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LOG-LOGISTIC WEIBULL-GEOMETRIC DISTRIBUTION WITH APPLICATION TO LIFETIME DATA

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Abstract

In the present paper, a new generalized distribution labeled as log-logistic Weibull-Geometric (LLoGWG) distribution has been presented. The distribution subsumes the log-logistic Rayleigh (LLoGR), log-logistic Weibull (LLoGW), log-logistic (LLoG), log-logistic exponential (LLoGE), distributions and many more as particular cases. The derived structural properties of this distribution include the hazard, reverse hazard and quantile functions. Maximum likelihood estimation is used for parameter estimation of this new distribution. Also, the real data example is presented to depict the applicability of the model.

Key Words: Weibull Distribution, Maximum Likelihood Estimation, Log-logistic Distribution.

1. Introduction

Univariate distributions have several generalizations including those that were familiarized by Gurvichet al. (1997) while dealing with univariate distributions. Gurvich et al. in 1997 gave the c.d.f and p.d.fas

$$F(x;\alpha,\Theta)=1-\exp\left(-\alpha G(x;\Theta)\right),\quad x\in C,\alpha>0 \tag{1}$$

$$g(x;\alpha,\Theta)=\alpha\exp\left(-\alpha G(x;\Theta)\right)g(x;\Theta),\quad x\in C,\alpha>0 \tag{2}$$

where, $C\subseteq R$ and $G(x;\Theta)$is non-negative non decreasing function depending on parameter vectorΘ.$G'(x;\Theta)=g(x;\Theta)$. Different models can be obtained for various choices of the function $G(x;\Theta)$.Rayleigh distribution is obtained from $G(x;\Theta)=x^3$, exponential distribution with $G(x;\Theta)=x$, and Pareto distribution from setting $G(x;\Theta)=log(x/k)$.

In the literature there are considerable ways of obtaining new probability distributions. Nelson in 1982 mentioned distributions with failure rates heaped bath tub that are adequately complicated and, thus not easy to obtain, for example, Hjorth in 1980 proposed such distribution. Later in 1988, revision of these distributions was presented by Rajarshi and Rajarshi, and in 1992 Haupt and Schabe brought forward bathtub-shaped failure rates new lifetime model. But these are not complete enough to justify various situations, so on the modifications of Weibull distribution, new distributions were introduced to fit failure rate of non-monotonic type. Mudholkar and Srivastava in 1993 and Pham and Lai in 2007 compiled few generalizations of Weibull distribution. More generalizations cover Exponentiated Weibull (EW) introduced in 2001by Gupta, Modified Weibull (MW) by Laiet al.in 2003. Khan and Jan in 2016 developed two new distributions, Modified Generalized Linear Failure Rate Distribution and another Inverse Weibull-Geometric Distribution. The former generalizes various distributions and the later generalizes a number of distributions for example Inverse Weibull, Inverse Exponential-Geometric, Inverse Exponential and Inverse Rayleigh distributions.

In this section, graphical properties together with different statistical peculiarities, pdf, cdf, quantile function, etc. of new LLoGWGD are debated. We firstly begin with Burr-XII, Weibull Geometric and log-logistic distributions. Type III and XII Burr distributions grand debate as these cover distinct band of distributions with different extents of skewness and kurtosis. Also these type of distributions have vast utilizations in applied mathematics and statistics including events linked with life testing, fracture roughness, software reliability growth and reliability. The distribution and density functions of Burr XII distribution are

$$F_B(x) = 1 - \left(1 + \left(\frac{x}{s}\right)^{\mu}\right)^{-k} \tag{3}$$

$$f_B(x) = \frac{k\mu}{s}\left(\frac{x}{s}\right)^{\mu-1}\left(1 + \left(\frac{x}{s}\right)^{\mu}\right)^{-k-1}; \; s, \mu, k \text{ and } x \geq 0 \tag{4}$$

respectively, where s is a scale, μ and k are shape parameters. The survival and hazard rate functions are

$$R_B(x) = \left(1 + \left(\frac{x}{s}\right)^{\mu}\right)^{-k} \text{ and } h_B(x) = \frac{k\mu}{s}\left(\frac{x}{s}\right)^{\mu-1}\left(1 + \left(\frac{x}{s}\right)^{\mu}\right)^{-1} \tag{5}$$

respectively. The non-centralr^{th} moment is

$$E(X^r) = ks^r\, B(k - r\mu^{-1}, 1 + r\mu^{-1})\,; \text{for } r < \mu k$$

The pdf of Burr XII distribution is unimodal with mode at$x_0 = \left(\frac{\mu-1}{\mu k+1}\right)^{1/\mu}$when $\mu > 1$ and L-shaped for $\mu = 1$. For $k = 1$ log-logistic distribution is obtained. The Weibull geometric distribution (2008) with parameters $p \in (0,1), \alpha > 0$ *and* $\beta > 0$ is defined by its pdf as

$$f_{GW}(x) = \alpha\beta(1-p)x^{\beta-1}e^{-\alpha x^{\beta}}\left(1 - pe^{-\alpha x^{\beta}}\right)^{-2}; x > 0 \tag{6}$$

and c.d.f as

$$F_{GW}(x) = \frac{1 - e^{-\alpha x^{\beta}}}{1 - pe^{-\alpha x^{\beta}}} \tag{7}$$

The survivor and hazard rate functions are

$$R_{GW}(x) = \frac{(1-p)e^{-\alpha x^{\beta}}}{1 - pe^{-\alpha x^{\beta}}} \text{ and } h_{GW}(x) = \alpha\beta x^{\beta-1}\left(1 - pe^{-\alpha x^{\beta}}\right)^{-1}$$

respectively.

Now, taking $R_1(x) = R_B(x) = \left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-1}$ and $R_1(x) = R_{GW}(x) = \frac{(1-p)e^{-\alpha x^{\beta}}}{1-pe^{-\alpha x^{\beta}}}$ in equation (1) to obtain the cdf of new LLoGWGDas

$$G(x) = 1 - \frac{(1-p)e^{-\alpha x^{\beta}}\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-1}}{1-pe^{-\alpha x^{\beta}}}\;;\;\; \alpha,\mu,s,\beta \geq 0, \qquad x \geq 0 \tag{8}$$

associated pdf of LLoGWGD is

$$g(x) = \frac{(1-p)e^{-\alpha x^{\beta}}\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-1}}{1-pe^{-\alpha x^{\beta}}}\left[\frac{\alpha\beta x^{\beta-1}}{1-pe^{-\alpha x^{\beta}}} + \frac{\mu x^{\mu}-1}{s^{\mu}+x^{\mu}}\right];\; \alpha,\mu,s,\beta \geq 0, x \geq 0 \tag{9}$$

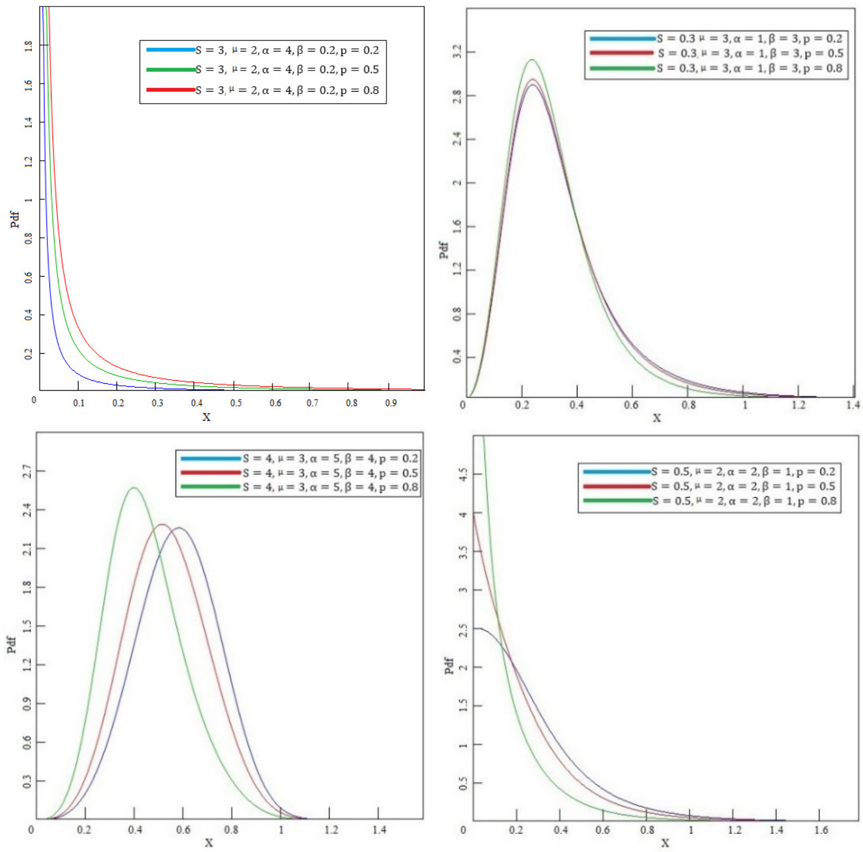

Fig. 1: Plots of LLoGWG pdf

1.1 Sub models

From the LLoGWGD some well known and new distributions can be obtained by assigning some particular values to the parameters involved in LLoGWGD.

Parameter Value	Distribution Obtained
$p \to 0$	Log-Logistic Weibull (LLoGW)
$p \to 0, \beta = 1$	Log-Logistic Exponential (LLoGE)
$p \to 0, \beta = 2$	Log-Logistic Rayleigh (LLoGR)
$p \to 0, \alpha \to 0$	Log-Logistic Distribution (LLoG)
$\beta = 1$	Log-Logistic Exponential-Geometric (LLoEG)
$\beta = 2$	Log-Logistic Rayleigh-Geometric (LLoGRG)
$\alpha \to 0$	Log-Logistic Geometric (LLoGG)
$p \to 0, \mu = 1, s \to \infty$	Weibull
$p \to 0, \mu = 1, \beta = 1, s \to \infty$	Exponential
$p \to 0, \mu = 1, \beta = 2, s \to \infty$	Rayleigh
$\mu = 1, s \to \infty$	Weibull-Geometric (EG)
$\mu = 1, \beta = 1, s \to \infty$	Exponential-Geometric (EG)
$\mu = 1, s \to \infty, \beta = 2$	Rayleigh-Geometric (RG)

Table 1: Special cases of LLOGWGD

2. Quantile Function

The qth quantile of the LLoGWG (9) is

$$q = \int_0^{x_q} \frac{(1-p)e^{-\alpha x^\beta}\left(1+\left(\frac{x}{s}\right)^\mu\right)^{-1}}{1-pe^{-\alpha x^\beta}}\left[\frac{\alpha\beta x^{\beta-1}}{1-pe^{-\alpha x^\beta}}+\frac{\mu x^{\mu}-1}{s^\mu+x^\mu}\right]dx$$

$$q = 1-\frac{(1-p)e^{-\alpha x_q^\beta}\left(1+\left(\frac{x_q}{s}\right)^\mu\right)^{-1}}{1-pe^{-\alpha x_q^\beta}}$$

$$log(1-p)-log\left(1+\left(\frac{x_q}{s}\right)^\mu\right)-\alpha x_q^\beta-log\left(1-pe^{-\alpha x_q^\beta}\right)-log(1-q)=0 \qquad (10)$$

Solving the equation (10) using the numerical methods to obtain quantile function of LLoGWD. Quantile of LLoGWD for some particular values of the parameters is given in Table 2.

	$(\mathbf{s}, \boldsymbol{\mu}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{p})$							
u	(1.5,1.5, 1.5,1.5, 0.2)	(1.5,1.5, 1.5,1.5, 0.8)	(0.5,1.5, 0.5,1.5, 0.4)	(0.5,1.5, 0.5,1.5, 0.8)	(0.5,2.0, 2.0,1.0, 0.3)	(0.5,2.0, 2.0,1.0, 0.7)	(2.0, 0.4, 0.7, 1.5, 0.3)	(2.0, 0.4, 0.7, 1.5, 0.7)
0.1	0.1245	0.0569	0.0959	0.0743	0.0969	0.0572	0.0081	0.0079
0.2	0.2067	0.0965	0.1621	0.1248	0.1671	0.1114	0.0534	0.0462
0.3	0.2845	0.1360	0.2276	0.1741	0.2319	0.1655	0.1464	0.1107
0.4	0.3643	0.1789	0.2981	0.2266	0.2981	0.2222	0.2721	0.1900
0.5	0.4505	0.2282	0.3785	0.2857	0.3706	0.2848	0.4216	0.2837
0.6	0.5482	0.2882	0.4756	0.3563	0.4557	0.3579	0.5980	0.3979
0.7	0.6656	0.3665	0.6013	0.4471	0.5644	0.4501	0.8151	0.5465
0.8	0.8203	0.4800	0.7828	0.5781	0.7219	0.5816	1.1056	0.7620
0.9	1.0654	0.6826	1.1094	0.8175	1.0169	0.8229	1.5711	1.1481

Table 2: LLoGWD quantile for elected values of parameters

3. Hazard and Reverse Hazard functions

Let the lifetime of the component having pdf (9) and cdf (8) be T. The probability component survives after the time t is called the reliability $R(t, \alpha, \beta, \gamma, \delta, \theta)$ of that component. Thus,

$$R(t) = 1 - G(t), \qquad t > 0$$

$$R(t) = \frac{(1-p)e^{-\alpha t^{\beta}}\left(1+\left(\frac{t}{s}\right)^{\mu}\right)^{-1}}{1-pe^{-\alpha t^{\beta}}} \; ; s, \mu, \alpha, \beta, x \geq 0 \tag{11}$$

The hazard rate function of LLoGWD is

$$h(x) = \frac{g(x)}{\bar{G}(x)}$$

$$h(x) = \frac{\alpha\beta x^{\beta-1}}{1-pe^{-\alpha x^{\beta}}} + \frac{\mu x^{\mu} - 1}{s^{\mu} + x^{\mu}} \tag{12}$$

The reversed hazard rate function of LLoGWD is

$$r(x) = \frac{g(x)}{G(x)}$$

$$= \frac{(1-p)e^{-\alpha x^{\beta}}\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-1}}{1-pe^{-\alpha x^{\beta}}}\left[\frac{\alpha\beta x^{\beta-1}}{1-pe^{-\alpha x^{\beta}}} + \frac{\mu x^{\mu} - 1}{s^{\mu} + x^{\mu}}\right]\left[1 - \frac{(1-p)e^{-\alpha x^{\beta}}\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-1}}{1-pe^{-\alpha x^{\beta}}}\right]^{-1} \tag{13}$$

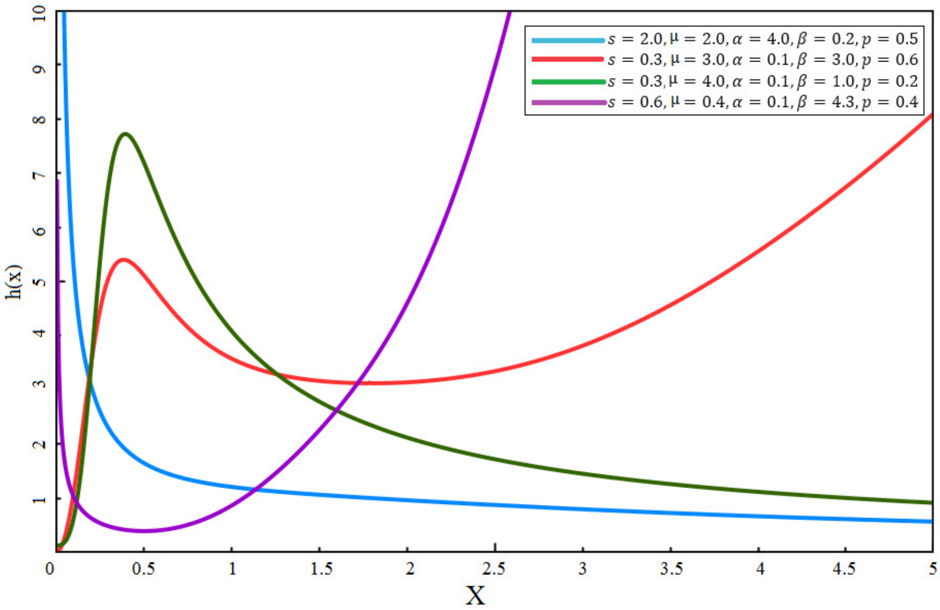

Fig. 2: Plots of Hazard Function

4. Moments

The probability weighted moments in 1979 by Greenwood et al. of LLoLWG distribution is

$$E\left(X^r G^n(X)\bar{G}^m(X)\right)=\int_0^\infty x^r G^n(x)\bar{G}^m(x)g(x)dx=\int_0^\infty x^r(1-\bar{G}(x)))^n\bar{G}^m(x)g(x)dx$$

$$E\left(X^r G^n(X)\bar{G}^m(X)\right)=\sum_{j=0}^\infty\frac{(-1)^j\Gamma(n+1)}{\Gamma(j+1)\Gamma(n+1-j)}E\left(X^r(\bar{G}(X))^{j+m}\right) \tag{14}$$

Now, $$\mathrm{E}\left(\mathrm{X}^{\mathrm{r}}\left(\bar{\mathrm{G}}(\mathrm{x})\right)^{\mathrm{j+m}}\right)=\int_0^\infty \mathrm{x}^{\mathrm{r}}\left(\bar{\mathrm{G}}(\mathrm{x})\right)^{\mathrm{j+m}}\mathrm{g}(\mathrm{x})\mathrm{dx}$$

$$=\int_0^\infty x^r\frac{(1-p)^w\left(1+\left(\frac{x}{s}\right)^\mu\right)^{-w}e^{-\alpha w x^\beta}}{\left(1-pe^{-\alpha x^\beta}\right)^w}\left[\frac{\alpha\beta x^{\beta-1}}{1-pe^{-\alpha x^\beta}}+\left(\frac{\mu}{s}\right)\left(\frac{x}{s}\right)^{\mu-1}\left(1+\left(\frac{x}{s}\right)^\mu\right)^{-1}\right]dx$$

Where, $w=j+m+1$

$$=\sum_{t=0}^\infty\frac{\alpha\beta(1-p)^w p^t\Gamma(w+t+1)}{\Gamma(t+1)\Gamma(w+1)}\int_0^\infty x^{r+\beta-1}\left(1+\left(\frac{x}{s}\right)^\mu\right)^{-w}e^{-\alpha(w+t)x^\beta}dx$$

$$+\sum_{z=0}^\infty\frac{(1-p)^w p^z\Gamma(w+z)}{\Gamma(z+1)\Gamma(w)}\int_0^\infty x^r\left(\frac{\mu}{s}\right)\left(\frac{x}{s}\right)^{\mu-1}\left(1+\left(\frac{x}{s}\right)^\mu\right)^{-(w+1)}e^{-\alpha(w+z)x^\beta}dx$$

$$= \sum_{t=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \alpha\beta(1-p)^w p^t \Gamma(w+t+1)[\alpha(w+t)]^k}{\Gamma(t+1)\Gamma(+1)k!} \int_0^{\infty} x^{r+(k+1)\beta-1}\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-w} dx$$

$$+ \sum_{z=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^l (1-p)^w p^z \Gamma(w+z)[\alpha(w+z)]^l}{\Gamma(z+1)\Gamma(w)l!} \int_0^{\infty} x^{r+l\beta}\left(\frac{\mu}{s}\right)\left(\frac{x}{s}\right)^{\mu-1}\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-(w+1)} dx$$

Put $\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-1} = y$, we get

$$= \sum_{t=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \alpha\beta s^{r+(k+1)\beta}(1-p)^w p^t \times \Gamma(w+t+1)[\alpha(w+t)]^k}{\mu\,\Gamma(t+1)\Gamma(w+1)\,k!} \int_0^1 y^{\left(w-\frac{r+(k+1)\beta}{\mu}\right)-1}(1-y)^{\frac{r+(k+1)\beta}{\mu}-1} dy$$

$$+ \sum_{z=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^l s^{r+l\beta}(1-p)^w p^z \Gamma(w+z)[\alpha(w+z)]^l}{\Gamma(z+1)\Gamma(w)l!} \int_0^1 y^{\left(w-\frac{r+l\beta}{\mu}\right)-1}(1-y)^{\frac{r+l\beta+\mu}{\mu}-1} dy$$

$$= \sum_{t=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \alpha\beta s^{r+(k+1)\beta}(1-p)^w p^t \times \Gamma(w+t+1)[\alpha(w+t)]^k}{\mu\,\Gamma(t+1)\Gamma(w+1)\,k!} B\left[w-\frac{r+(k+1)\beta}{\mu}, \frac{r+(k+1)\beta}{\mu}\right]$$

$$+ \sum_{z=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^l s^{r+l\beta}(1-p)^w p^z \Gamma(w+z)[\alpha(w+z)]^l}{\Gamma(z+1)\Gamma(w)l!} B\left[w-\frac{r+l\beta}{\mu}, \frac{r+l\beta+\mu}{\mu}\right]$$

Using this in (14), the PWMs of the LLoGWGD is

$$E\left(X^r G^n(X)\bar{G}^m(X)\right) = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(n+1)}{\Gamma(j+1)\Gamma(n+1-j)}$$

$$\times\left[\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k}\alpha\beta s^{r+(k+1)\beta}(1-p)^{w}p^{t}\times\Gamma(w+t+1)[\alpha(w+t)]^{k}}{\mu\,\Gamma(t+1)\Gamma(w+1)\,k!}B\left(w-\frac{r+(k+1)\beta}{\mu},\frac{r+(k+1)\beta}{\mu}\right)+\sum_{z=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{l}s^{r+l\beta}(1-p)^{w}p^{z}\Gamma(w+z)[\alpha(w+z)]^{l}}{\Gamma(z+1)\Gamma(w)l!}B\left(w-\frac{r+l\beta}{\mu},\frac{r+l\beta+\mu}{\mu}\right)\right]$$

4.1 Special Cases

When $m = n = 0$ we get the rthnon-central moment μ_r'as

$$E(X^r)=\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k}\beta(1-p)s^{r+(k+1)\beta}p^{t}\times[\alpha(t+1)]^{k+1}}{\mu\ k!}B\left(1-\frac{r+(k+1)\beta}{\mu},\frac{r+(k+1)\beta}{\mu}\right)+\sum_{z=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{l}s^{r+l\beta}(1-p)p^{z}[\alpha(z+1)]^{l}}{l!}B\left(1-\frac{r+l\beta}{\mu},\frac{r+l\beta+\mu}{\mu}\right)$$

For $r = 0, n = 0$

$$E(\bar{G}^m(X))=\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k}\alpha\beta s^{(k+1)\beta}(1-p)^{m+1}p^{t}\times\Gamma(m+t+2)[\alpha(m+t+1)]^{k}}{\mu\,\Gamma(t+1)\Gamma(m+2)\,k!}B\left(m+1-\frac{(k+1)\beta}{\mu},\frac{(k+1)\beta}{\mu}\right)+\sum_{z=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{l}s^{l\beta}(1-p)^{m+1}p^{z}\Gamma(m+z+1)[\alpha(m+z+1)]^{l}}{\Gamma(z+1)\Gamma(m+1)l!}B\left(m+1-\frac{l\beta}{\mu},\frac{l\beta+\mu}{\mu}\right)$$

For $n = 0$ the LLoGWG PWMs reduces to

$$E(X^r\bar{G}^m(X))=\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k}\alpha\beta s^{r+(k+1)\beta}(1-p)^{m+1}p^{t}\times\Gamma(m+t+2)[\alpha(m+t+1)]^{k}}{\mu\,\Gamma(t+1)\Gamma(m+2)\,k!}B\left(m+1-\frac{r+(k+1)\beta}{\mu},\frac{r+(k+1)\beta}{\mu}\right)$$

$$+\sum_{z=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{l}s^{r+l\beta}(1-p)^{m+1}p^{z}\Gamma(m+z+1)[\alpha(w+z)]^{l}}{\Gamma(m+2)\Gamma(m+1)l!}B\left(m+1-\frac{r+l\beta}{\mu},\frac{r+l\beta+\mu}{\mu}\right)$$

when $m=0$ the LLoGWG PWMs is

$$E\left(X^{r}G^{n}(X)\right)=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(n+1)}{\Gamma(j+1)\Gamma(n+1-j)}$$

$$\times\left[\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k}\alpha\beta s^{r+(k+1)\beta}(1-p)^{J+1}p^{t}\times\Gamma(j+t+2)[\alpha(w+t)]^{k}}{\mu\,\Gamma(t+1)\Gamma(j+2)\,k!}B\left(j+1-\frac{r+(k+1)\beta}{\mu},\frac{r+(k+1)\beta}{\mu}\right)\right.$$
$$\left.+\sum_{z=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{l}s^{r+l\beta}(1-p)^{j+1}p^{z}\Gamma(j+z+1)[\alpha(w+z)]^{l}}{\Gamma(z+1)\Gamma(j+1)l!}B\left(j+1-\frac{r+l\beta}{\mu},\frac{r+l\beta+\mu}{\mu}\right)\right]$$

when $r=m=0$ the LLoGWG PWMs is

$$E\left(G^{n}(X)\right)=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(n+1)}{\Gamma(j+1)\Gamma(n+1-j)}$$

$$\times\left[\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k}\alpha\beta s^{(k+1)\beta}(1-p)^{j+1}p^{t}\times\Gamma(j+t+2)[\alpha(j+t+1)]^{k}}{\mu\,\Gamma(t+1)\Gamma(j+2)\,k!}B\left(j+1-\frac{(k+1)\beta}{\mu},\frac{(k+1)\beta}{\mu}\right)\right.$$
$$\left.+\sum_{z=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{l}s^{l\beta}(1-p)^{j+1}p^{z}\Gamma(j+z+1)[\alpha(j+z+1)]^{l}}{\Gamma(z+1)\Gamma(j+1)l!}B\left(j+1-\frac{l\beta}{\mu},\frac{l\beta+\mu}{\mu}\right)\right]$$

when $r=0$ the LLoGWG PWMs is

$$E\left(G^{n}(X)\bar{G}^{m}(X)\right)=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(n+1)}{\Gamma(j+1)\Gamma(n+1-j)}$$

$$\times\left[\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\frac{\begin{array}{c}(-1)^{k}\alpha\beta s^{(k+1)\beta}(1-p)^{w}p^{t}\\ \times\Gamma(w+t+1)[\alpha(w+t)]^{k}\end{array}}{\mu\,\Gamma(t+1)\Gamma(w+1)\,k!}B\left(w-\frac{(k+1)\beta}{\mu},\frac{(k+1)\beta}{\mu}\right)\right.$$

$$\left.+\sum_{z=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{l}s^{l\beta}(1-p)^{w}p^{z}\Gamma(w+z)[\alpha(w+z)]^{l}}{\Gamma(z+1)\Gamma(w)l!}B\left(w-\frac{l\beta}{\mu},\frac{l\beta+\mu}{\mu}\right)\right]$$

Theorem: The r^{th} raw moment of the LLoGWGD is

$$E(X^{r})=\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\frac{\begin{array}{c}(-1)^{k}s^{r+(k+1)\beta}(1-p)p^{t}\\ \times\beta[\alpha(t+1)]^{k+1}\end{array}}{\mu k!}B\left[1-\frac{r+(k+1)\beta}{\mu},\frac{r+(k+1)\beta}{\mu}\right]$$

$$+\sum_{z=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{l}s^{r+l\beta}(1-p)p^{z}[\alpha(z+1)]^{l}}{l!}B\left[1-\frac{r+l\beta}{\mu},\frac{r+l\beta+\mu}{\mu}\right]$$

Proof: $E(X^{r})=\int_{0}^{\infty}x^{r}g(x)dx$

$$=\alpha\beta(1-p)\int_{0}^{\infty}\frac{x^{r+\beta-1}e^{-\alpha x^{\beta}}\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-1}}{\left(1-pe^{-\alpha x^{\beta}}\right)^{2}}dx$$

$$+(1-p)\int_{0}^{\infty}\frac{x^{r}\left(\frac{\mu}{s}\right)\left(\frac{x}{s}\right)^{\mu-1}e^{-\alpha x^{\beta}}\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-2}}{1-pe^{-\alpha x^{\beta}}}dx$$

$$=\sum_{t=0}^{\infty}\alpha\beta(1-p)p^{t}(t+1)\int_{0}^{\infty}x^{r+\beta-1}e^{-\alpha(t+1)x^{\beta}}\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-1}dx$$

$$+\sum_{z=0}^{\infty}(1-p)p^{z}\int_{0}^{\infty}x^{r}\left(\frac{\mu}{s}\right)\left(\frac{x}{s}\right)^{\mu-1}e^{-\alpha(z+1)x^{\beta}}\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-2}dx$$

$$=\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k}\alpha\beta(1-p)p^{t}(t+1)[\alpha(t+1)]^{k}}{k!}\int_{0}^{\infty}x^{r+(k+1)\beta-1}\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-1}dx$$

$$+\sum_{z=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{l}(1-p)p^{z}[\alpha(z+1)]^{l}}{l!}\int_{0}^{\infty}x^{r+l\beta}\left(\frac{\mu}{s}\right)\left(\frac{x}{s}\right)^{\mu-1}\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-2}dx$$

Put $\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-1}=y$, we get

$$E(X^r)=\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k s^{r+(k+1)\beta}(1-p)p^t\times\beta[\alpha(t+1)]^{k+1}}{\mu k!}\int_0^1 y^{\left(1-\frac{r+(k+1)\beta}{\mu}\right)-1}(1-y)^{\frac{r+(k+1)\beta}{\mu}-1}dy$$

$$+\sum_{z=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^l s^{r+l\beta}(1-p)p^z[\alpha(z+1)]^l}{l!}\int_0^1 y^{\left(1-\frac{r+l\beta}{\mu}\right)-1}(1-y)^{\frac{r+l\beta+\mu}{\mu}-1}dy$$

$$E(X^r)=\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k s^{r+(k+1)\beta}(1-p)p^t\times\beta[\alpha(t+1)]^{k+1}}{\mu k!}B\left[1-\frac{r+(k+1)\beta}{\mu},\frac{r+(k+1)\beta}{\mu}\right]$$

$$+\sum_{z=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^l s^{r+l\beta}(1-p)p^z[\alpha(z+1)]^l}{l!}B\left[1-\frac{r+l\beta}{\mu},\frac{r+l\beta+\mu}{\mu}\right]$$

5. Order Statistics

Let from the sample, $X_1 \leq X_2 \leq \cdots \leq X_n$ be order statistics, then pdf of X_i is

$$g_{i:n}(x)=\frac{n!\,g(x)}{(i-1)!\,(n-1)!}[G(x)]^{i-1}[1-G(x)]^{n-i}$$

$$=\sum_{j=0}^{n-i}\frac{(-1)^j n!\,g(x)}{(i-1)!\,(n-1)!}\binom{n-i}{j}[G(x)]^{i+j-1}$$

The t^{th} moment of i^{th} order statistics of LLoGWGD is obtained using conclusion of (Barakat and Abdelkader 2004)

$$E\left(X_{i:n}^t\right)=t\sum_{q=n-i+1}^{n}(-1)^{q-n+i-1}\binom{q-1}{n-i}\binom{n}{q}\int_0^{\infty}x^{t-1}[1-G(x)]^q dx$$

Here,

$$\int_0^{\infty}x^{t-1}[1-G(x)]^q dx=\int_0^{\infty}x^{t-1}\frac{(1-p)^q e^{-\alpha q x^{\beta}}\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-q}}{\left(1-pe^{-\alpha x^{\beta}}\right)^q}dx$$

$$=\sum_{k=0}^{\infty}\frac{(1-p)^q\Gamma(q+\mathrm{k})}{\Gamma(k+1)\Gamma(q)}\int_0^{\infty}x^{t-1}e^{-\alpha(q+k)x^{\beta}}\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-q}dx$$

$$=\sum_{k=0}^{\infty}\sum_{z=0}^{\infty}\frac{(-1)^z(1-p)^q\Gamma(q+\mathrm{k})[\alpha(q+k)]^z}{\Gamma(k+1)\Gamma(q)z!}\int_0^{\infty}x^{t+z\beta-1}\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-q}dx$$

$$= \sum_{k=0}^{\infty}\sum_{z=0}^{\infty} \frac{(-1)^z s^{z\beta+t}(1-p)^q \Gamma(q+k)[\alpha(q+k)]^z}{\mu\Gamma(k+1)\Gamma(q)z!} B\left[q-\frac{t+z\beta}{\mu}, \frac{t+z\beta}{\mu}\right]$$

Now,

$$E\left(X_{i:n}^t\right) = t\sum_{q=n-i+1}^{n}\sum_{k=0}^{\infty}\sum_{z=0}^{\infty} (-1)^{q+z-n+i-1}\binom{q-1}{n-i}\binom{n}{q}\frac{s^{z\beta+t}(1-p)^q\Gamma(q+k)[\alpha(q+k)]^z}{\mu\Gamma(k+1)\Gamma(q)z!} \times B\left[q-\frac{t+z\beta}{\mu}, \frac{t+z\beta}{\mu}\right]$$

6. Entropy

Here, we have derived Rényi entropy of LLoGWGD and Rényi entropy is obtained as

$$I_R(v) = \frac{1}{1-v} log\left(\int_0^{\infty} g^v(x)dx\right); v > 0, v \neq 1,$$

as $v \to 1$we get Shannon entropy.

Now,

$$g(x) = \frac{(1-p)e^{-\alpha x^\beta}\left(1+\left(\frac{x}{s}\right)^\mu\right)^{-1}}{1-pe^{-\alpha x^\beta}}\left[\frac{\alpha\beta x^{\beta-1}}{1-pe^{-\alpha x^\beta}} + \frac{\mu x^{\mu}-1}{s^\mu + x^\mu}\right]; ; s, \mu, \alpha, \beta, x \geq 0$$

$$\int_0^{\infty} g^v(x)dx = \int_0^{\infty} \frac{(1-p)^v\left(1+\left(\frac{x}{s}\right)^\mu\right)^{-v} e^{-\alpha v x^\beta}}{\left(1-pe^{-\alpha x^\beta}\right)^v}\left[\frac{\alpha\beta x^{\beta-1}}{1-pe^{-\alpha x^\beta}} + \frac{\mu x^{\mu}-1}{s^\mu + x^\mu}\right]^v dx$$

$$= \int_0^{\infty} (1-p)^v e^{-\alpha v x^\beta}\left[\frac{\mu x^{\mu}-1}{s^\mu}\frac{\left(1+\left(\frac{x}{s}\right)^\mu\right)^{-2}}{1-pe^{-\alpha x^\beta}} + \frac{\alpha\beta x^{\beta-1}\left(1+\left(\frac{x}{s}\right)^\mu\right)^{-1}}{\left(1-pe^{-\alpha x^\beta}\right)^2}\right]^v dx$$

$$= \sum_{k=0}^{v}\binom{v}{k}\frac{(1-p)^v\mu^k(\alpha\beta)^{v-k}}{s^{\mu k}}\int_0^{\infty}\frac{x^{\mu k-\beta k+\beta v-v}\left(1+\left(\frac{x}{s}\right)^\mu\right)^{-v-k}}{\left(1-pe^{-\alpha x^\beta}\right)^{2v-k}} e^{-\alpha v x^\beta} dx$$

$$= \sum_{k=0}^{v}\sum_{m=0}^{\infty}\binom{v}{k}\frac{p^m(1-p)^v\mu^k(\alpha\beta)^{v-k}\Gamma(2v-k+m)}{s^{\mu k}\Gamma(m+1)\Gamma(2v-k)}\int_0^{\infty} x^{\mu k-\beta k+\beta v-v}\left(1+\left(\frac{x}{s}\right)^\mu\right)^{-v-k} e^{-\alpha(v+m)x^\beta} dx$$

$$= \sum_{k=0}^{v}\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\binom{v}{k}\frac{(-1)^t p^m(1-p)^v\mu^k(\alpha\beta)^{v-k}\times\Gamma(2v-k+m)\left(\alpha(v+m)\right)^t}{s^{\mu k}\Gamma(m+1)\Gamma(2v-k)t!}\int_0^{\infty} x^{\mu k-\beta k+\beta v-v}\left(1+\left(\frac{x}{s}\right)^\mu\right)^{-v-k} dx$$

$$= \sum_{k=0}^{v}\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\binom{v}{k}\frac{(-1)^t p^m (1-p)^v \mu^{k-1}(\alpha\beta)^{v-k}\Gamma(2v-k+m)\left(\alpha(v+m)\right)^t}{\Gamma(m+1)\Gamma(2v-k)\,t!}s^{t\beta-\beta k+\beta v-v+1}$$
$$\times B\left[v+\frac{v+k\beta-v\beta-t\beta-1}{\mu},1+\frac{t\beta+v\beta-k\beta-v+1}{\mu}\right]$$

therefore, Rényi entropy of the LLoGWGD is

$$I_R(v)=\frac{1}{1-v}\log\left(\sum_{k=0}^{v}\sum_{m=0}^{\infty}\sum_{t=0}^{\infty}\binom{v}{k}\frac{(-1)^t p^m (1-p)^v \mu^{k-1}(\alpha\beta)^{v-k}\,\Gamma(2v-k+m)\left(\alpha(v+m)\right)^t s^{t\beta-\beta k+\beta v-v+1}}{\Gamma(m+1)\Gamma(2v-k)\,t!}B\left[v+\frac{v+k\beta-v\beta-t\beta-1}{\mu},1+\frac{t\beta-k\beta+v\beta-v+1}{\mu}\right]\right);\ v>0, v\neq 1$$

7. Estimation

Let $x = x_1, x_2, \ldots, x_n$ be randomly selected sample of the LLoGWGD with parameter vector $\varphi = (p, \alpha, \beta, \mu, s)^T$ unknown.The log likelihood $l = l(\varphi; x)$ for φ is

$$l = \log(1-p) - \alpha x^{\beta} - \log\left(1+\left(\frac{x}{s}\right)^{\mu}\right) - \log\left(1-pe^{-\alpha x^{\beta}}\right) + \log\left[\frac{\alpha\beta x^{\beta-1}}{1-pe^{-\alpha x^{\beta}}}+\frac{\mu x^{\mu}-1}{s^{\mu}+x^{\mu}}\right] \qquad (15)$$

The components of score function $U(\varphi) = \left(\frac{\partial l}{\partial p}, \frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \beta}, \frac{\partial l}{\partial \mu}, \frac{\partial l}{\partial s}\right)^T$ are

$$\frac{\partial l}{\partial p} = \frac{1}{p-1} + \frac{e^{-\alpha x^{\beta}}}{1-pe^{-\alpha x^{\beta}}} + \frac{\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\left(1-pe^{-\alpha x^{\beta}}\right)^{-2}}{\alpha\beta x^{\beta-1}\left(1-pe^{-\alpha x^{\beta}}\right)^{-1} + \mu x^{\mu}-1(s^{\mu}+x^{\mu})^{-1}}$$

$$\frac{\partial l}{\partial \alpha} = -x^{\beta} + \frac{\alpha\beta p x^{\beta-1}e^{-\alpha x^{\beta}}}{1-pe^{-\alpha x^{\beta}}} + \frac{\beta x^{\beta-1}\left(1-pe^{-\alpha x^{\beta}}\right)^{-2}\left(1-pe^{-\alpha x^{\beta}}-\alpha^2\beta p x^{\beta-1}e^{-\alpha x^{\beta}}\right)}{\alpha\beta x^{\beta-1}\left(1-pe^{-\alpha x^{\beta}}\right)^{-1} + \mu x^{\mu}-1(s^{\mu}+x^{\mu})^{-1}}$$

$$\frac{\partial l}{\partial \beta} = -\alpha x^{\beta}\log x - \frac{\alpha p x^{\beta}e^{-\alpha x^{\beta}}\log x}{1-pe^{-\alpha x^{\beta}}} + \frac{\alpha\left(1-pe^{-\alpha x^{\beta}}\right)\left(\beta x^{\beta-1}\log x + x^{\beta-1}\right) - p\alpha^2\beta p x^{2\beta-1}e^{-\alpha x^{\beta}}\log x}{\alpha\beta x^{\beta-1}\left(1-pe^{-\alpha x^{\beta}}\right)^{-1} + \mu x^{\mu}-1(s^{\mu}+x^{\mu})^{-1}}$$

$$\frac{\partial l}{\partial s}=\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-1}\left(\frac{\mu}{s}\right)\left(\frac{x}{s}\right)^{\mu}-\frac{\mu^2(xs)^{\mu-1}(s^{\mu}+x^{\mu})^{-2}}{\alpha\beta x^{\beta-1}\left(1-pe^{-\alpha x^{\beta}}\right)^{-1}+\mu x^{\mu}-1(s^{\mu}+x^{\mu})^{-1}}$$

$$\frac{\partial l}{\partial \mu}=-\left(1+\left(\frac{x}{s}\right)^{\mu}\right)^{-1}\left(\frac{x}{s}\right)^{\mu}\log x+\frac{(s^{\mu}+x^{\mu})^{-1}(x^{\mu-1}+\mu x^{\mu-1})-\mu x^{\mu-1}(s^{\mu}+x^{\mu})^{-2}(s^{\mu}\log s+x^{\mu}\log x)}{\alpha\beta x^{\beta-1}\left(1-pe^{-\alpha x^{\beta}}\right)^{-1}+\mu x^{\mu}-1(s^{\mu}+x^{\mu})^{-1}}$$

The log-likelihood based on sample $x1, x2, \ldots, xn$ of n observations from the LLoGWGD is given by$l_n = \sum_{i=1}^{n} l_i$, where $l_i$is given by the equation (15).

Solving non-linear equations $U(\hat{\varphi}) = 0$ the MLE $\hat{\varphi}$ of φ is obtained. $U(\hat{\varphi}) = 0$ is not possible to solve analytically but there are statistical software which are helpful to solve these type of equations numerically, for example, through the R-language or any iterative methods like, Newton-Raphson.

For the hypothesis testing and interval estimation observed 5×5 information matrix for parameters in φ is

$$K = K(\varphi) = -\begin{pmatrix} K_{p,p}K_{p,\alpha}K_{p,\beta}K_{p,\mu}K_{p,s} \\ K_{\alpha,\alpha}K_{\alpha,\beta}K_{\alpha,\mu}K_{\alpha,s} \\ K_{\beta,\beta}K_{\beta,\mu}K_{\beta,s} \\ K_{\mu,\mu}K_{\mu,s} \\ K_{s,s} \end{pmatrix}$$

K(φ) is observed and but not the expected information matrix as the mathematics involved turns out to be troublesome for including in the matrix. The mathematics for the elements of K are partial derivatives of second order. Under regularity condition that parameters aren't on the boundary but within the parameter space

$$\sqrt{n}(\hat{\varphi}-\varphi) \sim N_5(0, I(\varphi)^{-1})$$

Therefore $K(\varphi)$ can be replaced by $I(\varphi)$and is called the expected information matrix. $I(\varphi)$isused for parameters appropriate confidence regions and construction of tests of hypotheses.

8. Simulation

In this section, we study the conduct of ML estimators for different sample sizes (n=25, 75, 100, 200, 300, 600) using inverse CDF technique for data simulation for log-logistic Weibull-geometric distribution using R software. The practice was repeated 500 times for calculation of bias, variance and MSE. For two parameter combinations $(s = 0.1, \lambda = 1.2, \alpha = 0.1, \beta = 0.9, p = 0.3)$ and $(s = 1.2, \lambda = 2.5, \alpha = 3.5, \beta = 1.3, p = 0.8)$ for log-logistic Weibull-geometric distribution, decreasing trend is being observed in average bias, variance and MSE as we increase size of the sample. Hence, the performance of ML estimators is quite well, consistent in case of log-logistic Weibull-geometric distribution.

Parameter	n	$s = 0.1, \mu = 1.2, \alpha = 0.1, \beta = 0.9, p = 0.3$		
		Bias	Variance	MSE
s	25	0.291345	0.2935412	0.3784231
μ		0.11345	0.335412	0.3482829
α		0.181345	0.1835412	0.2164272
β		0.291345	0.2935412	0.3784231
p		0.191345	0.1935412	0.2301541
s	75	0.181345	0.1735412	0.2064272
μ		0.171345	0.1735412	0.2029003
α		0.11345	0.1635412	0.1764121
β		0.1827054	0.2620161	0.2953974
p		0.0827054	0.1620161	0.1688563
s	100	0.0727054	0.1520161	0.1573022
μ		0.0627054	0.1420161	0.1459481
α		0.0327054	0.1320161	0.1330857
β		0.1306202	0.1448995	0.1619611
p		0.0306202	0.0448995	0.0458371
s	300	0.0206202	0.0348995	0.0353247
μ		0.0106202	0.0248995	0.0250123
α		0.06202	0.0148995	0.0187460
β		0.0619261	0.0224973	0.0263321
p		0.0380739	0.0175027	0.0189523
s	600	0.0280739	0.0075027	0.0082908
μ		0.0180739	0.0024973	0.0028240
α		0.0807387	0.0075027	0.0140214
β		0.0269206	0.0107587	0.0114834
p		0.0730794	0.0019241	0.0072647

Parameter	n	$s = 1.2, \mu = 2.5, \alpha = 3.5, \beta = 1.3, p = 0.8$		
		Bias	Variance	MSE
s	25	0.351345	0.2835412	0.4069845
μ		0.10345	0.325412	0.3361139
α		0.181345	0.1735412	0.2064272
β		0.281345	0.2835412	0.3626962

p		0.181345	0.1835412	0.2164272
s	75	0.171345	0.1635412	0.1929003
μ		0.161345	0.1635412	0.1895734
α		0.10345	0.1535412	0.1642431
β		0.1727054	0.2520161	0.2818433
p		0.0727054	0.1520161	0.1573022
s	100	0.0627054	0.1420161	0.1459481
μ		0.0527054	0.1320161	0.1347940
α		0.0227054	0.1220161	0.1225316
β		0.1206202	0.1348995	0.1494487
p		0.0206202	0.0348995	0.0353247
s	300	0.0106202	0.0248995	0.0250123
μ		0.0006202	0.0148995	0.0148999
α		0.05202	0.0048995	0.0076056
β		0.0519261	0.0124973	0.0151936
p		0.0280739	0.0075027	0.0082908
s	600	0.0180739	0.0024973	0.0028240
μ		0.0080739	0.0075027	0.0075679
α		0.0707387	0.0024973	0.0075013
β		0.0169206	0.0007587	0.0010450
p		0.0130794	0.0020759	0.0022470

Table 3: Simulation Study of ML estimators for log-logistic Weibull-geometric distribution

9. Real Data Illustration

The results of fitting LLoGWG, LLoGEG, LLoG,LLoGE, LLoGW, LLoGR and LLoGRG are compared using −2log(L), AIC (Akaike Information Criterion) to the data set of carbon fibers from Nichols and Padgett of hundred observations in 2006.

MODEL	$-2\text{Log}(l)$	AIC	Estimates	St. Error
LLoGWG	**281.918**	**291.918**	$\hat{p} = 0.73244$	0.45172
			$\hat{s} = 3.55879$	0.56373
			$\hat{\mu} = 6.13300$	1.97481
			$\hat{\alpha} = 0.01388$	0.02291
			$\hat{\beta} = 2.69143$	0.54679

LLoGEG	290.017	298.017	$\hat{p} =$ 0.00010	0.34181
			$\hat{s} =$ 2.62240	0.64828
			$\hat{\mu} =$ 4.59626	0.39512
			$\hat{\alpha} =$ 0.03113	0.03567
LLoG	292.559	296.5591	$\hat{s} =$ 2.49836	0.10540
			$\hat{\mu} =$ 4.11794	0.34411
LLoGE	289.952	295.952	$\hat{s} =$ 2.62240	0.14819
			$\hat{\mu} =$ 0.03113	0.54680
			$\hat{\alpha} =$ 4.59626	0.02979
LLoGW	284.373	292.373	$\hat{s} =$ 3.48402	0.76016
			$\hat{\mu} =$ 4.36507	2.03185
			$\hat{\alpha} =$ 0.04181	0.02729
			$\hat{\beta} =$ 2.51951	0.57925
LLoGR	287.482	293.482	$\hat{s} =$ 3.17358	0.30426
			$\hat{\mu} =$ 5.32597	1.147059
			$\hat{\alpha} =$ 0.05952	0.02125
LLoGRG	283.962	291.962	$\hat{p} =$ 0.00010	2.15469
			$\hat{s} =$ 3.17367	0.32914
			$\hat{\mu} =$ 5.32619	1.92816
			$\hat{\alpha} =$ 0.05952	0.10848

Table 4: Estimates of models

Table 4 shows that values of $-2log(L)$, AIC are lowest for LLoGWG. So, we can accomplish that LLoGWGD performs significantly better than its sub-models.

10. Conclusion

The proposed new model, so-called LLoGWGD performs better than its sub-models. A pronounced reason for generalization of distribution is that generalized form shows resilience in modeling real data.

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