Reliability Structures Consisting of Weighted Components: Synopsis and New Advances

Ioannis S. Triantafyllou

Department of Statistics and Insurance Science, University of Piraeus, 80 Karaoli and Dimitriou str., 18534 Piraeus, Greece
Department of Management Studies, Graphic Era Deemed to be University, Dehradun, 248002, Uttarakhand, India
E-mail: itriantafyllou@unipi.gr

Received 10 February 2023; Accepted 25 March 2023;
Publication 15 May 2023

Abstract

Throughout the lines of the present article the family of reliability systems consisting of weighted components is examined. A synoptic exposition of the existing reliability structures with n weighted components is carried out. Among others, some newfangled and crucial points for every structure that belongs to this class are reported. In addition, a new weighted consecutive-type structure is introduced, while its characteristics are also studied in some detail.

Keywords: Reliability function, coherent systems, weighted structures, consecutive-type systems.

Journal of Reliability and Statistical Studies, Vol. 16, Issue 1 (2023), 25–56.
doi: 10.13052/jrss0974-8024.1612

1 Introduction

In the topic of Reliability Modeling, the *k*–out–of–*n* structures have quite important position. The particular models contain *n* units, while several different modifications of them appear in the field. The so-called *k*–out–of–*n: F* stops operating then and only then the overall number of non-operating units is equal to or more than *k*. In addition, the *k*–out–of–*n: G* operates then and only then the overall number of operating units is equal to or more than *k* (see, e.g. Barlow and Heidtmann (1984) or Pham and Upadhyaya (1988)).

Additionally, reliability systems which operate (or fail) under consecutive-type requirements have been studied assiduously in the last decades. It is evident that these reliability systems find practical applications in several fields, such as the optimization of telecommunication chains, complicated infrared locating devices and vacuum plexuses in accelerators. For instance, the linear consecutive–*k*–out–of–*n*: *F* model has *n* units which are placed in a line (Derman et al. (1982), Kumar, Singh and Ram (2019)). Among its generalizations, one may distinguish the *m*–consecutive– *k*–out–of–*n*: *F* models (Griffith (1986)), the *r*-within-consecutive-*k*–out–of–*n*: *F* structures (Tong (1985) or Triantafyllou and Koutras (2011)) and the consecutive-k_1 and k_2-out-of-*n* models established by Zhao et al. (2006) (see, also Triantafyllou (2020)). On the other hand, there exist sevreral reliabiltiy models which utilize more than one failure conditions, such as the ones established by Cui, Kuo and Xie (2006), Eryilmaz and Zuo (2010), Triantafyllou (2021a) and Triantafyllou (2021b). A through study on the reliability models which fail upon a run or scan rule, is provided by Triantafyllou (2015) or Kuo and Zuo (2003). A nice examination of existing statistical methodologies in distinct fields of Reliability Modeling is offered by Ram (2013).

According to a popular reliability framework, the units of a system share the same portion with regard to the operating (or failure) state of it. Consequently, we can readily verbalize that all components share the same weight. On the other hand, one may consider some situations in practice, where the above-mentioned assumption about the equality of weights of all components does not stand for. In other words, it is quite usual that in practical circumstances we can see that some units are more significant than others. That practically means that if these units fail, it may cost, with larger probability, the failure of the whole system. Under such modeling, we need to accredit larger weights to some units than the remaining ones. Some intriguing applications of the weighted consecutive-type systems can be found in the fields of Statistical Process Control, project management, networks, military vehicles or Automotive Engineering.

In the present work, the class of reliability systems consisting of weighted components is under investigation. Section 2 displays the necessary nomenclature for making the rest of the manuscript more readable. In Section 3, the weighted *k*–out–of–*n* structures including units with binary states (Fail/Operate) are studied in detail, where several fundamental results accompanied with some recent advances are also presented. Section 4 refers to the weighted consecutive–*k*–out–of–*n*: *F* structures. A new weighted consecutive-type system is established in Section 5, while its performance attributes are contemplated in some detail.

2 Symbols and Notations

$R_G(n,k)$: reliability of weighted *k*–out–of–*n: G* system.
$R_F(n,k)$: reliability of weighted *k*–out–of–*n: F* system.
X_z: state of the z–th unit of the structure.
p_z: reliability of the z–th unit of the structure.
q_z: unreliability of the z–th unit of the structure.
w_z: weight of the z–th unit of the structure.
T_z: lifetime of the z–th unit of the structure.
I_z: Birnbaum (marginal) importance of the z–th unit of the structure.
$W_n(t)$: total weight (capacity) of the structure at time *t*.
$m(t,s)$: mean capacity loss of the structure between time points *t* and *s*.
$h(s)$: mean residual capacity at time *s*.
$R_{G,C}(n,k)$: reliability of the weighted *k*–out–of–*n: G* system having 2 different kinds of units and a cold standby unit at time *t*.
$T_{r:n}$: $r-$th smallest lifetime among $T_1, T_2, \ldots, T_n$.

3 Weighted *k*–out–of–*n* Structures with Binary Components

Let us next examine a weighted *k*–out–of–*n* having *n* units. We next accredit to the i–th unit a distinct integer-valued weight $w_i, i = 1, 2, \ldots, n$, while the overall summation of weights with regard to the whole structure is given as $w = \sum_{i=1}^{n} w_i$. The system as well as its components have two possible states. They can either work or fail. Denoting by $X_z, z = 1, 2, \ldots, n$ the status of the z–th unit, we write $X_z = 1(0)$ if the component works (has failed).

The weighted *k*–out–of–*n: F* (*G*) model stops operating (operates) then and only then the cumulative weight of the disrupted (operating) units equals to or is larger than *k*. Notice that the reliability of the weighted *k*–out–of–*n: F* model corresponds to the complementary probability with regard to the unreliability of the weighted (w-k+1)–out–of–*n: G* structure. Moreover, the common *k*–out–of–*n: F* (*G*) model corresponds to a particular member of the family of the weighted *k*–out–of–*n: F* (*G*) structures, if we assume $w_i = 1$, $i = 1, 2, \ldots, n$ (see, e.g. Wu and Chen (1994)). It is noticeable that the so-called threshold structures having binary weighted units, which have been established by Rushdi (1990), coincide to the weighted *k*–out–of–*n*: *F* mentioned before (Rushdi and Alturki (2018)). However, the subsequent relative literature followed the term "weighted *k*–out–of–*n*" instead of "threshold" structures for describing the particular reliability mechanism.

If $p_i, i = 1, 2, \ldots, n$ corresponds to the probability that the i–th unit of the weighted *k*–out–of–*n: F* (*G*) structure operates and $q_i = 1 - p_i, i = 1, 2, \ldots, n$ the possibility that its i–th component fails. If $R_G(n, k)$ expresses the reliability of the weighted *k*–out–of–*n: G* structure, Proposition 3.1 offers some recurrences for computing $R_G(n, k)$ (Wu and Chen (1994)).

Proposition 3.1. The reliability function $R_G(n, k)$ of the weighted *k*–out–of–*n: G* model with binary components obey the next recursive relations

$$R_G(n,k) = \begin{cases} p_i \cdot R_G(k-1, n-w_i) + q_i \cdot R_G(k-1, n), & \text{if } n - w_i \geq 0 \\ p_i + q_i \cdot R_G(k-1, n), & \text{otherwise.} \end{cases} \tag{1}$$

The necessary initial conditions for performing the aforementioned recursive formula are given below

$$R_G(i,0) = 1, \text{ for } i = 0, 1, \ldots, n, R_G(0,j) = 0, \text{ for } j = 1, 2, \ldots, k,$$
$$R_G(i,j) = 1, \text{ for } i = 0, 1, \ldots, n \text{ and } j < 0. \tag{2}$$

Note that the time complexity and space complexity of the algorithmic procedure proposed by Wu and Chen (1994) equals to $O(n \cdot k)$.

In addition, Eryilmaz and Bozbulut (2014) investigated the well-known Birnbaum importance of binary units of the weighted *k*–out–of–*n: G* model. Note first that Birnbaum (marginal) importance of the i–th unit of a reliability structure is defined as (see, e.g. Birnbaum (1969))

$$I_i = P(E|X_i = 1) - P(E|X_i = 0), \ i = 1, 2, \ldots, n, \tag{3}$$

where E is the event that the system operates. The technique of universal generating function can be implemented for computing marginal and joint Birnbaum importance of the components of such structures. Proposition 3.2

furnishes a nice way for delivering the marginal importance of all components of weighted *k*–out–of–*n: G* models (see Eryilmaz and Bozbulut (2014)).

Proposition 3.2. The Birnbaum marginal importance of the i–th unit of the weighted *k*–out–of–*n: G* structure consisting of binary components is given by

$$I_i = \sum_{m \in \{0,1,\ldots,M_1\}} P^1_{m,i} \cdot a(z^{H^1_{m,i}} - k) - \sum_{m \in \{0,1,\ldots,M_0\}} P^0_{m,i} \cdot a(z^{H^0_{m,i}} - k), \tag{4}$$

where $M_1(M_0)$ is the largest possible state of the structure, $P^1_{m,i}(P^0_{m,i})$ expresses the probability of state m, $H^1_{m,i}(H^0_{m,i})$ corresponds to the total performance associated with the state m, when the i–th unit is working. Moreover, note that $a(x) = 1(0)$, if $x \geq 0(x < 0)$. It is worth mentioning that an alternative importance measure has been introduced by Rahmani, Izadi and Khaledi (2016). In their framework, the so-called weighted importance becomes larger as the corresponding weight of the component increases. Some advances on the same topic are also provided by Meshkat and Mahmoudi (2017).

Two additional characteristics of the weighted *k*–out–of–*n: G* structure are examined by Eryilmaz (2015a). More precisely, the capacity loss and residual capacity of the structure are investigated. If we denote by $W_n(t)$ the total weight (capacity) of the weighted *k*–out–of–*n: G* model at time *t* and by *T* its lifetime, the following proposition delivers closed expressions for determining the mean capacity loss and the mean residual capacity of such structures.

Proposition 3.3. The mean capacity loss $m(t, s)$ and the mean residual capacity $h(s)$ of the weighted *k*–out–of–*n: G* model consisting of binary components are given by

$$m(t, s) = \sum_{a=0}^{\sum_{i=1}^{n} w_i - k} aP(W_n(t) - W_n(s) = a | T > s)$$

and

$$h(s) = \sum_{i=1}^{n} w_i - m(0, s) \tag{5}$$

respectively.

Moreover, Eryilmaz (2015b) proposed a Monte-Carlo algorithmic approach for delivering the mean time to failure of the weighted *k*–out–of–*n: G* models. According to his framework, the structure's total weight at time *t* is represented by an appropriately defined stochastic process. Under a different approach, Higashiyama (2001) proposed an algorithmic procedure for evaluating the reliability of the weighted *k*–out–of–*n: G* models with intriguing computing time.

Eryilmaz (2013) studied the mean instantaneous behavior (capacity) of a reliability model having weighted independent units with randomly distributed lifetimes $T_i, i = 1, 2, \ldots, n$. If $F(t)$ corresponds to the unreliability function of the i–th unit at time point *t*, the overall performance (capacity) of the structure is given by

$$W_n(t) = \sum_{i=1}^{n} w_i I(T_i > t). \tag{6}$$

Note that Li, You and Fang (2016) delivered an extensive investigation of $W_n(t)$ by the aid of its ordering properties and the vector of the corresponding components' weights w_i. Zhang, Ding and Zhao (2018) utilized distinct sets of requirements in order to achieve the maximization of overall capacity. Among others, the common stochastic order and the expectation order are two approaches used therein. In addition, the authors investigated, by the aid of the corresponding expectations of random weights, stochastic attributes of the underlying weighted model.

If $T_{r:n}, r = 1, 2, \ldots, n$ denotes the r–th smallest among lifetimes $T_1, T_2, \ldots, T_n$, the random variable $W_n(T_{r:n})$ corresponds to the instantaneous performance of the structure after the time where the r–th ordered component stopped working. Therefore, the mean instantaneous performance of a reliability model with weighted units seems to give evidence about the dynamic ability of the underlying structure. Eryilmaz (2013) determined the mean instantaneous performance of any reliability model with weighted units (including the weighted *k*–out–of–*n: G*). Note that $w_{[d]}, d = 1, 2, \ldots, n$ corresponds the weight which is related to that unit having the d–th smallest lifetime.

Proposition 3.4. The mean instantaneous performance at the r–th ordered component's failure of reliability structure with weighted binary units is expressed via the next formula

$$E(W_n(T_{r:n})) = \sum_{i=1}^{n} w_i - \sum_{u=1}^{r} E(w_{[u]}), \tag{7}$$

where

$$E(w_{[u]}) = \sum_{i=1}^{n} w_i \cdot \int_0^{\infty} \sum \prod_{i=1}^{u} F_{j_i}(x) \prod_{i=1}^{u} \overline{F}_{j_i}(x) dF_i(x),$$

while the summation takes into account all permutations $j_1, j_2, \ldots, j_{u-1}$ of $\{1, 2, \ldots, i = 1, i+1, \ldots, n\}$ for which $j_1 < j_2 < \cdots < j_{u-1}$ and $j_{u+1} < j_{u+2} < \cdots < j_n$.

It would be an oversight if we do not mention that Eryilmaz (2014) implemented a copula-based approach for studying the dynamic behavior of the weighted *k*–out–of–*n: G* structure having dependent units.

Rahmani, Izadi and Khaledi (2016) investigated the effect of units lifetimes and weights on the total capacity of the weighted *k*–out–of–*n: G* model. According to the classical stochastic comparison ($\leq_{st}$) between random variables, the next proposition offers an interesting stochastic ordering result referring to the lifetimes of the members of a class of weighted *k*–out–of–*n: G* systems.

Proposition 3.5. Let us take into account the weighted *k*–out–of–*n: G* models having independent unit lifetimes $T_{\pi_1}, T_{\pi_2}, \ldots, T_{\pi_n}$, where $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ is a permutation of $\{1, 2, \ldots, n\}$ and weight coefficients $(w_1, w_2, \ldots, w_n)$, such that $w_1 \leq w_2 \leq \ldots \leq w_n$. If $T_1 \leq_{st} T_2 \leq_{st} \ldots \leq_{st} T_n$ and $W_n^{\pi}(t)$ denotes the total capacity of the structure with the components' lifetimes $T_{\pi_1}, T_{\pi_2}, \ldots, T_{\pi_n}$ at time *t*, then

$$W_n^{\pi_1}(t) \leq_{st} W_n^{\pi}(t) \leq_{st} W_n^{\pi_2}(t), \tag{8}$$

where

$$\boldsymbol{\pi_1} = (1, 2, \ldots, n), \boldsymbol{\pi_1} = (n, n-1, \ldots, 1).$$

Eryilmaz and Sarikaya (2014) provided some interesting results for the weighted *k*–out–of–*n: G* model having 2 distinct kinds of independent binary units. The structure is supposed to consist of *n* units, but not all of them are identical. In other words, the units are grouped into 2 distinct collections (Group 1 and Group 2) according to their weight and their operating probability. The structure operates then and only then the overall weight of all working units exceeds *k*. In their framework, each component of the same collection shares the same weight and operating probability. More precisely, the *m* components of Group 1 have common reliability p_1 and common weight w_1, while the corresponding reliability and weight of the (*n-m*) components of Group 2 are expressed as p_2 and w_2 respectively. Denoting by $C_1(C_2)$ the set of components of Group 1, it is straightforward that $p_1 = P(X_i = 1)$ for $i \in C_1$ and $p_2 = P(X_i = 1)$ for $i \in C_2$. Proposition 3.6 furnishes a nice way for delivering the reliability of the aforementioned models.

Proposition 3.6. The reliability function $R_G(n, k, w_1, w_2)$ of the weighted k–out–of–n: *G* model having 2 distinct kinds of binary units can be viewed as

$$R_G(n, k, w_1, w_2) = \sum_{\substack{w_1 y + \\ 0 \le y \le m}} \sum_{\substack{w_2 z \ge k \\ 0 \le z \le n-m}} \binom{m}{y} p_1^y (1-p_1)^{m-y} \binom{n-m}{z} p_2^z (1-p_2)^{n-m-z} \tag{9}$$

where w_1, w_2 express the weight of the units of each type respectively.

In addition, Birnbaum importance measures of units of the weighted k–out–of–n: *G* structure having 2 distinct types of units is of some interest. Proposition 3.7 furnishes a nice way for delivering the importance of the units of such models (see Eryilmaz and Sarikaya (2014)).

Proposition 3.7. The Birnbaum importance of the i–th unit of the weighted k–out–of–n: *G* structure having 2 distinct types of binary units can be expressed as

$$\begin{aligned} I_i = &\sum_{\substack{w_1 y + \\ 1 \le y \le m}} \sum_{\substack{w_2 z \ge k \\ 0 \le z \le n-m}} \binom{m-1}{y-1} p_1^{y-1} (1-p_1)^{m-y} \binom{n-m}{z} \\ &\times p_2^z (1-p_2)^{n-m-z} \\ &- \sum_{\substack{w_1 y + \\ 0 \le y \le m-1}} \sum_{\substack{w_2 z \ge k \\ 0 \le z \le n-m}} \binom{m-1}{y} p_1^{y} (1-p_1)^{m-y-1} \binom{n-m}{z} \\ &\times p_2^z (1-p_2)^{n-m-z}, \ i \in C_1 \end{aligned}$$

and

$$\begin{aligned} I_i = &\sum_{\substack{w_1 y + \\ 0 \le y \le m}} \sum_{\substack{w_2 z \ge k \\ 0 \le z \le n-m}} \binom{m}{y} p_1^{y} (1-p_1)^{m-y} \binom{n-m-1}{z-1} \\ &\times p_2^{z-1} (1-p_2)^{n-m-z} \end{aligned}$$

$$- \sum_{\substack{w_1 y + \\ 0 \le y \le m}} \sum_{\substack{w_2 z \ge k \\ 0 \le z \le n-m-1}} \binom{m}{y} p_1^y (1-p_1)^{m-y} \binom{n-m-1}{z}$$

$$\times p_2^z (1-p_2)^{n-m-z-1}, \ i \in C_2. \tag{10}$$

It would be a delinquency not to mention that Franco, Tutuncu and Eryilmaz (2017) focused on the effect of a cold standby unit to the operating probability of weighted *k*–out–of–*n: G* structures having 2 distinct types of units. The single cold standby unit has its own distinct capacity (weight) w_c. Since a cold standby unit is present, it is possible for the structure to remain at operating status having active the rest working components and the cold standby component. We next denote by F, G the cumulative distribution function of the units of Group 1 and Group 2 respectively, while $T_{r:n}$ corresponds to the r–th ordered component's lifetime. Proposition 3.8 provides a nice expression for determining the operating probability of the weighted *k*–out–of–*n: G* structure with 2 distinct kinds of units and a cold standby unit at time *t*. Note that $T_k^{(m,n-m)}$ expresses the lifetime of the aforementioned structure under the assumption that there is no cold standby unit.

Proposition 3.8. The reliability of the weighted *k*–out–of–*n: G* structure having 2 distinct kinds of units and a cold standby unit at time *t* can be viewed as

$$R_{G,C}(t) = \sum_{\substack{w_1 i + w_2 j \ge k \\ 0 \le i \le m}} \sum_{0 \le zj \le n-m} \binom{m}{i} (1-F(t))^i (F(t))^{m-i} \binom{n-m}{j}$$

$$\times (G(t))^j (1-G(t))^{n-m}$$

$$+ \sum_{r=1}^{n} \sum_{u=\max(0,a)}^{\min(n-1,b_1)} \Bigg(\int_0^\infty m f(x) \binom{m-1}{u}$$

$$\times (F(x))^u \binom{n-m}{r-1-u} (G(x))^{r-1-u}$$

$$\times (1-F(x))^{m-1-u} (1-G(t))^{n-m-r+u+1} dx$$

$$\times \int_0^t (1-S(t-x)) P(T_{k-w_c}^{(m-u-1,n-m-r+u+1)}$$

$$
\begin{aligned}
&> t - x | T_{r:n} = x, M = u+1) h_{(r)}(x) dx \Big) \\
&\quad + \sum_{r=1}^{n} \sum_{u=\max(0,a)}^{\min(m,b_2)} \Big(\int_0^{\infty} (n-m) g(x) \binom{m}{u} \\
&\quad \times (F(x))^u \binom{n-m-1}{r-1-u} (G(x))^{r-1-u} \\
&\quad \times (1-G(x))^{n-m-r+u} dx \\
&\quad \times \int_0^t (1-S(t-x)) P(T_{k-w_c}^{(m-u,n-m-r+u)} \\
&> t - x | T_{r:n} = x, M = u) h_{(r)}(x) dx \Big). \qquad (11)
\end{aligned}
$$

where

- $S(t)$ corresponds to the survival function of the cold standby component,
- $h_{(r)}(x)$ expresses the probability density function of $T_{r:n}$,
- $a = [\frac{k-mw_1-(n-m+1-r)w_2}{w_2-w_1}]$,
- $b_1 = [\frac{k-(m-1)w_1-(n-m+1-r)w_2}{w_2-w_1} - 1]$,
- $b_2 = [\frac{k-mw_1-(n-m-r)w_2}{w_2-w_1} - 1]$.

On the other hand, Mahmoudi and Meshkat (2020) extended the case of the weighted *k*–out–of–*n: G* structure with 2 distinct kinds of units (Group 1 and Group 2), by assuming that the units are nonidentical. Proposition 3.9 provides a nice formula for determining its operating probability.

Proposition 3.9. Let us consider the weighted *k*–out–of–*n: G* structure with 2 distinct kinds of independent and nonidentical units. If $p_{1i}(w_{1i}), i \in C_1$ and $p_{2i}(w_{2i})$, $i \in C_2$ denote the reliability (weight) of the i–th unit of Group 1 and Group 2 respectively, the operating probability of the whole structure is computed via the next formula

$$
R_G = \sum_{s=0}^{m} \sum_{u=0}^{n-m} \sum_{E_s} \sum_{E_u} \prod_{d=1}^{s} p_{1l_d} \prod_{d=s+1}^{m} (1-p_{1l_d}) \prod_{d=1}^{u} p_{2l_d} \prod_{d=u+1}^{n-m} (1-p_{2l_d}), \qquad (12)
$$

where $s = \sum_{i \in C_1} X_i$, $u = \sum_{i \in C_2} X_i$, while the summation E_s extends over all combinations $l_1, l_2, \dots, l_m$ of $\{1, 2, \dots, m\}$ and the summation

E_u extends over all combinations $l_1, l_2, \ldots, l_u$ of $\{1, 2, \ldots, n - m\}$. Moreover, $(E_s, E_u) \in E$, where $E = \{(E_s, E_u) : \sum_{d=1}^{s} w_{1l_d} X_{l_d} + \sum_{d=1}^{u} w_{2l_d} X_{l_d} \geq k\}$.

Cook (2018) investigated the operating probability of a structure with functional redundancy amongst weighted units and variable demand. The demand upon the structure's processing capability is moved towards by structure's mode. The capacity (weight) of each unit of the structure changes over time or at least changes upon the underlying configuration of the components. It is straightforward that the overall demand of the structure is not constant too.

Let us next assume that a system configuration is obtained as a plausible combination of unit states for all units in the structure. Afterwards, determine the amount of all combinations of unit states the structure can deliver and mark those configurations as $j = 1, 2, \ldots, 2^n$. In such a framework, the status of each unit given as $X_{i,j}$, while i expresses the specific unit and j corresponds to the underlying configuration. If we denote by C_j the overall demand of the structure when the j–th configuration has been activated, the following result holds true

$$C_j = \sum_{i=1}^{n} X_{i,j} w_i. \tag{13}$$

The next proposition offers a nice formula for evaluating the operating probability of the aforementioned structure (Cook (2018)).

Proposition 3.10. Let us consider the weighted *k*–out–of–*n: G* structure with independent and nonidentical units with weight $w_i, i = 1, 2, \ldots, n$. If C_j denotes the overall demand of the structure when the j–th configuration has been activated ($j = 1, 2, \ldots, 2^n$) and the demand follows Normal distribution with known parameters μ, σ^2 respectively, the operating probability of the whole structure is given by

$$R_{G,j} = \Phi\left(\frac{C_j - \mu}{\sigma}\right). \tag{14}$$

In a different approach, Zhang (2018) examined optimal allocation of active redundancies for weighted *k*-out-of-*n* structures. In his fine work, 1, 2 and multiple active redundancies have been considered with regard to the common stochastic order under the assumption that the structure has independent and heterogeneous units with distinct weights.

In more recent developments on the topic, Zhang (2021) carried out a survey on *k*–out–of–*n* structures with heterogeneous units and random demands. Among others, some ordering outcomes on the total capacities of randomly weighted *k*–out–of–*n* structures are investigated. Note that distinct criteria for guaranteeing randomness have been applied when the units are drawn from 2 distinct groups. In addition, Hamdan, Tavangar and Asadi (2021) utilized the average cost and some common availability requirements for introducing optimal models, with respect to preventive maintenance, for the family of weighted *k*–out–of–*n* structures.

Finally, it is worth mentioning that Chen and Yang (2005) expanded the already known one-stage weighted-*k*–out–of–*n* structure to the corresponding one with 2 stages and also having common units. An algorithmic approach had been developed for delivering the operating probability of the resulting structure. The minimal cut sets and minimal path sets of the proposed two-stage structure were also studied. A fuzzy reliability study of the structure has been provided by Chaube and Singh (2016).

4 Consecutive *k*–out–of–*n* Structures

We next concentrate on the consecutive weighted *k*–out–of–*n* having *n* binary units. We accredit to each unit a distinct integer-valued weight w_i, $i = 1, 2, \ldots, n$, while the overall weight of the whole structure can be expressed as $w = \sum_{i=1}^{n} w_i$. The system as well as its components have two possible states. They can either work or fail. Denoting by $X_z, z = 1, 2, \ldots, n$ the status of the z–th unit, we write $X_z = 1(0)$ if the unit works (has failed).

The consecutive weighted *k*–out–of–*n: F* model fails then and only then the cumulative weight of some consecutive disrupted units is at least equal to *k*. Notice that the common consecutive *k*–out–of–*n: F* structure is actually a member of the family of the weighted consecutive *k*–out–of–*n: F* structures (for $w_i = 1, i = 1, 2, \ldots, n$, see, for instance Wu and Chen (1994b) or Chang, Chen and Hwang (1998)). If the components are linearly (circularly) arranged, then the resulting scheme is known as linear (circular) consecutive weighted *k*–out–of–*n: F* structure. Wu and Chen (1994b) developed algorithms for evaluating the operating probability of the (either linear or circular) consecutive weighted *k*–out–of–*n: F* structure. If we denote by $p_i(q_i = 1 - p_i), i = 1, 2, \ldots, n$ the reliability (unreliability) of the i–th unit and by $S_i, i = 1, 2, \ldots, n$ the minimum set of requirements which results in the underachievement of the whole structure, Proposition 4.1 furnishes a nice way for delivering the unreliability of the linear consecutive weighted *k*–out–of–*n: F* structure (Wu and Chen (1994b).

Proposition 4.1. The unreliability $F_L(i,j)$ of the linear consecutive weighted *k*–out–of–*n: F* structure having units $i, i+1, \ldots, j$ can be viewed as

$$\begin{aligned} F_L(1, End(j)) = F_L(1, End(j-1)) \\ &+ \sum_{i=0}^{Beg(j)-Beg(j-1)-1} R_L(1, Beg(j-1)+i-1) \\ &\times p_{Beg(j-1)+i} \cdot \left[\prod_{i=Beg(j-1)+i+1}^{Beg(j)-1} q_i \right] \\ &\cdot Q(j),\ j = 2, 3, \ldots, m, \end{aligned} \tag{15}$$

where $Beg(i)(End(i)), i = 1, 2, \ldots, n$ refers to the first (last) component of S_i and

$$Q(j) = \prod_{h=Beg(j)}^{End(j)} q_h.$$

Chang, Chen and Hwang (1998) proposed an alternative algorithmic approach for delivering the operating possibility of the circular consecutive weighted *k*–out–of–*n: F* structure. Their algorithm seems to be simpler and more efficient algorithm in comparison with the one introduced by Wu and Chen (1994b). The next proposition deals with the reliability of the circular consecutive weighted *k*–out–of–*n: F* structure and provides some relation of it to the corresponding reliability of the linear model.

Proposition 4.2. The reliability $R_C(i,j)$ of the circular consecutive weighted *k*–out–of–*n: F* structure having components $i, i+1, \ldots, j$ w is given by

$$R_C(1,n) = \begin{cases} 1, & T > n \\ \displaystyle\sum_{i=1}^{T} \left[R_L(i+1, n+i-1) \cdot \left[\prod_{j=1}^{i-1} q_j \right] \cdot p_i \right], & T \le n \end{cases} \tag{16}$$

where $R_C(i,j)$ is the corresponding one of the linear consecutive weighted *k*–out–of–*n: F* system.

In addition, Samaniego and Shaked (2008) described (among others) the minimal cut sets of the linear consecutive weighted *k*–out–of–*n: F* systems. Moreover, Eryilmaz and Tutuncu (2009) studied the linear consecutive weighted-*k*-out-of-*n:F* structures having independent and nonhomogeneous Markov dependent units. Proposition 4.3 furnishes a nice way for delivering the reliability of the structure under the restriction $2k \geq \sum_{i=1}^{n} w_i$.

Proposition 4.3. The reliability $R_w(n,k)$ of the linear consecutive weighted *k*–out–of–*n: F* structure consisting of independent units is calculated via

$$R_w(n,k) = 1 - \sum_{i=1}^{n} p_{i+1}(1 - f(i,k)), \tag{17}$$

where $p_{n+1} = 1$ and

$$f(i,k) = \begin{cases} \displaystyle\sum_{m=0}^{j} p_{i-m} \prod_{s=i-m+1}^{i} (1-p_s), & \text{if } \displaystyle\sum_{r=i-j+1}^{i} w_r < k \leq \sum_{r=i-j}^{i} w_r \\ 1, & \text{if } \displaystyle\sum_{r=1}^{i} w_r < k. \end{cases}$$

Despite the fact that recurrences provide a nice tool for determining the reliability of the underlying model, their application could be proved quite inconveniencing due to their complexity in terms of computing time. A common alternative is provided by delivering simple approximation formulae for the same quantity of the underlying system. Proposition 4.4 offers an approximation for determining the reliability of the model (Eryilmaz and Tutuncu (2009)).

Proposition 4.4. The reliability $R_w(n,k)$ of the linear consecutive weighted *k*–out–of–*n: F* structure consisting of independent units is approximated by

$$R_w(n,k) \cong \frac{\prod_{i=1}^{n-1} g(i,k)}{\prod_{i=2}^{n-1} f(i,k)}, \tag{18}$$

where

$$g(i,k) = \begin{cases} p_{i+1} f(i,k) + (1 - p_{i+1} f(i, k - w_{i+1}), & \text{if } k > w_{i+1} \\ p_{i+1} f(i,k), & \text{if } k \leq w_{i+1} \end{cases}$$

and

$$f(i,k) = \begin{cases} \displaystyle\sum_{m=0}^{j} p_{i-m} \prod_{s=i-m+1}^{i} (1-p_s), & \text{if } \displaystyle\sum_{r=i-j+1}^{i} w_r < k \leq \sum_{r=i-j}^{i} w_r \\ 1, & \text{if } \displaystyle\sum_{r=1}^{i} w_r < k. \end{cases}$$

5 A New Weighted Consecutive-type Structure

In what follows, we define a new model having *n* independent and not necessarily identically distributed units. We next assume that all units are ordered in a line. Denote by $X_1, X_2, \ldots, X_n$ the state of each one of the components, where

$$X_i = \begin{cases} 1, & \text{if the } i\text{–th component has failed,} \\ 0, & \text{if the } i\text{–th component is working.} \end{cases} \tag{19}$$

We next assume that the contribution of the i–th unit to the operation of the whole structure can be expressed as a positive weight $w_i, i = 1, 2, \ldots, n$. Therefore, the so-called consecutive-weighted-k_1 and k_2-out-of-*n*: *F* model fails then and only then there exist non-overlapping disrupted units with overall weight at least equal to k_1 and non-overlapping failed components (different from the ones involved to the first criterion) with total weight at least equal to k_2. To capture the operation of such a system, we first define the random variable T_i as follows

$$T_i = w_i \cdot X_i,\ i = 1, 2, \ldots, n. \tag{20}$$

It is straightforward that the resulting scheme has binary components, while the whole system could be either in operating or in failure state. The consecutive-weighted-k_1 and k_2-out-of-*n*: *F* model shall be actualized in terms of a suitable Markov chain. Let us first define the random variables $D_1, D_2, \ldots, D_n$ as follows

$$D_i = \begin{cases} D_{i-1} + T_i, & \text{if } T_i \neq 0 \\ 0, & \text{if } T_i = 0, \end{cases} \tag{21}$$

for all $i = 1, 2, \ldots, n$. Note that $D_1, D_2, \ldots, D_n$ take on only non-negative values. Under the convention that $D_0 = 0, w_0 = 0$, D_i could be equal to

- 0, if $T_i = 0$,
- w_i, if $T_{i-1} = 0$ and $T_i \neq 0$,
- $w_{i-1} + w_i$, if $T_{i-2} = 0$, $T_{i-1} \neq 0$ and $T_i \neq 0$,
- $w_{i-2} + w_{i-1} + w_i$, if $T_{i-3} = 0, T_{i-2} \neq 0$, $T_{i-1} \neq 0$ and $T_i \neq 0$,

$$\vdots$$

- $w_1 + w_2 + \cdots + w_i$, if $T_1 \neq 0, T_2 \neq 0, \ldots,\ T_{i-1} \neq 0$ and $T_i \neq 0$.

In fact, the above-mentioned values correspond to all possible states of the underlying Markov chain. Moreover, the sequence $D_i, i \geq 1$ is directly connected to the reliability of the underlying consecutive-weighted-k_1 and k_2-out-of-*n*: *F* model. More specifically, the structure works if and only if

- either all $D_1, D_2, \ldots, D_n$ are less than $\max(k_1, k_2)$,
- or exactly one $D_i, i \geq 1$ is equal to or greater than $\max(k_1, k_2)$ and the remaining ones are less than $\min(k_1, k_2)$.

In other words, the structure works if and only if the following ensue

$$D_1 < \max(k_1, k_2),\ D_2 < \max(k_1, k_2), \ldots, D_n < \max(k_1, k_2)$$

or

$$\begin{gathered} D_\xi \geq \max(k_1, k_2),\ D_1 < \min(k_1, k_2), \\ D_2 < \min(k_1, k_2), \ldots, D_{\xi-1} < \min(k_1, k_2), \ldots, D_{\xi+1} = 0, \\ D_{\xi+2} < \min(k_1, k_2), \ldots, D_n < \min(k_1, k_2), \quad \text{for } \xi = 1, 2, \ldots, n-1 \end{gathered}$$

or

$$\begin{gathered} D_1 < \min(k_1, k_2),\ D_2 < \min(k_1, k_2), \ldots, D_{n-1} < \min(k_1, k_2), \\ D_n \geq \max(k_1, k_2). \end{gathered} \tag{22}$$

Consequently, the reliability function $R(k_1, k_2.n)$ of the consecutive-weighted-k_1 and k_2-out-of-*n*: *F* model can be expressed via the following formula

$$\begin{aligned} R(k_1, k_2 \cdot n) = P(&D_1 < \max(k_1, k_2), \\ &D_2 < \max(k_1, k_2), \ldots, D_n < \max(k_1, k_2)) \end{aligned}$$

$$\begin{aligned}
&+ P(D_\xi \geq \max(k_1, k_2),\ D_1 < \min(k_1, k_2),\\
&D_2 < \min(k_1, k_2), \ldots, D_{\xi-1} < \min(k_1, k_2), \ldots, D_{\xi+1} = 0,\\
&D_{\xi+2} < \min(k_1, k_2), \ldots, D_n < \min(k_1, k_2),\\
&\text{for } \xi = 1, 2, \ldots, n-1)\\
&+ P(D_1 < \min(k_1, k_2),\\
&D_2 < \min(k_1, k_2), \ldots, D_{n-1} < \min(k_1, k_2),\\
&D_n \geq \max(k_1, k_2)). \qquad (23)
\end{aligned}$$

Denoting by $p_i = P(X_i = 0), i = 1, 2, \ldots, n$ the possibility that the i–th unit is operating, the Markovian sequence $D_i, i \geq 1$ transitions from state a to state b $(a, b \geq 0)$ with probability equal to

$$P(D_{i+1} = b | D_i = a) = \begin{cases} p_{i+1}, & \text{if } b = 0 \\ 1 - p_{i+1}, & \text{if } b = a + w_{i+1}. \end{cases} \qquad (24)$$

where $i = 1, 2, \ldots, n-1$.

For illustration purposes, we next provide an introductory example of such a reliability structure. Let us consider the consecutive-weighted-3 and 6-out-of-5:F structure having independent units with weights $w_1 = 2$, $w_2 = 1$, $w_3 = 3, w_4 = 3, w_5 = 4$ respectively. The resulting structure $(n = 5, k_1 = 3, k_2 = 6)$ fails then and only then there exist non-overlapping failed units with overall weight equal to or more than 3 and non-overlapping failed components (different from the ones involved to the first criterion) with total weight at least equal to 6. Figure 1 illustrates some scenarios that result in the operation disruption of the underlying structure.

Kindly mention that a grey-drawn block corresponds to a non-operating unit, while a blank block corresponds to an operating unit. Both schemes displayed in Figure 1, lead to the operation disruption of the system. This is actually evident since the stoppage criterion of consecutive-weighted-3 and

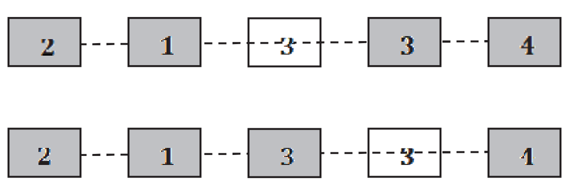


Figure 1 Failure scenarios for the consecutive-weighted-3 and 6-out-of-5: F system.

6-out-of-5: *F* system is met under both configurations. For instance, under the first scenario the failed components are placed at 1st, 2nd, 4th and 5th position in the line. The system fails since

- components 1 and 2 are consecutive with total weight $w_1 + w_2 = 3$. Consequently, their failure led to the fulfillment of the first failure criterion ($k_1 = 3$).
- components 4 and 5 are consecutive with total weight $w_4 + w_5 = 7$. Consequently, their failure led to the fulfillment of the second failure criterion ($k_2 = 6$).

We next study the reliability function of the consecutive-weighted-k_1 and k_2-out-of-*n*: *F* model having units with failure probability $(1 - p_i)$, $i = 1, 2, \ldots, n$ under the assumption of independence. A recursive scheme for evaluating the reliability function of the consecutive-weighted-k_1 and k_2-out-of-*n*: *F* model is derived, while a nice expression for the determination of the corresponding reliability is also provided. Based on (23), the reliability function of the consecutive-weighted-k_1 and k_2-out-of-*n*: *F* model could be alternatively written as

$$\begin{aligned} R(k_1, k_2 \cdot n) = & \sum_{i_1=0}^{\max(k_1,k_2)\ -1} \sum_{i_2=0}^{\max(k_1,k_2)-1} \cdots \sum_{i_n=0}^{\max(k_1,k_2)-1} \prod_{b=1}^{n-1} \\ & \times P(D_{b+1} = i_{b+1} | D_b = i_b) P(D_1 = i_1) \\ & + \sum_{i_1=0}^{\min(k_1,k_2)\ -1} \sum_{i_2=0}^{\min(k_1,k_2)\ -1} \cdots \sum_{i_{\xi-1}=0}^{\min(k_1,k_2)-1} \sum_{i_\xi=\max(k_1,k_2)}^{w_1+w_2+\cdots+w_\xi} \\ & \sum_{i_{\xi+1}=0}^{0} \sum_{i_{\xi+2}=0}^{\min(k_1,k_2)-1} \cdots \sum_{i_n=0}^{\min(k_1,k_2)-1} \prod_{b=1}^{n-1} \\ & \times P(D_{b+1} = i_{b+1} | D_b = i_b) P(D_1 = i_1) \\ & + \sum_{i_1=0}^{\min(k_1,k_2)-1} \sum_{i_2=0}^{\min(k_1,k_2)-1} \cdots \sum_{i_{n-1}=0}^{\min(k_1,k_2)\ -1} \sum_{i_n=\max(k_1,k_2)}^{w_1+w_2+\cdots+w_n} \prod_{b=1}^{n-1} \\ & \times P(D_{b+1} = i_{b+1} | D_b = i_b) P(D_1 = i_1) \end{aligned} \tag{25}$$

For example, let us consider the consecutive-weighted-2 and 3-out-of-4: *F* structure having independent and identically distributed (*i.i.d.*) units with weights $w_1 = 2, w_2 = 1, w_3 = 1, w_4 = 3$. The random variables D_i, $1 \leq i \leq 4$ take on values as follows (see Section 2)

- $D_1 \in \{0, 2\}$,
- $D_2 \in \{0, 1, 3\}$,
- $D_3 \in \{0, 1, 2, 4\}$,
- $D_4 \in \{0, 3, 4, 5, 7\}$.

The reliability function of the aforementioned consecutive-weighted-2 and 3-out-of-4: *F* structure is now expressed as

$$\begin{aligned}
R(k_1, k_2.n) &= \sum_{i_1=0}^{2}\sum_{i_2=0}^{2}\sum_{i_3=0}^{2}\sum_{i_4=0}^{2}\prod_{b=1}^{3} P(D_{b+1} = i_{b+1}|D_b = i_b)P(D_1 = i_1) \\
&\quad + \sum_{i_1=3}^{2}\sum_{i_2=0}^{0}\sum_{i_3=0}^{1}\sum_{i_4=0}^{1}\prod_{b=1}^{3} P(D_{b+1} = i_{b+1}|D_b = i_b)P(D_1 = i_1) \\
&\quad + \sum_{i_1=0}^{1}\sum_{i_2=3}^{3}\sum_{i_3=0}^{0}\sum_{i_4=0}^{1}\prod_{b=1}^{3} P(D_{b+1} = i_{b+1}|D_b = i_b)P(D_1 = i_1) \\
&\quad + \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\sum_{i_3=3}^{4}\sum_{i_4=0}^{0}\prod_{b=1}^{3} P(D_{b+1} = i_{b+1}|D_b = i_b)P(D_1 = i_1) \\
&\quad + \sum_{i_1=0}^{0}\sum_{i_2=0}^{0}\sum_{i_3=0}^{0}\sum_{i_4=3}^{7}\prod_{b=1}^{3} P(D_{b+1} = i_{b+1}|D_b = i_b)P(D_1 = i_1) \\
&= \sum_{i_1=0}^{2}\sum_{i_2=0}^{2}\sum_{i_3=0}^{2}\sum_{i_4=0}^{2} P(D_2 = i_2|D_1 = i_1)P(D_3 = i_3|D_2 = i_2) \\
&\quad \times P(D_4 = i_4|D_3 = i_3)P(D_1 = i_1) \\
&\quad + \sum_{i_1=3}^{2}\sum_{i_2=0}^{0}\sum_{i_3=0}^{1}\sum_{i_4=0}^{1} P(D_2 = i_2|D_1 = i_1) \\
&\quad \times P(D_3 = i_3|D_2 = i_2)P(D_4 = i_4|D_3 = i_3)P(D_1 = i_1) \\
&\quad + \sum_{i_1=0}^{1}\sum_{i_2=3}^{3}\sum_{i_3=0}^{0}\sum_{i_4=0}^{1} P(D_2 = i_2|D_1 = i_1)
\end{aligned}$$

$$\begin{aligned}
&\times P(D_3 = i_3|D_2 = i_2)P(D_4 = i_4|D_3 = i_3)P(D_1 = i_1)\\
&+ \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\sum_{i_3=3}^{4}\sum_{i_4=0}^{0} P(D_2 = i_2|D_1 = i_1)\\
&\times P(D_3 = i_3|D_2 = i_2)P(D_4 = i_4|D_3 = i_3)P(D_1 = i_1)\\
&+ \sum_{i_1=0}^{0}\sum_{i_2=0}^{0}\sum_{i_3=0}^{0}\sum_{i_4=3}^{7} P(D_2 = i_2|D_1 = i_1)\\
&\times P(D_3 = i_3|D_2 = i_2)P(D_4 = i_4|D_3 = i_3)P(D_1 = i_1)
\end{aligned} \tag{26}$$

Since the structure consists of $n = 4$ *i.i.d.* units, we have $p_1 = p_2 = p_3 = p_4 = p$. By the aid of (24), we readily deduce that

$$P(D_1 = 0) = p,\ P(D_1 = 2) = 1 - p,$$

$$P(D_2 = 0|D_1 = 0) = P(D_2 = 0|D_1 = 2) = p,$$

$$P(D_2 = 1|D_1 = 0) = P(D_2 = 3|D_1 = 2) = 1 - p,$$

$$P(D_3 = 0|D_2 = 0) = P(D_3 = 0|D_2 = 1) = P(D_3 = 0|D_2 = 3) = p,$$

$$P(D_3 = 1|D_2 = 0) = P(D_3 = 2|D_2 = 1) = P(D_3 = 4|D_2 = 3) = 1 - p,$$

$$\begin{aligned}
P(D_4 = 0|D_3 = 0) &= P(D_4 = 0|D_3 = 1) = P(D_4 = 0|D_3 = 2)\\
&= P(D_4 = 0|D_3 = 4) = p,\\
P(D_4 = 3|D_3 = 0) &= P(D_4 = 4|D_3 = 1) = P(D_4 = 5|D_3 = 2)\\
&= P(D_4 = 7|D_3 = 4) = 1 - p.
\end{aligned}$$

The first term of (26) can be rewritten as

$$\begin{aligned}
R(k_1, k_2.n) &= \sum_{i_1=0}^{2}\sum_{i_2=0}^{2}\sum_{i_3=0}^{2} P(D_2 = i_2|D_1 = i_1)P(D_3 = i_3|D_2 = i_2)\\
&\quad \times P(D_4 = 0|D_3 = i_3)P(D_1 = i_1)\\
&= \sum_{i_1=0}^{2}\sum_{i_2=0}^{2} P(D_2 = i_2|D_1 = i_1)P(D_3 = 0|D_2 = i_2)\\
&\quad \times P(D_4 = 0|D_3 = 0)P(D_1 = i_1)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i_1=0}^{2} \sum_{i_2=0}^{2} P(D_2 = i_2 | D_1 = i_1) P(D_3 = 1 | D_2 = i_2) \\
&\times P(D_4 = 0 | D_3 = 1) P(D_1 = i_1) \\
&+ \sum_{i_1=0}^{2} \sum_{i_2=0}^{2} P(D_2 = i_2 | D_1 = i_1) P(D_3 = 2 | D_2 = i_2) \\
&\times P(D_4 = 0 | D_3 = 2) P(D_1 = i_1) \\
= &\sum_{i_1=0}^{2} P(D_2 = 0 | D_1 = i_1) P(D_3 = 0 | D_2 = 0) \\
&\times P(D_4 = 0 | D_3 = 0) P(D_1 = i_1) \\
&+ \sum_{i_1=0}^{2} P(D_2 = 1 | D_1 = i_1) P(D_3 = 0 | D_2 = 1) \\
&\times P(D_4 = 0 | D_3 = 0) P(D_1 = i_1) \\
&+ \sum_{i_1=0}^{2} P(D_2 = 0 | D_1 = i_1) P(D_3 = 1 | D_2 = 0) \\
&\times P(D_4 = 0 | D_3 = 1) P(D_1 = i_1) \\
&+ \sum_{i_1=0}^{2} P(D_2 = 1 | D_1 = i_1) P(D_3 = 1 | D_2 = 1) \\
&\times P(D_4 = 0 | D_3 = 1) P(D_1 = i_1) \\
&+ \sum_{i_1=0}^{2} P(D_2 = 0 | D_1 = i_1) P(D_3 = 2 | D_2 = 0) \\
&\times P(D_4 = 0 | D_3 = 2) P(D_1 = i_1) \\
&+ \sum_{i_1=0}^{2} P(D_2 = 1 | D_1 = i_1) P(D_3 = 2 | D_2 = 1) \\
&\times P(D_4 = 0 | D_3 = 2) P(D_1 = i_1)
\end{aligned}
$$

$$
\begin{aligned}
&= P(D_2 = 0|D_1 = 0)P(D_3 = 0|D_2 = 0)\\
&\quad \times P(D_4 = 0|D_3 = 0)P(D_1 = 0)\\
&\quad + P(D_2 = 0|D_1 = 2)P(D_3 = 0|D_2 = 0)\\
&\quad \times P(D_4 = 0|D_3 = 0)P(D_1 = 2)\\
&\quad + P(D_2 = 1|D_1 = 0)P(D_3 = 0|D_2 = 1)\\
&\quad \times P(D_4 = 0|D_3 = 0)P(D_1 = 0)\\
&\quad + P(D_2 = 1|D_1 = 2)P(D_3 = 0|D_2 = 1)\\
&\quad \times P(D_4 = 0|D_3 = 0)P(D_1 = 2)\\
&\quad + P(D_2 = 0|D_1 = 0)P(D_3 = 1|D_2 = 0)\\
&\quad \times P(D_4 = 0|D_3 = 1)P(D_1 = 0)\\
&\quad + P(D_2 = 0|D_1 = 2)P(D_3 = 1|D_2 = 0)\\
&\quad \times P(D_4 = 0|D_3 = 1)P(D_1 = 2)\\
&\quad + P(D_2 = 1|D_1 = 0)P(D_3 = 1|D_2 = 1)\\
&\quad \times P(D_4 = 0|D_3 = 1)P(D_1 = 0)\\
&\quad + P(D_2 = 1|D_1 = 2)P(D_3 = 1|D_2 = 1)\\
&\quad \times P(D_4 = 0|D_3 = 1)P(D_1 = 2)\\
&\quad + P(D_2 = 0|D_1 = 0)P(D_3 = 2|D_2 = 0)\\
&\quad \times P(D_4 = 0|D_3 = 2)P(D_1 = 0)\\
&\quad + P(D_2 = 0|D_1 = 2)P(D_3 = 2|D_2 = 0)\\
&\quad \times P(D_4 = 0|D_3 = 2)P(D_1 = 2)\\
&\quad + P(D_2 = 1|D_1 = 0)P(D_3 = 2|D_2 = 1)\\
&\quad \times P(D_4 = 0|D_3 = 2)P(D_1 = 0)\\
&\quad + P(D_2 = 1|D_1 = 2)P(D_3 = 2|D_2 = 1)\\
&\quad \times P(D_4 = 0|D_3 = 2)P(D_1 = 2)\\
&= p^4 + p^3(1-p) + (1-p)p^3 + (1-p)p^3 + p^2(1-p)^2\\
&\quad + p^2(1-p)^2\\
&= -p^3 + 2p^2. \qquad (27)
\end{aligned}
$$

On the other hand, the second term of (26) equals to zero, while the third term can be viewed as

$$\begin{aligned}
&\sum_{i_1=0}^{1}\sum_{i_2=3}^{3}\sum_{i_3=0}^{0}\sum_{i_4=0}^{1} P(D_2=i_2|D_1=i_1)P(D_3=i_3|D_2=i_2)\\
&\qquad \times P(D_4=i_4|D_3=i_3)P(D_1=i_1)\\
&\quad = P(D_2=3|D_1=0)P(D_3=0|D_2=3)\\
&\qquad \times P(D_4=0|D_3=0)P(D_1=0)\\
&\quad = (1-p)p^3.
\end{aligned} \tag{28}$$

Moreover, the fourth term of (26) can be determined as

$$\begin{aligned}
&\sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\sum_{i_3=3}^{4}\sum_{i_4=0}^{0} P(D_2=i_2|D_1=i_1)P(D_3=i_3|D_2=i_2)\\
&\qquad \times P(D_4=i_4|D_3=i_3)P(D_1=i_1)\\
&\quad = P(D_2=0|D_1=0)P(D_3=4|D_2=0)\\
&\qquad \times P(D_4=0|D_3=4)P(D_1=0)\\
&\qquad + P(D_2=1|D_1=0)P(D_3=4|D_2=1)\\
&\qquad \times P(D_4=0|D_3=4)P(D_1=0)\\
&\quad = 0.
\end{aligned} \tag{29}$$

The last term of (26) is given by

$$\begin{aligned}
&\sum_{i_1=0}^{0}\sum_{i_2=0}^{0}\sum_{i_3=0}^{0}\sum_{i_4=3}^{7} P(D_2=i_2|D_1=i_1)P(D_3=i_3|D_2=i_2)\\
&\qquad \times P(D_4=i_4|D_3=i_3)P(D_1=i_1)\\
&\quad = P(D_2=0|D_1=0)P(D_3=0|D_2=0)\\
&\qquad \times P(D_4=3|D_3=0)P(D_1=0)\\
&\quad = (1-p)p^3.
\end{aligned} \tag{30}$$

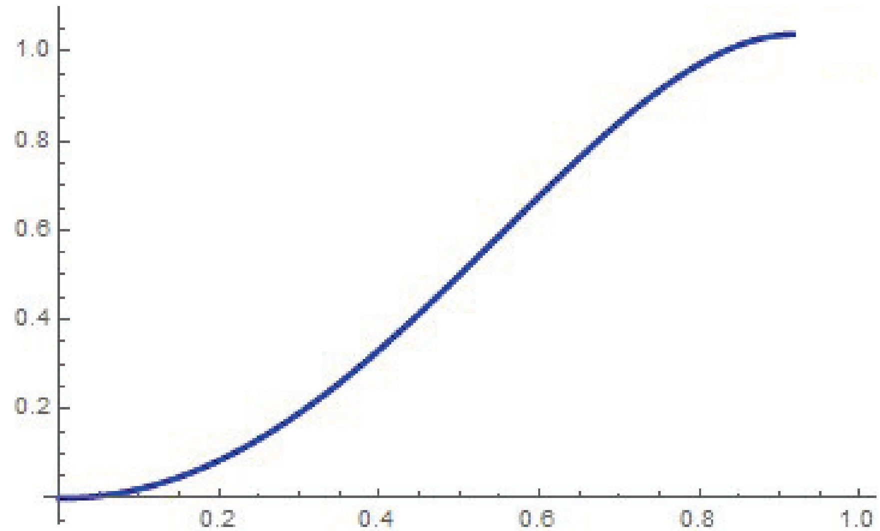


Figure 2 The reliability of the consecutive-weighted-2 and 3-out-of-4: *F* structure. ($w_1 = 2, w_2 = 1, w_3 = 1, w_4 = 3$).

Combining equations (26)–(30), the reliability function of the consecutive-weighted-2 and 3-out-of-4: *F* structure can be rewritten as

$$R(k_1, k_2 \cdot n) = -2p^4 + p^3 + 2p^2.$$

Figure 2 illustrates the above polynomial form in terms of components' reliability.

According to the latter argumentation, it is easily obtained that formula (25) seems to be computationally inefficient. Although we considered a specific member of the class of consecutive-weighted-k_1 and k_2-out-of-*n*: *F* structures with low-valued design parameters ($n = 4, k_1 = 2, k_2 = 3$) and identically distributed components, the computational effort for determining its reliability function was quite large. Therefore, we next develop alternative techniques for evaluating the reliability function of consecutive-weighted-k_1 and k_2-out-of-*n*: *F* structures. Proposition 5.1 offers a recursive scheme for this purpose.

Proposition 5.1. Let us consider a consecutive-weighted-k_1 and k_2-out-of-*n*: *F* model having independent units with reliability $p_i, i = 1, 2, \ldots, n$ respectively. If $w_1, w_2, \ldots, w_n$ are the respective weights of its units such that $\sum_{i=1}^{n} w_i \geq \max(k_1, k_2)$, the reliability function of the system obeys the next recurrences

$$R(k_1, k_2 \cdot n) = \begin{cases} R(k_1, k_2.n-1)p_n, & \text{if } w_1 \geq \min(k_1, k_2), \\ & w_n \geq \max(k_1, k_2) \\ & \text{and } p_1 = p_n \\ R(k_1, k_2.n-1)p_n & \text{if } w_n \geq \max(k_1, k_2), \\ \quad + (p_1 - p_n)(1 - p_n) & w_i \geq \min(k_1, k_2), \\ \quad \times \prod_{i=2}^{n-1} p_i, & \text{for } i = 2, 3, \ldots, n-1. \end{cases} \tag{31}$$

Proof. By conditioning on D_n, the first term of the reliability function of the consecutive-weighted-k_1 and k_2-out-of-*n*: *F* model (see (23)) can be expressed as

$$\begin{aligned} &P(D_1 < \max(k_1, k_2),\ D_2 < \max(k_1, k_2), \ldots, D_n < \max(k_1, k_2)) \\ &\quad = \sum_{v < \max(k_1, k_2)} P(D_1 < \max(k_1, k_2), \\ &\quad D_2 < \max(k_1, k_2), \ldots, D_n = v), \end{aligned} \tag{32}$$

where the random variable D_n is obliged to be zero. Indeed, D_n is supposed to take on values smaller than $\max(k_1, k_2)$ (see (32)). Therefore, the only value that is allowed to be equal to, is the zero one, since the remaining possible values for D_n are greater than $\max(k_1, k_2)$. That practically means that the reliability function of the consecutive-weighted-k_1 and k_2-out-of-*n*: *F* system can be rewritten as (see (32))

$$\begin{aligned} &P(D_1 < \max(k_1, k_2),\ D_2 < \max(k_1, k_2), \ldots, D_n < \max(k_1, k_2)) \\ &\quad = P(D_1 < \max(k_1, k_2), \\ &\qquad D_2 < \max(k_1, k_2), \ldots, D_{n-1} < \max(k_1, k_2), D_n = 0) \\ &\quad = P(D_1 < \max(k_1, k_2), \\ &\qquad D_2 < \max(k_1, k_2), \ldots, D_{n-1} < \max(k_1, k_2)) \cdot P(D_n = 0) \\ &\quad = P(D_1 < \max(k_1, k_2), \\ &\qquad D_2 < \max(k_1, k_2), \ldots, D_{n-1} < \max(k_1, k_2)) \cdot P(X_n = 0) \\ &\quad = P(D_1 < \max(k_1, k_2), \\ &\qquad D_2 < \max(k_1, k_2), \ldots, D_{n-1} < \max(k_1, k_2)) \cdot p_n \end{aligned} \tag{33}$$

Following a similar argumentation, the second term of the reliability function of the consecutive-weighted-k_1 and k_2-out-of-n: F system (see (23)) can be viewed as ($w_1 \geq \min(k_1, k_2)$)

$$\begin{aligned}
&P(D_\xi \geq \max(k_1,k_2), D_1 < \min(k_1,k_2),\\
&\quad D_2 < \min(k_1,k_2), \ldots, D_{\xi-1} < \min(k_1,k_2), \ldots, D_{\xi+1} = 0,\\
&\quad D_{\xi+2} < \min(k_1,k_2), \ldots, D_n < \min(k_1,k_2),\\
&\quad \text{for } \xi = 1,2,\ldots,n-1)\\
&\quad = \sum_{v<\min(k_1,k_2)} P(D_\xi \geq \max(k_1,k_2), D_1 < \min(k_1,k_2),\\
&\quad\quad D_2 < \min(k_1,k_2), \ldots, D_{\xi-1} < \min(k_1,k_2), \ldots, D_{\xi+1} = 0,\\
&\quad\quad D_{\xi+2} < \min(k_1,k_2), \ldots, D_{n-1} < \min(k_1,k_2),\\
&\quad\quad \text{for } \xi = 1,2,\ldots,n-1, \ldots, D_n = v)\\
&\quad = P(D_\xi \geq \max(k_1,k_2), D_1 < \min(k_1,k_2),\\
&\quad\quad D_2 < \min(k_1,k_2), \ldots, D_{\xi-1} < \min(k_1,k_2), \ldots, D_{\xi+1} = 0,\\
&\quad\quad D_{\xi+2} < \min(k_1,k_2), \ldots, D_{n-1} < \min(k_1,k_2),\\
&\quad\quad \text{for } \xi = 1,2,\ldots,n-1) \cdot p_n
\end{aligned} \tag{34}$$

By conditioning on D_1, the last term of $R(k_1, k_2.n)$ (see (23)) can be expressed as

$$\begin{aligned}
&P(D_1 < \min(k_1,k_2),\ D_2 < \min(k_1,k_2), \ldots, D_{n-1} < \min(k_1,k_2),\\
&\quad D_n \geq \max(k_1,k_2))\\
&\quad = \sum_{v<\min(k_1,k_2)} P(D_1 = \nu,\\
&\quad\quad D_2 < \min(k_1,k_2), \ldots, D_{n-1} < \min(k_1,k_2), D_n \geq \max(k_1,k_2))\\
&\quad = p_1 \cdot P(D_2 < \min(k_1,k_2), \ldots, D_{n-1} < \min(k_1,k_2),\\
&\quad\quad D_n \geq \max(k_1,k_2))
\end{aligned} \tag{35}$$

We next combine (32)–(35) and the desired result is readily derived under the assumption $p_1 = p_n$.

On the other hand, the second branch of the recursive scheme in (31) could be readily deduced by rewriting the last probability in (35) as

$$
\begin{aligned}
&P(D_2 < \min(k_1, k_2), \ldots, D_{n-1} < \min(k_1, k_2), D_n \geq \max(k_1, k_2)) \\
&\quad = \sum_{v < \min(k_1, k_2)} P(D_2 = \nu, \\
&\qquad D_3 < \min(k_1, k_2), \ldots, D_{n-1} < \min(k_1, k_2), D_n \geq \max(k_1, k_2)) \\
&\quad = p_2 \cdot \sum_{v < \min(k_1, k_2)} P(D_3 = \nu, \\
&\qquad D_4 < \min(k_1, k_2), \ldots, D_{n-1} < \min(k_1, k_2), D_n \geq \max(k_1, k_2)) \\
&\quad = p_2 \cdot p_3 \cdot \sum_{v < \min(k_1, k_2)} P(D_4 = \nu, \\
&\qquad D_5 < \min(k_1, k_2), \ldots, D_{n-1} < \min(k_1, k_2), D_n \geq \max(k_1, k_2)) \\
&\quad \vdots \\
&\quad = p_2 \cdot p_3 \cdot p_{n-1}(1 - p_n).
\end{aligned}
$$

We next replace the last expression in (35) and the scheme we are chasing for is immediately delivered by combining the latter result with Equations (32)–(34). □

It is worth mentioning that Eyilmaz and Tutuncu (2009) followed a parallel argumentation and delivered some recurrences for the reliability function of traditional consecutive-weighted-*k*-out-of-*n*: *F* structure (see also Wu and Chen (1994)).

6 Discussion

At the first part of the present manuscript, an up-to date presentation of the developments on the field of weighted reliability structures is provided. Some fundamental theoretical results are presented, while their applicability is also highlighted. At the second part of the article, a new weighted structure is introduced. The main motivation for establishing the proposed reliability structure is that in several practical situations, the contribution of each element which takes part in a bigger device or system, is not the same.

For handling those applications, we should assume that each element has a distinct contribution with regard to the operating (or non-operating) status of the device (or system). To sum up, some intriguing thoughts for future research include definitely the relaxation of the assumptions, such as components' independence. This extension seems to improve the applicability of the underlying reliability models. Finally, the weighed systems seem to be useful in several alternative scientific fields, such as Financial Engineering, Artificial Intelligence and their dynamic development in such modern areas is strongly encouraged.

References

Barlow, R. E. and K. D. Heidtmann. 1984. Computing *k*-out-of-*n* system reliability. *IEEE Transactions on Reliability,* 33: 322–323.

Birnbaum, Z. W. 1969. On the importance of different components in a multicomponent system. In *Multivariate analysis II* (ed. P. R. Krishnaiah), Academic Press, New York, pp. 581–592.

Chang, J.-C., R.-J. Chen, and Hwang, F. K. 1998. A fast reliability algorithm for the circular consecutive-weighted-*k*-out-of-*n: F* systems. *IEEE Transactions on Reliability*, 47: 472–474.

Chaube, S. and S. B. Singh. 2016. Fuzzy reliability of two-stage weighted-*k*-out-of-*n* systems with common components. *International Journal of Mathematical, Engineering and Management Sciences*. 1: 41–51.

Chen, Y. and Q. Yang. 2005. Reliability of two-stage weighted-*k*-out-of-*n* systems with components in common. *IEEE Transactions on Reliability*. 54: 431–440.

Cook, J. L. 2018. Reliability analysis of weighted *k*-out-of-*n* systems with variable demand. *Quality Engineering*. 30: 687–693.

Cui,L., W. Kuo, and M. Xie.. 2006. On the dual reliability systems of (*n,f,k*) and <*n,f,k*>. *Statistics and Probability Letters*. 76: 1081–1088.

Derman, C., G. J. Lieberman, and S. M. Ross. 1982. On the consecutive-*k*-out-of-*n*: *F* system. *IEEE Transactions on Reliability*. 31: 57–63.

Eryilmaz, S. 2013. Mean instantaneous performance of a system with weighted components that have arbitrarily distributed lifetimes. *Reliability Engineering and System Safety*. 119: 290–293.

Eryilmaz, S. 2014. Multivariate copula based dynamic reliability modeling with application to weighted *k*-out-of-*n*: *G* systems of dependent components. *Structural Safety*. 51: 23–28.

Eryilmaz, S. 2015a. Capacity loss and residual capacity in weighted k-out-of-n: G systems, *Reliability Engineering and System Safety*. **136**, 140–144.

Eryilmaz, S. 2015b. Mean Time to Failure of weighted k-out-of-n: G systems. *Communication in Statistics: Simulation and Computation*. 44: 2705–2713.

Eryilmaz, S. and K.Sarikaya. 2014. Modeling and analysis of weighted k-out-of-$n{:}G$ system consisting of two different types of components. *Proceedings of the Institution of Mechanical Engineers: Part* 0: *Journal of Risk and Reliability*. 228: 265–271.

Eryilmaz, S. and M. J. Zuo. 2010. Constrained (k,d)-out-of-n systems. *International Journal of Systems Science*. 41: 679–685.

Eryilmaz, S. and G. Y. Tutuncu. 2009. Reliability evaluation of linear consecutive-weighted-k-out-of-n: F system. *Asia-Pacific Journal of Operational Research.* 26: 805–816.

Franco, C., G. Y. Tutuncu, and S. Eryilmaz. 2017. Reliability of weighted k-out-of-n: G systems consisting of two types of components and a cold standby component. *Communication in Statistics: Simulation and Computation*. 46: 4067–4081.

Griffith, W. S. 1986. On consecutive–k–out–of–n: failure systems and their generalizations. In *Reliability and Quality Control* Basu, A.P. (ed.). Elsevier, Amsterdam, pp. 157–165.

Hamdan, K., M. Tavangar, and M. Asadi, 2021. Optimal preventive maintenance for repairable weighted k-out-of-n systems. *Reliability Engineering and System Safety*. 205: 107267.

Higashiyama, Y. 2001. A factored reliability formula for weighted-k-out-of-n system. *Asia-Pacific Journal of Operational Research*. 18: 61–66.

Kumar, A., S. B. Singh, and M. Ram. 2019. Reliability appraisal for consecutive-k-out-of-n: F system of non-identical components with intuitionistic fuzzy set. *International Journal of Operational Research*. 36: 362–374.

Kuo, W. and M. J. Zuo 2003. *Optimal Reliability Modeling: Principles and Applications*, John Wiley & Sons, N.J.

Li, X., Y. You, and R.Fang 2016. On weighted k-out-of-n systems with statistically dependent component lifetimes. *Probability in the Engineering and Informational Sciences*. 30: 533–546.

Mahmoudi, E. and R. Meshkat, 2020. Reliability analysis of weighted-k-out-of-n: G system consisting of two different types of nonidentical components each with its own positive integer-valued weight. *Journal of Statistical Theory and Applications*. 19: 408–414.

Meshkat, R. and E. Mahmoudi, 2017. Joint reliability and weighted importance measures of a k-out-of-n system with random weights for components. *Journal of Computational and Applied Mathematics*. 326: 273–283.

Pham, H. and S. J. Upadhyaya 1988. The efficiency of computing the reliability of k-out-of-n systems. *IEEE Transactions on Reliability.* 36: 78–79.

Rahmani, R.-A., M. Izadi, and B.-E. Khaledi. 2016. Stochastic comparisons of total capacity of weighted- k-out-of-n systems. *Statistics and Probability Letters*. 117: 216–220.

Rahmani, R.-A., M. Izadi, and B.-E. Khaledi. 2016. Importance of components in k-out-of-n system with components having random weights. *Journal of Computational and Applied Mathematics.* 296: 1–9.

Ram, M. 2013. On system reliability approaches: a brief survey. *International Journal of System Assurance Engineering and Management*. **4**: 101–117.

Rushdi, A. M. 1990. Threshold systems and their reliability. *Microelectronics Reliability*. 30: 299–312.

Rushdi, A. M. and A. M. Alturki. 2018. Novel representations for a coherent threshold reliability system: a tale of eight signal flow graphs. *Turkish Journal of Electrical Engineering & Computer Sciences.* 26: 257–269.

Samaniego, F. J. and M. Shaked. 2008. Systems with weighted components, *Statistics & Probability Letters*. 78: 815–823.

Tong, Y. L. 1985. A rearrangement inequality for the longest run, with an application to network reliability. *Journal of Applied Probability*. 22: 386–393.

Triantafyllou, I. S. and M. V. Koutras. 2011. Signature and *IFR* preservation of 2-within-consecutive k-out-of-n:F systems. *IEEE Transactions on Reliability.* 60: 315–322.

Triantafyllou, I. S. 2015. Consecutive-type reliability systems: an overview and some applications. *Journal of Quality and Reliability Engineering*. 2015. Article ID 212303: 20 pages.

Triantafyllou, I. S. 2020. On the consecutive k_1- and k_2-out-of-n reliability systems. *Mathematics.* 8: 630.

Triantafyllou, I. S. 2021a. Reliability Study of $<n, f, 2>$ systems: A Generating Function Approach. *International Journal of Mathematical, Engineering and Management Sciences*. 6: 44–65.

Triantafyllou, I. S. 2021b. On the lifetime and signature of the constrained (*k,d*) out-of-*n: F* reliability systems. *International Journal of Mathematical, Engineering and Management Sciences*. 6: 66–78.

Wu, J. S. and R. J.Chen, 1994a. An algorithm for computing the reliability of a weighted-*k*-out-of-*n* system. *IEEE Transactions on Reliability.* 43: 327–328.

Wu, J. S. and R. J. Chen 1994b. Efficient algorithms for *k*-out-of-*n* & consecutive weighted-*k*-out-of-*n* system. *IEEE Transactions on Reliability.* 43: 650–655.

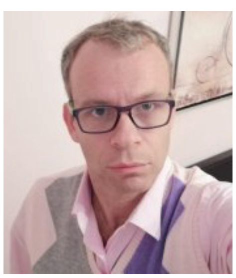
Zhang, Y. 2018. Optimal allocation of active redundancies in weighted *k*-out-of-*n* systems. *Statistics and Probability Letters.* 135: 110–117.

Zhang, Y., W. Ding, and P. Zhao. 2018. On total capacity of *k*-out-of-*n* systems with random weights. *Naval Research Logistics*. 65: 347–359.

Zhang, Y. 2021. Reliability analysis of randomly weighted-*k*-out-of-*n* systems with heterogeneous components. *Reliability Engineering and System Safety*. 205: 107184.

Zhao, X., L. Cui, and X. Yang. 2006. A consecutive k_1 and k_2-out-of-*n* system and its reliability. In *Advanced Reliability Modeling II.* Yun, W. Y. & Dohi, T. (eds), pp. 97–103.

Biography



Ioannis S. Triantafyllou is an Assistant Professor at the Department of Statistics and Insurance Science of the University of Piraeus. His research interests focus on the area of Statistical Reliability Theory and Non-Parametric Statistical Quality Control. More than 60 research papers of him in the field of Applied Probability and Statistics have been published in international scientific journals and edited volumes. In the literature, there exist more than 500 citations of his research work, while he has served as a referee for more than 35 international scientific journals. He serves as an Associate

Editor for *International Journal of Mathematical Engineering and Management Sciences* and as Reviewer for *Mathematical Reviews/MathSciNet* (by *American Mathematical Society*). He has served as Guest Editor for special issues published in two scientific journals indexed by Web of Science and Scopus.