLF $\dfrac{\left(\psi-\widehat{\psi}\right)^2}{\widehat{\psi}}$	$\sqrt{E(\psi^2\mid \mathrm{x})}$	$2\left\{\sqrt{E(\psi^2\mid \mathrm{x})}-E(\psi\mid \mathrm{x})\right\}$
WSELF $\dfrac{\left(\psi-\widehat{\psi}\right)^2}{\widehat{\psi}}$	$\left[E(\psi^{-1}\mid \mathrm{x})\right]^{-1}$	$E(\psi\mid \mathrm{x})-\left[E(\psi^{-1}\mid \mathrm{x})\right]^{-1}$
SLELF $\left(\ln\psi-\ln\widehat{\psi}\right)^2$	$\exp\left\{E(\ln\psi\mid \mathrm{x})\right\}$	$E\left\{(\ln\psi\mid \mathrm{x})\right\}^2-\left\{E(\ln\psi\mid \mathrm{x})\right\}^2$
ELF $b\left\{\left(\dfrac{\widehat{\psi}}{\psi}\right)-\ln\left(\dfrac{\widehat{\psi}}{\psi}\right)-1\right\}$	$\left[E(\psi^{-1}\mid \mathrm{x})\right]^{-1}$	$\ln\left\{E(\psi^{-1}\mid \mathrm{x})\right\}+E\left(\ln\psi\mid \mathrm{x}\right)$

Table 1: Bayes Estimator and Posterior Risks under different Loss Functions

To derive the BE and PR under different loss functions, as mentioned in Table 1, we give some details here. For example, to derive the Bayes estimator and posterior risk under the WSELF

$$L\left(\lambda,\hat{\lambda}\right)=\frac{\left(\lambda-\hat{\lambda}\right)^2}{\lambda} \tag{10}$$

we take expectation of Eq.10 with respect to the posterior of parameter λ, i.e.,

$$E_{\lambda|\mathbf{x}}\left\{L\left(\lambda,\hat{\lambda}\right)\right\}=E_{\lambda|\mathbf{x}}\left\{\left(\lambda-\hat{\lambda}\right)^2\Big/\lambda\right\}$$

Now, take the partial derivative with respect to $\hat{\lambda}$ and equate the resulting expression equals zero, we have

$$\frac{\partial E_{\lambda|\mathbf{x}}\left\{L\left(\lambda,\hat{\lambda}\right)\right\}}{\partial\hat{\lambda}}=\frac{2}{\lambda}E_{\lambda|\mathbf{x}}\left(\lambda-\hat{\lambda}\right)\left(-1\right)=0$$

After simplifying for $\hat{\lambda}$, we get $\hat{\lambda}_{WSELF}=\left\{E\left(\lambda^{-1}\mid\mathrm{X}\right)\right\}^{-1}$

Next, to derive the posterior risk, we substitute $\hat{\lambda}_{WSELF}$ in $E_{\lambda|\mathbf{x}}\left\{L\left(\lambda,\hat{\lambda}\right)\right\}=E(\lambda\mid\mathbf{x})+E(\lambda^{-1}\mid\mathbf{x})\hat{\lambda}^2-2\hat{\lambda}$ and get the posterior risk, i.e., $E\left\{\lambda\mid\mathrm{x}\right\}-E\left\{\lambda^{-1}\mid\mathrm{x}\right\}^{-1}$. The Bayes estimators and posterior risks under the other loss functions can be derived in a similar fashion.

3. Posterior Summaries-Markov Chain Monte Carlo Algorithm

Since the posterior distributions, derived in the previous section assuming different priors, are complicated and to obtain different summaries, like posterior mean, risk and credible intervals, we need an MCMC approach. Hence, we propose the following MCMC algorithm:

Algorithm-1:

- Generate $\gamma\sim f(\gamma\mid\alpha,\beta,p)$ by acceptance-rejection sampling after generating $\alpha\sim f(\alpha\mid x)$, $\beta\sim f(\beta\mid\mathrm{x})$, $p\sim f(p\mid\mathrm{x})$ and $\gamma\sim f(\gamma\mid\alpha,\beta,p)$.
- Repeat the above step N (=100000) times to obtain $\left(\psi_1,...,\psi_N\right)=(\alpha_1,\beta_1,\gamma_1,p_1),...,(\alpha_N,\beta_N,\gamma_N,p_N)$
- The Bayes estimate of ψ can be calculated as $\sum_{i=M+1}^{N}\psi_i\Big/\left(N-M\right)$, where M denotes the burn-in period.

To assess the effect of different sample sizes on the Bayes estimates under different loss functions, we consider two sets of parameters, i.e., $\left(\alpha,\beta,\gamma,p\right)=\left\{\left(1,1,1,0.5\right),\left(4.5,6,5,0.5\right)\right\}$, and *n=25, 50* and *75*. We used the aforementioned algorithm to compute the posterior summaries and the resulting study has been listed in Tables 2-4. The Bayes estimates and the respective posterior risk under different loss functions assuming the noninformative and informative priors have been tabulated in Tables 5-7. It is worth mentioning that to compute Bayes estimates, we used mild informative priors, i.e., hyperparameters were selected in such a way that provide the mean of the prior distribution equal to the nominal value of the parameter with large variance.

From the tables, it is clear that as the sample size increased, the Bayes estimates converged to their nominal value. Also, the standard deviation and MCMC error decreased with the increase of sample size. Thus, all the Bayes estimates show the property of consistency, i.e., the MCMC error decreases as the sample size increases. In addition to Bayes estimates, 95% credible intervals and median of each parameter have been reported in Tables 2-4 and we observed that with the increase in sample size, the credible intervals became narrow. For a small sample size, like 25, it was observed that BE of α, γ and p were underestimated while overestimated for β. Moreover, the Bayes estimates and posterior median were observed approximately the same for the larger sample sizes. As far as the choice of a suitable prior is concerned, we noticed that posterior risk under UP is the highest, leading ILP the second while the GP has the lowest. Thus, GP is the most suitable informative prior. We observed that for the first set of parameters, the ELF has the least risk while SLELF is suitable for the second set of parameters. More specifically, we found that the ELF has the least risk for the estimation of α, SLELF for β, and ELF for γ and *p,* respectively.

(α,β,γ,p)	**n**	**Estimate**	**Mean**	**SD**	**Posterior Risk**	**MC Error**
(1,1,1,0.5)	25	α	0.93655	0.03958	0.0015664	0.0001252
		β	1.01549	0.02781	0.0007734	0.0000879
		γ	0.96107	0.02829	0.0008005	0.0000895
		p	0.43650	0.03506	0.0012293	0.0001109
	50	α	0.98083	0.02894	0.0008376	0.0000915
		β	1.01338	0.01901	0.0003615	0.0000601
		γ	0.97550	0.01775	0.0003150	0.0000561
		p	0.47013	0.02696	0.0007266	0.0000852
	75	α	1.01258	0.01598	0.0002554	0.0000505
		β	1.00453	0.01638	0.0002682	0.0000518
		γ	1.00396	0.01520	0.0002310	0.0000481
		p	0.47147	0.01301	0.0001692	0.0000411
(4.5,6,5,0.5)	25	α	4.44478	0.02255	0.0005087	0.0000713
		β	5.97626	0.01211	0.0001467	0.0000383
		γ	4.94786	0.03243	0.0010518	0.0001026
		p	0.43650	0.03506	0.0012293	0.0001109
	50	α	4.48844	0.01108	0.0001227	0.0000350
		β	5.99895	0.01141	0.0001302	0.0000361
		γ	4.95800	0.01875	0.0003517	0.0000593
		p	0.47013	0.02696	0.0007266	0.0000852
	75	α	4.49176	0.00764	0.0000583	0.0000241
		β	6.00471	0.00955	0.0000912	0.0000302
		γ	4.98345	0.00936	0.0000877	0.0000296
		p	0.47147	0.01301	0.0001692	0.0000411

(α,β,γ,p)	n	Estimate	2.5%	Median	97.5%
(1,1,1,0.5)	25	α	0.86633	0.94109	0.99148
		β	0.97164	1.01946	1.05420
		γ	0.92656	0.94701	1.00393
		p	0.38853	0.42607	0.49785
	50	α	0.93164	0.98830	1.01896
		β	0.97669	1.01527	1.04134
		γ	0.94391	0.97750	1.00165
		p	0.41340	0.46996	0.50879
	75	α	0.97748	1.01390	1.04006
		β	0.97980	1.00411	1.03822
		γ	0.97228	1.00628	1.03090
		p	0.44666	0.47343	0.49371
(4.5,6,5,0.5)	25	α	4.41387	4.43848	4.49612
		β	5.95451	5.97701	5.99859
		γ	4.89672	4.94099	4.99640
		p	0.38853	0.42607	0.49785
	50	α	4.46765	4.48928	4.50688
		β	5.97720	6.00105	6.02075
		γ	4.92536	4.95420	4.99430
		p	0.41340	0.46996	0.50879
	75	α	4.47850	4.49186	4.50942
		β	5.98277	6.00651	6.02005
		γ	4.96611	4.98285	5.00243
		p	0.44666	0.47343	0.49371

Table 2: Bayes Estimates and other Posterior Summaries for different Sample Sizes under Uniform Prior

(α,β,γ,p)	n	Estimate	Mean	SD	Posterior Risk	MC Error
(1,1,1,0.5)	25	α	0.95362	0.02115	0.0004472	0.0000669
		β	1.06417	0.02972	0.0008833	0.0000940
		γ	1.04481	0.02012	0.0004046	0.0000636
		p	0.47169	0.02234	0.0004989	0.0000706
	50	α	0.99257	0.01199	0.0001437	0.0000379

		β	1.06299	0.02770	0.0007673	0.0000876
		γ	1.03427	0.01976	0.0003904	0.0000625
		p	0.48098	0.01350	0.0001822	0.0000427
	75	α	1.00109	0.01173	0.0001376	0.0000371
		β	1.01638	0.01261	0.0001590	0.0000399
		γ	1.02719	0.01540	0.0002372	0.0000487
		p	0.49285	0.01152	0.0001328	0.0000364
(4.5,6,5,0.5)	25	α	4.54813	0.03678	0.0013529	0.0001163
		β	5.97446	0.02465	0.0006077	0.0000780
		γ	5.01253	0.01040	0.0001081	0.0000329
		p	0.47169	0.02234	0.0004989	0.0000706
	50	α	4.51551	0.01389	0.0001929	0.0000439
		β	5.99426	0.01226	0.0001503	0.0000388
		γ	5.01067	0.01028	0.0001057	0.0000325
		p	0.48098	0.01350	0.0001822	0.0000427
	75	α	4.50057	0.01144	0.0001308	0.0000362
		β	6.02570	0.01083	0.0001174	0.0000343
		γ	5.00877	0.00946	0.0000895	0.0000299
		p	0.49285	0.01152	0.0001328	0.0000364

(α,β,γ,p)	**n**	**Estimate**	**2.5%**	**Median**	**97.5%**
(1,1,1,0.5)	25	α	0.91790	0.95095	1.00175
		β	1.00813	1.06900	1.11015
		γ	1.00148	1.04633	1.08376
		p	0.43894	0.47119	0.50400
	50	α	0.97229	0.99206	1.01424
		β	1.00870	1.06328	1.11076
		γ	1.00360	1.03130	1.07362
		p	0.46029	0.47929	0.51105
	75	α	0.98005	0.99840	1.02090
		β	0.99594	1.01557	1.04091
		γ	1.00106	1.02670	1.05830
		p	0.47461	0.49131	0.51484

(4.5,6,5,0.5)	25	α	4.49839	4.53652	4.60993
		β	5.92994	5.98048	6.01040
		γ	4.98987	5.01261	5.03293
		p	0.43894	0.47119	0.50400
	50	α	4.49590	4.51140	4.54363
		β	5.97241	5.99421	6.01583
		γ	4.99407	5.00972	5.03096
		p	0.46029	0.47929	0.51105
	75	α	4.48138	4.50000	4.52981
		β	6.00366	6.02662	6.04344
		γ	4.99192	5.00838	5.02682
		p	0.47461	0.49131	0.51484

Table 3: Bayes Estimates and other Posterior Summaries for different Sample Sizes under GP

(α,β,γ,p)	n	Estimate	Mean	SD	Posterior Risk	MC Error
(1,1,1,0.5)	25	α	0.94275	0.03412	0.0011639	0.0001079
		β	1.03067	0.02345	0.0005498	0.0000741
		γ	0.96440	0.03022	0.0009135	0.0000956
		p	0.47169	0.02234	0.0004989	0.0000706
	50	α	0.97143	0.02481	0.0006155	0.0000785
		β	1.02879	0.01275	0.0001625	0.0000403
		γ	0.96988	0.01396	0.0001949	0.0000441
		p	0.48098	0.01350	0.0001822	0.0000427
	75	α	0.99234	0.01421	0.0002019	0.0000449
		β	1.00588	0.01223	0.0001497	0.0000387
		γ	1.00100	0.01205	0.0001451	0.0000381
		p	0.49285	0.01152	0.0001328	0.0000364
(4.5,6,5,0.5)	25	α	4.46623	0.02235	0.0004996	0.0000707
		β	5.98316	0.03180	0.0010112	0.0001006
		γ	4.95844	0.02567	0.0006592	0.0000812
		p	0.47169	0.02234	0.0004989	0.0000706
	50	α	4.48414	0.01666	0.0002774	0.0000527
		β	5.98816	0.01183	0.0001399	0.0000374

		γ	4.99051	0.02207	0.0004871	0.0000698
		p	0.48098	0.01350	0.0001822	0.0000427
	75	α	4.49854	0.01097	0.0001203	0.0000347
		β	5.99500	0.01011	0.0001022	0.0000320
		γ	5.00592	0.01171	0.0001372	0.0000370
		p	0.49285	0.01152	0.0001328	0.0000364

(α,β,γ,p)	**n**	**Estimate**	**2.5%**	**Median**	**97.5%**
(1,1,1,0.5)	25	α	0.88478	0.93095	1.00788
		β	0.98839	1.03459	1.07446
		γ	0.92394	0.95888	1.01553
		p	0.43894	0.47119	0.50400
	50	α	0.91518	0.98123	1.00341
		β	0.99942	1.02880	1.04932
		γ	0.94583	0.96767	1.00443
		p	0.46029	0.47929	0.51105
	75	α	0.96457	0.99285	1.01416
		β	0.98547	1.00491	1.03031
		γ	0.97706	1.00499	1.01739
		p	0.47461	0.49131	0.51484
(4.5,6,5,0.5)	25	α	4.42522	4.46487	4.50466
		β	5.93451	5.98522	6.03456
		γ	4.92120	4.95546	5.00868
		p	0.43894	0.47119	0.50400
	50	α	4.45618	4.48970	4.50771
		β	5.96876	5.98726	6.01216
		γ	4.94921	4.99825	5.01736
		p	0.46029	0.47929	0.51105
	75	α	4.47894	4.49883	4.51435
		β	5.97642	5.99487	6.01251
		γ	4.98141	5.00538	5.02581
		p	0.47461	0.49131	0.51484

Table 4: Bayes Estimates and other Posterior Summaries for different Sample Sizes under ILP

Loss Function	Estimate	(1,1,1,0.5)		
		25	50	75
SELF	$\hat{\alpha}$	0.93655 **(0.0015664)**	0.98083 **(0.0008376)**	1.01258 **(0.0002554)**
	$\hat{\beta}$	1.01549 **(0.0007734)**	1.01338 **(0.0003615)**	1.00453 **(0.0002682)**
	$\hat{\gamma}$	0.96107 **(0.0008005)**	0.97550 **(0.0003150)**	1.00396 **(0.0002310)**
	$\hat{p}$	0.43650 **(0.0012293)**	0.47013 **(0.0007266)**	0.47147 **(0.0001692)**
PLE	$\hat{\alpha}$	0.92735 **(0.0019698)**	0.95895 **(0.0003899)**	0.96644 **(0.0001934)**
	$\hat{\beta}$	0.97186 **(0.0003275)**	0.99424 **(0.0002031)**	1.00097 **(0.0000874)**
	$\hat{\gamma}$	0.93888 **(0.0008965)**	0.95597 **(0.0007751)**	0.99365 **(0.0001850)**
	$\hat{p}$	0.43215 **(0.0024565)**	0.44380 **(0.0025405)**	0.45831 **(0.0007249)**
WSELF	$\hat{\alpha}$	0.90086 **(0.0025336)**	0.96670 **(0.0004545)**	1.00740 **(0.0001964)**
	$\hat{\beta}$	1.03432 **(0.0003197)**	1.03255 **(0.0002840)**	1.02411 **(0.0000772)**
	$\hat{\gamma}$	0.96779 **(0.0006034)**	0.96220 **(0.0004529)**	0.99108 **(0.0001866)**
	$\hat{p}$	0.43431 **(0.0026943)**	0.46100 **(0.0012626)**	0.47624 **(0.0003621)**
SLELF	$\hat{\alpha}$	0.95020 **(0.0009014)**	0.96518 **(0.0006486)**	1.00016 **(0.0001964)**
	$\hat{\beta}$	1.03125 **(0.0004929)**	1.01475 **(0.0001714)**	0.99004 **(0.0001375)**
	$\hat{\gamma}$	0.94048 **(0.0012644)**	0.95490 **(0.0004982)**	1.00741 **(0.0004677)**
	$\hat{p}$	0.45366 **(0.0025071)**	0.46561 **(0.0012244)**	0.47917 **(0.0011473)**
ELF	$\hat{\alpha}$	0.91898 **(0.0008216)**	0.96233 **(0.0004044)**	0.99951 **(0.0000326)**
	$\hat{\beta}$	1.04641 **(0.0001297)**	1.03193 **(0.0001104)**	1.01139 **(0.0000904)**
	$\hat{\gamma}$	0.93661 **(0.0004799)**	0.96033 **(0.0002593)**	0.99730 **(0.0001043)**
	$\hat{p}$	0.42866 **(0.0031930)**	0.45828 **(0.0026001)**	0.46143 **(0.0011465)**

Loss Function	Estimate	(4.5,6,5,0.5)		
		25	50	75
SELF	$\hat{\alpha}$	4.44478 **(0.0005087)**	4.48844 **(0.0001227)**	4.49176 **(0.0000583)**
	$\hat{\beta}$	5.97626 **(0.0001467)**	5.99895 **(0.0001302)**	6.00471 **(0.0000912)**
	$\hat{\gamma}$	4.94786 **(0.0010518)**	4.95800 **(0.0003517)**	4.98345 **(0.0000877)**
	$\hat{p}$	0.43650 **(0.0012293)**	0.47013 **(0.0007266)**	0.47147 **(0.0001692)**
PLE	$\hat{\alpha}$	4.45196 **(0.0003060)**	4.47026 **(0.0001099)**	4.51183 **(0.0000125)**
	$\hat{\beta}$	6.05371 **(0.0000928)**	6.01860 **(0.0000243)**	6.01752 **(0.0000160)**
	$\hat{\gamma}$	4.93095 **(0.0001590)**	4.97276 **(0.0000377)**	5.00053 **(0.0000135)**
	$\hat{p}$	0.43215 **(0.0024565)**	0.44380 **(0.0025405)**	0.45831 **(0.0007249)**
WSELF	$\hat{\alpha}$	4.43075 **(0.0002067)**	4.45568 **(0.0001779)**	4.47806 **(0.0001425)**
	$\hat{\beta}$	6.04873 **(0.0000640)**	6.01623 **(0.0000311)**	5.99840 **(0.0000132)**
	$\hat{\gamma}$	4.96223 **(0.0001268)**	4.99312 **(0.0002175)**	5.00087 **(0.0000788)**
	$\hat{p}$	0.43431 **(0.0026943)**	0.46100 **(0.0012626)**	0.47624 **(0.0003621)**
SLELF	$\hat{\alpha}$	4.46201 **(0.0000586)**	4.46447 **(0.0000116)**	4.49376 **(0.0000092)**
	$\hat{\beta}$	6.02977 **(0.0000111)**	6.01706 **(0.0000194)**	6.00936 **(0.0000028)**
	$\hat{\gamma}$	4.96151 **(0.0000188)**	4.96374 **(8.502×10^{-6})**	4.99198 **(6.909×10^{-6})**
	$\hat{p}$	0.45366 **(0.0025071)**	0.46561 **(0.0012244)**	0.47917 **(0.0011473)**
ELF	$\hat{\alpha}$	4.42178 **(0.0000799)**	4.42611 **(0.0000652)**	4.43655 **(0.0000293)**
	$\hat{\beta}$	6.03442 **(0.0000064)**	6.02296 **(0.0000044)**	6.01187 **(0.0000021)**
	$\hat{\gamma}$	4.97747 **(6.387×10^{-6})**	4.97991 **(5.154×10^{-6})**	4.99975 **(4.871×10^{-6})**
	$\hat{p}$	0.42866 **(0.0031930)**	0.45828 **(0.0026001)**	0.46143 **(0.0011465)**

Table 5: Bayes Estimates of EWG distribution and respective posterior Risk (give in parenthesis) under uniform prior

Loss Function	Estimate	(1,1,1,0.5)		
		25	50	75
SELF	$\hat{\alpha}$	0.95362 **(0.0004472)**	0.99257 **(0.0001437)**	1.00109 **(0.0001376)**
	$\hat{\beta}$	1.06417 **(0.0008833)**	1.06299 **(0.0007673)**	1.01638 **(0.0001590)**
	$\hat{\gamma}$	1.04481 **(0.0004046)**	1.03427 **(0.0003904)**	1.02719 **(0.0002372)**
	$\hat{p}$	0.47169 **(0.0004989)**	0.48098 **(0.0001822)**	0.49285 **(0.0001328)**
PLF	$\hat{\alpha}$	1.04299 **(0.0002844)**	1.02879 **(0.0003481)**	1.01855 **(0.0001157)**
	$\hat{\beta}$	1.08938 **(0.0020220)**	1.03945 **(0.0003683)**	1.01311 **(0.0001680)**
	$\hat{\gamma}$	1.05312 **(0.0002845)**	1.03567 **(0.0002070)**	1.02302 **(0.0001552)**
	$\hat{p}$	0.53370 **(0.0007762)**	0.51155 **(0.0006596)**	0.49951 **(0.0005304)**
WSELF	$\hat{\alpha}$	1.02553 **(0.0009781)**	1.01872 **(0.0001253)**	1.00695 **(0.0001167)**
	$\hat{\beta}$	1.04119 **(0.0007768)**	1.02827 **(0.0005741)**	1.02372 **(0.0002407)**
	$\hat{\gamma}$	1.03143 **(0.0004423)**	1.02712 **(0.0005375)**	1.00919 **(0.0001478)**
	$\hat{p}$	0.47831 **(0.0003126)**	0.48809 **(0.0002881)**	0.49171 **(0.0002080)**
SLELF	$\hat{\alpha}$	0.97241 **(0.0002296)**	0.99223 **(0.0001679)**	1.02186 **(0.0001037)**
	$\hat{\beta}$	1.04012 **(0.0008413)**	1.02996 **(0.0011804)**	1.01899 **(0.0008421)**
	$\hat{\gamma}$	1.06058 **(0.0020873)**	1.04014 **(0.0003586)**	1.02755 **(0.0001746)**
	$\hat{p}$	0.53320 **(0.0011155)**	0.51706 **(0.0002528)**	0.50019 **(0.0001264)**
ELF	$\hat{\alpha}$	0.95194 **(0.0006510)**	0.99919 **(0.0000534)**	1.01098 **(0.0000320)**
	$\hat{\beta}$	1.01183 **(0.0003717)**	1.01179 **(0.0003112)**	1.00699 **(0.0004130)**
	$\hat{\gamma}$	1.06305 **(0.0003717)**	1.03128 **(0.0001340)**	1.00051 **(0.0000647)**
	$\hat{p}$	0.53561 **(0.0011421)**	0.50584 **(0.0004436)**	0.50267 **(0.0003471)**

Loss Function	Estimate	(4.5,6,5,0.5)		
		25	50	75
SELF	$\hat{\alpha}$	4.54813 **(0.0013529)**	4.51551 **(0.0001929)**	4.50057 **(0.0001308)**
	$\hat{\beta}$	5.97446 **(0.0006077)**	5.99426 **(0.0001503)**	6.02570 **(0.0001174)**
	$\hat{\gamma}$	5.01253 **(0.0001081)**	5.01067 **(0.0001057)**	5.00877 **(0.0000895)**
	$\hat{p}$	0.47169 **(0.0004989)**	0.48098 **(0.0001822)**	0.49285 **(0.0001328)**
PLF	$\hat{\alpha}$	4.46163 **(0.0002325)**	4.48142 **(0.0000321)**	4.48587 **(0.0000165)**
	$\hat{\beta}$	5.97340 **(0.0001670)**	5.99881 **(0.0000364)**	6.01694 **(0.0000338)**
	$\hat{\gamma}$	5.06389 **(0.0001225)**	5.03894 **(0.0001185)**	5.00650 **(0.0000499)**
	$\hat{p}$	0.53370 **(0.0007762)**	0.51155 **(0.0006596)**	0.49951 **(0.0005304)**
WSELF	$\hat{\alpha}$	4.53294 **(0.0001137)**	4.52622 **(0.0000678)**	4.51272 **(0.0000255)**
	$\hat{\beta}$	5.95809 **(0.0000853)**	5.97517 **(0.0000232)**	5.99308 **(0.0000171)**
	$\hat{\gamma}$	5.04254 **(0.0000575)**	5.02849 **(0.0000446)**	5.00003 **(0.0000283)**
	$\hat{p}$	0.47831 **(0.0003126)**	0.48809 **(0.0002881)**	0.49171 **(0.0002080)**
SLELF	$\hat{\alpha}$	4.48907 **(0.0000083)**	4.49293 **(0.0000049)**	4.50440 **(0.0000006)**
	$\hat{\beta}$	5.97092 **(0.0000066)**	6.00216 **(0.0000053)**	6.00059 **(0.0000023)**
	$\hat{\gamma}$	4.92099 **(0.0000718)**	4.98634 **(5.378×10^{-6})**	5.00334 **(4.793×10^{-6})**
	$\hat{p}$	0.53320 **(0.0011155)**	0.51706 **(0.0002528)**	0.50019 **(0.0001264)**
ELF	$\hat{\alpha}$	4.47752 **(0.0000122)**	4.49108 **(0.0000063)**	4.49208 **(0.0000017)**
	$\hat{\beta}$	5.97539 **(0.0000105)**	5.97954 **(0.0000033)**	5.99451 **(0.0000024)**
	$\hat{\gamma}$	4.95602 **(8.785×10^{-6})**	4.98634 **(5.378×10^{-6})**	4.99070 **(2.409×10^{-6})**
	$\hat{p}$	0.53561 **(0.0011421)**	0.50584 **(0.0004436)**	0.50267 **(0.0003471)**

Table 6: Bayes Estimates of EWG distribution and their posterior Risk (in parenthesis) under Gamma Prior

Loss Function	**Estimate**	**(1,1,1,0.5)**		
		25	**50**	**75**
SELF	$\hat{\alpha}$	0.94275 **(0.0011639)**	0.97143 **(0.0006155)**	0.99234 **(0.0002019)**
	$\hat{\beta}$	1.03067 **(0.0005498)**	1.02879 **(0.0001625)**	1.00588 **(0.0001497)**
	$\hat{\gamma}$	0.96440 **(0.0009135)**	0.96988 **(0.0001949)**	1.00100 **(0.0001451)**
	$\hat{p}$	0.47169 **(0.0004989)**	0.48098 **(0.0001822)**	0.49285 **(0.0001328)**
PLF	$\hat{\alpha}$	0.98157 **(0.0004284)**	0.98364 **(0.0001381)**	0.99414 **(0.0000867)**
	$\hat{\beta}$	1.08375 **(0.0007076)**	1.03277 **(0.0002907)**	1.01078 **(0.0002661)**
	$\hat{\gamma}$	1.03309 **(0.0008486)**	1.02206 **(0.0003698)**	1.00435 **(0.0000919)**
	$\hat{p}$	0.45616 **(0.0011351)**	0.46019 **(0.0007409)**	0.50798 **(0.0003755)**
WSELF	$\hat{\alpha}$	0.97861 **(0.0010678)**	0.98058 **(0.0004308)**	0.99928 **(0.0002680)**
	$\hat{\beta}$	1.03484 **(0.0006471)**	1.03269 **(0.0002651)**	1.02375 **(0.0001233)**
	$\hat{\gamma}$	0.97402 **(0.0008865)**	0.97987 **(0.0003190)**	0.99786 **(0.0001027)**
	$\hat{p}$	0.57564 **(0.0038404)**	0.52355 **(0.0005664)**	0.50616 **(0.0001663)**
SLELF	$\hat{\alpha}$	0.95946 **(0.0021970)**	0.96485 **(0.0014660)**	1.01979 **(0.0004651**)
	$\hat{\beta}$	1.05174 **(0.0004849)**	1.03675 **(0.0003366)**	1.02397 **(0.0001605)**
	$\hat{\gamma}$	0.93205 **(0.0019196)**	0.97981 **(0.0009997)**	0.99250 **(0.0007861)**
	$\hat{p}$	0.54765 **(0.0007141)**	0.53426 **(0.0007094)**	0.50955 **(0.0006328)**
ELF	$\hat{\alpha}$	0.97578 **(0.0002010)**	0.98641 **(0.0001885)**	0.99964 **(0.0000377)**
	$\hat{\beta}$	1.06265 **(0.0005464)**	1.03424 **(0.0004674)**	1.01649 **(0.0003210)**
	$\hat{\gamma}$	0.93357 **(0.0009070)**	0.96998 **(0.0001741)**	1.00518 **(0.0000518)**
	$\hat{p}$	0.47086 **(0.0007311)**	0.47281 **(0.0005283)**	0.51264 **(0.0001571)**

Loss Function	Estimate	(4.5,6,5,0.5)		
		25	50	75
SELF	$\hat{\alpha}$	4.46623 **(0.0004996)**	4.48414 **(0.0002774)**	4.49854 **(0.0001203)**
	$\hat{\beta}$	5.98316 **(0.0010112)**	5.98816 **(0.0001399)**	5.99500 **(0.0001022)**
	$\hat{\gamma}$	4.95844 **(0.0006592)**	4.99051 **(0.0004871)**	5.00592 **(0.0001372)**
	$\hat{p}$	0.47169 **(0.0004989)**	0.48098 **(0.0001822)**	0.49285 **(0.0001328)**
PLF	$\hat{\alpha}$	4.46471 **(0.0001684)**	4.47315 **(0.0000805)**	4.48156 **(0.0000776)**
	$\hat{\beta}$	5.94154 **(0.0001272)**	5.95827 **(0.0001261)**	6.02155 **(0.0000224)**
	$\hat{\gamma}$	4.96840 **(0.0000834)**	4.99392 **(0.0000498)**	4.99912 **(0.0000172)**
	$\hat{p}$	0.45616 **(0.0011351)**	0.46019 **(0.0007409)**	0.50798 **(0.0003755)**
WSELF	$\hat{\alpha}$	4.45634 **(0.0001196)**	4.46781 **(0.0000848)**	4.48918 **(0.0000173)**
	$\hat{\beta}$	6.06449 **(0.0003253)**	6.01175 **(0.0000545)**	6.00660 **(0.0000430)**
	$\hat{\gamma}$	4.96638 **(0.0000632)**	4.98243 **(0.0000381)**	4.99775 **(0.0000412)**
	$\hat{p}$	0.57564 **(0.0038404)**	0.52355 **(0.0005664)**	0.50616 **(0.0001663)**
SLELF	$\hat{\alpha}$	4.43016 **(0.0001034)**	4.48069 **(0.0000483)**	4.48167 **(0.0000107)**
	$\hat{\beta}$	6.05739 **(0.0000212)**	6.00046 **(0.0000100)**	6.00019 **(0.0000064)**
	$\hat{\gamma}$	4.97512 **(0.0000371)**	4.97993 **(0.0000312)**	4.99283 **(5.370×10^{-6})**
	$\hat{p}$	0.54765 **(0.0007141)**	0.53426 **(0.0007094)**	0.50955 **(0.0006328)**
ELF	$\hat{\alpha}$	4.43413 **(0.0000213)**	4.43819 **(0.0000198)**	4.49249 **(0.0000036)**
	$\hat{\beta}$	6.06924 **(0.0000076)**	6.02540 **(0.0000067)**	6.00003 **(0.0000010)**
	$\hat{\gamma}$	4.96126 **(0.0000105)**	4.99443 **(4.839×10^{-6})**	4.99999 **(3.336×10^{-6})**
	$\hat{p}$	0.47086 **(0.0007311)**	0.47281 **(0.0005283)**	0.51264 **(0.0001571)**

Table 7: Bayes Estimates of EWG distribution and their posterior Risk (in parenthesis) under Inverse Levy Prior

4. An Application

For illustration purpose, a real data set about the gauge lengths of 10mm is taken from Kundu and Gupta (2006). The data represent the strength measured in GPA, for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 1, 10, 20 and 50 mm. For illustrative purpose, we consider the single fibers of 10 mm gauge length. The total observations are sixty three and the data set is reproduced in Table 8.

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397
2.445	2.474	2.454	2.518	2.522	2.525	2.532	2.575	2.614
2.616	2.618	2.624	2.738	2.659	2.675	2.740	2.856	2.917
2.928	2.937	2.937	2.977	2.996	3.145	3.030	3.125	3.139
3.220	3.223	3.235	3.243	3.264	3.272	3.294	3.435	3.332
3.346	3.377	3.408	3.493	3.501	3.537	3.554	3.562	3.628
4.027	3.852	3.871	3.886	3.971	4.024	4.225	4.395	5.020

Table 8: Gauge lengths of 10mm

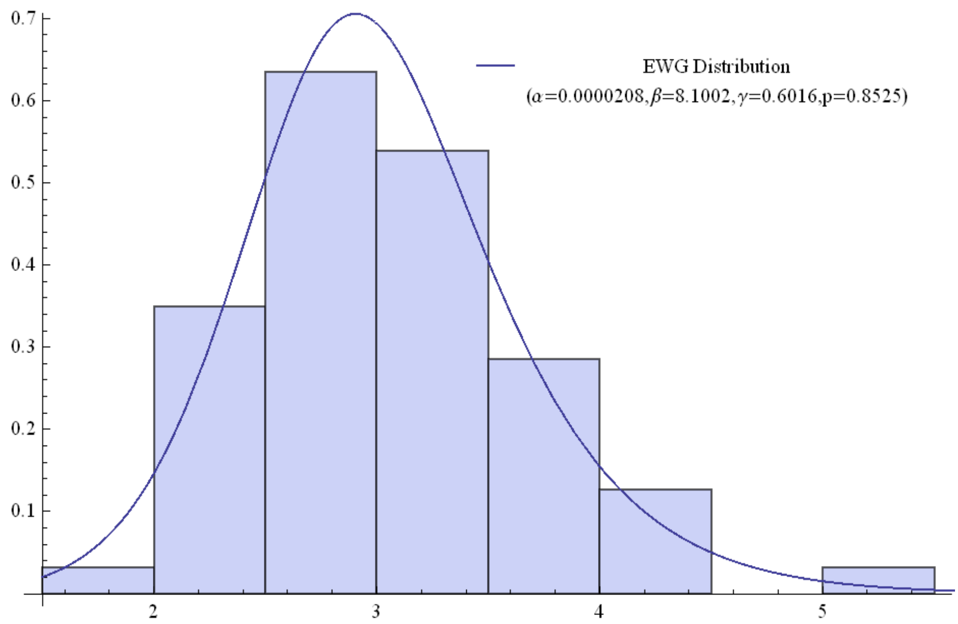


Figure 2: Histogram and the fitted EWG

First, we fitted the EWG distribution to this data set and found, by Kolmogrov Smirnove test (p-value=0.85), that it fits well. Next, we estimated its parameters by Bayesian method and the results have been tabulated in Table 9. It is worth mentioning that we used UP to compute the summaries in Table 9. From the table, it is clear that the proposed MCMC algorithm efficiently estimates the unknown parameters and results are inline as discussed in the simulation study section.

Estimate	Mean	SD	MC error	2.5%	Median	97.5%
α	0.0000208	5.175×10^{-6}	1.636×10^{-8}	0.0000126	0.0000202	0.0000326
β	8.10020	0.01192	0.0000376	8.07384	8.09954	8.12101
γ	0.60163	0.01648	0.0000521	0.56909	0.59962	0.62912
p	0.85252	0.10171	0.0000540	0.82224	0.85319	0.88245

Table 9: Posterior summaries of Gauge data

5. Conclusion

In this article, Bayesian approach to estimate the parameters of EWG distribution has been presented. In particular, posterior distributions using UP, GP and ILP have been derived. The Bayes estimates have been calculated assuming different loss functions. Since the posterior distributions were not in simplified form, we proposed an MCMC algorithm to estimate the unknown parameters. From a comprehensive simulation study assuming different sample sizes, it is found that the gamma prior has the least posterior risk than the rest. For a small parameter values, it is noticed that the ELF has the least risk while SLELF for the second set. More specifically, it is noticed noticed that the ELF has the least risk for the estimation of α , SLELF for β, and ELF for γ and *p*. In the future, one can consider the Bayesian analysis of the EWG distribution assuming lower and upper record values (Yousaf et al., 2019). Furthermore, the mixture of EWG can be introduced (Sultana and Aslam, 2017, Sultana et al., 2017).

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