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ON THE MODIFIED SINGH-MADDALA DISTRIBUTION: DEVELOPMENT, PROPERTIES, CHARACTERIZATIONS AND APPLICATION

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Abstract

In this paper, a flexible Burr XII distribution with one additional shape parameter and one scale parameter called the MSM distribution is derived from the generalized differential equation (GDE).Basic structural properties are studied. Moments, mean deviations, conditional moments, incomplete moments, inequality curves, L-moments, and TL- moments, reliability and uncertainty measures are theoretically presented. We characterize the MSM distribution via various techniques. We adopt maximum likelihood estimation technique for model parameters. We assess the behavior of the maximum likelihood estimates (MLEs) through a simulation study. We illustrate the significance and tractability of the MSM distribution by its application of serum-reversal times.

Key Words: Moments, L-Moments, TL- Moments, Characterization, Maximum Likelihood Method.

1. INTRODUCTION

During recent years, a lot of continuous distributions have been derived but data sets from sciences such as biometry, reliability, engineering ecology, hydrology and financial don't follow these distributions. So modified distributions and their applications to such issues are essential requirements.

Pearson (1895, 1901, and 1916) derived Pearson system of distributions to model skew data from the clue of limiting situation of hypergeometric distribution. Shakil and Singh (2015) derived Michaelismenten class of distributions from GDE with varieties of coefficients.

Burr (1942) proposed Burr family but Burr XII (BXII) distribution has significant applications in sciences such as biometry, ecology, hydrology, finance, reliability, engineering, survival analysis and quality control plans.

Many modified types of BXII distribution are presented in statistical literature such as Burr (Takahasi;1965),income model (Singh and Maddala;1976), extended three-parameter BXII (Shao et al.; 2004), drought models (Nadarajah and Kibria;

2006),beta BXII (Paranaíba et al.; 2011), Kumaraswamy BXII (Paranaíba et al.; 2013), Singh Maddala (Shahzad and Asgher;2013), BXII-Geometric (Korkmaz and Erisoglu; 2014), BXII power series (Silver and Cordeiro ;2015), McDonald BXII (Gomes et al.; 2015), generalized BXII-Poisson (Muhammad; 2016),extended BXII (Ghosh and Bourguignon; 2017), BXII modified Weibull (Mdlongwa et al.;2017), gamma BXII (Guerra et al.; 2017), BXII modified Weibull (Mdlongwa et al.; 2017), five-parameter BXII (Mead and Afify; 2017), new BXII (Yari and Tondpour; 2017), four-parameter BXII (Afify et al.; 2018), generalized log BXII (Bhatti et al.;2018), modified log BXII distribution (Bhatti et al.;2018) and modified BXII-Power (Bhatti et al.;2019).

We derive and study a flexible BXII distribution with one additional shape parameter and one scale parameter called the MSM distribution. The MSM density can be arc, J, reverse-J, exponential, symmetrical, left-skewed and right-skewed shaped. The flexible failure rate for MSM model can produce many shapes such as increasing, decreasing, constant, inverted tub and modified bathtub. The MSM distribution recommends good fits for data sets in survival analysis, reliability, finance, business and economics.

In this article, we study the MSM distribution. Section 2 deals with development of the MSM distribution from the GDE. We also present some structural properties and sub models. We plot graphs of density and failure rate functions of the proposed model. Section 3 takes up moments, conditional moments, incomplete moments, mean deviations, inequality curves, life expectancy and mean inactivity time, L-Moments and TL- moments. Section 4 deals with the study of various reliability measures. In Section 5, various uncertainty measures are presented. Section 6 characterizes the MSM distribution via different techniques. In Section 7, the parameters for the MSM distribution are assessed via maximum likelihood estimation technique. We clarify the consistency of the maximum likelihood estimates (MLEs) through a simulation study. We test the significance and utility of the MSM distribution through different goodness of fit criteria by application to serum-reversal times. The conclusion and remarks about the MSM distribution are presented in last Section.

2. The MSM Distribution

Now, we derive the MSM distribution from the GDE (Dunning and Hanson; 1977) given as

$$\frac{df}{dx} = \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m}{b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n} f\left(x\right), \quad x > 0,\ m, n = 1, 2\ldots \qquad (1)$$

Taking $a_0 = \beta - 1, a_1 = a_2 = \cdots a_{m-1} = 0, a_m = -\left[\alpha\beta + \gamma\right]\lambda^{-\beta}$, $b_0 = 0$, $b_1 = 1$,

$b_i = 0,\ i = 2, 3, \ldots n-1, b_n = \gamma\lambda^{-\beta}, m = \beta, n = \beta + 1$ in (1), we obtain

$$\frac{d}{dx}\left[\ln f\left(x\right)\right] = \frac{\left(\beta - 1\right)\lambda^{\beta} - \left(\alpha\beta + \gamma\right)x^{\beta}}{\lambda^{\beta} x + \gamma x^{\beta+1}}. \qquad (2)$$

Integrate (2), we obtain as

$$f(x)=\kappa\left(\frac{x}{\lambda}\right)^{\beta-1}\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1},$$

using property $\int_0^{\infty} f(x)\,dx=1$, we obtain $\kappa=\frac{\alpha\beta}{\lambda}$. Therefore probability density function (pdf) of the MSM distribution with shape parameters α,β,γ and scale parameter λ, is

$$f(x)=\frac{\alpha\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\left(1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right)^{-\frac{\alpha}{\gamma}-1},\ x>0. \tag{3}$$

Therefore the cumulative distribution function (cdf) for the MSM distribution, is

$$F(x)=1-\left(1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right)^{-\frac{\alpha}{\gamma}},\quad x\geq 0. \tag{4}$$

It is simple to observe that $F(x)$ is differential and strictly increasing in $(0,\infty)$. The cdf of MSM also shows that $\lim_{x\to\infty}F(x)=1$ and $\lim_{x\to 0}F(x)=0$. It means that $F(x)$ is absolutely continuous cdf.

2.1 Basic Structural Properties

The survival, failure rate, reverse failure rate, cumulative failure rate, elasticity functions and Mills ratio of X with the MSM distribution are specified, respectively, by

$$S(x)=\left(1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right)^{-\frac{\alpha}{\gamma}},\qquad x\geq 0, \tag{5}$$

$$h(x)=\frac{\alpha\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\left(1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right)^{-1}, \tag{6}$$

$$r(x)=\frac{d}{dx}\ln\left[1-\left(1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right)^{-\frac{\alpha}{\gamma}}\right], \tag{7}$$

$$H(x)=\frac{\alpha}{\gamma}\ln\left[1-\left(1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right)\right], \tag{8}$$

$$m(x)=\left[\frac{\alpha\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\left(1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right)^{-1}\right]^{-1}, \tag{9}$$

and

$$e(x)=\frac{d}{d\ln x}\ln\left[1-\left(1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right)^{-\frac{\alpha}{\gamma}}\right]. \tag{10}$$

The mode of the MSM distribution after simplifying $\frac{d}{dx}\left(\ln\left(f(x)\right)\right)=0$, is

$$Mode=\lambda^{-1}\left[(\beta-1)(\alpha\beta+\gamma)^{-1}\right]^{\frac{1}{\beta}}. \tag{11}$$

The MSM distribution has the following quantile function

$$x_q=\lambda\left[\gamma^{-1}\left((1-q)^{-\frac{\gamma}{\alpha}}-1\right)\right]^{\frac{1}{\beta}}. \tag{12}$$

The random number generator for MSM model is

$$X=\lambda\left[\gamma^{-1}\left((1-U)^{-\frac{\gamma}{\alpha}}-1\right)\right]^{\frac{1}{\beta}}, \tag{13}$$

where U is the uniform the random variable on $(0,1)$.

2.2 Plotsfor the MSM Density and Failure Rate Functions

Fig. 1 displays that the MSM density can produce the shapes such as arc, J, reverse J, exponential, symmetrical, positively skewed and negatively skewed. The flexible failure rate for MSM distribution produces various shapes such as increasing, decreasing, constant, inverted tub shapes and modified bathtub (Fig. 2).

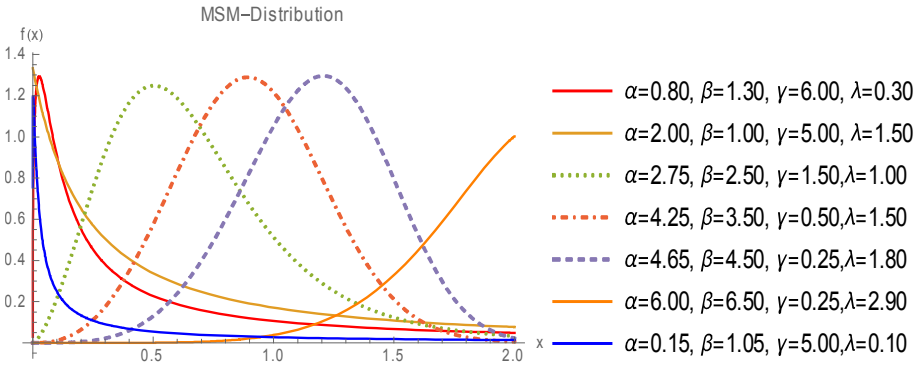

Fig. 1: Plots for pdf of the MSM Distribution

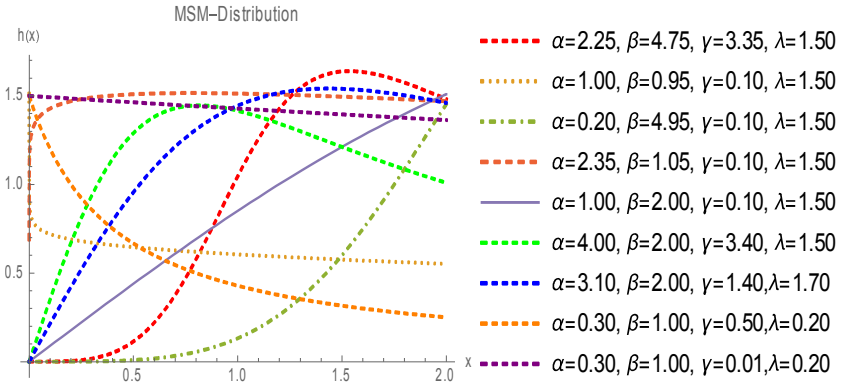

Fig. 2: Plots for failure rate function of the MSM Distribution

The MSM distribution has the following sub models.

1	X	α	β	γ	λ	Modified Singh Maddala (MSM)
2	X	α	β	1	λ	Singh Maddala (SM)
3	X	α	β	γ	1	Modified Burr XII (MBXII)
4	X	α	β	1	1	Burr XII (BXII)
5	X	α	1	γ	λ	Modified Lomax (ML)
6	X	α	1	1	1	Lomax
7	X	1	β	1	1	Log-logistic (Log-logistic)
8	X	α	β	$\gamma \to 0$	λ	Generalized Weibull (GW)
9	X	1	β	$\gamma \to 0$	**1**	Weibull (W)
10	X	α	1	$\gamma \to 0$	**1**	Generalized exponential (GE)
11	X	1	1	$\gamma \to 0$	**1**	Exponential (E)
12	X	α	2	$\gamma \to 0$	**1**	Generalized Rayleigh (GR)
13	X	1	2	$\gamma \to 0$	**1**	Rayleigh (R)
14	$\frac{1}{X}$	α	β	γ	λ	Modified Dagum (MD)
15	$\frac{1}{X}$	α	β	1	λ	Dagum (D)
16	$\frac{1}{X}$	α	β	γ	1	Modified Burr III (MBIII)
17	$\frac{1}{X}$	α	β	1	1	Burr III (BIII)
18	$\frac{1}{X}$	α	β	$\gamma \to 0$	λ	Generalized inverse Weibull (GIW)

Table 1: Sub-Models of the MSM Distribution

3. Estimation

We take up moments related properties such as moments, characteristic function, conditional moments, incomplete moments, mean deviations, inequality curves, life expectancy and mean inactivity time, L-Moments and TL- moments.

3.1 Moments

The moments are significant for statistical analysis in practical sciences. The descriptive statistics such as mean $\left(\mu_1'\right)$, median $\left(\tilde{\mu}\right)$, standard deviation $\left(\sigma\right)$, skewness (γ_1) and kurtosis (γ_2) can be calculated through moments. The r[th] ordinary moment for X with MSM distribution is

$$\mu_r' = E\left(X^r\right) = \int_0^\infty x^r \frac{\alpha\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\left(1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right)^{\frac{\alpha}{\gamma}-1} dx,$$

$$\mu_r' = \alpha\lambda^r\gamma^{-\left(1+\frac{r}{\beta}\right)} B\left(1+\frac{r}{\beta}, \frac{\alpha}{\gamma}-\frac{r}{\beta}\right), \quad r = 1,2,3,... \qquad (14)$$

Mean and Variance of X with the MSM distribution are

$$E\left(X\right) = \frac{\alpha\lambda}{\gamma^{\frac{1}{\beta}+1}} B\left(\frac{1}{\beta}+1, \frac{\alpha}{\gamma}-\frac{1}{\beta}\right),$$

and

$$Var\left(X\right) = \frac{\alpha\lambda^2}{\gamma^{\frac{2}{\beta}+1}} B\left(\frac{2}{\beta}+1, \frac{\alpha}{\gamma}-\frac{2}{\beta}\right) - \left(\frac{\alpha\lambda}{\gamma^{\frac{1}{\beta}+1}} B\left(\frac{1}{\beta}+1, \frac{\alpha}{\gamma}-\frac{1}{\beta}\right)\right)^2$$

.

The characteristic function $\phi_X\left(t\right) = E\left[e^{itX}\right]$ for X with the MSM distribution is

$$\phi_X\left(t\right) = \sum_{r=1}^{\infty}\frac{\left(it\right)^r}{\mathrm{r!}}E\left(X^r\right) = \alpha\sum_{r=1}^{\infty}\frac{\left(it\right)^r}{\mathrm{r!}}\lambda^r\gamma^{-\left(1+\frac{r}{\beta}\right)} B\left(1+\frac{r}{\beta}, \frac{\alpha}{\gamma}-\frac{r}{\beta}\right), \qquad (15)$$

where i is the imaginary number.

The m[th] central moment, Pearson's coefficient of skewness and kurtosis of X with the MSM distribution are computed from the relations

$$\mu_m = \sum_{\ell=1}^{m}\binom{m}{\ell}\left(-1\right)^{\ell}\mu_{\ell}'\mu_{\ell-m}', \ \gamma_1 = \frac{\mu_3}{\left(\mu_2\right)^{\frac{3}{2}}} \text{ and } \beta_2 = \frac{\mu_4}{\left(\mu_2\right)^2}.$$

We study numerically central tendency (median and mean), dispersion (standard deviation), skewness and kurtosis of the MSM distribution based on selected values of parameters. We also depict the effect of parameter values on the descriptive measures.

Parameters $\alpha,\beta,\gamma,\lambda$	Median	Mean	Standard Deviation	Skewness	Kurtosis
4,1.5,1.5,1.5	0.5085	0.6860	0.6935	11.0374	1020.85
5,1.5,1.5,1.5	0.4304	0.5588	0.5027	0.5027	122.106
5.75,1.5,1.5,1.5	0.3885	0.4952	0.4241	2.9730	42.3833
5,6.5,1.5,1.5	1.1245	1.1222	0.2285	0.0085	3.2869
5.4,6,1.5,1.5	1.0824	1.0809	0.2351	0.0304	3.2430
5.5,5,1,1	0.6690	0.6677	0.1652	0.0148	3.0482
5,5,0.9,05	0.3409	0.3402	0.8411	0.0102	3.0432
5,5,0.5,0.5	0.3390	0.3367	0.0804	-0.1131	2.9315
5,5,0.3,0.5	0.3380	0.3350	0.0786	-0.1707	2.9010
5,5,0.3,0.5	0.3371	0.3334	0.0770	-0.2260	2.8841
5,6,0.05,5	3.5976	3.5495	0.6904	-0.3584	3.0293
5,10,0.05,5	4.1039	4.0511	0.4890	-0.6233	3.5502
5,5,2.5,5	3.4880	3.5582	1.0323	0.6630	5.0678
5,5,3.5,5	3.5400	3.6723	1.1973	1.3240	12.0698
5,5,5,5	3.6215	3.8718	1.5497	4.1028	161.485
3,5,3,5	4.0110	4.2883	1.7163	4.1346	166.29

Table 2: Median, mean, standard deviation, skewness and Kurtosis of the MSM Distribution

3.2 Conditional Moments

The k[th] conditional moments $E\left(X^k \middle| X>z\right)=\frac{1}{S(z)}\left[E\left(X^k\right)-E_{X\le z}\left(X^s\right)\right]$ of X for the MSM distribution are

$$E\left(X^k \middle| X>z\right)=\frac{1}{S(z)}\frac{\alpha\lambda^k}{\gamma^{\frac{k}{\beta}+1}}\left[B\left(\frac{k}{\beta}+1,\frac{\alpha}{\gamma}-\frac{k}{\beta}\right)-B\left(\frac{\gamma z^\beta}{\lambda^\beta};\frac{k}{\beta}+1,\frac{\alpha}{\gamma}-\frac{k}{\beta}\right)\right]. \quad (16)$$

where $B\left(t,.,.\right)$ is incomplete beta function.

The upper incomplete s[th] incomplete ordinary moment of X for the MSM distribution is

$$E_{X>z}\left(X^k\right)=\frac{\alpha\lambda^k}{\gamma^{\frac{k}{\beta}+1}}\left[B\left(\frac{k}{\beta}+1,\frac{\alpha}{\gamma}-\frac{k}{\beta}\right)-B\left(\frac{\gamma z^\beta}{\lambda^\beta};\frac{k}{\beta}+1,\frac{\alpha}{\gamma}-\frac{k}{\beta}\right)\right]. \quad (17)$$

The s[th] reversed conditional moments $E\left(X^k \middle| X\le z\right)=\frac{1}{F(z)}\left[E_{X\le z}\left(X^k\right)\right]$ of X for the MSM distribution is

$$E\left(X^k \middle| X\le z\right)=\frac{1}{F(t)}\frac{\alpha\lambda^k}{\gamma^{\frac{k}{\beta}+1}}B\left(\frac{\gamma z^\beta}{\lambda^\beta};\frac{k}{\beta}+1,\frac{\alpha}{\gamma}-\frac{k}{\beta}\right). \quad (18)$$

The lower incomplete s[th] incomplete ordinary moment of X for the MSM distribution is

$$M_k'(z) = E_{X\le z}\left(X^k\right) = \frac{\alpha\lambda^k}{\gamma^{\frac{k}{\beta}+1}} B\left(\frac{\gamma z^\beta}{\lambda^\beta}; \frac{k}{\beta}+1, \frac{\alpha}{\gamma}-\frac{k}{\beta}\right). \quad (19)$$

The residual life functions play vital role in reliabilty and survival analysis.

The mean for residual life $R_{(z)} = X - z \,|\, X > z, z \ge 0$ of a component at time z or life expectancy known as mean residual life (MRL) function is

$$E\left[R_{(z)}\right] = E\left(X - z \,|\, X > z\right) = \frac{1}{S(z)}\left[E(X) - E_{X\le z}(X)\right] - z,$$

$$E\left[R_{(z)}\right] = \frac{1}{S(z)} \frac{\alpha\lambda^s}{\gamma^{\frac{s}{\beta}+1}} \left[B\left(\frac{s}{\beta}+1, \frac{\alpha}{\gamma}-\frac{s}{\beta}\right) - B\left(\frac{\gamma z^\beta}{\lambda^\beta}; \frac{s}{\beta}+1, \frac{\alpha}{\gamma}-\frac{s}{\beta}\right)\right] - z \quad (20)$$

The mean for reverse residual life $\bar{R}_{(z)} = z - X \,|\, X \le z, z \ge 0$ or mean waiting time or mean inactivity timeis

$$E\left[\bar{R}_{(z)}\right] = E\left(z - X \,|\, X \le z\right) = z - \frac{1}{F(z)}\left[E_{X\le z}(X)\right],$$

$$E\left[\bar{R}_{(z)}\right] = z - \frac{1}{F(z)} \frac{\alpha\lambda^k}{\gamma^{\frac{k}{\beta}+1}} B\left(\frac{\gamma z^\beta}{\lambda^\beta}; \frac{k}{\beta}+1, \frac{\alpha}{\gamma}-\frac{k}{\beta}\right). \quad (21)$$

The mean deviation about the mean $\left(\delta_1 = E|X-\mu|\right)$ and about the median $\left(\delta_2 = E|X-\tilde{M}|\right)$ can be written as $\delta_1 = 2\mu F(\mu) - 2\mu M_1'(\mu)$ and $\delta_2 = \mu - 2M_1'(\tilde{M})$ respectively, where $\mu = E(X)$ and $\tilde{M} = Q_{0.5}$ .The quantities $M_1'(\mu)$ and $M_1'(\tilde{M})$ can be obtained from (19). For specific probability p, Lorenz and Bonferroni curves are computed as $L(p) = \frac{M_1'(q)}{\mu'}$, $B(p) = L(p)\,|\, p$ and, where $q = Q(p)$.

3.3 L-Moments and related Descriptive Measures

Hosking (1990) developed L-moments to estimate model parameters. L-moments are less affected due to outlier in data. The r[th] L-moments are computed as

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k C_k^{r-1} E\left(X_{r-k:r}\right). \quad (22)$$

The mean of $X_{r:n}$ is computed as

$$E\left(X_{r:n}\right) = r\binom{n}{r}\int x\left(F(x)\right)^{r-1}\left(1-F(x)\right)^{n-r} f(x)\,\mathrm{d}x. \quad (23)$$

The mean of $X_{r:n}$ for the MSM distribution is

$$E\left(X_{r:n}\right)=r\binom{n}{r}\int_0^{\infty}x_{r:n}\left[1-\left[1+\gamma\left(\frac{x_{r:n}}{\lambda}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}}\right]^{r-1}\left(\left[1+\gamma\left(\frac{x_{r:n}}{\lambda}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}}\right)^{n-r}\times$$

$$\left(\frac{\alpha\beta}{\lambda}\left(\frac{x_{r:n}}{\lambda}\right)^{\beta-1}\left(1+\gamma\left(\frac{x_{r:n}}{\lambda}\right)^{\beta}\right)^{\frac{\alpha}{\gamma}-1}\right)dx_{r:n},$$

$$E\left(X_{r:n}\right)=r\binom{n}{r}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\frac{\alpha\lambda}{\gamma^{1+\frac{1}{\beta}}}B\left(1+\frac{1}{\beta},\frac{\alpha}{\gamma}(n-r+i+1)-\frac{1}{\beta}\right). \quad (24)$$

The first four L-moments are obtained from relationships

$$\lambda_1=E\left(X\right)=\int_0^1 x(\mathrm{F})\,\mathrm{dF}\,,$$

$$\lambda_2=\frac{1}{2}E\left(X_{2:2}-X_{1:2}\right)=\int_0^1 x(\mathrm{F})\left(2F-1\right)\mathrm{dF},$$

$$\lambda_3=\frac{1}{3}E\left(X_{3:3}-2X_{2:3}+X_{1:3}\right)=\int_0^1 x(\mathrm{F})\left(6F^2-6F-1\right)\mathrm{dF},$$

$$\lambda_4=\frac{1}{4}E\left(X_{4:4}-3X_{3:4}+3X_{2:4}-X_{1:4}\right)=\int_0^1 x(\mathrm{F})\left(20F^3-30F^2+12F-1\right)\mathrm{dF},$$

$L-CV=\frac{\lambda_2}{\lambda_1}$ is coefficient of variation. $\tau_r=\frac{\lambda_r}{\lambda_2}$, $r\geq 3$, $\tau_3=\frac{\lambda_3}{\lambda_2}$ and $\tau_4=\frac{\lambda_4}{\lambda_2}$ are measures for skewness and kurtosis respectively . The main four L-moments, L-CV, L-skewness and L-kurtosis for the MSM distribution are

$$\lambda_1=E\left(X\right)=\frac{\lambda}{\gamma^{\frac{1}{\beta}}}\frac{\Gamma\left(\frac{1}{\beta}+1\right)\Gamma\left(\frac{\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{\alpha}{\gamma}\right)}, \quad (25)$$

$$\lambda_2=\frac{\lambda\Gamma\left(\frac{1}{\beta}+1\right)}{\gamma^{\frac{1}{\beta}}}\left[\frac{\Gamma\left(\frac{\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{\alpha}{\gamma}\right)}-\frac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)}\right], \quad (26)$$

$$\lambda_3=\frac{\lambda\Gamma\left(\frac{1}{\beta}+1\right)}{\gamma^{\frac{1}{\beta}}}\left(\frac{\Gamma\left(\frac{\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{\alpha}{\gamma}\right)}-3\frac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)}+2\frac{\Gamma\left(\frac{3\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)}\right), \quad (27)$$

$$\lambda_4 = \frac{\lambda\Gamma\left(\frac{1}{\beta}+1\right)}{\gamma^{\frac{1}{\beta}}}\left[\begin{array}{l}\frac{\Gamma\left(\frac{\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{\alpha}{\gamma}\right)} - 6\frac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)} + \\ \qquad 10\frac{\Gamma\left(\frac{3\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)} - 5\frac{\Gamma\left(\frac{4\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{4\alpha}{\gamma}\right)}\end{array}\right], \tag{28}$$

$$L-CV = \frac{\lambda_2}{\lambda_1} = 1 - \left[\frac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)}\right]\frac{\left[\Gamma\left(\frac{\alpha}{\gamma}\right)\right]}{\Gamma\left(\frac{\alpha}{\gamma}-\frac{1}{\beta}\right)}, \tag{29}$$

where $0 < L-CV < 1$,

$$\tau_3 = L-CS = \frac{\left(\frac{\Gamma\left(\frac{\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{\alpha}{\gamma}\right)} - 3\frac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)} + 2\frac{\Gamma\left(\frac{3\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)}\right)}{\left[\frac{\Gamma\left(\frac{\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{\alpha}{\gamma}\right)} - \frac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)}\right]}, \tag{30}$$

is a measure of skewness.

$$\tau_4 = L-CK = \frac{\left(\frac{\Gamma\left(\frac{\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{\alpha}{\gamma}\right)} - 6\frac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)} + 10\frac{\Gamma\left(\frac{3\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)} - 5\frac{\Gamma\left(\frac{4\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{4\alpha}{\gamma}\right)}\right)}{\left[\frac{\Gamma\left(\frac{\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{\alpha}{\gamma}\right)} - \frac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)}\right]}, \tag{31}$$

is measure of kurtosis.

3.4 TL-Moments and related Descriptive Measures

Elamir and Seheult (2003) introduced trimmed L-moments(TL-moments). TL-moments trim unwanted influences. TL-moments provide best estimates of model parameters from TL-moments. The *r*th TL-moments as follows:

$$\lambda_r^{(1)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k C_k^{r-1} E\left(X_{r+k+1\,:\,r+2}\right). \tag{32}$$

The main four TL-moments, TL-CV, TL-skewness and TL-kurtosis are determined from following relationships

$$\lambda_1^{(1)} = E\left(X_{2:3}\right) = \int_0^1 x(\mathrm{F})\,\mathrm{dF}\,,$$

$$\lambda_2^{(1)} = \frac{1}{2}E\left(X_{3:4} - X_{2:4}\right) = 6\int_0^1 x(\mathrm{F})\left(1-F\right)\left(2F-1\right)\mathrm{dF}\,,$$

$$\lambda_3^{(1)} = \frac{1}{3}E\left(X_{4:5} - 2X_{3:5} + X_{2:5}\right) = \frac{20}{3}\int_0^1 x(\mathrm{F})\left(1-F\right)\left(5F^2 - 5F + 1\right)\mathrm{dF},$$

$$\lambda_4^{(1)} = \frac{1}{4}E\left(X_{5:6} - 3X_{4:6} + 3X_{3:6} - X_{2:6}\right) = \int_0^1 x(\mathrm{F})\left(1-F\right)\left(14F^3 - 21F^2 + 9F - 1\right)\mathrm{dF},$$

$\tau_r^{(1)} = \dfrac{\lambda_r^{(1)}}{\lambda_2^{(1)}}$, $r \geq 3$, $\tau_3^{(1)} = \dfrac{\lambda_3^{(1)}}{\lambda_2^{(1)}}$ and $\tau_4^{(1)} = \dfrac{\lambda_4^{(1)}}{\lambda_2^{(1)}}$ are measures of skewness and kurtosis respectively. The main four TL-moments, TL-CV, TL-skewness, TL-kurtosis for the MSM distribution are

$$\lambda_1^{(1)} = \frac{\lambda\Gamma\left(\frac{1}{\beta}+1\right)}{\gamma^{\frac{1}{\beta}}}\left[3\frac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)} - \frac{2\Gamma\left(\frac{3\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)}\right], \tag{33}$$

$$\lambda_2^{(1)} = \frac{3\lambda\Gamma\left(\frac{1}{\beta}+1\right)}{\gamma^{\frac{1}{\beta}}}\left[\frac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)} - \frac{2\Gamma\left(\frac{3\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)} + \frac{\Gamma\left(\frac{4\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{4\alpha}{\gamma}\right)}\right], \tag{34}$$

$$\lambda_3^{(1)} = \frac{10}{3}\frac{\lambda\Gamma\left(\frac{1}{\beta}+1\right)}{\gamma^{\frac{1}{\beta}}}\left[\begin{array}{l}\dfrac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)} - \dfrac{4\Gamma\left(\frac{3\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)} + \\ \qquad\qquad \dfrac{5\Gamma\left(\frac{4\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{4\alpha}{\gamma}\right)} - \dfrac{2\Gamma\left(\frac{5\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{5\alpha}{\gamma}\right)}\end{array}\right], \tag{35}$$

$$\lambda_4^{(1)} = \frac{15\lambda\Gamma\left(\frac{1}{\beta}+1\right)}{\gamma^{\frac{1}{\beta}}}\left[\begin{array}{l}\dfrac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{4\Gamma\left(\frac{2\alpha}{\gamma}\right)} - \dfrac{5\Gamma\left(\frac{3\alpha}{\gamma}-\frac{1}{\beta}\right)}{3\Gamma\left(\frac{3\alpha}{\gamma}\right)} + \\ \qquad \dfrac{15\Gamma\left(\frac{4\alpha}{\gamma}-\frac{1}{\beta}\right)}{4\Gamma\left(\frac{4\alpha}{\gamma}\right)} - \dfrac{7\Gamma\left(\frac{5\alpha}{\gamma}-\frac{1}{\beta}\right)}{2\Gamma\left(\frac{5\alpha}{\gamma}\right)} + \dfrac{7\Gamma\left(\frac{6\alpha}{\gamma}-\frac{1}{\beta}\right)}{6\Gamma\left(\frac{6\alpha}{\gamma}\right)}\end{array}\right], \tag{36}$$

$$TL-CV=\frac{\lambda_2^{(1)}}{\lambda_1^{(1)}}=\frac{\left[\frac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)}-\frac{2\Gamma\left(\frac{3\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)}+\frac{\Gamma\left(\frac{4\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{4\alpha}{\gamma}\right)}\right]}{\left[3\frac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)}-\frac{2\Gamma\left(\frac{3\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)}\right]}, \quad (37)$$

$$\tau_3^{(1)}=TL-CS=\frac{\frac{10}{9}\left[\frac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)}-\frac{4\Gamma\left(\frac{3\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)}+\frac{5\Gamma\left(\frac{4\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{4\alpha}{\gamma}\right)}-\frac{2\Gamma\left(\frac{5\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{5\alpha}{\gamma}\right)}\right]}{\left[\frac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)}-\frac{2\Gamma\left(\frac{3\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)}+\frac{\Gamma\left(\frac{4\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{4\alpha}{\gamma}\right)}\right]}, \quad (38)$$

is a measure of skewness.

$$\tau_4^{(1)}=TL-CK=\frac{5\left[\frac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{4\Gamma\left(\frac{2\alpha}{\gamma}\right)}-\frac{5\Gamma\left(\frac{3\alpha}{\gamma}-\frac{1}{\beta}\right)}{3\Gamma\left(\frac{3\alpha}{\gamma}\right)}+\frac{15\Gamma\left(\frac{4\alpha}{\gamma}-\frac{1}{\beta}\right)}{4\Gamma\left(\frac{4\alpha}{\gamma}\right)}-\frac{7\Gamma\left(\frac{5\alpha}{\gamma}-\frac{1}{\beta}\right)}{2\Gamma\left(\frac{5\alpha}{\gamma}\right)}+\frac{7\Gamma\left(\frac{6\alpha}{\gamma}-\frac{1}{\beta}\right)}{6\Gamma\left(\frac{6\alpha}{\gamma}\right)}\right]}{\left[\frac{\Gamma\left(\frac{2\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{2\alpha}{\gamma}\right)}-\frac{2\Gamma\left(\frac{3\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{3\alpha}{\gamma}\right)}+\frac{\Gamma\left(\frac{4\alpha}{\gamma}-\frac{1}{\beta}\right)}{\Gamma\left(\frac{4\alpha}{\gamma}\right)}\right]}, \quad (39)$$

is a measure of Kurtosis.

4. Reliability Measures

We present two reliability measures such as stress-strength reliability and multi-component stress-strength reliability.

4.1 Stress-Strength Reliability of the MSM Distribution

Let X_1 be strength and X_2 be stress and X_1 follows the MSM distribution $(\alpha_1,\beta,\gamma,\lambda)$ and X_2 follows the MSM distribution $(\alpha_2,\beta,\gamma,\lambda)$, then $R=\Pr(X_2<X_1)=\int_0^\infty f_{x_1}(x)F_{x_2}(x)\,dx$ is reliability parameter (Kotz et al.; 2003).

The Stress-Strength reliability of the component is calculated as

$$R=\int_0^\infty \frac{\alpha_1\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-\frac{\alpha_1}{\gamma}-1}\left[1-\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-\frac{\alpha_2}{\gamma}}\right]dx=\frac{\alpha_2}{(\alpha_1+\alpha_2)}. \quad (40)$$

So, R does not depend upon β, γ and λ.

4.2 Multi-component Stress-Strength Reliability Estimator $R_{s,\kappa}$ Based on the MSM Distribution

Consider a system that has κ identical components out of which *s* components are functioning. The strengths of κ components are $X_i, i = 1,2...\kappa$ with common cdf F while, the stress Y imposed on the components has cdf G. The strengths $X_i, i = 1,2...\kappa$ and stress Y are i.i.d. distributed. The probability that system operates properly is reliability of the system i.e.

$$R_{s,\kappa} = P[strengths\,(X_i, i = 1,2...\kappa) > stress\,(Y)],$$

$$R_{s,\kappa} = P[\text{at the minimum "}s\text{" of } (X_i, i = 1,2...\kappa)\, exceed\, Y].$$

$$R_{s,\kappa} = \sum_{l=s}^{\kappa} \binom{\kappa}{\ell} \int_0^{\infty} [1-F(y)]^l \, [F(y)]^{\kappa-l} \, dG(y), \text{ (Bhattacharyya and Johnson;1974) (41)}$$

Let X_i follows the MSM distribution $(\alpha_1, \beta, \gamma, \lambda)$ and Y follows the MSM distribution $(\alpha_2, \beta, \gamma, \lambda)$ with common shape parameters β, γ and unknown shape parameters α_1 *and* α_2. The reliability that system operates properly in multi-component stress- strength for the MSM distribution is

$$R_{s,\kappa} = \sum_{\ell=s}^{\kappa} \binom{\kappa}{\ell} \int_0^{\infty} \left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-\frac{\ell\alpha_1}{\gamma}} \left[1-\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-\frac{\alpha_1}{\gamma}}\right]^{(\kappa-\ell)} d\left[1-\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-\frac{\alpha_2}{\gamma}}\right].$$

Letting $u = \left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-\frac{\alpha_2}{\gamma}}$, then we obtain

$$R_{s,\kappa} = \sum_{\ell=s}^{\kappa} \binom{\kappa}{\ell} \int_0^1 \left(u^{\Upsilon}\right)^{\ell} \left(1-u^{\Upsilon}\right)^{(\kappa-\ell)} du.$$

Let $w = u^{\Upsilon}$,

$$R_{s,k} = \frac{1}{\Upsilon} \sum_{\ell=s}^{k} \binom{k}{\ell} \int_0^1 w^{\ell+\frac{1}{\Upsilon}-1} (1-w)^{(k-\ell)} \, dw,$$

$$R_{s,\kappa} = \frac{1}{\Upsilon} \sum_{\ell=s}^{\kappa} \binom{\kappa}{\ell} B\left(\ell+\frac{1}{\Upsilon}, \kappa-\ell+1\right), \quad where\ \Upsilon = \frac{\alpha_1}{\alpha_2}. \qquad (42)$$

The probability $R_{s,k}$ in (42) is multi-component stress- strength reliability model.

5. Uncertainty Measures

Information generating function, Shannon entropy, Renyi entropy, Q-entropy and other entropies for the MSM distribution are studied.

5.1 Information Generating Function

The differentiation of information generating function at point 0 or 1, help to derive the measures of information which are otherwise tough to characterize and compute. The information generating function for X with the MSM model is computed as

$$H(f)=E\left[f^{\Upsilon-1}(X)\right]=\int_0^\infty f^{\Upsilon}(x)dx, \tag{43}$$

$$H(f)=\alpha^{\Upsilon}\left(\frac{\beta}{\lambda}\right)^{\Upsilon-1}\gamma^{\frac{(1-\beta)\Upsilon-1}{\beta}}B\left(\Upsilon+\frac{1-\Upsilon}{\beta},\frac{\alpha\beta\Upsilon-\Upsilon\gamma+\gamma}{\gamma\beta}\right). \tag{44}$$

The Shannon entropy can be found by $\frac{d}{d\Upsilon}H(f)\Big|_{\Upsilon=1}$.

5.2 Shannon, Awad, Renyi and Other Entropies

Entropy is a device to measure the amount of uncertainty or randomness confined in random observation about its mother distribution. The smaller uncertainty in the data is smaller entropy. The Shannon entropy $h(X)=E\left[I(X)\right]$ for X with the MSM model is

$$h(X)=-\int\left\{\ln\left[f(x)\right]\right\}f(x)dx. \tag{45}$$

$$h(X)=-\ln\left(\frac{\alpha\beta}{\lambda}\right)-(\beta-1)\int_0^\infty\ln\left(\frac{x}{\lambda}\right)\frac{\alpha\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1}dx+$$

$$\left(\frac{\alpha}{\gamma}+1\right)\int_0^\infty\ln\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]\frac{\alpha\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1}dx,$$

$$h(X)=1+\frac{\gamma}{\alpha}-\ln\left(\frac{\alpha\beta}{\lambda}\right)+\left(1-\frac{1}{\beta}\right)\left[\psi\left(\frac{\alpha}{\gamma}\right)+(\ln\gamma)-\psi(1)\right]. \tag{46}$$

Awad (1987) extended the Shannon entropy as

$$A(X)=\int_0^\infty\ln\left(\frac{\delta}{f(x)}\right)f(x)dx,$$

where δ is maximum value of $f(x)$ for the MSM distribution in the domain of X.

Awad entropy for X with the MSM distribution is

$$A(X)=\left\{\begin{array}{l}\ln\delta+1+\dfrac{\gamma}{\alpha}-\ln\left(\dfrac{\alpha\beta}{\lambda}\right)+\\ \qquad\left(1-\dfrac{1}{\beta}\right)\left[\psi\left(\dfrac{\alpha}{\gamma}\right)+(\ln\gamma)-\psi(1)\right]\end{array}\right\}. \tag{47}$$

The Renyi entropy $I_R(v)$ for X with the MSM distribution is

$$I_R(\vartheta)=\frac{1}{1-\vartheta}\log\left\{\int_{-\infty}^{\infty}[f(x)]^{\vartheta}dx\right\},\vartheta>0,\vartheta\neq1$$

$$\int_0^{\infty}f^{\vartheta}(x)dx=\int_0^{\infty}\left[\frac{\alpha\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}-1}\right]^{\vartheta}dx,$$

we obtain

$$I_R=\frac{1}{1-\vartheta}\log\left\{\frac{\alpha^{\vartheta}\beta^{\vartheta-1}}{\lambda^{\vartheta-1}\gamma^{\vartheta+\frac{1-\vartheta}{\beta}}}B\left(\frac{\vartheta\beta-\vartheta+1}{\beta},\frac{\alpha\beta\vartheta+\gamma-\vartheta\gamma}{\gamma\beta}\right)\right\}. \tag{48}$$

The Q-entropy $H_q(f)$ for X with the MSM distribution is

$$H_q(f)=\frac{1}{1-q}\log\left[1-I(q)\right],$$

$$H_q(f)=\frac{1}{1-q}\log\left[1-\frac{\alpha^{q}\beta^{q-1}}{\lambda^{q-1}\gamma^{q+\frac{1-q}{\beta}}}B\left(\frac{1-q+q\beta}{\beta},\frac{\alpha\beta q-q\gamma+\gamma}{\gamma\beta}\right)\right]. \tag{49}$$

Havrda and Chavrat entropy $I_{HC}(v)$ for X with the MSM distribution is

$$I_{HC}(v)=\frac{1}{v-1}\log\left\{\int_{-\infty}^{\infty}[f(x)]^{v}dx\right\},v>0,v\neq1, \tag{50}$$

$$I_{HC}=\frac{1}{v-1}\log\left\{\alpha\frac{\alpha^{v}\beta^{v-1}}{\lambda^{v-1}\gamma^{v+\frac{1-v}{\beta}}}B\left(\frac{1-v++v\beta}{\beta},\frac{\alpha\beta v-v\gamma+\gamma}{\gamma\beta}\right)\right\},v\neq1. \tag{51}$$

Tsallis entropy $S_q(f(x))$ forX with the MSM distribution is

$$S_q(f(x))=\frac{1}{q-1}\left(1-\int f^{q}(x)dx\right),$$

$$S_q\left[f\left(x\right)\right]=\frac{1}{q-1}\left(1-\frac{\alpha^q\beta^{q-1}}{\lambda^{q-1}\gamma^{q+\frac{1-q}{\beta}}}B\left(\frac{1-q+q\beta}{\beta},\frac{\alpha\beta q-q\gamma+\gamma}{\gamma\beta}\right)\right)\quad q\neq 1. \tag{52}$$

Entropies are useful to study daily temperature fluctuations (climatic), anomalous diffusion, DNA sequences, information content signals, heart rate variability (HRV) and cardiac autonomic neuropathy (CAN).

6. Characterizations

In this section, we characterize the MSM distribution via (i) Truncated moment of a variable; (ii) ratio of truncated moments; (iii) Hazard function and (iv) Mills ratio.

6.1 Truncated moment of a Variable

Proposition 6.1: Let $X:\Omega\to\left(0,\infty\right)$ be a random variable of a continuous distribution, the function $F\left(x\right)$ in (4) is cdf of X, if and only if

$$E\left[X^\beta\middle|X>z\right]=\frac{\lambda^\beta}{\left(\alpha-\gamma\right)}\left[1+\alpha\left(\frac{z}{\lambda}\right)^\beta\right]\ with\ \alpha>\gamma. \tag{53}$$

Proof. If (3) is pdf of X, then (53) holds, it is easy to show

$$E\left[X^\beta\middle|X>z\right]=\left[1-F\left(z\right)\right]^{-1}\int_z^\infty x^\beta f\left(x\right)dx,$$

$$E\left[X^\beta\middle|X>z\right]=\left[1-F\left(z\right)\right]^{-1}\int_z^\infty x^\beta\frac{\alpha\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\left[1+\gamma\left(\frac{x}{\lambda}\right)^\beta\right]^{-\frac{\alpha}{\gamma}-1}dx,$$

$$E\left[X^\beta\middle|X>z\right]=\frac{\lambda^\beta}{\left(\alpha-\gamma\right)}\left[1+\alpha\left(\frac{z}{\lambda}\right)^\beta\right].$$

Conversely if (53) holds, then

$$\frac{1}{\bar{F}\left(z\right)}\int_z^\infty x^\beta f\left(x\right)dx=\frac{\lambda^\beta}{\left(\alpha-\gamma\right)}\left[1+\alpha\left(\frac{z}{\lambda}\right)^\beta\right],$$

$$\int_z^\infty x^\beta f\left(x\right)dx=\frac{\left(1-F\left(z\right)\right)\lambda^\beta}{\left(\alpha-\gamma\right)}\left[1+\alpha\left(\frac{z}{\lambda}\right)^\beta\right]. \tag{54}$$

Differentiating (54), we obtain

$$-z^\beta f\left(z\right)=\frac{\left(1-F\left(z\right)\right)}{\left(\alpha-\gamma\right)}\left\{\alpha\beta z^{\beta-1}\right\}-\frac{f\left(z\right)\lambda^\beta}{\left(\alpha-\gamma\right)}\left[1+\alpha\left(\frac{z}{\lambda}\right)^\beta\right],$$

$$F(z)=1-\left[1+\gamma\left(\frac{z}{\lambda}\right)^{\beta}\right]^{-\frac{\alpha}{\gamma}},\ z\geq 0,$$

which is cdf of the MSM distribution.

6.2 Ratio of Truncated Moments

Now, we characterize the MSM distribution on the basis of two truncated moments of X using Theorem G (Glänzel; 1990).

Proportion 6.2.1:.Let $\xi_1(x)=\frac{\alpha+\gamma}{\alpha}\left(1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right)^{-1}$

and $\xi_2(x)=\frac{\alpha+2\gamma}{\alpha}\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-2}$, $x>0$. Let $X:\Omega\to(0,\infty)$ be a random variable of a continuous distribution, the pdf of X is (3) if and only if $\upsilon(x)$ has the form $\upsilon(x)=\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]$.

Proof: For random variable X with the MSM distribution with pdf (3), then

$$E\left(\xi_1(x)\middle|X\geq x\right)=\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-1} \text{ and}$$

$$E\left(\xi_2(x)\middle|X\geq x\right)=\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-2},\ x>0,$$

$$\frac{E[\xi_1(X)|\mathrm{X}\geq \mathrm{x}]}{E[\xi_2(X)|\mathrm{X}\geq \mathrm{x}]}=\upsilon(x)=\left(1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right) \text{ and } \upsilon'(x)=\frac{\beta\gamma}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}.$$

The differential equation $s'(x)=\frac{\upsilon'(x)\xi_2(x)}{\upsilon(x)\xi_2(x)-\xi_1(x)}=\frac{\left(\frac{\alpha}{\gamma}+2\right)\frac{\beta\gamma}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}}{\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]}$

has solution $s(x)=\ln\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{\left(\frac{\alpha}{\gamma}+2\right)}$.

Now, according to the theorem G, X has pdf (3).

Corollary 6.2.1. Let $X:\Omega\to(0,\infty)$ be a random variable of a continuous distribution and $\xi_2(x)=\frac{\alpha+2\gamma}{\alpha}\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-2}$, $x>0$.The pdf of X is (3) if and only if the functions $\upsilon(x)$ and $\xi_1(x)$ fulfills the equation

$$\frac{\upsilon'(x)}{\upsilon(x)\xi_2(x)-\xi_1(x)}=\frac{\alpha\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]. \tag{55}$$

Remarks 6.2.1. The general solution (55) is

$$\upsilon(x)=\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{\left(\frac{\alpha}{\gamma}+2\right)}\int\left(-\frac{\alpha\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]^{-\left(\frac{\alpha}{\gamma}+1\right)}\xi_1(x)\right)+E$$

where E is constant.

6.3 Hazard Function

Definition 6.3.1: Let $X:\Omega\to(0,\infty)$ be a random variable of a continuous distribution, the function $f(x)$ is pdf of X, if and only if the hazard function $h_F(x)$, of a twice differentiable function F, satisfying differential equation

$$\frac{d}{dx}\left[\ln f(x)\right]=\frac{h_F'(x)}{h_F(x)}-h_F(x). \tag{56}$$

Proposition 6.3.1: Let $X:\Omega\to(0,\infty)$ be a random variable of a continuous distribution, the function $f(x)$ in (3) is pdf of X, if and only if the hazard function of X, $h_F(x)$ satisfies the first order differential equation

$$h_F'(x)\left(1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right)+h_F(x)\frac{\gamma\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}=\frac{\alpha\beta(\beta-1)}{\lambda^2}\left(\frac{x}{\lambda}\right)^{\beta-2}. \tag{57}$$

Proof: If (3) is pdf of X, then (57) holds. Now if (57) holds, then

$$\frac{d}{dx}\left[h_F(x)\left(1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right)\right]=\frac{\alpha\beta}{\lambda}\frac{d}{dx}\left[\left(\frac{x}{\lambda}\right)^{\beta-1}\right],$$

or

$$h_F(x)=\frac{\alpha\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}\left(1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right)^{-1},$$

which is hazard function of MSM distribution.

6.4 Mills Ratio

Definition 6.4.1: Let $X:\Omega\to(0,\infty)$ be a random variable of a continuous distribution. The function $f(x)$ is pdf of X, if and only if the Mills ratio $m(x)$, of a twice differentiable function F, satisfies equation

$$\frac{1+m'(x)}{m(x)}+\frac{d\left[\ln f(x)\right]}{dx}=0. \tag{58}$$

Proposition 6.4.1: Let $X:\Omega\to(0,\infty)$ be a random variable of a continuous distribution. The function $f(x)$ in (3) is pdf of X, if and only if the Mills ratio of X, $m_F(x)$ satisfies the first order differential equation

$$m_F'(x)\left(\frac{x}{\lambda}\right)^{\beta-1}+m_F(x)\frac{(\beta-1)}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-2}=\frac{\lambda}{\alpha\beta}\frac{\gamma\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}. \tag{59}$$

Proof: If(3) is pdf of X, then (59) holds. Now if (59) holds, then

$$\frac{d}{dx}\left[m_F(x)\left(\frac{x}{\lambda}\right)^{\beta-1}\right]=\frac{\lambda}{\alpha\beta}\frac{d}{dx}\left(1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right),$$

or

$$m_F(x)=\frac{\lambda}{\alpha\beta}\left(\frac{x}{\lambda}\right)^{-\beta+1}\left(1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right),$$

which is Mills ratio of MSM distribution.

7. Maximum Likelihood Estimation

Here, we adopt maximum likelihood estimation technique for MSM parameters. Let $\zeta=(\alpha,\beta,\gamma,\lambda)^{\tau}$ be unknown parameter vector. The log likelihood function $\ell(\zeta)$ for the MSM distribution is

$$\ell(\zeta)=\ln L(\alpha,\beta,\gamma,\lambda)=n\ln\alpha+n\ln\beta-n\beta\ln\lambda+(\beta-1)\sum\ln x-\left(\frac{\alpha}{\gamma}+1\right)\sum\ln\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right] \tag{60}$$

We can compute the maximum likelihood estimators (MLEs) of $\alpha,\beta,\gamma,\lambda$ by simplifying the following nonlinear equations either directly or using quasi-Newton procedure, computer packages/ software such as R, SAS, Ox, MATHEMATICA, MATLAB and MAPLE.

$$\frac{\partial}{\partial\alpha}\left(\ln L(\alpha,\beta,\gamma,\lambda)\right)=\frac{n}{\alpha}-\frac{1}{\gamma}\sum\ln\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]=0, \tag{61}$$

$$\frac{\partial}{\partial\beta}\left(\ln L\left(\alpha,\beta,\gamma,\lambda\right)\right)=\frac{n}{\beta}-n\ln\lambda+\sum\ln x-\left(\gamma+\alpha\right)\sum\frac{\left(\frac{x}{\lambda}\right)^{\beta}\ln\left(\frac{x}{\lambda}\right)}{\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]}=0, \tag{62}$$

$$\frac{\partial}{\partial\gamma}\left(\ln L\left(\alpha,\beta,\gamma,\lambda\right)\right)=\frac{\alpha}{\gamma^{2}}\sum\ln\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]-\left(1+\frac{\alpha}{\gamma}\right)\sum\frac{\left(\frac{x}{\lambda}\right)^{\beta}}{\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]}=0, \tag{63}$$

$$\frac{\partial}{\partial\lambda}\left(\ln L\left(\alpha,\beta,\gamma,\lambda\right)\right)=-\frac{n\beta}{\lambda}+\frac{\gamma\beta}{\lambda}\left(1+\frac{\alpha}{\gamma}\right)\sum\frac{\left(\frac{x}{\lambda}\right)^{\beta}}{\left[1+\gamma\left(\frac{x}{\lambda}\right)^{\beta}\right]}=0. \tag{64}$$

7.1 Simulation Study

Here, we consider the behavior of the MLEs of the MSM parameters regarding sample size n. To assess the behavior, the steps for simulation are as follows.

- Generate 10000 samples of sizes *n* from the MSM distribution using the inverse cdf method.
- Calculate the MLEs for 10000 samples, say $(\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\lambda})$ for *i*=1, 2,..., 10000 from non-linear optimization technique with constraint matching to range of parameters. (1.25, 1.5, 1.75, 1.0), (1.5,1.7, 1.8,1.5) and (2, 2, 2, 1.8) are taken as the true parameter values $(\alpha,\beta,\gamma,\lambda)$.
- Calculate the means, biases and mean squared errors (MSE) of MLEs.

For this purpose, we choose various arbitrarily parameters and *n*=50,100,200,300,500 sample sizes. We summarize all the results in Table 3. The results clearly indicate that when the sample size n increases, the estimated MSE decrease and estimated biases drop to zero. MSE of estimated parameters increases, as shape parameter rises. This reveals that MLEs for MSM distribution are reliable.

Sample	Statistics	$\alpha = 1.25$	$\beta = 1.5$	$\gamma = 1.75$	$\lambda = 1.0$
n=50	Means	1.2636	1.6569	1.6919	1.1586
	Bias	0.0136	0.1569	-0.0581	0.1586
	MSE	0.3423	3.5327	1.6134	1.9019
n=100	Means	1.275	1.5518	1.7491	1.0695
	Bias	0.025	0.0518	-9e-04	0.0695
	MSE	0.0596	0.0836	0.1432	0.1194
n=200	Means	1.2914	1.5176	1.7752	1.0524
	Bias	0.0414	0.0176	0.0252	0.0524
	MSE	0.0262	0.033	0.0748	0.0521
n=300	Means	1.2881	1.5158	1.7945	1.0358
	Bias	0.0381	0.0158	0.0445	0.0358
	MSE	0.0174	0.0217	0.0534	0.032
n=500	Means	1.289	1.5093	1.7992	1.0304
	Bias	0.039	0.0093	0.0492	0.0304
	MSE	0.0108	0.0125	0.0345	0.0186

Sample	Statistics	$\alpha = 1.5$	$\beta = 1.7$	$\gamma = 1.8$	$\lambda = 1.5$
n=50	Means	1.5008	1.839	1.7198	1.6609
	Bias	8e-04	0.139	-0.0802	0.1609
	MSE	0.1778	4.0598	4.7372	0.7976
n=100	Means	1.5211	1.7537	1.7882	1.5739
	Bias	0.0211	0.0537	-0.0118	0.0739
	MSE	0.0587	0.0957	0.201	0.1794
n=200	Means	1.534	1.7263	1.8293	1.5486
	Bias	0.034	0.0263	0.0293	0.0486
	MSE	0.0375	0.0426	0.1138	0.0813
n=300	Means	1.5383	1.7163	1.8367	1.5417
	Bias	0.0383	0.0163	0.0367	0.0417
	MSE	0.0223	0.0259	0.0666	0.0521
n=500	Means	1.5398	1.7102	1.8444	1.5353
	Bias	0.0398	0.0102	0.0444	0.0353
	MSE	0.0136	0.0153	0.0438	0.0311

Sample	Statistics	$\alpha = 2.0$	$\beta = 2.0$	$\gamma = 2.0$	$\lambda = 1.8$
n=50	Means	2.0406	2.1235	1.8599	2.0007
	Bias	0.0406	0.1235	-0.1401	0.2007
	MSE	0.8962	0.4469	0.9227	1.1532
n=100	Means	2.0419	2.0522	1.971	1.8883
	Bias	0.0419	0.0522	-0.029	0.0883
	MSE	0.1298	0.1238	0.3622	0.2121
n=200	Means	2.057	2.0221	2.0243	1.8585
	Bias	0.057	0.0221	0.0243	0.0585
	MSE	0.0608	0.0528	0.1544	0.094
n=300	Means	2.0552	2.0185	2.0392	1.8431
	Bias	0.0552	0.0185	0.0392	0.0431
	MSE	0.0414	0.0347	0.1082	0.0607
n=500	Means	2.0573	2.009	2.0466	1.8378
	Bias	0.0573	0.009	0.0466	0.0378
	MSE	0.0249	0.0199	0.07	0.0343

Table 3: Means, Bias and MSEs of the MSM distribution (1.25, 1.5, 1.75, 1.0), (1.5, 1.7, 1.8, 1.5) and (2, 2, 2, 1.8)

7.2 Application: Serum-Reversal Time of Children Data

We now consider data set such as serum-reversal times (in days) for authentication the flexibility, utility and potentiality of the MSM distribution. For serum-reversal times, we compare the results of fitting the MSM distribution with Singh Maddala (SM), modified Burr XII (MBXII), Burr XII (BXII), Lomax, Log-logistic (Log-log) distributions. For selection of the optimum distribution, we compute Cramer-von Mises (W*), Anderson Darling (A*), and Kolmogorov- Smirnov statistics with p-values [K-S(p-values] statistics for MSM distribution and its sub-models. We compute the MLEs and their standard errors (in parentheses). We summarize all the results such as the MLEs **(**standard errors in parentheses) and measures W*, A*, K-S with p-values in table 4.

The data set related to Serum-reversal times of children born from HIV-infected mothers (Lee, 1992)are

2, 2, 2, 5,9, 9, 19, 32, 32, 46, 50, 56, 56, 78, 91, 95, 106, 129, 129, 148, 149, 156, 175, 176, 191, 192, 204, 209, 211, 225, 229, 230, 238, 254, 271, 274, 276, 290, 291, 292, 297, 297, 322, 334, 334, 334, 344, 346, 353, 353, 359, 365, 366, 367, 370, 378, 378, 382, 382, 385, 398, 400, 402, 414, 422, 424, 428, 434, 435, 440, 443, 446, 448, 448, 451, 454, 459, 460, 461, 473, 480, 481, 484, 487, 493, 497, 498, 502, 511, 511, 513, 514, 516, 521, 524, 526, 537, 538, 541, 543, 544, 544, 545, 549, 551, 553, 553, 554, 556, 559, 571, 576, 577, 578, 582, 588, 590, 596, 609, 610, 615, 619, 626, 627, 648, 653, 678, 680, 687, 696, 729, 744, 748, 777, 847, 848, 867, 874, 894, 901, 907, 974 and 1021.

Model	α	β	γ	λ	W*	A*	K-S (p-value)
MSM	940715.9 (2310.858)	1.9324 (0.0138)	90390.20 (12420.61)	564861.0 (14213.58)	0.8301	4.6726	0.1125 (0.0535)
SM	1112.3637 (536.3601)	1.6349 (0.1144)	-	33818.8522 (1690.8385)	0.8542	4.7958	0.1425 (0.0060)
MBXII	0.0110 (0.0317)	86.6803 (1067.626)	5.4828 (83.1094)	-	3.9246	20.5629	0.4542 (<2.2e-16)
BXII	0.0233 (0.0259)	7.5074 (8.3425)	-	-	3.9366	20.6203	0.454 (<2.2e-16)
Lomax	3.5492 (0.0146)	-	-	-	3.5492	18.7682	0.4534 (<2.2e-16)
Log-log	-	0.2665 (0.0172)	-	-	2.7239	14.7181	0.6721 (<2.2e-16)

Table 4: MLEs (standard errors),W*, A*, K-S(p-values) for Serum-reversal times

We infer from the table4 that MSM distribution is best model, with smallest values for (W*, A* and K-S) statistics and maximum p-value.

8. Conclusion and Final Remarks

We derive the MSM distribution from the GDE. We present some structural and main descriptive properties such as quantile function, sub-models, ordinary moments, characteristic function, conditional moments, incomplete moments, mean deviations, inequality curves, moments for residual life functions, L-moments, L-coefficient of variation, L- coefficient of skewness and L- coefficient of kurtosis, T L-moments, TL-coefficient of variation, TL- coefficient of skewness and TL- coefficient of kurtosis and reliability measures. We characterize the MSM distribution via different techniques. We address maximum likelihood estimation technique for MSM parameters. We assess the behavior of the maximum likelihood estimates (MLEs) through a simulation study. We test the importance of the MSM distribution with goodness of fit statistics via its applications to serum-reversal time's data. The values of goodness of fit statistics suggest that that the MSM distribution is best fitted model.

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