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LINDLEY-RAYLEIGH DISTRIBUTION WITH APPLICATION TO LIFETIME DATA

Selen Cakmakyapan* and Gamze Ozel
Department of Statistics, Istanbul Medeniyet University, Istanbul, Turkey
Department of Statistics, Hacettepe University, Ankara, Turkey

E Mail: *selen.cakmakyapan@medeniyet.edu.tr; gamzeozl@gmail.com

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Abstract

The analysis of lifetime data has become a popular topic in many fields such as engineering, medicine and social science. In this study, the Lindley-Rayleigh (LR) distribution is proposed and several properties are studied including ordinary moments, quantile function, generating function, asymptotes, entropy measures and order statistics. Parameter estimations are obtained via the maximum likelihood method. Then, the flexibility of the proposed distribution is shown by the use of lifetime data.

Key Words: Lindley Distribution, Rayleigh Distribution, Lifetime.

1. Introduction

Lindley (1958) proposed the Lindley distribution for analyzing lifetime data which belongs to an exponential family. The Lindley distribution is an alternative to the exponential lifetime distributions when the failure rate is unimodal or bathtub shaped (Bakouch et al., 2012). The statistical properties and the parameter estimation of the Lindley distribution were obtained by Ghitany et al. (2008, 2011). Mazucheli and Achcar (2011) obtained that the Lindley distribution was more flexible than the exponential distribution and proposed it as an alternative to exponential or Weibull distributions. Ghitany et al. (2012) obtained Marshall-Olkin extended Lindley distribution. Then, the gamma Lindley distribution was proposed by Nedjar and Zeghdoudi (2016). An extended new generalized Lindley distribution was obtained by Shibuand Irshad (2016). Recently, Irshad and Maya (2017) have obtained two different extensions of the Lindley distribution. Then, Cakmayapan and Ozel (2017) have introduced the Lindley family of distributions.

The Rayleigh distribution was introduced by Rayleigh (1880) as a special case of the Weibull distribution. The probability density function (pdf) of the Rayleigh distribution with a shape parameter $\sigma > 0$ is given by

$$g(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, & x > 0 \\ 0, & otherwise \end{cases} \tag{1.1}$$

and corresponding cumulative distribution function (cdf) is obtained as

$$G(x) = \begin{cases} 1 - e^{-x^2/2\sigma^2}, & x > 0 \\ 0, & otherwise \end{cases} \quad (1.2)$$

There have been many forms for the Rayleigh distribution to provide flexibility for modeling data. Dyer and Whisenand (1973) used Rayleigh distribution in an application for communication engineering. Voda (1975) derived the generalized Rayleigh distribution and obtain statistical properties, also introduced the left-truncated Rayleigh distribution. Voda (1976 a, b) obtained the maximum likelihood estimates (MLEs) of the Rayleigh distribution. Bhattacharya and Tyagi (1990) found that some clinical data follow the Rayleigh distribution. Fernandez (2010) mentioned the parameter estimation problems of the Rayleigh distribution. Gomes et al. (2014) obtained the Kumaraswamy generalized Rayleigh distribution for analyzing lifetime data. The transmuted Rayleigh distribution was developed by Merovci (2013). Leao et al. (2013) introduced the beta inverse Rayleigh distribution. Cordeiro et al. (2013) proposed beta generalized Rayleigh distribution then provided its mathematical properties. Ahmad et al. (2014) introduced the transmuted inverse Rayleigh distribution and its properties. Recently, slashed generalized Rayleigh distribution has been introduced by Iriarte et al. (2017).

In this paper, we propose the Lindley Rayleigh ("LR" for short) distribution to increase the flexibility of the Lindley distribution using Lindley generator by adding an extra parameter. The proposed distribution is a member of Lindley-G family introduced by Cakmakyapan and Ozel (2017). Here, we consider the Lindley distribution as a generator and the Rayleigh is considered as a baseline distribution.

The main motivations of this study are: (i) toderive more flexible model with the less parameters for the lifetime datasets, (ii) to make the kurtosis moreflexible (iii) to generate symmetric, left-skewed, right-skewed, J and reversed-J shapeddistributions (iv) to provide better fits than other existing models.

The cdf and pdf of the Lindley-G family with a parameter $\theta > 0$ are, respectively, given by

$$F(x) = \begin{cases} 1 - \left[1 - \frac{\theta}{\theta+1}[\log(1 - G(x))]\right][1 - G(x)]^{\theta}, & x > 0 \\ 0, \quad otherwise \end{cases} \quad (1.4)$$

$$f(x) = \begin{cases} \frac{\theta^2}{\theta+1} g(x)\left[1 - \log\left(1 - G(x)\right)\right][1 - G(x)]^{\theta-1}, & x > 0, \\ 0, \quad otherwise \end{cases} \quad (1.5)$$

where G(x) is a baseline cdf and g(x) is the baseline pdf. We can define the Lindley-Rayleigh distribution using the pdf in (1.5). Then, the random variable X is denoted by X~LR(θ, σ).

The paper is organized as follows: The LR distribution is proposed and figures of the density, survival and hazard rate functions are presented in Section 2. Asymptotes, shapes, quantile function, skewness, kurtosis, moments, moment generating functions are given in Section 3. The Rényi entropy, reliability function and order statistics are determined in Section 4. MLEs and an expression for the observed information matrix are given in Section 5. An application on the lifetime data set is considered in Section 6. Concluding remarks are presented in Section 7.

2. Lindley-Rayleigh Distribution

2.1The cumulative density and probability density functions

The cdf of the LR distribution is obtained by inserting (1.2) in (1.4) as

$$F(x)=\begin{cases}1-\left[1+\frac{\theta x^2}{(\theta+1)2\sigma^2}\right]e^{-\frac{\theta x^2}{2\sigma^2}}, & x>0\\ 0, & otherwise\end{cases} \tag{2.1}$$

where $\theta > 0$ and $\sigma > 0$. Then, the pdf of the LR distribution is given by

$$f(x)=\begin{cases}\frac{\theta^2 x}{(\theta+1)\sigma^2}e^{-\frac{\theta x^2}{2\sigma^2}}\left(1+\frac{x^2}{2\sigma^2}\right), & x>0\\ 0, & otherwise\end{cases} \tag{2.2}$$

Figure 1 shows the plots forpdf and cdffor LR distribution with various parameter values. As seen in Figure 1, the density function of the LR distribution has several different shapes according to the values of the parameters. This proves that LR distribution is more flexible than Rayleigh distribution. The shape parameter θ allows great flexibility to the LR distribution. For a fixed σ, as θ increased, the right tail of the LR distribution becomes longer. Hence, the proposed model can be important to model positive real data sets.

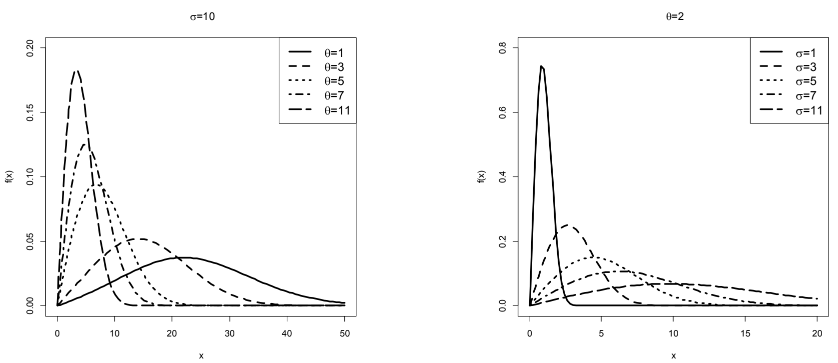

Figure 1: Plots for pdf of LR distribution with various values of parameters

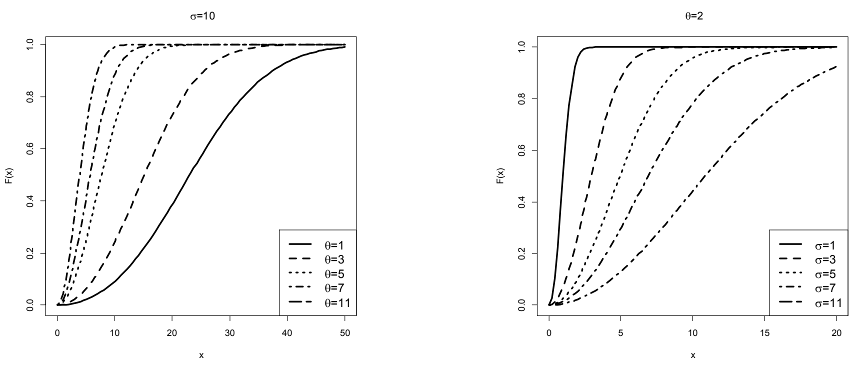

Figure 2: Plots for the cdf for the LR distribution with various values of parameters

2.2 Survival and hazard rate functions

We obtain corresponding survival function (sf) as

$$S(x) = \left[1 + \frac{\theta x^2}{(\theta+1)2\sigma^2}\right] e^{-\frac{\theta x^2}{2\sigma^2}} \tag{2.3}$$

The LR distribution can be used in survival analysis, lifetime analysis, economics, etc. The hazard rate function (hrf) is the other important function for a random variable for characterizing life time data. Then, the hrf of the LR distribution is given by

$$h(x) = \frac{2\theta^2\sigma^2 x + \theta^2 x^3}{2\sigma^4\theta + 2\sigma^4 + \theta\sigma^2 x^2} \tag{2.4}$$

Figure 2 shows several shapes of the sf for the LR distribution, respectively.

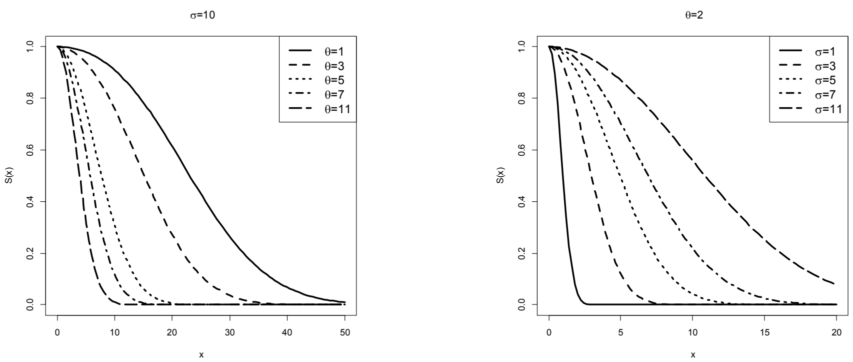

Figure 3: Plots for sfs of LR distribution withvarious parameter values

3. Properties of LR distribution

Now, the main properties of LR distribution are studied including quantile function, moments, skewness, kurtosis.

3.1 Quantile Function

Let G(x) be the cdf for Rayleigh distribution and $F^{-1}(u)$, 0<u<1, be the quantile function of the LR distribution. Then, $F^{-1}(u)$is given by

$$F^{-1}(u) = G^{-1}\left(1 - e^{-L^{-1}(u)}\right) \tag{3.1}$$

where L is the cdf of the Lindley distribution. Inverting G(x)=u, the quantile function for the Rayleigh distribution is obtained as

$$G^{-1}(u) = [-2\sigma^2 \log(1-u)]^{1/2} \tag{3.2}$$

for 0<u<1.

If X~LR(σ,θ), then quantile function of LR distribution is obtained as

$$F^{-1}(u) = \left(-2\sigma^2 \frac{W[(u-1)(\theta+1)e^{-(\theta+1)}]+\theta+1}{\theta}\right)^{\frac{1}{2}} \tag{3.3}$$

where W is the Lambert function.

The effect of the shape parameter σ on the skewness and kurtosis is obtained by quantile measures. The Bowley's skewness is given using the quartiles as follows:

$$S = -\frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}$$

and the Moors' kurtosis based on octiles is given by

$$K = -\frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)}$$

where Q(.) represents the quantile function of X. For $F^{-1}(u) = Q(u)$ in (3.3), skewness and kurtosis of the LR distribution can be obtained formulas given above. They are less sensitive to outliers and they exist even for distributions without moments. Skewness is a measure of degree of long tail and kurtosis measures the degree of tail heaviness. For a symmetric distribution, S equals to 0. For right-skewed distribution, S is greater than zero. Similarly, S is smaller than zero for left-skewed distribution. The tail of the distribution becomes heavier when K increases.

3.2 Moments

The ordinary moments μ'_n, n=1,2,..., of the LR distribution are obtained as

$$\mu'_n = \frac{\theta^2}{(\theta+1)\sigma^2}\int_0^\infty x^{n+1}\, e^{-\frac{\theta x^2}{2\sigma^2}}\left(1+\frac{x^2}{2\sigma^2}\right)dx$$
$$= \frac{\theta^2}{(\theta+1)\sigma^2}\left[\int_0^\infty x^{n+1}\, e^{-\frac{\theta x^2}{2\sigma^2}}dx + \frac{1}{2\sigma^2}\int_0^\infty x^{n+3}\, e^{-\frac{\theta x^2}{2\sigma^2}}dx\right] \quad (3.4)$$
$$= \frac{\theta^2}{(\theta+1)\sigma^2}\left[2^{\frac{n}{2}}\theta^{-\frac{(n+2)}{2}}\sigma^{n+2}\Gamma\left(\frac{n+2}{2}\right) + \frac{1}{2\sigma^2}2^{\frac{n+2}{2}}\theta^{-\frac{(n+4)}{2}}\sigma^{n+4}\Gamma\left(\frac{n+4}{2}\right)\right]$$

Further, the central moments (μ_n) and cumulants (κ_n), $n = 1,2,\ldots$, of the LR distribution can be obtained from$\mu_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\mu'^{n}_1\mu'_{n-k}$ and $\kappa_n = \mu'_n - \sum_{k=0}^{n}(-1)^k\binom{n}{k}\mu'^{k}_1\mu'_{n-k}$. Here, $\kappa_1 = \mu'_1$, $\kappa_2 = \mu'_2 - \mu'^2_1$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1$, $\kappa_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu'^2_2 + 12\mu'_2{\mu'_1}^2 - 6{\mu'_1}^4$ etc. The skewness $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and kurtosis $\gamma_2 = \kappa_4/\kappa_2^2$ are also computed from the second, third and fourth cumulants. Table 1 gives moments, skewness, and kurtosis of LR distribution for different parameter values.

	Parameters			
	$\theta = 2, \sigma = 1$	$\theta = 0.5, \sigma = 10$	$\theta = 7, \sigma = 0.5$	$\theta = 0.1, \sigma = 0.1$
μ'_1	1.033931	23.63272	0.251658	0.576484
μ'_2	1.333333	666.6667	0.080357	0.381818
μ'_3	1.994011	21269.45	0.030135	0.281036
μ'_4	3.333333	746666.7	0.012755	0.225455
μ'_5	6.09281	28359262	0.005948	0.194563
μ'_6	1.20E+01	1.15E+09	0.003007	0.178909
Skewness	0.506726	0.357383	0.604859	0.351557
Kurtosis	0.011381	-0.06948	0.180827	0.027313

Table 1: Skewness, Kurtosis and Moments for the LR distribution

Table 1 presents that the LR distribution is right-skewed and has thin or heavy tails for the selected parameter values.

3.3 Moment Generating Function

Now, we give a formula for the mgf $M(t)$of the LR distribution as

$$M(t) = \frac{\theta^2}{(\theta+1)\sigma^2}\int_0^\infty e^{tx}\, xe^{-\frac{\theta x^2}{2\sigma^2}}\left(1+\frac{x^2}{2\sigma^2}\right)dx$$
$$= \frac{\theta^2}{(\theta+1)\sigma^2}\left[\int_0^\infty xe^{tx-\frac{\theta x^2}{2\sigma^2}}\,dx + \frac{1}{2\sigma^2}\int_0^\infty x^3e^{tx-\frac{\theta x^2}{2\sigma^2}}\,dx\right]$$
$$= \frac{\theta^2}{(\theta+1)\sigma^2}\begin{bmatrix} 1 + \frac{\sigma}{\sqrt{\theta}}te^{\frac{\sigma^2t^2}{2\theta}}\sqrt{\frac{\pi}{2}}\left(erf\left(\frac{\sigma t}{\sqrt{2\theta}}\right)+1\right) \\ -\frac{1}{4}\sqrt{\frac{2\pi}{\theta}}\frac{\partial^3}{\partial t^3}\left(e^{\frac{\sigma^2t^2}{2\theta}}erfc\left(-\frac{\sigma t}{\sqrt{2\theta}}\right)\right)\end{bmatrix} \quad (3.5)$$

where mgf of X exists, if there exists a positive constant b such that $M(t)$is finite for all t$\in [-\mathrm{b},\mathrm{b}]$.

4. Other Measures

4.1 Rényi Entropy

The Rényi entropy of a random variable X with the pdf f(x) is defined as

$$\mathrm{I}_R(\gamma) = \frac{\gamma}{1-\gamma}\log\int_0^\infty f^\gamma(x)dx \tag{4.1}$$

for $\gamma > 0$ and which implies that

$$\begin{aligned}\int_0^\infty f^\gamma(x)dx &= \int_0^\infty \left[\frac{\theta^2 x}{(\theta+1)\sigma^2} e^{-\frac{\theta x^2}{2\sigma^2}}\left(1+\frac{x^2}{2\sigma^2}\right)\right]^\gamma dx \\ &= \frac{\theta^{2\gamma}}{(\theta+1)^\gamma\sigma^{2\gamma}}\sum_{i=0}^{\gamma}\binom{\gamma}{i}\frac{1}{2^i\sigma^{2i}}\int_0^\infty x^{\gamma+2i}e^{-\frac{\theta\gamma x^2}{2\sigma^2}}dx \\ &= \frac{\theta^{2\gamma}}{(\theta+1)^\gamma\sigma^{2\gamma}}\sum_{i=0}^{\gamma}\binom{\gamma}{i}\frac{1}{2^{i+1}\sigma^{2i}}\left(\frac{\theta\gamma}{2\sigma^2}\right)^{\frac{-(\gamma+2i+1)}{2}}\Gamma\left(\frac{\gamma+2i+1}{2}\right)\end{aligned} \tag{4.2}$$

Then, the Rényi entropy of LR distribution is obtained as

$$\mathrm{I}_R(\gamma) = \frac{\gamma}{1-\gamma}\log\left[\frac{\theta^{2\gamma}}{(\theta+1)^\gamma\sigma^{2\gamma}}\sum_{i=0}^{\gamma}\binom{\gamma}{i}\frac{1}{2^{i+1}\sigma^{2i}}\left(\frac{\theta\gamma}{2\sigma^2}\right)^{\frac{-(\gamma+2i+1)}{2}}\Gamma\left(\frac{\gamma+2i+1}{2}\right)\right] \tag{4.3}$$

4.2 Stress-Strength Reliability

In reliability, the stress-strength shows a component life which has a random strengthX_1that is subjected to a random stress X_2. Therefore, $\mathrm{R} = \mathrm{P}(\mathrm{X}_2 < \mathrm{X}_1)$shows component reliability measure. In this section, we derive reliability functionR if$\mathrm{X}_1 \sim \mathrm{LR}(\theta_1, \sigma_1)$ and $\mathrm{X}_2 \sim \mathrm{LR}(\theta_2, \sigma_2)$ are independent random variables.

Let f_i denote the pdf of X_i and F_i denote the cdf of X_i for $i = 1,2$, then the reliability function for the LR distribution is obtained as

$$\begin{aligned}R &= \int_0^\infty f_1(x)F_2(x)dx \\ &= \int_0^\infty \frac{\theta_1^2 x}{(\theta_1+1)\sigma_1^2} e^{-\frac{\theta_1 x^2}{2\sigma_1^2}}\left(1+\frac{x^2}{2\sigma_1^2}\right)\left[1-\left[1+\frac{\theta_2 x^2}{(\theta_2+1)2\sigma_2^2}\right]e^{-\frac{\theta_2 x^2}{2\sigma_2^2}}\right]dx\end{aligned} \tag{4.4}$$

After some algebra, we obtain

$$R=\frac{\theta_1^2}{(\theta_1+1)\sigma_1^2}\left[\begin{array}{l}\int_0^\infty xe^{-\frac{\theta_1x^2}{2\sigma_1^2}}dx-\int_0^\infty xe^{-x^2\left(\frac{\theta_1}{2\sigma_1^2}+\frac{\theta_2}{2\sigma_2^2}\right)}dx\\ -\frac{\theta_2}{(\theta_2+1)2\sigma_2^2}\int_0^\infty x^3e^{-x^2\left(\frac{\theta_1}{2\sigma_1^2}+\frac{\theta_2}{2\sigma_2^2}\right)}dx\end{array}\right]+\frac{\theta_1^2}{(\theta_1+1)2\sigma_1^4}\left[\begin{array}{l}\int_0^\infty x^3e^{-\frac{\theta_1x^2}{2\sigma_1^2}}dx-\int_0^\infty x^3e^{-x^2\left(\frac{\theta_1}{2\sigma_1^2}+\frac{\theta_2}{2\sigma_2^2}\right)}\\ -\frac{\theta_2}{(\theta_2+1)2\sigma_2^2}\int_0^\infty x^5e^{-x^2\left(\frac{\theta_1}{2\sigma_1^2}+\frac{\theta_2}{2\sigma_2^2}\right)}dx\end{array}\right] \quad (4.5)$$

and hence the reliability function of the LR distribution takes the form

$$R=\frac{\theta_1^2}{(\theta_1+1)\sigma_1^2}\left[\frac{\sigma_1^2}{\theta_1}-\frac{1}{2}\left(\frac{\theta_1}{2\sigma_1^2}+\frac{\theta_2}{2\sigma_2^2}\right)^{-1}-\frac{\theta_2}{(\theta_2+1)2\sigma_2^2}\frac{1}{2}\left(\frac{\theta_1}{2\sigma_1^2}+\frac{\theta_2}{2\sigma_2^2}\right)^{-2}\right]+\frac{\theta_1^2}{(\theta_1+1)2\sigma_1^4}\left[2\left(\frac{\sigma_1^2}{\theta_1}\right)^2-\frac{1}{2}\left(\frac{\theta_1}{2\sigma_1^2}+\frac{\theta_2}{2\sigma_2^2}\right)^{-2}-\frac{\theta_2}{(\theta_2+1)2\sigma_2^2}\left(\frac{\theta_1}{2\sigma_1^2}+\frac{\theta_2}{2\sigma_2^2}\right)^{-3}\right]. \quad (4.6)$$

4.3 Order Statistics

Order statistics is especially important for the estimation and hypothesis tests. Hence, the order statistics for the proposed distribution is studied now. Let $X_{i:n}$ denote the ith order statistic. Then, pdf $f_{i:n}(x)$of the ith order statistic for a random sample $X_1, X_2, \ldots, X_n$ from Lindley-G family is given by

$$f_{i:n}(x)=\frac{n!}{(i-1)!\,(n-i)!}f(x)F(x)^{i-1}[1-F(x)]^{n-i}$$
$$=\frac{n!}{(i-1)!(n-i)!}(-1)^j\binom{n-i}{j}f(x)F(x)^{j+i-1}. \quad (4.7)$$

Then, the pdf of $X_{i:n}$ for the LR distribution is obtained as

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-1}(-1)^j\binom{n-i}{j}\left(1+\sum_{k=1}^{\infty}\frac{G^k(x)}{k}\right)\sum_{\ell=0}^{\theta-1}(-1)^\ell\binom{\theta-1}{\ell}\times G^\ell(x)\frac{\theta^2}{\theta+1}g(x)\sum_{z=0}^{i+j-1}(-1)^z\binom{i+j-1}{z} \quad (4.8)$$

where $g(.)$and$G(.)$arepdf and cdf of the Rayleigh distribution, respectively.

5. Maximum Likelihood Estimation

Now, parameter estimatorss of the LR distribution using maximum likelihood method are obtained. Let $x_1, x_2, \ldots, x_n$ be observed values from LR distribution with parameters θ and σ. The likelihood function for (θ, σ) can be written as

$$L=\prod_{i=1}^{n}\left\{\frac{\theta^2x_i}{(\theta+1)\sigma^2}e^{-\frac{\theta x_i^2}{2\sigma^2}}\left(1+\frac{x_i^2}{2\sigma^2}\right)\right\} \quad (5.1)$$

and the log-likelihood function is obtained as

$$\log L = n\log\theta + \sum_{i=1}^{n}\log x_i - \frac{\theta}{2\sigma^2}\sum_{i=1}^{n}x_i^2 - n\log(\theta+1) - 4n\log\sigma + \sum_{i=1}^{n}\log(2\sigma^2 + x_i^2) - n\log 2. \quad (5.2)$$

The first derivatives of the log-likelihood function with respect to the parameters are given by

$$\frac{\partial \log L}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{\theta+1} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}x_i^2 \quad (5.3)$$

$$\frac{\partial \log L}{\partial \sigma} = \frac{\theta}{\sigma^3}\sum_{i=1}^{n}x_i^2 - \frac{4n}{\sigma} + \sum_{i=1}^{n}\frac{4\sigma}{2\sigma^2 + x_i^2} \quad (5.4)$$

The MLEs of (θ, σ), say $(\hat{\theta}, \hat{\sigma})$, are the simultaneous solutions of the equations $\frac{\partial \log L}{\partial \theta} = 0$ and $\frac{\partial \log L}{\partial \sigma} = 0$. Maximizing of (5.2) is performed using *nlm* or *optimize* in R statistical package. The Fisher information matrix is required for the interval estimation of (θ, σ). The observed Fisher information matrix for (θ, σ)can be written as

$$\mathrm{I} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} \quad (5.5)$$

The elements of the Fisher information matrix are as follows:

$$I_{11} = \frac{\partial \log L}{\partial\theta\,\partial\theta} = -\frac{2n}{\theta^2} + \frac{n}{(\theta+1)^2}, \quad (5.6)$$

$$I_{22} = \frac{\partial \log L}{\partial\sigma\,\partial\sigma} = -\frac{3\theta}{\sigma^4}\sum_{i=1}^{n}x_i^2 + \frac{4n}{\sigma^2} + \sum_{i=1}^{n}\frac{4x_i^2 - 8\sigma^2}{(2\sigma^2 + x_i^2)^2}, \quad (5.7)$$

$$I_{12} = I_{21} = \frac{\partial \log L}{\partial\theta\,\partial\sigma} = \frac{1}{\sigma^3}\sum_{i=1}^{n}x_i^2. \quad (5.8)$$

6. Simulation Study

Now, a Monte Carlo simulation is used to perform of the MLEs of the parameters for the LR distribution. The results of all simulation are obtained from 5000 replications using R-Project. In each replication, a random sample of size N is obtained from $X \sim LR(\sigma, \theta)$. The random number generation for the LR distribution is performed by the inversion method using the quantile function Q(u) given in Section 3.1.

Five different combinations of true parameter values in the first row in Table 2 are used for the data generating processes. Table 2 lists the MSEs (MSE1 for σ, MSE2 for θ) for parameter estimation with their corresponding biases with five different sample sizes.

Parameters	$\sigma = 2,$ $\theta = 5$	$\sigma = 7,$ $\theta = 3$	$\sigma = 19,$ $\theta = 13$	$\sigma = 0.1,$ $\theta = 0.5$	$\sigma = 5,$ $\theta = 5$
		N=50			
MSE1	0.25999	2.94723	0.76714	0.00701	0.78544
MSE2	0.20728	1.67598	0.54967	0.12122	1.909517
$\hat{\sigma}$	1.92024	6.70869	18.86398	0.07520	4.67605
$\hat{\theta}$	4.93235	2.94664	12.96974	0.53773	4.56897
		N=100			
MSE1	0.22587	2.39933	0.53002	0.00529	0.79180
MSE2	0.40159	1.40631	0.50191	0.03014	1.56879
$\hat{\sigma}$	1.92116	6.99192	18.90658	0.08075	4.89462
$\hat{\theta}$	4.91285	3.11305	12.93325	0.50350	4.96854
		N=200			
MSE1	0.22372	0.96272	0.41254	0.00208	1.41657
MSE2	0.05541	0.53115	0.50468	0.04566	0.74738
$\hat{\sigma}$	1.94561	6.85442	18.91327	0.09494	4.83748
$\hat{\theta}$	4.99978	2.94426	12.93074	0.52102	4.97364
		N=500			
MSE1	0.13814	0.99659	0.13544	0.00288	1.30531
MSE2	0.16281	0.62720	0.15705	0.01252	0.88236
$\hat{\sigma}$	1.96107	7.10561	18.97481	0.08087	4.89864
$\hat{\theta}$	4.98047	3.12999	12.98116	0.45952	5.04560
		N=1000			
MSE1	0.03645	0.23258	0.14443	0.00022	0.85197
MSE2	0.01337	0.13205	0.20389	0.00142	0.34684
$\hat{\sigma}$	1.99123	6.97853	18.96817	0.09974	4.92507
$\hat{\theta}$	4.99906	2.997881	12.97179	0.50711	5.02179

Table 2: MSEs for the estimated parameters of the LR distribution for several values

Table 2 shows that the estimates are close to true values for a large sample size. As seen in Table 2, MSE values decrease if n increases.

7. Application

Now, a real data set is used to compare LR distribution with other distributions presented in Table 3.

Distribution	**Abb.**	**References**
Lindley Rayleigh	LR	Proposed
Exponentiated Weibull	EW	Mudholkar and Srivastava (1993)
Kumaraswamy-Generalized Exponentiated Exponential	KGEE	Mohammed (2014)
Kumaraswamy Fréchet	KF	Nead and Abd-Eltawab (2014)
Beta Fréchet	BF	Nadarajah and Gupta (2004)
Exponentiated Fréchet	EF	Nadarajah and Kotz (2003)
Exponentiated Exponential	EE	Gupta and Kundu (1999)
Kumaraswamy inverse exponential	KIE	Oguntunde et al. (2014)
Zografos-Balakrishnan log-logistic	ZBLL	Zografos and Balakrishnan (2009)
Kumaraswamy Pareto	KP	Bourguignon et al., 2013)
Kumaraswamy inverse Rayleigh	KIR	Roges et al. (2014)
Lindley	L	Lindley (1958)
Exponential	E	
Fréchet	F	

Table 3: Fitted distributions and their abbreviations

We use one hundred observations on breaking stress of carbon fibres (in Gba) in Nichols and Padgett (2006). The observations are as follows:

3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

The parameter estimates are obtained using the maximum likelihood method. Then, we present comparison criteria values: Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC) and Bayesian Information Criterion (BIC). The smaller are the values of these statistics, the better is the fit to the data. Furthermore, we give results of Kolmogorov-Smirnov (K-S) statistics for the best three models.

Distribution	Estimated Parameters				AIC	CAIC	BIC
$LR(\theta,\sigma)$	0.075738	0.3943670	-	-	286.729	286.852	291.939
$EW(\alpha,\lambda,\gamma)$	1.24278	2.31161	2.76753	-	289.913	290.163	297.728
$KGEE(\lambda,\alpha,a,b)$	0.110961	4.95024	0.660032	66.246	290.637	291.058	301.057
$KF(a,b,\theta,\beta)$	6.76357	904.3435	2.90998	0.38843	292.926	291.035	303.347
$BF(a,b,\theta,\beta)$	0.42934	138.0664	34.38484	0.72474	293.733	291.842	304.154
$EF(b,\theta,\beta)$	52.0491	26.173	0.6181	-	296.174	294.755	303.989
$EE(\lambda,\alpha)$	1.01317	7.78824	-	-	296.365	298.474	301.575
$KIE(a,b,\theta)$	2.34352	9.06108	2.6446	-	308.482	307.064	316.298
$ZBLL(a,\theta,\beta)$	1.55009	1.90903	3.61259	-	331.826	330.408	339.642
$KP(a,b,\theta,\beta)$	4.69523	236.2335	0.39	0.19204	339.502	338.084	347.318
$KF(\theta,\beta)$	1.89156	1.76902	-	-	350.288	349.342	355.498
$KIR(a,b,\theta)$	1.35232	1.13974	1.62816	-	355.681	354.263	363.497
$L(\theta)$	0.617342	-	-	-	365.507	366.561	368.112
$E(\lambda)$	0.381476	-	-	-	394.742	395.796	397.347

Table 4: MLEs and AIC, CAIC and BIC statistics

Distribution	K-S Statistics
$LR(\theta,\sigma)$	D = 0.062338, p-value = 0.8319
$EW(\alpha,\lambda,\gamma)$	D = 0.082516, p-value = 0.5038
$KGEE(\lambda,\alpha,a,b)$	D = 0.064496, p-value = 0.7998

Table 5: K-S statistics for the best three models

Tables 4 and 5 show that LR distribution is a good fit and can be the considered best model for this data set. We provide more information from a histogram of the data given in Figure4 with fitted lines which are the best three models: LR, EW and KGEE. In Figure 5, the plots of empirical cdf with cdf of LR are presented.

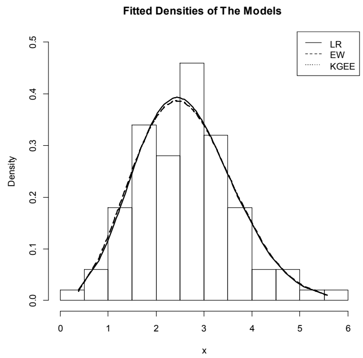

Figure 4: Fitted densities of LR, EW and KGEE distributions

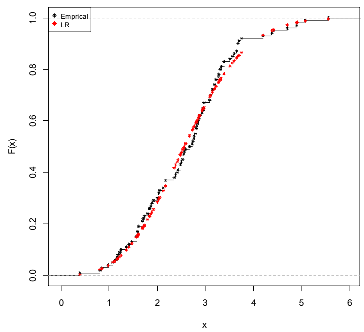

Figure 5: Plot of empirical cdf and cdf for LR distribution

7. Conclusion

In this study, we introduce the Lindley Rayleigh distribution as a lifetime distribution with two parameters by extending the Rayleigh distribution. We provide properties of this distribution including moments, moment generating function, reliability, and order statistics. The maximum likelihood method is used for parameter estimation of the LR distribution. A simulation study and a real data set are used to show performance distribution of the LR distribution. The results present that the LR distribution provide better fits than existing distributions.

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Appendix
Application Code

```
lr_pdf<- function(par,x){
sigma = par[1]
t = par[2]
f=x*exp(-x^2/(2*sigma^2))/sigma^2
F=1-exp(-x^2/(2*sigma^2))
g=t^2/(t+1)*f*(1-F)^(t-1)*F*(1-log(1-F))
return(g)
}
lr_cdf<- function(par,x){
sigma = par[1]
t = par[2]
f=x*exp(-x^2/(2*sigma^2))/sigma^2
F=1-exp(-x^2/(2*sigma^2))
G=1-(t+1-t*log(1-F))*(1-F)^t/(t+1)
return(G)
}
goodness.fit(pdf=lr_pdf,cdf=lr_cdf, starts=c(1,1),data=x, method="C",
domain=c(0,Inf),mle=NULL)
```

Generate LR code

```
j=1
while (j<5001){
for(i in 1:5){
m=matrix(0,5000,2)
N=c(50,100,200,500,1000)
u=runif(N[i])
sigma=7
teta=3
z=(u-1)*(teta+1)*exp(-teta-1)
lamda=1-exp((lambertW_base(z, b = -1, maxiter = 1000, eps = .Machine$double.eps,
min.imag =1e-09)+(teta+1))/teta)
x_lr=(-2*sigma^2*log(1-lamda))^0.5
s=goodness.fit(pdf=lr_pdf,cdf = lr_cdf, starts=c(1,1),data=x_lr, method="C",
domain=c(0,Inf),mle=NULL)
m[j,]=s$mle
j=j+1
real.par=c(sigma,teta)
mse1[i]=sum((m[1:j,1]-real.par[1])^2)/j
mse2[i]=sum((m[1:j,2]-real.par[2])^2)/j
est.par[i]=c(mean(m[,1]),mean(m[,2]))

}
}
```