Journal of Reliability and Statistical Studies; ISSN (Print): 0974-8024, (Online): 2229-5666
Vol. 12, Issue 1 (2019): 01-22

THE WEIBULL-MOMENT EXPONENTIAL DISTRIBUTION: PROPERTIES, CHARACTERIZATIONS AND APPLICATIONS

[1]Sharqa Hashmi, [2,3*]Muhammad Ahsan Ul Haq, [3]Rana Muhammad Usman and [4]Gamze Ozel
[1]Department of Statistics, Lahore College for Women University, Pakistan
[2]Quality Enhancement Cell, National College of Arts, Lahore, Pakistan
[3]College of Statistical & Actuarial Sciences, University of the Punjab, Pakistan
[4]Department of Statistics, Hacettepe University, Turkey
*E Mail: ahsanshani36@gmail.com

Received August 02, 2018
Modified November 30, 2018
Accepted December 29, 2018

Abstract

In this article, another three-parameter Weibull moment exponential (WME) distribution is derived and studied. The proposed distribution is more flexible because of various different shapes of hazard rate function including monotone and non-monotone. Mathematical properties of the WME model are derived such as moments, m.g.f, quantile function, and Rényi entropy. The pdf's of its order statistics are also obtained. Characterizations on two aspects are also presented including the ratio of truncated moments and hazard rate function. Model parameters are estimated by the method of maximum likelihood. Monte Carlo simulations are used to show the consistency of parameters. Some real data applications are given to illustrate the flexibility of the proposed distribution among other competitive models. It may be concluded that the proposed model is entirely adaptable for lifetime datasets with either monotone or non-monotone shape of hazard rate function.

Key Words: Moment Exponential, Moments, Order Statistics, Maximum Likelihood.

1. Introduction

Several univariate continuous distributions have been extensively used in environmental, engineering, financial, biomedical sciences, among other areas for modeling lifetime data. There is still a strong need for generalized distributions and a significant improvement has been made to generalize classical distributions in recent years. These distributions are useful in lifetime analysis (reliability analysis), insurance, economy, finance, and engineering. The exponential distribution has been used to deal with lifetime datasets in reliability analysis (Epstein, 1958). Its generalizations include the double exponential (Norton, 1984), exponentiated exponential (Gupta and Kundu, 2001), transmuted exponentiated exponential (Merovci, 2013), moment exponential (Dara and Ahmed, 2012) and moment exponential (ME) distribution is obtained by assigning linear weights to well-knownexponential distribution.

Some generalizations of probability distributions have been proposed to introduce new families which are more flexible than baseline models. For example, odd generalized exponential (Tahir et al. 2015), exponential (Shushi 2017), new exponentiated extended-G (Elgarhy et al. 2017), odd Frèchet-G (Haq et al. 2018) and moment exponential-G (Haq et al. 2018) family of distributions. One of these generalizations is Weibull-G family of distribution suggested by Bourguignon et al. (2014).

A random variable (r.v.) Y is said to have moment exponential distribution with a scale parameter β, if its cdf is given as

$$G(y;\beta)=1-\left(1+\frac{y}{\beta}\right)e^{-\left(\frac{y}{\beta}\right)},\quad y>0,\beta>0,$$

and its associated pdf is given by

$$g(y;\beta)=\frac{y}{\beta^2}e^{-\left(\frac{y}{\beta}\right)},\ y>0,\beta>0. \tag{1}$$

For different value of the shape parameter, the pdf can take different shapes. Adding parameters to a well-defined distribution is a way of obtaining more flexible new families of distributions. A few extensions of moment exponential distribution are accessible, for example exponentiated moment exponential (EME) (Hasnain and Ahmad, 2013), generalized exponentiated moment exponential (GEME) (Iqbal et al. 2014) and Marshall-Olkin length-biased exponential (MOLBE) (Haq et al. 2017) distributions.

The first parameter induction to the moment exponential was proposed by Hasnain and Ahmed (2013) taking the power of cdf of moment exponential distribution. The two-parameter exponentiated moment exponential cdf is

$$H(y;\alpha,\beta)=\left[1-\left(1+\frac{y}{\beta}\right)e^{-\left(\frac{y}{\beta}\right)}\right]^{\alpha},\quad y>0,\alpha,\beta>0,$$

where α>0 is a shape parameter and its associated density function is

$$h(y;\alpha,\beta)=\frac{\alpha y}{\beta^2}\left[1-\left(1+\frac{y}{\beta}\right)e^{-\left(\frac{y}{\beta}\right)}\right]^{\alpha-1}e^{-\left(\frac{y}{\beta}\right)},\quad y>0,\alpha,\beta>0.$$

The second parameter extension to the moment exponential distribution is GEME suggested by Iqbal et al. (2014) and its cdf is

$$G(y;\alpha,\beta,\gamma)=\left[1-\left(1+\frac{y^{\gamma}}{\beta}\right)e^{-\left(\frac{y^{\gamma}}{\beta}\right)}\right]^{\alpha},\quad y>0,\alpha,\beta,\gamma>0,$$

where $\gamma>0$ is another shape parameter and its pdf reduces to

$$g(y;\alpha,\beta,\gamma)=\frac{\alpha\gamma y^{2\gamma-1}}{\beta^2}\left[1-\left(1+\frac{y^{\gamma}}{\beta}\right)e^{-\left(\frac{y^{\gamma}}{\beta}\right)}\right]^{\alpha-1}e^{-\left(\frac{y^{\gamma}}{\beta}\right)},\quad y>0,\alpha,\beta,\gamma>0.$$

Another second parameter extension of moment exponential distribution is Marshall-Olkin length-biased exponential (MOLBE) distribution presented by Haq et al. (2017).

In this research paper, we propose and study mathematical properties of this extension of the moment exponential (ME) distribution called Weibull moment exponential (WME) distribution. This new distribution would be useful for modeling lifetime datasets. For example, time until disease recurrence, time until cancer patients' death after some treatment intervention or time until a part of a machine fails can fit the proposed distribution.

Let $g(y;\gamma)$ and $G(y;\gamma)$ denote the pdf and cdf of the baseline probability distribution with parameter vector γ. The cdf of the Weibull distribution is $F(y)=1-e^{-(ay^b)}$ with positive parameters a and b and $Y>0$. Bourguignon et al. (2014) considered the upper limit $G(y,\gamma)/\bar{G}(y,\gamma)$ in this cdf, where $\bar{G}(y,\gamma)=1-G(y,\gamma)$ and the cdf of the Weibull-G family is as follows:

$$F(y,a,b,\gamma)=ab\int_0^{G(y,\gamma)/\bar{G}(y,\gamma)} t^{b-1}e^{-(at^b)}dt$$

$$F(y,a,b,\gamma)=1-\exp\left\{-a\left[\frac{G(y,\gamma)}{\bar{G}(y,\gamma)}\right]^b\right\}, y>0 \tag{2}$$

Then, the density function of the Weibull-G distribution becomes

$$f(y;a,b,\gamma)=abg(y;\gamma)\left[\frac{G(y,\gamma)^{b-1}}{\bar{G}(y,\gamma)^{b+1}}\right]\exp\left\{-a\left[\frac{G(y,\gamma)}{\bar{G}(y,\gamma)}\right]^b\right\}. \tag{3}$$

Motivated by the Weibull-G family, many probability distributions have been proposed and studied in the literature. Few examples include Weibull-Pareto by Alzaatreh et al. (2013) and Nasiru and Luguterah (2015), transmuted New Weibull-Pareto (Tahir et al. 2018), Weibull-Dagum distribution by Tahir et al. (2016), Weibull-Power Function distribution by Tahir et al. (2014), transmuted Weibull Power Function (Haq et al. 2018) and transmuted Weibull-Fréchet distribution (Haq et al. 2017). Weibull-Lomax distribution by Tahir et al. (2015), Weibull-Rayleigh by Merovci and Elbatal (2015), Weibull-exponential distribution by Oguntunde et al. (2015). However, these distributions are not flexible enough for the lifetime datasets. Thus, the main purpose of this paper is to derive a new three-parameter model named as the Weibull-moment exponential (WME) distribution based on this idea.

This article is arranged in following manner: In Section 2, the expressions of cumulative distribution function and density functions of proposed model are defined. Then, we derive the linear representation of the pdf and its important main mathematical properties. In Section 3, the entropy functions of the WME distribution are also given. The mean residual life and mean inactivity time are presented in Section 4. Characterization of distribution based on truncated moments and hazard function are given in Section 5. The order statistics of the WME distribution are given in Section 6. In Section 7, we considered maximum likelihood estimation (MLE) method to estimate the unknown parameters. Monte Carlo simulations are used to show the performance of the WME distribution is presented in Section 8. The flexibility of the derived distribution is assessed by few real data applications in Section 9. Finally, final comments are given in Section 10.

2. Properties of the Weibull Moment Exponential Distribution

2.1. Some important functions

By putting Eq. (1) in Eq. (2), the cdf of the WME distribution is

$$F(y)=1-\exp\left\{-a\left[\frac{1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}\right]^{b}\right\}. \tag{4}$$

The corresponding pdf is obtained as

$$f(y)=ab\frac{y}{\beta^{2}}e^{-\frac{y}{\beta}}\frac{\left[1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right]^{b-1}}{\left[\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right]^{b+1}}\exp\left\{-a\left[\frac{1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}\right]^{b}\right\} \tag{5}$$

Henceforth we denote $Y\sim WME(a,b,\beta)$ a r. v. having pdf (5) and its survival function is

$$S(y)=\exp\left\{-a\left[\frac{1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}\right]^{b}\right\} \tag{6}$$

In lifetime data analysis, the expression of hazard rate function (hrf) has a vital significance. That's why, we will study hrf of the WME distribution and it is defined as

$$h(x)=ab\frac{x}{\beta^{2}}e^{-\frac{x}{\beta}}\frac{\left[1-\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{b-1}}{\left[\left(1+\frac{x}{\beta}\right)e^{-\frac{x}{\beta}}\right]^{b+1}} \tag{7}$$

Figures 1 (a) and 1 (b) show the graphs of pdf and hrf of the WME distribution for some specific parametric values. The unimodal and monotonically decreasing shapes can be seen in figure 1 (a), which shows the WME distribution more flexible than the ME distribution. Furthermore, Figure 1(b) shows different shapes of hrf of the WME distribution such as increasing and bathtub. This makes the WME demonstrate valuable and appropriate when non-monotone sample hazard behaviors are shown.

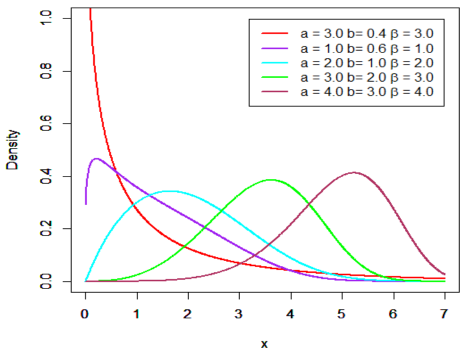

Figure 1(a). The pdf plots

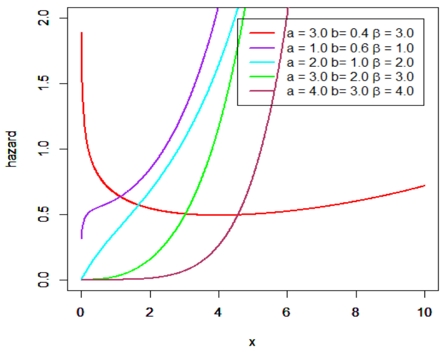

Figure 1(b). The hrf plots

Further, Table 1 displays some special models of WME distributions.

a	b	β	Reduced Model	Authors
-	-	1	Weibull distribution	Weibull (1951)
-	2	-	Rayleigh moment exponential distribution	New
-	1	-	Exponential moment exponential distribution	New
1	1	-	Moment exponential distribution	Dara and Ahmed (2012)
-	2	1	Rayleigh distribution	Rayleigh (1880)

Table 1: Some special models

2.2. Shape characteristics of the pdf and hrf of WME Distribution

The limiting behavior of the density and hrf are presented in following theorem.

Theorem: Let Y be ar.v. $Y \sim WME(a,b,\beta)$ then the limit of pdf and hrf of WME distribution at origin and as y approaches to infinity are given by

$$\lim_{y\to 0} f(y) = \lim_{y\to 0} h(y) = \begin{cases} \infty & b < 0.5 \\ \dfrac{a}{\sqrt{2\beta}} & b = 0.5 \\ 0 & b > 0.5 \end{cases}$$

and $\lim_{y\to\infty} f(y) = 0$, $\lim_{y\to\infty} h(y) = \infty$

Proof: For the given relation

f(y) = h(y)S(y), the limits are obvious.

From the above theorem, it can be observe that shape of density function may assume following shapes

- Decreasing when b<0.5, starting from $a/\sqrt{2\beta}$ and increase or decrease when b=0.5, and starting from origin and form modal shape if b>0.5.

- The density function always touches x-axis as x approaches to ∞.
- The hazard curve always form increasing trend or bathtub shape for all combinations of parametric values.

2.3. Expansion for the density function

We describe density function of WME model as a mixture representation in terms of power series expansion:

$$\exp\left\{-a\left[\frac{1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}\right]^{b}\right\}=\sum_{i=0}^{\infty}\frac{(-1)^{i}a^{i}}{i!}\left[\frac{1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}\right]^{bi}$$

Then, the pdf of the WME distribution reduces to

$$f(y)=b\frac{y}{\beta^{2}}e^{-\frac{y}{\beta}}\sum_{i=0}^{\infty}\frac{(-1)^{i}a^{i+1}}{i!}\left[1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right]^{b(i+1)-1}\left[\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right]^{-(b(i+1)+1)}$$

where

$$\left[1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right]^{b(i+1)-1}=\sum_{k=0}^{\infty}(-1)^{k}\binom{b(i+1)-1}{k}\left(1+\frac{y}{\beta}\right)^{k}e^{-\frac{ky}{\beta}}.$$

Therefore, we can also write the pdf as follows:

$$f(y)=\frac{by\,e^{-\frac{y}{\beta}}}{\beta^{2}}\sum_{i,k=0}^{\infty}\frac{(-1)^{i+k}a^{i+1}}{i!}\binom{b(i+1)-1}{k}\left[\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right]^{k-(b(i+1)+1)}$$

where $\left(1+\frac{y}{\beta}\right)^{k-b(i+1)-1}=\sum_{j=0}^{\infty}\binom{k-b(i+1)-1}{j}\left(\frac{y}{\beta}\right)^{j}$. Hence, the mixture representation for the WME distribution is given by

$$f(y)=b\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+k}a^{i+1}}{i!}\binom{b(i+1)-1}{k}\binom{k-b(i+1)-1}{j}\frac{y^{j+1}}{\beta^{j+2}}e^{-\frac{y}{\beta}(k-b(i+1))}$$

$$f(y)=b\sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{y^{j+1}}{\beta^{j+2}}e^{-\frac{y}{\beta}(k-b(i+1))} \tag{8}$$

where $\delta_{i,j,k}=\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+k}a^{i+1}}{i!}\binom{b(i+1)-1}{k}\binom{k-b(i+1)-1}{j}$.

Many characteristics of the WME distribution can be obtained using pdf expansion given in Eq. (8).

2.4 Moments

In this subsection, we derive the moments of WME (a,b,β) distribution.

Theorem: Let Y be a r. v. has WME distribution. Then the r[th] raw moment is

$$\mu'_r = b\sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{\beta^r\,\Gamma(r+j+2)}{\left(k-b(i+1)\right)^{r+j+2}} \tag{9}$$

Proof: The r^{th} moment of X of a distribution can be obtained using following integral

$$\mu'_r = \int_0^{\infty} y^r f(x;a,b,\beta)\,dy$$

From Eq. (8), we have

$$\mu'_r = b\sum_{i,j,k=0}^{\infty}\frac{\delta_{i,j,k}}{\beta^{j+2}}\int_0^{\infty} y^{r+j+1}\mathrm{e}^{-\frac{y}{\beta}(k-b(i+1))}dy$$

Let Z be $\frac{y}{\beta}\left(k-b(i+1)\right)$. Then, we can write $y=\frac{z\beta}{\left(k-b(i+1)\right)}$ and $dy=\frac{\beta}{\left(k-b(i+1)\right)}dz.$

The above equation reduces to

$$\mu'_r = b\sum_{i,j,k=0}^{\infty}\frac{\delta_{i,j,k}}{\beta^{j+2}}\int_0^{\infty}\frac{z^{r+j+1}\beta^{r+j+1}}{\left(k-b(i+1)\right)^{r+j+1}}e^{-z}\frac{\beta}{\left(k-b(i+1)\right)}dz$$

$$= b\sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{\beta^r}{\left(k-b(i+1)\right)^{r+j+2}}\int_0^{\infty} z^{r+j+1}e^{-z}dz$$

and simplification completes the proof.

Both Skewness (γ_1) and kurtosis (β_2) both are used to evaluate the shape of the probability distribution. These measures can be easily determined by the following expressions based on the first four mean moments:

$$\gamma_1 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}}, \quad \beta_2 = \frac{\mu_4}{\mu_2^2}.$$

2.5. Incomplete Moments

In this subsection, we derive the incomplete moments of WME (a,b,β).

Theorem: Let Y has the WME density (8). The r^{th} incomplete moment is obtained as

$$\varphi(y) = b\sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{\beta^r}{\left(k-b(i+1)\right)^{r+j+2}}\gamma(r+j+2,y) \tag{10}$$

Proof: The r^{th} incomplete moment of WME distribution is

$$\varphi(y) = \int_0^{x} v^r f(v)\mathrm{d}v = b\sum_{i,j,k=0}^{\infty}\frac{\delta_{i,j,k}}{\beta^{j+2}}\int_0^{x} v^{r+j+1}\mathrm{e}^{-\frac{v}{\beta}(k-b(i+1))}dv$$

Let $z=\frac{v}{\beta}\left(k-b(i+1)\right)$. Then, we have $v=\frac{z\beta}{\left(k-b(i+1)\right)}$ and $dv=\frac{\beta}{\left(k-b(i+1)\right)}dz$

The above expression reduces to

$$\varphi(y) = b\sum_{i,j,k=0}^{\infty}\frac{\delta_{i,j,k}}{\beta^{j+2}}\int_0^{y}\frac{z^{r+j+1}\beta^{r+j+1}}{\left(k-b(i+1)\right)^{r+j+1}}e^{-z}\frac{\beta}{\left(k-b(i+1)\right)}dz$$

$$\varphi(y) = b\sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{\beta^r}{\left(k-b(i+1)\right)^{r+j+2}}\int_0^{y} z^{r+j+1}e^{-z}dz$$

Incomplete gamma function is substituted and result follows.

2.6. Moment generating function

Now we define the mgf of WME distribution as

$$M_y(t)=\int_0^\infty e^{ty} f(y;\alpha,\theta,\lambda)\,dy=\int_0^\infty \sum_{r=0}^\infty \frac{t^r}{r!} y^r f(y;\alpha,\theta,\lambda)\,dy$$

$$=b\sum_{i,j,k,r=0}^\infty \delta_{i,j,k}\frac{t^r}{r!}\int_0^\infty \frac{y^{r+j+1}}{\beta^{j+2}} e^{-\frac{y}{\beta}(k-b(i+1))}dy.$$

After integration and simplification, we get the mgf as

$$M_y(t)=b\sum_{i,j,k,r=0}^\infty \delta_{i,j,k}\frac{t^r\beta^r\Gamma(r+j+2)}{r!(k-b(i+1))^{r+j+2}} \tag{11}$$

a	b	β	Mode	Mean	μ_2	$\gamma_1(sk)$	β_2
0.5	1	0.5	1.00861	0.978333	0.199257	0.0914216	2.35074
0.5	1	1	2.01722	1.95667	0.797027	0.0914216	2.35074
0.5	1	5	10.0861	9.78333	19.9257	0.0914216	2.35074
0.5	2	0.5	0.9525	0.88112	005489	-0.338926	2.80357
0.5	2	1	1.9050	1.77622	0.21956	-0.338926	2.80357
0.5	2	1.5	2.8575	2.66434	0.494011	-0.338926	2.80357
0.5	2	5	9.525	8.88112	5.48901	-0.338926	2.80357
0.5	5	0.5	0.8905	0.852701	0.00967	-0.76855	3.87153
0.5	5	1	2.6795	1.7054	0.038705	-0.76855	3.87153
0.5	5	5	8.90543	8.52701	0.967637	-0.76855	3.87153
1	1	0.5	0.64899	0.72278	0.128717	0.297641	2.49294
1	1	1	1.2980	1.44556	0.514867	0.297641	2.49294
1	1	1.5	1.9470	2.16834	1.15845	0.297641	2.49294
1	2	0.5	0.802782	0.755606	0.0431472	-0.259824	2.73323
1	2	1	1.60556	1.51121	0.172589	-0.259824	2.73322
1	2	1.5	2.40835	2.26682	0.388324	-0.259824	2.73323
1	5	0.5	0.833826	0.798303	0.0087937	-0.748911	3.81622
1	5	1	1.66765	1.59661	0.035175	-0.748911	3.81622
1	5	1.5	2.50148	2.39491	0.0791436	-0.748911	3.81622
5	1	0.5	0.237042	0.3232	0.031738	0.65702	3.15674
5	1	1	0.474083	0.6464	0.12695	0.65702	3.15674
5	1	1.5	0.711125	0.9696	0.285645	0.65702	3.15674
5	2	0.5	0.523777	0.505034	0.0218724	-0.115308	2.67573
5	2	1	1.04755	1.01007	0.087489	-0.115308	2.67573
5	2	1.5	1.57133	1.5151	0.19685	-0.115308	2.67573
5	5	0.5	0.711947	0.681638	0.0068822	-0.707178	3.70568
5	5	1	1.42389	1.36328	0.0275288	-0.707178	3.70568
5	5	1.5	2.13584	2.04492	0.0619397	-0.707178	3.70568

Table 2: Central tendencies and measure of dispersions for Y at different parameter combination

It is observed from Table 2 that

i. The range of y increases as β or an increases or b decreases.
ii. Variance increases for higher values of β.
iii. Skewness and kurtosis remain same for same vale of b.
iv. Distribution is negatively skewed for b>1.

2.7. Quantile function and random number generation

If U is a uniform r. v. with $(0,1)$, then $Y = Q(U)$ follows density Eq.(5). The quantile function of Y corresponding to Eq.(4) is

$$y = F^{-1}(u) = \beta\left[\frac{e^{\frac{y}{\beta}}}{\left(1-\log(1-u)^{-\frac{1}{a}}\right)^{\frac{1}{b}}} - 1\right] \tag{12}$$

Since it is a complex equation, by iteration method, the equation provides the quartiles and random numbers of the WME distribution.

3. Entropies

Let us now consider the Rényi and q entropies. These entropies are measures of uncertainty of a random variable. We derive the Rényi entropy defined by

$$I_R(\delta) = \frac{1}{1-\delta}\log\int_0^\infty f^\delta(x)\,dx,\ \ \delta > 0 \text{ and } \delta \neq 1.$$

Theorem: If the r. v. Y follows WME distribution, then the Rényi entropy is given by

$$I_R(\delta) = \frac{\delta\log a}{1-\delta} + \frac{\delta\log b}{1-\delta} + \log\beta + \frac{1}{1-\delta}\log\left[\sum_{i,j,k=0}^{\infty}\pi_{i,j,k}\frac{\Gamma(\delta+j+1)}{\left(k-b(i+\delta)\right)^{j+\delta+1}}\right] \tag{13}$$

Proof: If Y has the WME distribution, then the Rényi entropy can be obtained from

$$I_R(\delta) = \frac{1}{1-\delta}\log\left[I(\delta)\right] \tag{14}$$

where $\delta > 0$ and $\delta \neq 1$. Eq.(14) becomes

$$I(\delta) = \int_0^\infty f^\delta(y)\,dy$$

$$= \int_0^\infty a^\delta b^\delta \frac{y^\delta}{\beta^{2\delta}} \mathrm{e}^{-\frac{\delta y}{\beta}} \frac{\left[1-\left(1+\frac{y}{\beta}\right)\mathrm{e}^{-\frac{y}{\beta}}\right]^{\delta b-\delta}}{\left[\left(1+\frac{y}{\beta}\right)\mathrm{e}^{-\frac{y}{\beta}}\right]^{\delta b+\delta}} \exp\left\{-a\delta\left[\frac{\mathrm{e}^{\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)}-1\right]^b\right\}dy.$$

Since we can write the following expansion

$$\exp\left\{-a\delta\left[\frac{1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}\right]^{b}\right\}=\sum_{i=0}^{\infty}\frac{(-1)^{i}(a\delta)^{i}}{i!}\left[\frac{1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}\right]^{bi},$$

We have

$$f^{\delta}(y)=a^{\delta}b^{\delta}\frac{y^{\delta}}{\beta^{2\delta}}e^{-\frac{\delta y}{\beta}}\sum_{i=0}^{\infty}\frac{(-1)^{i}(a\delta)^{i}}{i!}\left[1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right]^{b(i+\delta)-\delta}\left[\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right]^{-(b(i+\delta)+\delta)}$$

where

$$\left\{1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right\}^{b(i+\delta)-\delta}=\sum_{k=0}^{\infty}(-1)^{k}\binom{b(i+\delta)-\delta}{k}\left(1+\frac{y}{\beta}\right)^{k}e^{-\frac{ky}{\beta}}.$$

Then, we have

$$f^{\delta}(y)=a^{\delta}b^{\delta}\frac{y^{\delta}}{\beta^{2\delta}}e^{-\frac{\delta y}{\beta}}\sum_{i,k=0}^{\infty}\frac{(-1)^{i+k}(a\delta)^{i}}{i!}\binom{b(i+\delta)-\delta}{k}\left[\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right]^{k-(b(i+\delta)+\delta)}$$

Since $\left(1+\frac{y}{\beta}\right)^{k-(b(i+\delta)+\delta)}=\sum_{j=0}^{\infty}\binom{k-(b(i+\delta)+\delta)}{j}\left(\frac{y}{\beta}\right)^{j}$, we obtain

$$f^{\delta}(y)=a^{\delta}b^{\delta}\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+k}(a\delta)^{i}}{i!}\binom{b(i+\delta)-\delta}{k}\binom{k-(b(i+\delta)+\delta)}{j}\frac{y^{j+\delta}}{\beta^{j+2\delta}}e^{-\frac{y}{\beta}(k-b(i+\delta))},$$

$$f^{\delta}(y)=a^{\delta}b^{\delta}\sum_{i,j,k=0}^{\infty}\pi_{i,j,k}\frac{y^{j+\delta}}{\beta^{j+2\delta}}e^{-\frac{y}{\beta}(k-b(i+\delta))},$$

where $\pi_{i,j,k}=\frac{(-1)^{i+k}(a\delta)^{i}}{i!}\binom{b(i+\delta)-\delta}{k}\binom{k-(b(i+\delta)+\delta)}{j}$. The entropy is obtained as

$$I(\delta)=a^{\delta}b^{\delta}\sum_{i,j,k=0}^{\infty}\pi_{i,j,k}\int_{0}^{\infty}\frac{y^{j+\delta}}{\beta^{j+2\delta}}e^{-\frac{y}{\beta}(k-b(i+\delta))}dy$$

After simplification, the final expression for the Rényi entropy is given by

$$I(\delta)=\frac{a^{\delta}b^{\delta}}{\beta^{\delta-1}}\sum_{i,j,k=0}^{\infty}\pi_{i,j,k}\frac{\Gamma(\delta+j+1)}{(k-b(i+\delta))^{j+\delta+1}} \quad (15)$$

Substituting Eq. (15) in Eq. (14) completes the proof.

The q entropy (H_q) of the WME distribution is obtained by

$$H_q=\frac{1}{q-1}\log\left(1-(1-q)I_R(\delta)\right)$$

$$=\frac{1}{q-1}\log\left\{1-(1-q)\left[\frac{\delta\log a}{1-\delta}+\frac{\delta\log b}{1-\delta}+\log\beta+\frac{1}{1-\delta}\log\left(\sum_{i,j,k=0}^{\infty}\pi_{i,j,k}\frac{\Gamma(\delta+j+1)}{(k-b(i+\delta))^{j+\delta+1}}\right)\right]\right\}$$

Substitution of Eq (13) completes the proof.

4. Mean Residual Life (MRL) and Mean Inactivity Time (MIT) Functions

If τ is a continuous r. v. representing the life of an substance or a part having distribution function $F(t_0)$ defined in Eq. (5), the mean residual life is obtained from

$$\mu(t_0) = E\left(\tau - t_0 \,|\, \tau \rangle t_0\right) = \frac{1}{\overline{F}}\int_{t_0}^{\infty} S(k)\,\mathrm{dk}, \quad t_0 \geq 0 \tag{16}$$

where $\overline{F} = 1 - F = S(t_0)$ is the survival function.

Theorem 4.1: Let Y be a r. v. having the WME distribution, then MRL function is

$$\mu(t_0) = \frac{\left[1 - b\sum_{i,j,k=0}^{\infty l}\delta_{i,j,k}\frac{\beta\gamma(j+3,y)}{\left(k-b(i+1)\right)^{j+3}}\right]}{\exp\left\{-a\left[\frac{1-\left(1+\frac{t_0}{\beta}\right)e^{-\frac{t_0}{\beta}}}{\left(1+\frac{t_0}{\beta}\right)e^{-\frac{t_0}{\beta}}}\right]^b\right\}} - t_0 \tag{17}$$

Proof: From Eq. (5), we can write

$$\mu(t_0) = \frac{1}{S(t_0)}\int_{t_0}^{\infty} S(k)\,\mathrm{dk}$$

$$\mu(t_0) = \frac{1}{\exp\left\{-a\left[\frac{1-\left(1+\frac{t_0}{\beta}\right)e^{-\frac{t_0}{\beta}}}{\left(1+\frac{t_0}{\beta}\right)e^{-\frac{t_0}{\beta}}}\right]^b\right\}}\int_{t_0}^{\infty}\exp\left\{-a\left[\frac{1-\left(1+\frac{k}{\beta}\right)e^{-\frac{k}{\beta}}}{\left(1+\frac{k}{\beta}\right)e^{-\frac{k}{\beta}}}\right]^b\right\}dk$$

Furthermore, the MRL function can be obtained as

$$\mu(t_0) = \frac{\left[1-\varphi_1(t_0)\right]}{S(t_0)} - t_0 = \frac{\int_{t_0}^{\infty} k\, f(k)\,\mathrm{dk}}{S(t_0)} - t_0, \quad t_0 \geq 0, \tag{18}$$

where $\varphi_1(t_0) = \int_0^{t_0} k\, f(k)\,\mathrm{dk}$ is the first incomplete moment of K. Substituting of Eq. (9) in Eq. (18), completes the proof.

Theorem 4.2: Let Y be a r. v. with the WME distribution, the MIT function is expressed as

$$M\left(t_0\right)=t_0-\frac{b\sum_{i,j,k=0}^{\infty}\delta_{i,j,k}\frac{\beta}{\left(k-b\left(i+1\right)\right)^{j+3}}\gamma\left(j+3,y\right)}{\exp\left\{-a\left[\frac{1-\left(1+\frac{t_0}{\beta}\right)\mathrm{e}^{-\frac{t_0}{\beta}}}{\left(1+\frac{t_0}{\beta}\right)\mathrm{e}^{-\frac{t_0}{\beta}}}\right]^b\right\}} \tag{19}$$

Proof: The MIT function is defined by

$$M\left(t_0\right)=E\left(t_0-\tau\mid\tau\le t_0\right)\right)=t_0-\frac{\left[\varphi_1\left(t_0\right)\right]}{F\left(t_0\right)},\quad t_0>0, \tag{20}$$

by inserting Eq. (9) and Eq. (6), the result follows.

5. Characterizations

Characterization of distribution is a significant aspect which helps the researcher to see if the proposed model is the correct one.

5.1. Characterization based on two truncated moments

Characterization of the WME distribution is derived using theorem proposed by Glänzel (1987). The theorem is based on truncated moments given in ratio form.

Theorem 5.1: Let Y has the pdf given in Eq. (5) and

$$q_1\left(y\right)=1, \tag{21}$$

$$q_2\left(y\right)=q_1\left(y\right)\exp\left\{-a\left[\frac{1-\left(1+\frac{y}{\beta}\right)\mathrm{e}^{-\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)\mathrm{e}^{-\frac{y}{\beta}}}\right]^b\right\},\text{for } y>0 \tag{22}$$

The r.v. Y follows WME distribution iff the function η having following expression

$$\eta\left(y\right)=\frac{1}{2}\exp\left\{-a\left(\frac{\mathrm{e}^{y/\beta}\beta}{y+\beta}-1\right)^b\right\} \tag{23}$$

Proof: For y>0, it can be seen that

$$\left(1-F\left(y\right)\right)E\left[q_1\left(Y\right)|\mathrm{Y}\ge y\right]=\exp\left\{-a\left[\frac{1-\left(1+\frac{y}{\beta}\right)\mathrm{e}^{-\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)\mathrm{e}^{-\frac{y}{\beta}}}\right]^b\right\}$$

$$\left(1-F\left(y\right)\right)E\left[q_2\left(Y\right)|\mathrm{Y}\ge y\right]=\frac{1}{2}\exp\left\{-2a\left[\frac{1-\left(1+\frac{y}{\beta}\right)\mathrm{e}^{-\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)\mathrm{e}^{-\frac{y}{\beta}}}\right]^b\right\}$$

and so $\eta(y)=\frac{1}{2}\exp\left\{-a\left(\frac{e^{y/\beta}\beta}{y+\beta}-1\right)^b\right\}$ as

$$\eta(y)q_1(y)-q_2(y)=q_1(y)\left\{\frac{1}{2}\exp\left\{-a\left(\frac{e^{y/\beta}\beta}{y+\beta}-1\right)^b\right\}-\exp\left\{-a\left[\frac{1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}\right]^b\right\}\right\}\neq 0$$

for all values of y. This completes the proof. Conversely, for given expressions of $q_1(y)$, $q_2(y)$ and $\eta(y)$ we show that the r.v. Y has WME distribution. Here, we get

$$s'(y)=\frac{\eta'(y)q_1(y)}{\eta(y)q_1(y)-q_2(y)}$$

$$s'(y)=-\frac{abe^{y/\beta}y\left(\frac{e^{y/\beta}\beta}{y+\beta}-1\right)^b}{(y+\beta)\left(y+\beta-e^{y/\beta}\beta\right)}>0$$

and so $s(y)=a\left(\frac{e^{y/\beta}\beta}{y+\beta}-1\right)^b$, $y>0$. Now using theorem of Glänzel (1987), Y has density in Eq. (5).

Corollary 5.1: Let Y be a r. v. assuming $\Omega\to(0,\infty)$ and let $q_1(y)$ as defined in Theorem (5.1). For the density in Eq. (5), iff there exist functions $q_2(y)$ and η defined in Theorem (Glanzel 1987, 1990) satisfying the differential equation

$$\frac{\eta'(y)q_1(y)}{\eta(y)q_1(y)-q_2(y)}=-\frac{\dfrac{abe^{\frac{y}{\beta}-a\left(\frac{e^{y/\beta}\beta}{y+\beta}-1\right)^b}y\left(\frac{e^{y/\beta}\beta}{y+\beta}-1\right)^b}{(y+\beta)\left(y+\beta-e^{y/\beta}\beta\right)}}{e^{-a\left(-\frac{y+\beta-e^{y/\beta}\beta}{y+\beta}\right)^b}},y\in\mathbb{R} \tag{24}$$

The general solution of Eq. (24) is given by

$$\eta(y)=e^{a\left(-\frac{y+\beta-e^{y/\beta}\beta}{y+\beta}\right)^b}\left[\int_0^y\frac{abe^{\frac{y}{\beta}-a\left(\frac{e^{y/\beta}\beta}{y+\beta}-1\right)^b}y\left(\frac{e^{y/\beta}\beta}{y+\beta}-1\right)^b}{(y+\beta)\left(y+\beta-e^{y/\beta}\beta\right)}\left[q_2(y)\{q_1(y)\}^{-1}\right]dy+C\right]$$

where C is a constant. Note that a set of functions satisfying the differential Eq. (24) is given in Theorem (5.1) with $C=0$. However; other triplets (q_1,q_2,η) satisfying the condition of Theorem (Glanzel 1987, 1990) also exist.

5.2. The characterization based on hazard function

The hazard rate function $h_F(y)$,satisfies the following equation $\frac{f'(y)}{f(y)}=\frac{h'_F(y)}{h_F(y)}-h_F(y)$, y є support of distribution function (F)
where F is twice differentiable distribution function. For numerous continuous distributions, hazard function characterization is the only characterization that exists.

Theorem 5.2: For the pdf of WME distribution in Eq. (5) if its $h_F(y)$ satisfies the equation

$$\mathrm{h}'(y)-y^{-1}\mathrm{h}(y)=\mathrm{ab}\left(\frac{e^{-\frac{y}{\beta}}(y+\beta)}{\beta}\right)^{-1-b}\left(1-\frac{e^{-\frac{y}{\beta}}(y+\beta)}{\beta}\right)^{b}$$

$$\frac{\left\{-\beta\left(-2y+\left(-2+(1+b)e^{y/\beta}\right)\beta\right)-(y+\beta)\left(y+\beta-be^{y/\beta}\beta\right)\mathrm{Log}[e]\right\}}{\left(\beta^2(y+\beta)\left(y+\beta-e^{y/\beta}\beta\right)^2\right)} \tag{25}$$

Proof: For the pdf in Eq. (5), we get

$$\mathrm{h}'(y)-y^{-1}\mathrm{h}(y)=\mathrm{ab}\left(\frac{e^{-\frac{y}{\beta}}(y+\beta)}{\beta}\right)^{-1-b}\left(1-\frac{e^{-\frac{y}{\beta}}(y+\beta)}{\beta}\right)^{b}\times$$

$$\frac{\left[\left(\beta\left(y^2-be^{y/\beta}y\beta+\left(-1+e^{y/\beta}\right)\beta^2\right)-y(y+\beta)\left(y+\beta-be^{y/\beta}\beta\right)\mathrm{Log}[e]\right)\right]}{\left[\left(\beta^2(y+\beta)\left(y+\beta-e^{y/\beta}\beta\right)^2\right)\right]} \tag{26}$$

$$-\ y^{-1}\left(\mathrm{ab}\frac{y}{\beta^2}\mathrm{e}^{-\frac{y}{\beta}}\frac{\left(1-\left(1+\frac{y}{\beta}\right)\mathrm{e}^{-\frac{y}{\beta}}\right)^{b-1}}{\left(\left(1+\frac{y}{\beta}\right)\mathrm{e}^{-\frac{y}{\beta}}\right)^{b+1}}\right)$$

and simplification follows Eq. (26). Now if Eq. (26) holds then

$$\frac{d}{dx}\left[y^{-1}\mathrm{h}(y)\right]=\frac{d}{dy}\left(\frac{\mathrm{ab}e^{-\frac{y}{\beta}}\left(1-e^{-\frac{y}{\beta}}\left(1+\frac{y}{\beta}\right)\right)^{-1+b}\left(e^{-\frac{y}{\beta}}\left(1+\frac{y}{\beta}\right)\right)^{-1-b}}{\beta^2}\right)$$

and simplification results in Eq. (7).

6. Order Statistics

Suppose Y_1, Y_2, ...,Y_n be a random sample from WME distribution and its ordered values is denoted as $Y_{(1)},Y_{(2)},Y_{(3)},\ldots\ldots,Y_{(n)}$. The pdf of s^{th} order statistic $Y_{s:n}$is expressed using the following function

$$f_{s:n}(y)=\frac{n!}{(s-1)!(n-s)!}\left[F(y)\right]^{s-1}\left[1-F(y)\right]^{n-s}f(y)$$

The density function of the sth ordered statistics follows the WME distribution is derived as follows:

$$f_{s:n}(y)=\frac{n!}{(s-1)!(n-s)!}\left[1-\exp\left\{-a\left[\frac{e^{\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)}-1\right]^{b}\right\}\right]^{s-1}\left[\exp\left\{-a\left[\frac{e^{\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)}1\right]^{b}\right\}\right]^{n-s}$$
$$\left[ab\frac{y}{\beta^{2}}e^{-\frac{y}{\beta}}\frac{\left[1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right]^{b-1}}{\left[\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right]^{b+1}}\exp\left\{-a\left[\frac{e^{\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)}-1\right]^{b}\right\}\right] \quad (27)$$

The density of the smallest order statistic obtained as

$$f_{1:n}(y)=n\left[\exp\left\{-a\left[\frac{e^{\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)}-1\right]^{b}\right\}\right]^{n-1}\left[ab\frac{y}{\beta^{2}}e^{-\frac{y}{\beta}}\frac{\left[1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right]^{b-1}}{\left[\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right]^{b+1}}\exp\left\{-a\left[\frac{e^{\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)}-1\right]^{b}\right\}\right] \quad (28)$$

The density of the largest order statistic obtained as

$$f_{n:n}(y)=n\left[1-\exp\left\{-a\left[\frac{e^{\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)}-1\right]^{b}\right\}\right]^{n-1}\left\{ab\frac{y}{\beta^{2}}e^{-\frac{y}{\beta}}\frac{\left[1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right]^{b-1}}{\left[\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right]^{b+1}}\exp\left\{-a\left[\frac{e^{\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)}-1\right]^{b}\right\}\right\} \quad (29)$$

7. Estimation

Here, we express the MLEs of parameters of the WME model. Let $y_1, y_2, \ldots, y_n$ be a random sample from Y~ WME(a,b,β) of size n with parameters $(a, b, \beta)^T$. Then log likelihood function of the WME distribution based on the given random sample is obtained as

$$\ell=n\log a+n\log b+\Sigma\log(y)-2n\log(\beta)-\Sigma\frac{y}{\beta}+(b-1)\Sigma\log\left\{1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right\}$$
$$-(b+1)\Sigma\log\left\{\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right\}-a\Sigma\left\{\frac{e^{\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)}-1\right\}^{b} \quad (30)$$

Therefore, the MLEs of parameters can be obtained by maximizing the above log likelihood function Eq. (30). The first derivative of the Eq. (30) with respect to ‘a’, ‘b’ and β parameters and equate to zero respectively to the equations (31) to (33). We have

$$\frac{\partial \ell}{\partial a}=\frac{n}{a}-\Sigma\left\{\frac{e^{\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)}-1\right\}^{b}, \tag{31}$$

$$\frac{\partial \ell}{\partial b}=\frac{n}{b}+\Sigma\log\left(1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right)-\Sigma\log\left(\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right)-a\Sigma\log\left\{\frac{e^{\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)}-1\right\}\left\{\frac{e^{\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)}-1\right\}^{b}, \tag{32}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \beta}&=\frac{2n}{\beta}+\Sigma\frac{y}{\beta^2}-ab\Sigma\left\{\frac{e^{\frac{y}{\beta}}}{\left(1+\frac{y}{\beta}\right)}-1\right\}^{b-1}\left\{\frac{ye^{\frac{y}{\beta}}}{\beta^2\left(1+\frac{y}{\beta}\right)^2}-\frac{ye^{\frac{y}{\beta}}}{\beta^2\left(1+\frac{y}{\beta}\right)}\right\}\\&+(b-1)\Sigma\frac{ye^{-\frac{y}{\beta}}-ye^{-\frac{y}{\beta}}\left(1+\frac{y}{\beta}\right)}{\beta^2\left\{1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right\}}-(b+1)\Sigma\frac{\left\{y\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}-xe^{-\frac{y}{\beta}}\right\}}{\beta^2\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}}.\end{aligned} \tag{33}$$

Non-linear equations are solved for a, b and β by utilizing softwares MATHEMATICA (NMaximize) or R-Language (Adequacy Model or maxBFGS sub-routine). There exist many maximizing algorithms in R to solve nonlinear equations simultaneously.

8. Simulation

Now, we simulate n =30, 50, 100 and 300 times the WME distribution for β =0.6, 1, 2; a = 0.5 and b = 1.5. We compute the ML estimates of parameters for each sample size. 10000 repetition are obtained and then the Bias and MSE are computed. The values such obtained are used for comparison of the performance of ML estimators, for same values of ‘a’ and b and different values of β and are given in Table 3.

n		β =0.6		β = 1		β = 2	
		a = 0.5	b =1.5	a = 0.5	b=1.5	a = 0.5	b=1.5
30	Bias	0.0888	0.1193	0.1152	0.2451	0.1076	0.0302
	MSE	0.3602	0.5236	0.4018	0,4834	0.4123	0.3920
50	Bias	0.0293	0.0992	0.0301	0.2182	0.0394	0.0274
	MSE	0.2403	0.4452	0.2890	0.2831	0.2742	0.2511
100	Bias	0.0280	0.0723	0.0241	0.1092	0.0142	0.0192
	MSE	0.1832	0.3029	0.1342	0.1632	0.1720	0.2013
200	Bias	0.0156	0.0624	0.0118	0.0981	0.0092	0.0095
	MSE	0.0921	0.0251	0.0913	0.1024	0.0928	0.1293

Table 3: Estimated bias and MSE for several values of parameters

It is observed from table values that

- Mean square error decreases as the sample size increases (n↑).
- Bias also decreases as sample size n↑.

9. Application

In this part of research, applications of WME model to real data are presented to demonstrate the usefulness of the WME distribution. Then a comparative study is carried out which includes WME distribution along with ME, exponentiated exponential (EE), exponentiated ME (EME), beta exponential (BE), gamma exponentiated exponential (GEE), Weibull Fréchet (WFr), Kumaraswamy exponential (Kw-E), Kumaraswamy modified Weibull (Kw-MW) distributions.

The first data are about carbon fibers which give measure of tensile strength of 100 fibers (Flaih et al. 2012) and are given below:
3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, .84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65. This data set also studied by Haq et al. (2017).

The unknown parameters of each distribution are obtained by using the ML method and with these obtained estimates, some accuracy measures are also computed to compare the fitted models. The model with minimum values for these statistics could be chosen as the best model to fit the data. Model parameters of the WME distribution are estimated using *NMaximize* command in MATHEMATICA. The models considered for comparison are given as follows:

- The moment exponential distribution by Dara and Ahmed (2012) with the pdf

$$f(y)=\frac{y}{\beta^2}\mathrm{e}^{-\frac{y}{\beta}}.I_{(0,\infty)}(y),\ \ \beta>0.$$

- The exponentiated exponential distribution by Gupta and Kundu (2001) with the pdf

$$F(y)=\left(1-e^{-\beta y}\right)^{\alpha}.I_{(0,\infty)}(y),\ \beta,\alpha>0.$$

- The EME distribution by Hasnain and Ahmad (2013) with the pdf

$$f(y)=a\frac{y}{\beta^2}e^{-\frac{y}{\beta}}\left[1-\left(1+\frac{y}{\beta}\right)e^{-\frac{y}{\beta}}\right]^{a-1}.I_{(0,\infty)}(y)\ \ ,\beta,\alpha>0.$$

- The BE distribution by Srivastava and Kumar (2011) with the pdf

$$f(y)=\frac{\lambda}{B(a,b)}e^{-b\lambda y}\left[1-e^{-\lambda y}\right]^{a-1}.I_{(0,\infty)}(y),\ \beta,\gamma>0.$$

- The GEE distribution Ristić and Balakrishnan (2012) with the pdf

$$f(y)=\frac{\lambda}{\Gamma(\delta)}e^{-\lambda y}\left(1-e^{-\lambda y}\right)^{\alpha-1}\left(-\log\left(1-e^{-\lambda y}\right)\right)^{\delta-1}.I_{(0,\infty)}(y),\ \lambda,\alpha,\delta>0.$$

- The KwMW distribution by Cordeiro et al. (2014) with the pdf

$$f(y)=ab\alpha y^{\gamma-1}(\gamma+\lambda y)e^{\left(\lambda y-\alpha y^{\gamma}e^{\lambda y}\right)}\left(1-e^{\left(\lambda y-\alpha y^{\gamma}e^{\lambda y}\right)}\right)^{a-1}\left(1-\left(1-e^{\left(\lambda y-\alpha y^{\gamma}e^{\lambda y}\right)}\right)^{a}\right)^{b-1}.I_{(0,\infty)}(y),\ a,b,\alpha,\gamma>0,\ \lambda\geq 0.$$

- The WFr distribution by Afify et al. (2016) with the pdf

$$f(y)=ab\beta\alpha^{\beta}y^{-(\beta+1)}\mathrm{e}^{-b\left(\frac{\alpha}{y}\right)^{\beta}}\left\{1-\mathrm{e}^{-\left(\frac{\alpha}{y}\right)^{\beta}}\right\}^{-(b+1)}\exp\left(-a\left\{\mathrm{e}^{-\left(\frac{\alpha}{y}\right)^{\beta}}-1\right\}^{-b}\right).I_{(0,\infty)}(y),\ a,b,\alpha,\beta>0.$$

- The Kumaraswamy exponential (Kw-E) by Adepoju and Chukwu (2015)with the pdf

$$f(y)=ab\left(1-\exp(-\lambda y)\right)^{a-1}\left(\left(1-\exp(-\lambda y)\right)^{a}\right)^{b-1}.I_{(0,\infty)}(y),\ a,b,\lambda>0.$$

Table 4 involves the ML estimates of the fitted distributions for the tensile data. The values of accuracy measures are provided in Table 5.

Model	**Estimates**		
ME(β)	1.3057		
EE (α, β)	7.8780	1.02111	
EME(α, β)	3.5135	0.77534	
BE (a, b, λ)	5.9964	191.696	0.01182
GEE(λ, α, δ)	0.2719	8.13102	6.17272
WME(a, b, β)	**258.345**	**1.47285**	**12.7842**

Table 4: The MLEs for carbon fibre

Model	**The goodness of fit criteria**				
	AIC	**BIC**	**- ℓ**	**A***	**W***
ME(β)	333.857	336.462	-165.928	8.14797	1.45650
EE (α, β)	294.910	300.120	-145.464	1.25447	0.23611
*EME(**α, β**)*	467.588	472.798	-231.794	1.09176	0.21035
BE (a, b, λ)	291.056	298.871	-142.528	0.77823	0.15443
GEE(λ, α, δ)	292.054	299.87	-143.027	0.85420	0.16904
WME	**287.792**	**295.608**	**-140.896**	**0.45223**	**0.07073**

Table 5: The accuracy measures AIC, BIC, - ℓ, A and W for carbon fibre

It is observed from Table 5 that the WME model provides a very good fit to the first data set. The density plots of the fitted WME and other fitted distributions are displayed in Figure 2, and it shows that the WME distribution is the most suitable model for tensile data.

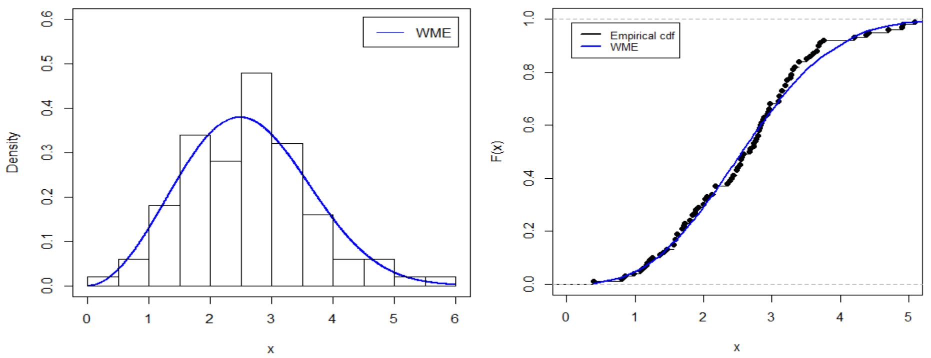

Figure 2: The tensile data is fitted using MLE method; left figure: The fitted pdf of WME and histogram of data. The right figure: The WME cdf estimates and empirical cdf

The second set consists of measurement of the strengths of 1.5 cm glass fibers, measued at National Physical Laboratory, England (Smith and Naylor, 1987). The estimates of unknown parameters of the distributions for the second data set are presented in Table 6. The values of the accuracy statistics AIC, BIC, A*, W* are listed in Table 7.

Model	Estimates				
ME(β)	0.7534				
EME(α,β)	12.925	0.312553			
WME(a, b, β)	0.1257	2.3562	0.6641		
BE(a, b, λ)	17.443	870.58	0.0132		
KwE*(a, b, λ)*	5.8184	$3.59696*10^{10}$	0.0095		
GEE(λ, α, δ)	0.4339	24.666	18.803		
WFr(a, b, α, β)	1.4762	16.856	0.3865	0.2436	
KwMW(a,b,α,γ,λ)	0.17111	0.64975	0.14981	1.79940	0.49987

Table 6: The MLEs for the glass fibre

Based on Table 6, we presume that the WME gives adequate fit when compared with considered models and WME distribution gives least values of AIC, BIC, A*, W*. The histogram of the glass fibre data with the estimated density and cdfs of the proposed model with sample cdf are given in Figure 3. It is revealed from the Figure 3 that proposed model fits the given data best and also it is supported by the results given in Table 6.

Model	The goodness of fit criteria				
	AIC	BIC	- ℓ	A*	W*
ME(β)	134.64	136.78	-66.317	13.325	2.6496
EME(α,β)	64.161	68.448	-30.081	17.611	3.6572
WME(a, b, β)	**35.017**	**41.447**	**-14.509**	**1.0254**	**0.1827**
BE(a, b, λ)	53.904	60.333	-23.952	3.1256	0.5703
KwE*(a, b, λ)*	36.455	42.884	-15.227	4.0379	0.7314
GEE(λ, α, δ)	55.019	61.448	-24.510	3.2248	0.5885
WFr(a, b, α, β)	39.000	47.573	-15.207	1.3410	0.2326
KwMW(a,b,α,γ,λ)	38.323	46.895	-15.161	1.2410	0.2158

Table 7: The statistics AIC, BIC, - ℓ, A and W for the glass fibre data

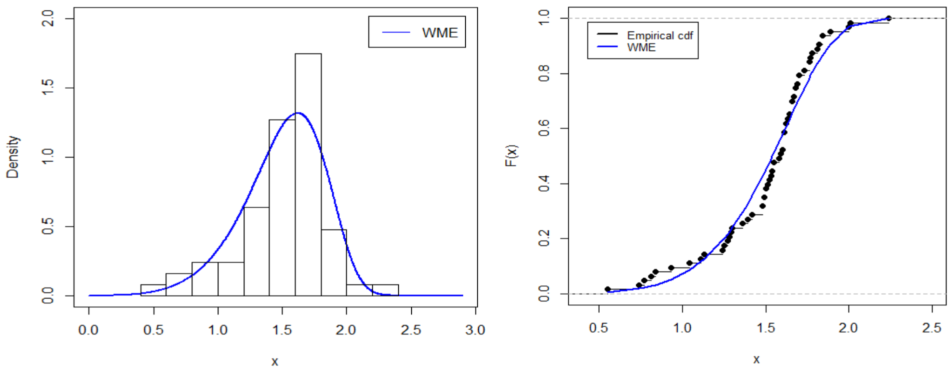

Figure 3: The glass fiber data is fitted using MLE method; left fig.: The fitted pdf of WME and histogram of data. The right fig.: The WME cdf estimates and empirical cdf

10. Conclusion

There has been an increasing interest to develop tractable lifetime models which fits survival data in better way. In this research article, we propose and study a new three-parameter distribution called Weibull moment exponential distribution and derives its some characteristics including ordinary moments, incomplete moments and quantile function along with its characterizations. The order statistics and explicit expressions for Renyi and q-entropies are also presented. The estimation of the model parameters is obtained using the method of maximum likelihood. We study the behavior of the estimators by means of simulation for sample size 30 to 200. We present applications to illustrate its usefulness.

References

1. Adepoju, K. and Chukwu, O.I. (2015). Maximum likelihood estimation of the Kumaraswamy exponential distribution with applications, Journal of Modern Applied Statistical Methods, 14(1), p. 208-214.
2. Alzaatreh, A., Famoye, F. and Lee, C. (2013). Weibull-Pareto distribution and its applications, Communication in Statistics: Theory and Methods, 42, p. 1673-1691.
3. Afify, A.Z., Yousof, H.M., Cordeiro, G.M., Ortega, E.M. and Nofal, Z.M. (2016). The Weibull Fréchet distribution and its applications, Journal of Applied Statistics, 43(14), p. 2608-2626.
4. Bourguignon, M., Silva, R.B. and Cordeiro, G.M. (2014). The Weibull-G family of probability distributions, Journal of Data Science, 12, p. 53-68.
5. Cordeiro, G.M., Ortega, E.M. and Silva, G.O. (2014). The Kumaraswamy modified Weibull distribution: theory and applications, Journal of Statistical Computation and Simulation, 84(7), p. 1387-1411.
6. Cousineau, D. (2009). Fitting the three-parameter Weibull distribution: review and evaluation of existing and new methods, IEEE Transactions on Dielectrics and Electrical Insulation, 16 (1), p. 281-288.
7. Dara, S. T. and Ahmad, M. (2012). Recent Advances in Moment Distribution and Their Hazard Rates, Lap Lambert Academic Publishing GmbH KG.
8. Elgarhy, M., Haq, M.A., & Ozel, G. (2017). A new exponentiated extended family of distributions with applications, Gazi University Journal of Science, 30(3), p. 101-115.
9. Epstein, B. (1958). The exponential distribution and its role in life testing, Technical Report, Wayne State University, Michigan.
10. Flaih, A., Elsalloukh, H., Mendi, E. and Milanova, M. (2012). The exponentiated inverted Weibull distribution, Applied Mathematics and Information Sciences, 6(2), p. 167-171.
11. Glänzel, W. (1987). A characterization theorem based on truncated moments and its application to some distribution families, In Mathematical Statistics and Probability Theory, Springer, Dordrecht, p. 75-84.
12. Glänzel, W. (1990). Some consequences of a characterization theorem based on truncated moments, Statistics, 21(4), p. 613-618.
13. Gupta, R. D.and Kundu, D. (2001). Exponentiated exponential family: an alternative to gamma and Weibull distributions, Biometrical Journal, 43(1), p. 117-130.
14. Haq, M. A., Yousof, H. M., and Hashmi, S. (2017). A new five-parameter Fréchet model for extreme values, Pakistan Journal of Statistics and Operation Research, 13(3), p. 617-632.
15. Haq, M.A., Usman, R.M., Hashmi, S. and Al-Omeri, A.I. (2017). The Marshall-Olkin length-biased exponential distribution and its applications, Journal of King Saud University-Science, https://doi.org/10.1016/ j.jksus. 2017.09.006.
16. Haq, M. A., Elgarhy, M., Hashmi, S., Ozel, G. and ul Ain, Q. (2018). transmuted Weibull power function distribution: its properties and applications, Journal of Data Science, 16(2), p. 397-418.
17. Haq, M. A. and Elgarhy, M. (2018). Odd Fréchet-G family of probability distributions, Journal of Statistics Applications and Probability, 7(1), p. 185-201.

18. Haq, M. A., Handique, L. and Chakraborty, S. (2018). The odd moment exponential family of distributions: its properties and applications, International Journal of Applied Mathematics and Statistics, 57(6), p. 47-62.
19. Iqbal, Z., Hasnain, S.A., Salman, M., Ahmad, M. and Hamedani, G. (2014). Generalized exponentiated moment exponential distribution, Pakistan Journal of Statistics, 30(4), p. 537-554.
20. Merovci, F. (2013). Transmuted exponentiated exponential distribution, Mathematical Sciences and Applications E-Notes, 1(2), p. 112-122.
21. Merovci, F. and Elbatal, I. (2015). Weibull Rayleigh distribution: theory and applications, Applied Mathematics and Information Sciences, 9(4), p. 21-27.
22. Nasiru, S. and Luguterah, A. (2015). The new Weibull-Pareto distribution, Pakistan Journal of Statistics and Operation Research, 11, p. 103-114.
23. Norton, R.M. (1984). The double exponential distribution: using calculus to find a maximum likelihood estimator, The American Statistician, 38(2), p. 135-136.
24. Oguntunde, P.E., Balogun, O.S., Okagbue, H.I. and Bishop, S.A. (2015). The Weibull exponential distribution: its properties and applications, Journal of Applied Sciences, 15 (11), p. 1305-1311.
25. Rayleigh, L. (1880). XII. On the resultant of a large number of vibrations of the same pitch and of arbitrary phase, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 10(60), p. 73-78.
26. Ristić, M.M. and Balakrishnan, N. (2012). The gamma-exponentiated exponential distribution, Journal of Statistical Computation and Simulation, 82(8), p. 1191-1206.
27. Smith, R.L. and Naylor, J. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution, Journal of the Royal Statistical Society: Series C (Applied Statistics), 36(3), p. 358–369.
28. Srivastava, A.K. and Kumar, V. (2011). Software reliability data analysis with Marshall-Olkin extended Weibull model using MCMC method for non-informative set of priors, International Journal of Computer Applications, 18(4), p. 31-39.
29. Tahir, M., Alizadeh, M., Mansoor, M., Cordeiro, G. M. and Zubair, M. (2014). The Weibull-power function distribution with applications, Hacettepe Journal of Mathematics and Statistics, 45 (1), p. 245 – 265
30. Tahir, M., Cordeiro, G. M., Mansoor, M. and Zubair, M. (2015). The Weibull-Lomax distribution: properties and applications, Hacettepe Journal of Mathematics and Statistics, 44(2), p. 461-480.
31. Tahir, M., Cordeiro, G.M., Mansoor, M., Zubair, M. and Alizadeh, M., (2016). The Weibull-Dagum distribution: properties and applications, Communication in Statistics-Theory and Methods, 45(24), p. 7396-7398.
32. Tahir, A., Akhter, A. S. and Haq, M. A. (2018). Transmuted new Weibull-Pareto distribution and its applications, Applications and Applied Mathematics-An International Journal, 13(1), p. 30-46.
33. Weibull, W. (1951). A statistical distribution function of wide applicability, Journal of Applied Mechanics, 103(730), p. 293-297.