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GENERALIZED DUAL ESTIMATORS FOR ESTIMATING MEAN USING SUB-SAMPLING THE NON-RESPONDENTS

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Abstract

In this paper an improved class of generalized dual estimators is proposed for estimating the population mean considering the prevalence of non-response in two different cases regarding the availability of population mean of the supportive variable. Expressions for the bias and the mean square error of the advised generalized class of dual estimators in each case are derived along with the optimum conditions which make the proposed generalized estimator more efficient than some of the existing estimators. Empirical studies have also been provided to establish the advantage of the advised generalized class of dual estimators. This paper illustrates that the proposed estimators perform better in each case than the existing estimators.

Key Words: Supporting Variable, Regression Estimator, Dual Estimator, Non-Response, Mean Squared Errors, Double Sampling.

1. Introduction

In surveys, generally this is very hard to have possible information from all the units at the initial attempt even after some callbacks. An estimate obtained from such missing information might be ambiguous especially when the respondents vary from the non-respondents which results in uncontainable bias.To avoid such type of bias, it is compulsory to interact again with those non-respondents through telephonic contact, personal interview, or using any other method to gain maximum response. There are different ways to tackle with non-respondents such as weighting adjustment and imputation procedures, randomized response technique and sub-sampling the non-respondents given by Hansen and Hurwitz (1946). Following Murthy (1964), Srivenkataramana (1980) provided a dual to ratio estimator assuming the existence of non-response for estimating the population mean. Kumar (2012) suggested some estimators for population mean in simple random sampling and Sanaullah et al. (2015) advised different improved exponential-type ratio and product estimators to estimate population mean. Saleem et al. (2018a, b) provided some generalized ratio-type estimators for estimating population mean in presence of the non-response. Following Sukhatme (1962) and Cochran (1977), many authors such as Sammiudin and Hanif (2006), and Sanaulah et al. (2014) advised an improved class of estimators for estimating the population mean in two-phase sampling.

2. Sampling Scheme and Notations

Let $U_N = (u_1, u_2, ..., u_N)$ be a finite population of size N in which y and x are the study and auxiliary characters having the non-negative i^{th} value of y_i and x_i on u_i. Let U_n be a sample of size n chosen from U_N by simple random sampling without replacement (S_{wor}). It is noticed that only n_1 units respond and remaining $n_2 (= n - n_1)$ units do not. In this problem, we suppose that entire population U_N is distributed into two non-overlapping assemblies in which U_{N_1} units belong to respondents and U_{N_2} units belong to non-respondents. The weights of responding and non-responding assemblies are $P_1 = \frac{N_1}{N}$ and $P_2 = \frac{N_2}{N}$ and their estimates are respectively given by $p_1 = \frac{n_1}{n}$ and $p_2 = \frac{n_2}{n}$. At the second-phase as suggested by Hansen & Hurwitz (1946), we take another sub-sample $U_{rn_{(2)}}$ of size $r = \frac{n_2}{k}, (k \geq 1)$ from U_{Rn_2} and obtain the information through personal interviews and assumed that all r units respond in 2nd attempt.Let $\mu_y = N^{-1} \sum_{i=1}^{N} y_i$ and $\sigma_y^2 = (N-1)^{-1} \sum_{i=1}^{N} (y_i - \mu_y)^2$ be the mean and variance of the study variable y from a population of size N. Let $\mu_{y(2)} = N_2^{-1} \sum_{i=1}^{N_2} y_i$ and $\sigma_{y(2)}^2 = (N_2 - 1)^{-1} \sum_{i=1}^{N_2} (y_i - \mu_{y(2)})^2$ be respectively the mean and variance from the non-respondent's group. Let $\hat{\mu}_{y(1)} = n_1^{-1} \sum_{i=1}^{n_1} y_i$ and $\hat{\mu}_{y(2)} = n_2^{-1} \sum_{i=1}^{n_2} y_i$ denote the sample means based on n_1 responding units and n_2 non-responding units respectively. Further, let $\hat{\mu}_{yr_{(2)}} = r^{-1} \sum_{i=1}^{r} y_i$ denote the mean of $r = k^{-1} n_2$ sub-sampled units.Thus, following Hansen and Hurwitz (1946) an estimator for assessing population mean of the variable of interest y is given by $\hat{\mu}_y^* = p_1 \hat{\mu}_{y_{(1)}} + p_2 \hat{\mu}_{yr_{(2)}}$ and the variance of $\hat{\mu}_y^*$ may be set by $\text{var}(\hat{\mu}_y^*) = \mu_y^2 (\theta C_y^2 + \gamma C_{y(2)}^2)$ or alternatively $C_{02}^* = (\mu_y^2)^{-1} \text{var}(\hat{\mu}_y^*)$ where $C_y^2 = (\mu_y^2)^{-1} \sigma_y^2, C_{y(2)}^2 = (\mu_{y(2)}^2)^{-1} \sigma_{y(2)}^2, \theta = n^{-1}(1-f), f = N^{-1} n$,and $\gamma = n^{-1} (P_2 (k-1))$. The covariance term between $\hat{\mu}_x^*$ and $\hat{\mu}_y^*$ can be given by

$\text{cov}\left(\hat{\mu}_x^*, \hat{\mu}_y^*\right) = \mu_x \mu_y \left(\theta C_x^2 + \gamma C_{x(2)}^2\right)$ or $C_{11}^* = \left(\mu_x \mu_y\right)^{-1} \text{cov}\left(\hat{\mu}_x^*, \hat{\mu}_y^*\right)$, where $\sigma_{xy} = (N-1)^{-1} \sum_{i=1}^{N} \left(y_i - \mu_y\right)\left(x_i - \mu_x\right)$, and $\sigma_{xy(2)} = (N_2 - 1)^{-1} \sum_{i=1}^{N_2} \left(y_i - \mu_{y(2)}\right)\left(x_i - \mu_{x(2)}\right)$.

Cochran (1977) suggested a ratio estimator considering the presence of non-response as,

$$t_1 = \hat{R}\mu_X \text{, where } \hat{R} = \frac{\hat{\mu}_y^*}{\hat{\mu}_x^*}. \tag{2.1}$$

Khare and Srivastava (1997) recommended the regression estimator ifpopulation mean is known assuming non-response on both y and x as,

$$t_2 = \hat{\mu}_y^* + \hat{\beta}_{yx}^* \left(\mu_X - \hat{\mu}_x^*\right), \tag{2.2}$$

where $\hat{\beta}_{yx}^* = \frac{\hat{\sigma}_{yx}^*}{\hat{\sigma}_x^{*2}}$, $\hat{\sigma}_{yx}^* = \frac{\sum_{i=1}^{n_1} y_i x_i + k \sum_{i=1}^{r} y_i x_i - n\hat{\mu}_y^* \hat{\mu}_x^*}{n-1}$ and $\hat{\sigma}_x^{*2} = \frac{\sum_{i=1}^{n_1} x_i^2 + k \sum_{i=1}^{r} x_i^2 - n\hat{\mu}_x^{*2}}{n-1}$.

Following Srivenkataramana's (1980) transformation, we modify Murthy (1964) product estimator into a dual-to-ratio estimator to the case if non-response is present as,

$$t_3 = \hat{\mu}_y^* . \frac{\tilde{\mu}_x}{\mu_x} \text{, where } \tilde{\mu}_x = \frac{N\mu_X - n\hat{\mu}_x^*}{N-n}. \tag{2.3}$$

Kumar and Bhougal (2011) suggested an exponential ratio-type estimator toassess population mean assuming the existence of non-responses,

$$t_4 = \hat{\mu}_y^* \exp\left(\frac{\mu_X - \hat{\mu}_x^*}{\mu_X + \hat{\mu}_x^*}\right). \tag{2.4}$$

Motivating from Singh et al. (2008), Kumar and Bhougal (2011) advised an estimator when there is missing information on both y and x is as,

$$t_5 = t_{KB} = \hat{\mu}_y^* \left(\alpha \exp\left(\frac{\mu_X - \hat{\mu}_x^*}{\mu_X + \hat{\mu}_x^*}\right) + (1-\alpha)\exp\left(\frac{\hat{\mu}_x^* - \mu_X}{\hat{\mu}_x^* + \mu_X}\right)\right). \tag{2.5}$$

Taking motivation from Kumar and Bhougal (2011), Chanu and Singh (2015) suggest dual to ratio estimator when non-response occur on both variable as,

$$t_{CS} = \hat{\mu}_y^* \left\{\alpha_2 \exp\left(\frac{\mu_X - \hat{\mu}_x^*}{\mu_X + \hat{\mu}_x^*}\right) + (1-\alpha_2)\exp\left(\frac{\tilde{\mu}_x - \mu_X}{\tilde{\mu}_x + \mu_X}\right)\right\}, \tag{2.6}$$

where $\alpha_2 = \frac{g}{g-1} - \frac{2}{(g-1)R} \frac{(\theta S_{yx} + \lambda S_{2yx})^2}{\theta S_x^2 + \lambda S_{2x}^2}$.

As estimators expressed in (2.1)-(2.6) can achieve the mean square error as minimum as that of the mean square error of the regression estimator. It is therefore in this paper, we advise more effective generalized class of dual estimators than some of the existing estimators including the regression estimator for retrieving the population mean of the variable of interest in the presence of non-response having non-sampled information on auxiliary variable. Further the two different situations, (i) information about the population mean of the auxiliary variable is available; and (ii) information about the population mean of the auxiliary variable is not available, are considered to suggest the class of generalized dual estimators.

3. The Suggested Class of Generalized Dual Estimators

We now suggest an improved class of generalized dual estimators for estimating the population mean as an alternative to some existing estimators, considering the non-response and availability of population mean of supportive variable in prior of survey as,

$$t_{6(G)} = \left[\hat{\mu}_y^* \left\{ \alpha \left(\frac{\mu_X}{\hat{\mu}_x^*} \right)^c + \beta \left(\frac{\tilde{\mu}_x}{\mu_X} \right)^c \right\} + k_1 \hat{\mu}_y^* + k_2 \left(\mu_X - \hat{\mu}_x^* \right) \right] \exp\left(\frac{\mu_X^c - \hat{\mu}_x^{*c}}{\mu_X^c + \hat{\mu}_x^{*c}} \right), \tag{3.1}$$

where $\tilde{\mu}_x = \dfrac{N\mu_X - n\hat{\mu}_x^*}{N-n}$ is based on the non-sampled units $N-n$ as its denominator shows the sum over x information from those units which are actually not selected in the sample. It is to note that $\tilde{\mu}_x$ is an unbiased estimator for μ_x and the correlation between $\hat{\mu}_y^*$ and $\tilde{\mu}_x$ is negative i.e $corr\left(\hat{\mu}_y^*, \tilde{\mu}_x\right) = -\rho_{xy}$. Also note that $\tilde{\mu}_X$ can be easily obtained once if $\tilde{\mu}_x$ is known. k_1, $k_2$are the constants which need to be estimated such that the proposed class of generalized dual estimators gives least *MSE*, whereas $(\alpha, \beta) \in [0,1]$ are the generalizing constants and c is used to show the power transformation on supporting variable.

3.1 The *Bias* and the *MSE* of the Proposed Class of Generalized Dual Estimators $t_{6(G)}$

To attain the expressions for the bias and the *MSE* of the advised class of generalized dual estimator we may consider,

$\hat{\mu}_y^* = \mu_Y\left(1+\delta_o^*\right)$, " $\hat{\mu}_x^* = \mu_X\left(1+\delta_1^*\right)$, such that," $E\left(\delta_o^*\right) = E\left(\delta_1^*\right) = 0$,

$E\left(\delta_o^{*2}\right) = \theta C_y^2 + \gamma C_{y(2)}^2 = C_{02}^*$," $E\left(\delta_1^{*2}\right) = \theta C_x^2 + \gamma C_{x(2)}^2 = C_{20}^*$,

and $E\left(\delta_o^*\delta_1^*\right) = \theta\rho_{xy}C_xC_y + \gamma\rho_{xy(2)}C_{x(2)}C_{y(2)} = C_{11}^*$,

where $\rho_{xy} = \dfrac{\sigma_{xy}}{\sigma_x\sigma_y}$ and $\rho_{xy(2)} = \dfrac{\sigma_{xy(2)}}{\sigma_{x(2)}\sigma_{y(2)}}$.

Now by expressing $t_{6(G)}$ in $\delta's$ we may have

$$t_{6(G)}=\left(\mu_Y\left(1+\delta_o^*\right)\left(\alpha\left(\frac{\mu_X}{\mu_X\left(1+\delta_1^*\right)}\right)^c+\beta\left(\frac{N\mu_X-\left(n\mu_X\left(1-\delta_1^*\right)\right)}{\mu_X}\right)^c\right)+\right.$$

$$\left.k_1\mu_Y\left(1+\delta_o^*\right)+k_2\left(\mu_X-\mu_X\left(1+\delta_1^*\right)\right)\right]\exp\left(\frac{\mu_X^c-\left(\mu_X\left(1+\delta_1^*\right)\right)^c}{\mu_X^c+\left(\mu_X\left(1+\delta_1^*\right)\right)^c}\right). \quad (3.2)$$

After expanding the equation (3.2) uptothe order $O\left(n^{-1}\right)$, we get

$$\left(t_{6(G)}-\mu_Y\right)=\mu_Y\left(\left(\alpha+\beta-1\right)+\delta_0^*(\alpha+\beta)-\delta_1^*\left(\alpha c+\beta Pc+\frac{\alpha c}{2}+\frac{\beta c}{2}\right)\right.$$
$$+\delta_1^{*2}\left(\frac{\alpha c(c+1)P^2}{2!}+\frac{\beta c(c-1)P^2}{2!}+\frac{\beta Pc^2}{2}+\frac{\alpha c^2}{2}+\frac{\alpha c}{4}\right.$$
$$\left.+\frac{\beta c}{4}+\frac{\alpha c^2}{8}+\frac{\beta c^2}{8}\right)-\delta_0^*\delta_1^*\left(\alpha c+\beta Pc+\frac{\alpha c}{2}+\frac{\beta c}{2}\right)+k_1\left(1+\delta_0^*\right.$$
$$\left.\left.-\frac{c\delta_1^*}{2}-\frac{c\delta_0^*\delta_1^*}{2}+\frac{c\delta_1^{*2}}{4}+\frac{c^2\delta_1^{*2}}{8}\right)\right)-k_2\mu_X\left(\delta_1^*-\frac{c\delta_1^{*2}}{2}\right)\Bigg),$$

where $P=\dfrac{n}{N-n}$.

(3.3)

Or alternatively one may write (3.3) as,

$$\left(t_{6(G)}-\mu_Y\right)=\left(\mu_Y\left(A_o+\delta_0^*A_1-\delta_1^*A_2+\delta_1^{*2}A_3-\delta_0^*\delta_1^*A_4+k_1\left(1+\delta_0^*-\frac{c\delta_1^*}{2}\right.\right.\right.$$
$$\left.\left.\left.-\frac{c\delta_0^*\delta_1^*}{2}+\frac{c\delta_1^{*2}}{4}+\frac{c^2\delta_1^{*2}}{8}-k_2\mu_X\left(\delta\varepsilon_1^*-\frac{c\delta_1^{*2}}{2}\right)\right)\right)\right), \quad (3.4)$$

where, $A_0=\alpha+\beta-1$, $A_1=\alpha+\beta$, $A_2=\alpha c+\beta Pc+\dfrac{\alpha c}{2}+\dfrac{\beta c}{2}$,

$A_3=\dfrac{\alpha c(c+1)P^2}{2!}+\dfrac{\beta c(c-1)P^2}{2!}+\dfrac{\beta Pc^2}{2}+\dfrac{\alpha c^2}{2}+\dfrac{\alpha c}{4}+\dfrac{\beta c}{4}+\dfrac{\alpha c^2}{8}+\dfrac{\beta c^2}{8}$ and

$A_4=\alpha c+\beta Pc+\dfrac{\alpha c}{2}+\dfrac{\beta c}{2}$.

The expression for the bias and the *MSE* of $t_{6(G)}$ up to the order $O\left(n^{-1}\right)$ may be obtained from (3.4) respectively as,

$$Bias(t_{6(G)}) = \left(\mu_Y \left(A_0 + C_{20}^* A_3 - C_{11}^* A_4 + k_1 \left(1 + \frac{cC_{20}^*}{4} + \frac{c^2 C_{20}^*}{8} - \frac{cC_{11}^*}{2} \right) \right) \right.$$

$$\left. + k_2 \mu_X \frac{cV_{20}^*}{2} \right) \tag{3.5}$$

and

$$MSE(t_{6(G)}) = \Big(\mu_Y^2 \Big(A_0^2 + C_{02}^* A_1^2 + C_{20}^* A_2^2 + 2C_{20}^* A_0 A_3 - 2C_{11}^* A_0 A_4 - 2C_{11}^* A_1 A_2$$

$$+ k_1^2 \left(1 + C_{02}^* + \frac{c^2 C_{20}^*}{4} + \frac{2cC_{20}^*}{4} + \frac{2c^2 C_{20}^*}{8} - \frac{2cC_{11}^*}{2} - \frac{2cC_{11}^*}{2} \right) + k_2^2 \mu_X^2 C_{20}^*$$

$$+ 2k_1 \left(A_0 \left(1 + \frac{cC_{20}^*}{4} + \frac{c^2 C_{20}^*}{8} - \frac{cC_{11}^*}{2} \right) + A_1 \left(C_{02}^* - \frac{cC_{11}^*}{2} \right) - A_2 \left(C_{11}^* - \frac{cC_{20}^*}{2} \right) + A_3 C_{20}^* - A_4 C_{11}^* \right) \Big)$$

$$- 2\mu_X \mu_Y k_2 \left(A_1 C_{11}^* - A_2 C_{20}^* - \frac{A_0 cC_{20}^*}{2} \right) - 2\mu_X \mu_Y k_1 k_2 \left(C_{11}^* - \frac{2cC_{20}^*}{2} \right) \Big). \tag{3.6}$$

Or alternatively the *MSE* of $t_{6(G)}$ can be taken as,

$$MSE(t_{6(G)}) = \Delta_o + k_1^2 \Delta_1 + k_2^2 \Delta_2 + k_1 \Delta_3 + k_2 \Delta_4 + k_1 k_2 \Delta_5 , \tag{3.7}$$

where,

$$\Delta_0 = \mu_Y^2 \left(A_0^2 + C_{02}^* A_1^2 + C_{20}^* A_2^2 + 2C_{20}^* A_0 A_3 - 2C_{11}^* A_0 A_4 - 2C_{11}^* A_1 A_2 \right),$$

$$\Delta_1 = \mu_Y^2 \left(1 + C_{02}^* + \frac{c^2 C_{20}^*}{4} + \frac{2cC_{20}^*}{4} + \frac{2c^2 C_{20}^*}{8} - 2cC_{11}^* \right),\ \Delta_2 = \mu_X^2 C_{20}^*,$$

$$\Delta_3 = 2\mu_Y^2 \left(A_0 \left(1 + \frac{cC_{20}^*}{4} + \frac{2c^2 C_{20}^*}{8} - \frac{cC_{11}^*}{2} \right) + A_1 \left(C_{02}^* - \frac{cC_{11}^*}{2} \right) \right.$$

$$\left. - A_2 \left(C_{11}^* - \frac{cC_{20}^*}{2} \right) + A_3 C_{20}^* - A_4 C_{11}^* \right),\ \Delta_4 = -2\mu_X \mu_Y \Big(A_1 C_{11}^* - A_2 C_{20}^*$$

$$\left. - \frac{A_0 cC_{20}^*}{2} \right) \text{and } \Delta_5 = -2\mu_X \mu_Y \left(C_{11}^* - \frac{2cC_{20}^*}{2} \right).$$

The optimum values for k_1 and k_2 and the least *MSE*($t_{6(G)}$)canbe obtained from (3.7) respectively as,

$$k_{1(opt)} = \frac{\Delta_4 \Delta_5 - 2\Delta_3 \Delta_2}{4\Delta_1 \Delta_2 - \Delta_5^2} \quad \text{and} \quad k_{2(opt)} = \frac{\Delta_3 \Delta_5 - 2\Delta_4 \Delta_1}{4\Delta_1 \Delta_2 - \Delta_5^2} \tag{3.8}$$

and

$$\min MSE(t_{6(G)}) = \Delta_0 - \frac{\Delta_3\Delta_4\Delta_5 - \Delta_1\Delta_4^2 - \Delta_2\Delta_3^2}{\Delta_5^2 - 4\Delta_1\Delta_2}\,. \tag{3.9}$$

It is to mention that for different values of c one can get different estimators directly from (3.1), for example,

for $c=1$, the estimator in (3.1) can be taken as,

$$t_{6(1)} = \left(\hat{\mu}_y^*\left(\alpha\left(\frac{\mu_X}{\hat{\mu}_x^*}\right)+\beta\left(\frac{\tilde{\mu}_x}{\mu_X}\right)\right)+k_1\hat{\mu}_y^*+k_2\left(\mu_X-\hat{\mu}_x^*\right)\right)\exp\left(\frac{\mu_X-\hat{\mu}_x^*}{\mu_X+\hat{\mu}_x^*}\right), \tag{3.10}$$

for $c=\frac{1}{2}$, the estimator in (3.1) can be taken as,

$$t_{6(2)} = \left(\hat{\mu}_y^*\left(\alpha\left(\frac{\mu_X}{\hat{\mu}_x^*}\right)^{\frac{1}{2}}+\beta\left(\frac{\tilde{\mu}_x}{\mu_X}\right)^{\frac{1}{2}}\right)+k_1\hat{\mu}_y^*+k_2\left(\mu_X-\hat{\mu}_x^*\right)\right)\exp\left(\frac{\mu_X^{\frac{1}{2}}-\hat{\mu}_x^{*\frac{1}{2}}}{\mu_X^{\frac{1}{2}}+\hat{\mu}_x^{*\frac{1}{2}}}\right), \tag{3.11}$$

for $c=\frac{1}{3}$, the estimator in (3.1) can be taken as,

$$t_{6(3)} = \left(\hat{\mu}_y^*\left(\alpha\left(\frac{\mu_X}{\hat{\mu}_x^*}\right)^{\frac{1}{3}}+\beta\left(\frac{\tilde{\mu}_x}{\mu_X}\right)^{\frac{1}{3}}\right)+k_1\hat{\mu}_y^*+k_2\left(\mu_X-\hat{\mu}_x^*\right)\right)\exp\left(\frac{\mu_X^{\frac{1}{3}}-\hat{\mu}_x^{*\frac{1}{3}}}{\mu_X^{\frac{1}{3}}+\hat{\mu}_x^{*\frac{1}{3}}}\right) \tag{3.12}$$

for $c=\frac{1}{4}$, the estimator in (3.1) can be taken as,

$$t_{6(4)} = \left(\hat{\mu}_y^*\left(\alpha\left(\frac{\mu_X}{\hat{\mu}_x^*}\right)^{\frac{1}{4}}+\beta\left(\frac{\tilde{\mu}_x}{\mu_X}\right)^{\frac{1}{4}}\right)+k_1\hat{\mu}_y^*+k_2\left(\mu_X-\hat{\mu}_x^*\right)\right)\exp\left(\frac{\mu_X^{\frac{1}{4}}-\hat{\mu}_x^{*\frac{1}{4}}}{\mu_X^{\frac{1}{4}}+\hat{\mu}_x^{*\frac{1}{4}}}\right), \tag{3.13}$$

for $c=\frac{1}{5}$, the estimator in (3.1) can be taken as,

$$t_{6(5)} = \left(\hat{\mu}_y^*\left(\alpha\left(\frac{\mu_X}{\hat{\mu}_x^*}\right)^{\frac{1}{5}}+\beta\left(\frac{\tilde{\mu}_x}{\mu_X}\right)^{\frac{1}{5}}\right)+k_1\hat{\mu}_y^*+k_2\left(\mu_X-\hat{\mu}_x^*\right)\right)\exp\left(\frac{\mu_X^{\frac{1}{5}}-\hat{\mu}_x^{*\frac{1}{5}}}{\mu_X^{\frac{1}{5}}+\hat{\mu}_x^{*\frac{1}{5}}}\right), \tag{3.14}$$

andfor $c=\frac{1}{6}$, the estimator in (3.1) can be taken as,

$$t_{6(6)} = \left(\hat{\mu}_y^* \left(\alpha \left(\frac{\mu_X}{\hat{\mu}_x^*} \right)^{\frac{1}{6}} + \beta \left(\frac{\tilde{\mu}_x}{\mu_X} \right)^{\frac{1}{6}} \right) + k_1 \hat{\mu}_y^* + k_2 \left(\mu_X - \hat{\mu}_x^* \right) \right) \exp \left(\frac{\mu_X^{\frac{1}{6}} - \hat{\mu}_x^{*\frac{1}{6}}}{\mu_X^{\frac{1}{6}} + \hat{\mu}_x^{*\frac{1}{6}}} \right). \tag{3.15}$$

Similarly, the expressions for the bias, the *MSE*, the optimum values and the *min MSE* for the estimator (3.10) to (3.15) can be obtained directly from (3.5), (3.7), (3.8) and (3.9) respectively.

3.2 Empirical Study of the Proposed Class of Generalized Dual $t_{6(G)}$ estimator:

In order to demonstrate the performance & application of the advised generalized estimator, we are considering two different populations and their explanation is given respectively as,

Population 1: Source [Khare and Srivastava (1993)]
Y: cultivated area (in acres); *X*:population of the villages

P_2 = 0.20, N = 70,, n = 35, μ_Y= 981.29, $\mu_{Y(2)}$= 597.29,, μ_X = 1755.53 , $\mu_{X(2)}$ = 1100.24, C_y= 0.6254, $C_{Y(2)}$= 0.4087, C_x= 0.8009, $C_{x(2)}$= 0.5739,, ρ_{xy} = 0.778, $\rho_{xy(2)}$ = 0.445.

Population 2: Source [Khare and Srivastava (1995)]
Y: measurements of turbine meter (in ml); *X*: measurements of displacement meter(incm^3)

N = 100, n =30, μ_X = 260.84, $\mu_{X(2)}$= 259.96, μ_Y= 3500.12, $\mu_{Y(2)}$= 3401.08, C_y= 0.5941, $C_{Y(2)}$ = 0.5075, C_x= 0.5996 , $C_{x(2)}$= 0.5168, ρ_{xy} = 0.985, $\rho_{xy(2)}$= 0.995, P_2 = 0.25.

	Estimators	$1/k$			
		$1/2$	$1/2.5$	$1/3$	$1/3.5$
	$\hat{\mu}_y^*$	6299.48	6759.03	7218.58	7678.14
Population 1	t_1	5065.70	5857.07	6648.45	7439.83
	$t_2 = t_{KB} = t_{CS}$	2987.80	3404.89	3814.91	4219.30
	t_3	4012.34	4277.04	4541.69	4806.39
	t_4	3023.579	3422.50	3821.43	4220.36

	$t_{6(G)}$ Suggested class of Estimator	$t_{6(1)}$, c=1	**2973.61**	**3384.78**	**3788.15**	**4184.88**
		$t_{6(2)}$,c=$\frac{1}{2}$	**2982.10**	**3396.57**	**3803.77**	**4204.84**
		$t_{6(3)}$, c=$\frac{1}{3}$	**2982.13**	**3396.83**	**3804.32**	**4205.74**
		$t_{6(4)}$,c=$\frac{1}{4}$	**2981.74**	**3396.44**	**3803.94**	**4205.40**
		$t_{6(5)}$,C=$\frac{1}{5}$	**2981.37**	**3396.12**	**3803.50**	**4204.94**
		$t_{6(6)}$,C=$\frac{1}{6}$	**2981.05**	**3395.77**	**3803.11**	**4204.52**
Population 2	$\hat{\mu}_y^*$		127187.20	140334.20	153481.20	166628.20
	t_1		3340.05	3478.34	3616.64	3754.93
	$t_2 = t_{KB} = t_{CS}$		3266.40	3397.55	3528.70	3659.85
	t_3		42290.59	46537.69	50784.06	55030.42
	t_4		32754.51	35988.84	39223.18	42457.51
	$t_{6(G)}$ Suggested class of Estimator	$t_{6(1)}$, c=1	**2980.30**	**3045.98**	**3164.889**	**3157.09**
		$t_{6(2)}$,c=$\frac{1}{2}$	**3207.46**	**3324.56**	**3440.15**	**3554.28**
		$t_{6(3)}$, c=$\frac{1}{3}$	**3243.94**	**3369.49**	**3494.40**	**3618.73**
		$t_{6(4)}$,c=$\frac{1}{4}$	**3255.54**	**3383.84**	**3511.80**	**3639.46**
		$t_{6(5)}$, C=$\frac{1}{5}$	**3260.46**	**3389.96**	**3519.25**	**3648.37**
		$t_{6(6)}$, C=$\frac{1}{6}$	**3262.91**	**3393.03**	**3523.06**	**3562.88**

Table 1: The *MSE* of the estimator for variousoptions of *k*

In this section, a comparison of the proposed class of generalized dual estimators $t_{6(G)}$ is shown in Table 1, with some estimators such as $\hat{\mu}_y^*$, t_1, t_2, t_3, t_4, t_{KB} and t_{CS}. We have computed the *MSE*s of the proposed generalized class of dual estimators considering four different values of *c* $(=1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6})$ and at value of c, four different choices of *k* (=2, 2.5,3,3.5) are used to acquire the mean square error of each estimator. From Table 1 it is established that by increasing the value of *k* the *MSEs* of existing estimators are increasing sharply but this increase is small in the *MSEs* of the proposed class of generalized dual estimators. Further it is noted that from the proposed class $t_{6(G)}$, some of the proposed estimators such as $t_{6(1)}$, $t_{6(2)}$, $t_{6(3)}$, $t_{6(4)}$, $t_{6(5)}$ and $t_{6(6)}$ attain less *MSEs* (presented in bold) than the *MSEs* of the existing estimators $\hat{\mu}_y^*, t_1, t_2, t_4, t_{KB}$ and t_{CS}. Moreover it is perceived that the $t_{6(1)}$ is the most effective estimator from the advised class of generalized dual estimators as it is attaining the minimum *MSE* than the *MSE*s of its own class of estimators. Therefore the proposed generalized class of dual estimator $t_{6(G)}$ is found to be an improved and more efficient class of estimators than the some other estimators.

4. Another Suggested Class of Generalized Dual Estimators

However, in certain practical conditions knowledge about the population mean of the auxiliary variable μ_x may not be readily available and due to the unavailability of such information one cannot use the estimator $t_{6(G)}$ as recommended in the former section. To tackle with such problem, the technique of two-phase sampling can be helpful. In two-phase sampling at initial-phase, we take a large sample of size $n'\left(n' \subset N\right)$ by SRSWOR to access μ_x as μ_x' and at second-phase a smaller sample is taken from initial-phase sample of size n'' where, $(n'' \subset n')$ by *SRSWOR* and it is assumed that n_1'' unit respond and n_2'' do not respond then following Hansen & Hurwitz(1946) another sub-sample of size $r'' = \frac{n_2''}{k''}$, $(k'' \geq 1)$ is taken from n_2'', and interviewed them. It is assumed that all r'' units respond while interviewing them for the study variable y.

Here we need to define some notations as previously we have define. Let μ_x' be the mean estimator based on n' and $\mu_{y(1)}''$ be the mean estimator from the respondents in second-phase sample of size n_1'' and $\mu_{yr_{(2)}}''$ be the mean estimator based on $r^{''}$ .Let $\hat{\mu}_{y(1)}^{''} = n_1^{''-1} \sum_{i=1}^{n_1^{''}} y_i$ and $\hat{\mu}_{y(2)}^{''} = n_2^{''-1} \sum_{i=1}^{n_2^{''}} y_i$ denote the sample means based on n_1 responding units and n_2 non-responding units respectively. Further, let $\hat{\mu}_{yr_{(2)}}^{''} = r^{''-1} \sum_{i=1}^{r^{''}} y_i$ denote the mean of $r^{''} = k^{''-1} n_2^{''}$ sub-sampled units.

An unbiased estimate of μ_y is defined by $\mu_y^{*''} = \frac{n_1^{''}}{n^{''}} \mu_{y(1)}^{''} + \frac{n_2^{''}}{n^{''}} \mu_{yr''(2)}^{''}$, and the variance of $\mu_y^{*''}$ is $\text{var}\left(\mu_y^{*''}\right) = \mu_y^2 \left(\theta^{''} C_y^2 + \gamma^{''} C_{y(2)}^2\right)$, or $C_{02}^{*''} = \left(\mu_y^2\right)^{-1} \text{var}\left(\mu_y^{*''}\right)$,where $\theta^{''} = \frac{1-f^{''}}{n^{''}}$, $f^{''} = \frac{n^{''}}{N}$ and $\gamma^{''} = \frac{P_2(k^{''}-1)}{n^{''}}$ .Similar expressions for any variable say x can be defined. Let the covariance term between $\mu_x^{*''}$ and $\mu_y^{*''}$ be $\text{cov}\left(\mu_x^{*''}, \mu_y^{*''}\right) = \mu_x \mu_y \left(\theta^{''} \sigma_{xy} + \gamma^{''} \sigma_{xy(2)}\right)$,or $C_{11}^{*''} = \left(\mu_x \mu_y\right)^{-1} \text{cov}\left(\mu_x^{*''}, \mu_y^{*''}\right)$, where $\varsigma_{20}^{*''} = C_{20}^{*''} - C_{20}^{'}$, and $\varsigma_{11}^{*''} = C_{11}^{*''} - C_{11}^{'}$.

Following Hansen & Hurwitz (1946), Cochran (1977) advised a ratio estimator following two-phase sampling and assuming population mean of the supporting variable is not available in advance as,

$$t_7 = \hat{R}^{''} \mu_x^{'} \text{, where } \hat{R}^{''} = \frac{\mu_y^{*''}}{\mu_x^{*''}} \text{ and } \mu_x^{'} = \frac{\sum_{i=1}^{n^{'}} x_i}{n^{'}}. \tag{4.1}$$

Following Srivenkataramana (1980) many authors have obtained a dual-to-ratio estimator in two-phase sampling when population mean of supporting variable is not available. As we are dealing with non-response a dual to ratio estimator is modified as;

$$t_8 = \mu_y^{*''} \frac{\tilde{\mu}_x^{*}}{\mu_x^{'}} \text{, where } \frac{n^{'} \mu_x^{'} - n \mu_x^{*}}{n^{'} - n}. \tag{4.2}$$

Similarly we can define a regression estimator of Cochran (1977) in two-phase sampling for dealing with non-response as,

$$t_9 = \mu_y^{*''} + \beta_{yx}^{*} \left(\mu_x^{'} - \mu_x^{*''}\right). \tag{4.3}$$

Singh et al (2010) following two-phase sampling proposed an exponential-type ratio estimator in the existence of non-response by,

$$t_{10} = \mu_y^{*''} \exp\left(\frac{\mu_x^{'} - \mu_x^{*''}}{\mu_x^{'} + \mu_x^{*''}}\right). \tag{4.4}$$

Following Singh et al. (2008), Kumar and Bhougal (2011) suggested an estimator in presence of non-response as,

$$t'_{KB} = \mu_y^{*''}\left(\alpha \exp\left(\frac{\mu_x^{'} - \mu_x^{*''}}{\mu_x^{'} + \mu_x^{*''}}\right) + (1-\alpha)\exp\left(\frac{\mu_x^{*''} - \mu_x^{'}}{\mu_x^{*''} + \mu_x^{'}}\right)\right). \quad (4.5)$$

Taking motivation from Kumar and Bhougal (2011), Chanu and Singh (2015) recommended an estimator utilizing information on the auxiliary variable assuming unknown population mean μ_x and existence of non-response in two-phase sampling as,

$$t'_{cs} = \hat{\mu}_y^{*}\left(\alpha_2 \frac{\hat{\mu}_x^{'}}{\hat{\mu}_x^{*}} + (1-\alpha_2)\frac{n'\hat{\mu}_x^{'} - n\hat{\mu}_x^{*}}{(n'-n)\hat{\mu}_x^{'}}\right). \quad (4.6)$$

Now motivating from (4.1) - (4.6) we suggest a more generalized and improved estimator for estimating the population mean in the presence of non-response when population mean of auxiliary variable *X* is unknown as,

$$t_{12(G)} = \left(\mu_y^{*''}\left(\alpha\left(\frac{\mu_x^{'}}{\mu_x^{*''}}\right)^c + \beta\left(\frac{\tilde{\mu}_x^{*}}{\mu_x^{'}}\right)^c\right) + k_1\mu_y^{*''} + k_2\left(\mu_x^{'} - \mu_x^{*''}\right)\right)\exp\left(\frac{\mu_x^{'c} - \mu_x^{*''c}}{\mu_x^{'c} + \mu_x^{*''c}}\right), \quad (4.7)$$

where $\tilde{\mu}_x^{*} = \dfrac{n^{'}\mu_x^{'} - n\mu_x^{*''}}{n^{'} - n}$. k_1 and k_2 are the constants which need to be estimated so that the generalized dual estimator gives the least *MSE*, and *c* is showing the power transformation on the auxiliary variable, and $(\alpha, \beta) \in [0,1]$ are the generalizing constants.

4.1 The *Bias* and the *MSE* of the Suggested Class of Generalized Dual Estimator $t_{12(G)}$

To attain the expressions for the bias and the *MSE* of the suggested class of generalized dual estimator we may consider,

$\mu_y^{*''} = \mu_Y\left(1+\delta_0^{*''}\right)$, $\mu_x^{'} = \mu_X\left(1+\delta_1^{'}\right)$, $\mu_x^{*''} = \mu_X\left(1+\delta_1^{*''}\right)$, such that $E\left(\delta_0^{*''}\right)$

$= E\left(\delta_1^{*''}\right) = E\left(\delta_1^{'}\right) = 0$, $E\left(\delta_0^{*''2}\right) = \theta^{''}C_y^2 + \gamma^{''}C_{y(2)}^2 = C_{02}^{*''}$

$E\left(\delta_1^{*''2}\right) = \theta^{''}C_x^2 + \gamma^{''}C_{x(2)}^2 = C_{20}^{*''}$, $E\left(\delta_1^{'2}\right) = E\left(\delta_1^{*''}\delta_1^{'}\right) = \theta^{'}C_x^2 = C_{20}^{'}$,

$E\left(\delta_0^{*''}\delta_1^{*''}\right) = \theta^{''}\rho_{xy}C_xC_y + \gamma^{''}\rho_{xy(2)}C_{x(2)}C_{y(2)} = C_{11}^{*''}$, $E\left(\delta_0^{*''}\delta_1^{'}\right) = \theta^{'}\rho_{xy}C_xC_y = C_{11}^{'}$,

$\varsigma_{20}^{*''} = C_{20}^{*''} - C_{20}^{'}$ and $\varsigma_{11}^{*''} = C_{11}^{*''} - C_{11}^{'}$.

Now by expressing $t_{12(G)}$ in terms of $\delta's$ we have,

$$t_{12(G)} = \left(\mu_Y\left(1+\delta_0^{*''}\right)\left(\alpha\left(\frac{\mu_X\left(1+\delta_1^{'}\right)}{\mu_X\left(1+\delta_1^{*''}\right)}\right)^c + \beta\left(\frac{n^{'}\left(\mu_X\left(1+\delta_1^{'}\right)\right) - \left(n\mu_X\left(1-\delta_1^{*''}\right)\right)}{\mu_X\left(1+\delta_1^{'}\right)}\right)^c\right) + k_1\mu_Y\left(1+\delta_0^{*''}\right)\right.$$

$$+k_2\left(\mu_X\left(1+\delta_1^{'}\right)-\mu_X\left(1+\delta_1^{*"}\right)\right)\Big)\exp\left(\frac{\left(\mu_X\left(1+\delta_1^{'}\right)\right)^c-\left(\mu_X\left(1+\delta_1^{*"}\right)\right)^c}{\left(\mu_X\left(1+\delta_1^{'}\right)\right)^c+\left(\mu_X\left(1+\delta_1^{*"}\right)\right)^c}\right) \quad (4.8)$$

After expanding the equation (4.8) upto $O(n^{'-1})$, we may get

$$\left(t_{12(G)}-\mu_Y\right)=\Bigg(\mu_Y\Bigg(\left(\alpha+\beta-1\right)+\delta_o^*(\alpha+\beta)-\delta_1^*\left(\alpha c+\beta Qc+\frac{\alpha c}{2}+\frac{\beta c}{2}\right)$$

$$+\delta_1^*\left(\alpha c-\beta c+\beta Qc+\frac{\alpha c}{2}+\frac{\beta c}{2}\right)+\delta_1^{*2}\Bigg(\frac{\alpha c(c+1)Q^2}{2!}-\frac{\beta c(c-1)P^2}{2!}+\frac{\beta Qc^2}{2}+\frac{\alpha c^2}{2}$$

$$+\frac{\alpha c}{4}+\frac{\beta c}{4}+\frac{\alpha c^2}{8}+\frac{\beta c^2}{8}\Bigg)+\delta_1^{'2}\Bigg(\frac{\alpha c(c-1)Q^2}{2!}+\frac{\beta c(c+1)P^2}{2!}-\beta Qc^2+\frac{\beta P^2c(c-1)}{2!}$$

$$+\frac{\beta Pc^2}{2}-\frac{\beta c^2}{2}+\frac{\alpha c^2}{2}-\frac{\alpha c}{4}-\frac{\beta c}{4}+\frac{\alpha c^2}{8}+\frac{\beta c^2}{8}\Bigg)-\delta_1^*\delta_1^{'}\Bigg(\alpha c^2-\beta Qc^2+\frac{\beta Qc^2}{2}$$

$$-\frac{\beta c^2}{2}+\frac{\alpha c^2}{2}+\frac{\beta Qc^2}{2}+\frac{\alpha c^2}{2}+\frac{2\alpha c^2}{8}+\frac{2\beta c^2}{8}\Bigg)-\delta_o^*\delta_1^*\left(\alpha c+\beta Qc+\frac{\alpha c}{2}+\frac{\beta c}{2}\right)$$

$$+\delta_o^*\delta_1^{'}\left(\alpha c-\beta c+\beta Pc+\frac{\alpha c}{2}+\frac{\beta c}{2}\right)+k_1\Bigg(1+\delta_o^{*"}-\frac{c\delta_1^{*"}}{2}+\frac{c\delta_1^{'}}{2}-\frac{c\delta_o^{*"}\delta_1^{*"}}{2}+\frac{c\delta_o^{*"}\delta_1^{'}}{2}+$$

$$\frac{c\delta_1^{*"2}}{4}-\frac{c\delta_1^{'2}}{4}+\frac{c^2\delta_1^{'2}}{8}+\frac{c^2\delta_1^{*"2}}{8}-\frac{2c^2\delta_1^{*"}\delta_1^{'}}{8}\Bigg)-k_2\mu_X\Bigg(\delta_1^{*"}-\delta_1^{'}-\frac{c\delta_1^{'2}}{2}$$

$$-\frac{c\delta_1^{*"2}}{2}+\frac{c\delta_1^{*"}\delta_1^{'}}{2}+\frac{c\delta_1^{*"}\delta_1^{'}}{2}\Bigg)\Bigg)\Bigg), \quad (4.9)$$

where $P=\frac{n}{n^{'}-n^{"}}$ and $Q=\frac{n^{'}}{n^{'}-n^{"}}$.

Or alternatively one can write (4.9) as,

$$\left(t_{12(G)}-\mu_Y\right)=\Big(\mu_Y\Big(A_0+\varepsilon_0^{*"}A_1-\varepsilon_1^{*"}A_2+\varepsilon_1^{'}A_2+\varepsilon_1^{*"2}A_3+\varepsilon_1^{'2}A_4-\varepsilon_0^{*"}\varepsilon_1^{*"}A_6$$

$$-\varepsilon_1^{'}\varepsilon_1^{*"}A_5+\varepsilon_1^{'}\varepsilon_0^{*"}A_6+k_1\Bigg(1+\varepsilon_0^{*"}+\frac{c\varepsilon_1^{'}}{2}-\frac{c\varepsilon_1^{*"}}{2}-\frac{c\varepsilon_0^{*"}\varepsilon_1^{*"}}{2}$$

$$-\frac{c\varepsilon_1^{'2}}{4}+\frac{c\varepsilon_1^{*"2}}{4}+\frac{c^2\varepsilon_1^{'2}}{8}+\frac{c^2\varepsilon_1^{*"2}}{8}+\frac{c\varepsilon_1^{'}\varepsilon_0^{*"}}{2}-\frac{c\varepsilon_1^{*"}\varepsilon_0^{*"}}{2}-\frac{2c^2\varepsilon_1^{*"}\varepsilon_1^{'}}{8}\Bigg)\Bigg)$$

$$-k_2\mu_X a\left(\varepsilon_1^{*"}-\varepsilon_1^{'}+\frac{c\varepsilon_1^{'2}}{2}-\frac{c\varepsilon_1^{*"}\varepsilon_1^{'}}{2}-\frac{c\varepsilon_1^{*"}\varepsilon_1^{'}}{2}+\frac{c\varepsilon_1^{*"2}}{2}\right)\Bigg), \quad (4.10)$$

where,

$$A_0=\alpha+\beta-1,\ A_1=\alpha+\beta,\text{“}\ A_2=\alpha c-\beta c+\beta Pc+\frac{\alpha c}{2}+\frac{\beta c}{2},$$

$$A_3=\frac{\alpha c(c+1)Q^2}{2!}-\frac{\beta c(c-1)P^2}{2!}+\frac{\beta Pc^2}{2}+\frac{\alpha c^2}{2}+\frac{\alpha c}{4}+\frac{\beta c}{4}+\frac{\alpha c^2}{8}+\frac{\beta c^2}{8},$$

$$A_4=\frac{\alpha c(c-1)Q^2}{2!}+\frac{\beta c(c+1)P^2}{2!}-\beta Qc^2+\frac{\beta P^2c(c-1)}{2!}+\frac{\beta Pc^2}{2}-\frac{\beta c^2}{2}+\frac{\alpha c^2}{2}$$

$$-\frac{\alpha c}{4}-\frac{\beta c}{4}+\frac{\alpha c^2}{8}+\frac{\beta c^2}{8},\ A_5=\alpha c^2-\beta Pc^2+\frac{\beta Pc^2}{2}-\frac{\beta c^2}{2}+\frac{\alpha c^2}{2}+\frac{\beta Qc^2}{2}$$

$$+\frac{\alpha c^2}{2}+\frac{2\alpha c^2}{8}+\frac{2\beta c^2}{8}\ A_6=\alpha c-\beta c+\beta Pc+\frac{\alpha c}{2}+\frac{\beta c}{2}.$$

One may get expression of the bias and *MSE* from (4.10) as,

$$Bias(t_{12(G)})=\mu_Y\left(A_0+\varsigma_{20}^{*"}A_3-\varsigma_{11}^{*"}A_6+k_1\left(1+\frac{c\varsigma_{20}^{*"}}{4}+\frac{c\varsigma_{20}^{*"}}{8h}-\frac{c\varsigma_{11}^{*"}}{2}\right)\right)$$

$$+k_2\mu_X\frac{c\varsigma_{20}^{*"}}{2}, \quad (4.11)$$

and

$$MSE(t_{12(G)})=\Big(\mu_Y^2\Big(A_0^2+\varsigma_{02}^{*}A_1^2+\varsigma_{20}^{*"}A_2^2+2\varsigma_{20}^{*"}A_0A_3-2\varsigma_{11}^{*"}A_0A_6-2\varsigma_{11}^{*"}A_1A_2$$

$$+k_1^2\left(1+C_{02}^{*}+\frac{c^2\varsigma_{20}^{*"}}{4}+\frac{2c\varsigma_{20}^{*"}}{4}+\frac{2c^2\varsigma_{20}^{*"}}{8}-\frac{2c\varsigma_{11}^{*"}}{2}-\frac{2c\varsigma_{11}^{*"}}{2}\right)+k_2^2\mu_X^2\varsigma_{20}^{*"}$$

$$+2k_1\left(A_0\left(1+\frac{c\varsigma_{20}^{*"}}{4}+\frac{c^2\varsigma_{20}^{*"}}{8}-\frac{c\varsigma_{11}^{*"}}{2}\right)+A_1\left(C_{02}^{*}-\frac{c\varsigma_{11}^{*"}}{2}\right)-A_2\left(\varsigma_{11}^{*"}-\frac{c\varsigma_{2o}^{*"}}{2}\right)\right.$$

$$\left.+A_3\varsigma_{20}^{*"}-A_6\varsigma_{11}^{*"}\right)\Big)-2\mu_X\mu_Yk_2\left(A_1\varsigma_{11}^{*"}-A_2\varsigma_{20}^{*"}-\frac{A_0c\varsigma_{20}^{*"}}{2}\right)$$

$$-2\mu_X\mu_Yk_1k_2\left(\varsigma_{11}^{*"}-\frac{2c\varsigma_{20}^{*"}}{2}\right)\Bigg). \quad (4.12)$$

Or alternatively we may write (4.12) as,

$$MSE(t_{12(G)})=\Delta_o+k_1^2\Delta_1+k_2^2\Delta_2+k_1\Delta_3+k_2\Delta_4+k_1k_2\Delta_5 \quad (4.13)$$

where $\Delta_0 = \mu_Y^2\left(A_0^2 + C_{02}^{*}A_1^2 + \varsigma_{20}^{*''}A_2^2 + 2\varsigma_{20}^{*''}A_0A_3 - 2\varsigma_{11}^{*''}A_0A_6 - 2\varsigma_{11}^{*}A_1A_2\right)$,

$$\Delta_1 = \mu_Y^2\left(1 + C_{02}^{*} + \frac{c^2\varsigma_{20}^{*''}}{4} + \frac{2c\varsigma_{20}^{*''}}{4} + \frac{2c^2\varsigma_{20}^{*''}}{8} - \frac{2c\varsigma_{11}^{*''}}{2} - \frac{2c\varsigma_{11}^{*''}}{2}\right),$$

$$\Delta_2 = \mu_X^2\varsigma_{20}^{*''}, \Delta_3 = 2\mu_Y^2\left(A_0\left(1 + \frac{c\varsigma_{20}^{*''}}{4} + \frac{2c^2\varsigma_{20}^{*''}}{8} - \frac{c\varsigma_{11}^{*''}}{2}\right)\right.$$

$$\left. + A_1\left(C_{02}^{*} - \frac{c\varsigma_{11}^{*''}}{2}\right) - A_2\left(\varsigma_{11}^{*''} - \frac{c\varsigma_{20}^{*''}}{2}\right) + A_3\varsigma_{20}^{*''} - A_6\varsigma_{11}^{*''}\right),$$

$$\Delta_4 = -2\mu_X\mu_Y\left(A_1\varsigma_{11}^{*''} - A_2\varsigma_{20}^{*''} - \frac{A_o c\varsigma_{20}^{*''}}{2}\right) \text{and } \Delta_5 = -2\mu_X\mu_Y\left(\varsigma_{11}^{*''} - \frac{2c\varsigma_{20}^{*''}}{2}\right).$$

The optimum values for k_1 and k_2 maybe obtained from (4.13) as,

$$k_{1(opt)} = \frac{\Delta_4\Delta_5 - 2\Delta_2\Delta_3}{4\Delta_1\Delta_2 - \Delta_5^2} \quad \text{and} \quad k_{2(opt)} = \frac{\Delta_3\Delta_5 - 2\Delta_4\Delta_1}{4\Delta_1\Delta_2 - \Delta_5^2}, \tag{4.14}$$

and expression for the minimum mean square errors may be obtained as,

$$\min MSE(t_{12(G)}) = \Delta_0 - \frac{\Delta_3\Delta_4\Delta_5 - \Delta_1\Delta_4^2 - \Delta_2\Delta_3^2}{\Delta_5^2 - 4\Delta_1\Delta_2}. \tag{4.15}$$

It is to mention that for different values of c one can get different estimators directly from (4.7), for example,

for c =1, the estimator in (4.7) can be taken as,

$$t_{12(1)} = \left(\mu_y^{*''}\left(\alpha\left(\frac{\mu_x^{'}}{\mu_x^{*''}}\right) + \beta\left(\frac{\tilde{\mu}_x^{*}}{\mu_x^{'}}\right)\right) + k_1\mu_y^{*''} + k_2\left(\mu_x^{'} - \mu_x^{*''}\right)\right)\exp\left(\frac{\mu_x^{'} - \mu_x^{*''}}{\mu_x^{'} + \mu_x^{*''}}\right), \tag{4.16}$$

for $c = \frac{1}{2}$, the estimator in (4.7) can be taken as,

$$t_{12(2)} = \left(\mu_y^{*''}\left(\alpha\left(\frac{\mu_x^{'}}{\mu_x^{*''}}\right)^{\frac{1}{2}} + \beta\left(\frac{\tilde{\mu}_x^{*}}{\mu_x^{'}}\right)^{\frac{1}{2}}\right) + k_1\mu_y^{*''} + k_2\left(\mu_x^{'} - \mu_x^{*''}\right)\right)\exp\left(\frac{\mu_x^{'\frac{1}{2}} - \mu_x^{*''\frac{1}{2}}}{\mu_x^{'\frac{1}{2}} + \mu_x^{*''\frac{1}{2}}}\right), \tag{4.17}$$

for $c = \frac{1}{3}$, the estimator in (4.7) can be taken as,

$$t_{12(3)} = \left(\mu_y^{*"} \left(\alpha \left(\frac{\mu_x^{'}}{\mu_x^{*"}} \right)^{\frac{1}{3}} + \beta \left(\frac{\tilde{\mu}_x^{*}}{\mu_x^{'}} \right)^{\frac{1}{3}} \right) + k_1 \mu_y^{*"} + k_2 \left(\mu_x^{'} - \mu_x^{*"} \right) \right) \exp \left(\frac{\mu_x^{'\frac{1}{3}} - \mu_x^{*"\frac{1}{3}}}{\mu_x^{'\frac{1}{3}} + \mu_x^{*"\frac{1}{3}}} \right),$$

(4.18)

for $c = \frac{1}{4}$, the estimator in (4.7) can be taken as,

$$t_{12(4)} = \left(\hat{\mu}_y^{*} \left(\alpha \left(\frac{\mu_x^{'}}{\hat{\mu}_x^{*}} \right)^{\frac{1}{4}} + \beta \left(\frac{\tilde{\mu}_x^{*}}{\mu_x^{'}} \right)^{\frac{1}{4}} \right) + k_1 \hat{\mu}_y^{*} + k_2 \left(\mu_x^{'} - \hat{\mu}_x^{*} \right) \right) \exp \left(\frac{\mu_x^{'\frac{1}{4}} - \hat{\mu}_x^{*\frac{1}{4}}}{\mu_x^{'\frac{1}{4}} + \hat{\mu}_x^{*\frac{1}{4}}} \right),$$

(4.19)

for $c = \frac{1}{5}$, the estimator in (4.7) can be taken as,

$$t_{12(5)} = \left(\hat{\mu}_y^{*} \left(\alpha \left(\frac{\mu_x^{'}}{\hat{\mu}_x^{*}} \right)^{\frac{1}{5}} + \beta \left(\frac{\tilde{\mu}_x^{*}}{\mu_x^{'}} \right)^{\frac{1}{5}} \right) + k_1 \hat{\mu}_y^{*} + k_2 \left(\mu_x^{'} - \hat{\mu}_x^{*} \right) \right) \exp \left(\frac{\mu_x^{'\frac{1}{5}} - \hat{\mu}_x^{*\frac{1}{5}}}{\mu_x^{'\frac{1}{5}} + \hat{\mu}_x^{*\frac{1}{5}}} \right),$$

(4.20)

andfor $c = \frac{1}{6}$, the estimator in (4.7) can be taken as,

$$t_{12(6)} = \left(\hat{\mu}_y^{*} \left(\alpha \left(\frac{\mu_x^{'}}{\hat{\mu}_x^{*}} \right)^{\frac{1}{6}} + \beta \left(\frac{\tilde{\mu}_x^{*}}{\mu_x^{'}} \right)^{\frac{1}{6}} \right) + k_1 \hat{\mu}_y^{*} + k_2 \left(\mu_x^{'} - \hat{\mu}_x^{*} \right) \right) \exp \left(\frac{\mu_x^{'\frac{1}{6}} - \hat{\mu}_x^{*\frac{1}{6}}}{\mu_x^{'\frac{1}{6}} + \hat{\mu}_x^{*\frac{1}{6}}} \right).$$

(4.21)

Similarly, the expressions for the bias, the *MSE*, the optimum values and the min *MSE* for the estimator (4.16) to (4.21) can be obtained directly from (4.11), (4.12), (4.14) and (4.15) respectively.

4.2 Empirical Study of the Proposed Class of Generalized Dual Estimator:
In order to demonstrate the performance & application of the advised generalized estimator we consider two different populations. Explanation is given for each population respectively as,

Population 1: Source [Khare and Srivastava (1993)]
Y: cultivated area (in acres); *X*: population of the villages

$P_2 = 0.20, N = 70,\ n'' = 35, n' = 50,\ \mu_Y = 981.29, \mu_{Y(2)} = 597.29, \mu_X = 1755.53,\ \mu_{X(2)} = 1100.24,\ C_y = 0.6254,\ C_{Y(2)} = 0.4087,\ C_x = 0.8009, C_{x(2)} = 0.5739,\ \rho_{xy} = 0.778,\ \rho_{xy(2)} = 0.445.$

Population 2: Source [Khare and Srivastava (1995)]
Y: measurements of turbine meter (in ml); *X*: measurements of displacement meter (in cm^3)

$N = 100,\ n'' = 30,\ n' = 50, \mu_Y = 3500.12, \mu_{Y(2)} = 3401.08, \mu_X = 260.84,\ \mu_{X(2)} = 259.96,\ C_y = 0.5941, C_{Y(2)} = 0.5075, C_x = 0.5996, C_{x(2)} = 0.5168,\ \rho_{xy} = 0.985,\ \rho_{xy(2)} = 0.995,\ P_2 = 0.25.$

	Estimators		$1/k''$			
			$1/2$	$1/2.5$	$1/3$	$1/3.5$
Population 1	$\mu_y^{*''}$		6299.48	6759.03	7218.58	7678.14
	t_7		5824.67	6616.04	7407.42	8198.80
	t_8		68801.14	72660.03	76518.90	80377.79
	$t_9 = t'_{KB} = t'_{CS}$		4277.50	4683.73	5082.29	5475.32
	t_{10}		64729.07	67963.41	71197.74	74432.07
	Suggested class of Estimator $t_{11(G)}$	$t_{12(1)}$, c=1	**4177.25**	**4560.87**	**4741.71**	**5300.79**
		$t_{12(2)}$, $c=\frac{1}{2}$	**4228.31**	**4623.59**	**4813.42**	**5390.40**
		$t_{12(3)}$, $c=\frac{1}{3}$	**4242.5**	**4641.04**	**4833.37**	**5415.29**
		$t_{12(4)}$, $c=\frac{1}{4}$	**4248.91**	**4648.89**	**4842.33**	**5426.48**
		$t_{12(5)}$, $c=\frac{1}{5}$	**4252.49**	**4653.30**	**4847.36**	**5432.75**
		$t_{12(6)}$, $c=\frac{1}{6}$	**4254.77**	**4656.10**	**4850.55**	**5436.73**

Population 2	$\mu_y^{*"}$		127187.17	140334	153481	166628.00
	t_7		45267.06	45405.4	45543.70	45682.00
	t_8		59989.93	68603.37	76507.87	85830.09
	$t_9 = t'_{KB} = t'_{CS}$		45218.86	45350	45481.20	45612.30
	t_{10}		4285.44	4684.368	5066.353	5482.224
	Suggested class of Estimator $t_{11(G)}$	$t_{12(1)}$, c=1	**43969.36**	**43833.21**	**43671.20**	**43483.27**
		$t_{12(2)}$, c=$\frac{1}{2}$	**44714.6**	**44773.62**	**44827.85**	**44877.24**
		$t_{12(3)}$, c=$\frac{1}{3}$	**44901.17**	**45002.48**	**45102.48**	**45201.10**
		$t_{12(4)}$, c=$\frac{1}{4}$	**44980.33**	**45097.66**	**45214.62**	**45331.15**
		$t_{12(5)}$ „ C=$\frac{1}{5}$	**45022.96**	**45148.14**	**45273.25**	**45398.24**
		$t_{12(6)}$, C=$\frac{1}{6}$	**45049.28**	**45178.93**	**45308.60**	**45438.22**

Table 2: The *MSE* of the estimator for various options of *k*

In Table 2, proposed class of generalized dual estimators $t_{12(G)}$ is compared for their performance with some existing estimators such as $\mu_y^{*"}$, t_7, t_8, t_9, t_{10}, t'_{KB} and t'_{CS}. We have computed the *MSE* of the proposed class of generalized dual estimators considering four different values of *c* (=1, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$) and at each value of c, four different choices of *k* (=2, 2.5,3,3.5) are used to get the mean square error of each estimator. It is to establish from Table 2 that by increasing the value of *k*, the *MSEs* of existing estimators are increasing sharply but this increase is small in the *MSEs* of the proposed class of generalized dual estimators. Further it is noted that from the proposed class $t_{12(G)}$, some estimators such as $t_{12(1)}, t_{12(2)}$, $t_{12(3)}$, $t_{12(4)}, t_{12(5)}$ and $t_{12(6)}$ attain less *MSEs* (presented in bold) than the *MSEs* existing estimators $\mu_y^{*"}$, t_7,

t_8, t_9, t_{10}, t'_{KB} and t'_{CS} . Furthermore it is perceived that the $t_{12(1)}$ is the most effective estimator among the advised class of generalized dual estimators as it attains the minimum *MSE* than the *MSE*s of its own class of estimators. Therefore the proposed generalized class of dual estimator $t_{12(G)}$ is found to be an improved and more effective than the existing estimators.

5. Conclusion

In the present study the difficulty of assessing the population mean of the study variable has been considered assuming the existence of non-response under the two different situations on availability of population mean of supportive variable. For the both situations, either the population mean of the supporting variable is available or not, we proposed an improved class of generalized dual estimators $t_{6(G)}$ and $t_{12(G)}$ respectively, and further properties of the proposed class of estimators have also been studied. The conditions under which the suggested class of estimators attained the *min MSE* have also been achieved. Furthermore it is witnessed by the empirical studies that the suggested generalized class of dual estimators $t_{6(G)}$ and $t_{12(G)}$ attains the minimum *MSE* in both situations and therefore it is found that the proposed generalized estimators are more efficient than the existing estimators.

References

1. Chanu and Singh (2015). Improved exponential ratio cum exponential dual to ratio estimator of finite population mean in presence of non-response, Journal of Statistics Applications and Probability, 4(1), p. 103-111.
2. Cochran, W.G. (1977). Sampling Techniques, 3rd Edition, New York, John Wiley and Sons.
3. Hansen, M.H. and Hurwitz, W.N., (1946). The Problem of non-response in sample Survey, Journal of the American Statistical Association, 41, p. 516-529.
4. Khare, B.B. and Srivastava, S. (1993). Estimation of population mean using auxiliary character in presence of non-response, The National Academy of Sciences, Letters, India, 16, p. 111-114.
5. Khare, B.B. and Srivastava, S. (1995). Study of conventional and alternative two phase sampling ratio product and regression estimators in presence of non-response, Proceedings of the National Academy of Sciences, 65, p. 195-203.
6. Khare, B. B. and Srivastava, S. (1997). Transformed ratio type estimators for the population mean in the presence of non-response, Communications in Statistics - Theory and Methods, 26, p. 1779-1791.
7. Kumar, S. (2012). Utilization of some known population parameters for estimating population mean in presence of non-response, Pakistan Journal of Statistics and Operation Research, 8(2), p. 233-244.
8. Kumar, S. and Bhougal, S. (2011). Estimation of the population mean in presence of non-response, Communications of the Korean Statistical Society, 18(4), p. 1-12.
9. Murthy, M. N. (1964). Product method of estimation, Sankhya, 26, p. 294-307.

10. Saleem, I., Sanaullah, A., and Hanif, M. (2018a). Generalized family of estimators in stratified random sampling using subsampling of non-respondents, Journal of Reliability and Statistical Studies, 11(2), p. 159-173.
11. Saleem, I., Sanaullah, A., and Hanif, M. (2018b). A generalized class of estimators for estimating population mean in the presence of non-response, Journal of Statistical Theory and Applications, 17(4), p. 616-626.
12. Samiuddin, M. and Hanif, M. (2006). Estimation in two phase sampling with complete and incomplete information, Proceedings: 8th Islamic Countries Conference on Statistical Science, 13, p. 479-495.
13. Sanaullah, A., Ali, H.A., Noor-ul-Amin, M. and Hanif, M. (2014). Generalized exponential chain ratio estimators under stratified two phase random sampling, Applied Mathematics and Computation, 226, p. 541-547.
14. Sanaullah, A., Noor-ul-Amin, M., and Hanif, M. (2015).Generalized exponential-type ratio-cum-ratio and product-cum-product estimators for population mean in the presence of non-response under stratified two-phase random sampling, Pakistan Journal of Statistics, 31(1), p. 71-94.
15. Singh, R., Chauhan, P. and Sawan, N. (2008). On linear combination of ratio and product type exponential estimator for estimating the finite population mean, Statistics in Transition, 9(1), p. 105-115.
16. Singh, H.P., and Bhougal, S. K. (2010). Improved estimation of population mean under two-phase sampling with sub-sampling the non-respondents, Journal of Statistical Planning and Inference, 140(9), p. 2536-2550.
17. Srivenkataramana, T. (1980). A dual to ratio estimator in sample surveys, Biometrika, 67, p. 194-204.
18. Sukhatme, B.V. (1962). Some ratio-type estimators in two-phase sampling, Journal of the American Statistics Association, 57, p. 628-632.