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STATISTICAL PROPERTIES AND ESTIMATION PROCEDURE FOR TRANSMUTED INVERTED EXPONENTIAL DISTRIBUTION: APPLICATION TO BLADDER CANCER DATA

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Abstract

In survival analysis, the inverted family of distributions is frequently used to analyze the data with non-monotone failure rate pattern. In this paper, the inverted exponential distribution called as transmuted inverted exponential distribution (TIED) is taken under study. The considered distribution is more flexible and admitting several attractive properties. Further, different estimation procedures have also been discussed to estimate the parameters of TIED and compared through the Monte Carlo simulation study. Also, the applicability of the TIED is shown for a bladder cancer data set.

Key Words: Transmuted Inverted Exponential Distribution, Inverse Moments, Generating Function And Different Methods Estimation.

AMS Subject Classifications: 60E05; 62F10; 62F15

1. Introduction

In reliability analysis, generalized exponential, gamma, and Weibull distributions are very popular lifetime distributions to analyze the survival data with monotone failure rate pattern. These generalized models are obtained by considering exponential distribution as a baseline model; thus these models also show enough flexibility to study the data with a constant failure rate. But sometimes these models become less advantageous to examine the characteristics of data which shows non-monotone failure rate behavior. For example, we may notice that the study of mortality associated with a particular decease is initially increasing, reaching a peak and then declining slowly. Here, the associated failure rate is non-monotone failure rate. Thus, in such situation, inverted exponential distribution (IED) seems to be the best choice to study the behavior of such data.

Inverted exponential distribution was proposed by Lin et al. (1989). The estimation of the parameter and reliability function for IED with censored information under Bayes paradigm is discussed by Singh et al. (2013). The hybrid censored form of inverted exponential distribution has been studied by Pundir et al. (2014). Singh et al. (2014) used IED for the estimation of parameter and reliability characteristics under the progressive type-II censoring scheme. Recently Singh et al. (2015) have used the same model as a stress-strength model and estimates its parameter. Oguntunde and Adejumo (2015) proposed a new probability distribution by introducing transmuted parameter by considering the IED as the baseline model. This is called as transmuted inverted exponential distribution (TIED). The cumulative distribution function (CDF) and probability density function (PDF) of transmuted inverted exponential distribution (TIED) are given by

$$G(x) = e^{-\frac{\alpha}{x}}[1+\beta(1-e^{-\frac{\alpha}{x}})], \quad x \geq 0, \ \alpha > 0, \ |\beta| \leq 1 \tag{1.1}$$

and

$$g(x) = \frac{\alpha}{x^2} e^{-\frac{\alpha}{x}}[1+\beta(1-2e^{-\frac{\alpha}{x}})], \quad x \geq 0, \alpha > 0, \ |\beta| \leq 1 \tag{1.2}$$

The above distribution is reduced to baseline distribution for $\beta = 0$. The hazard function of TIED is given as;

$$H(x) = \frac{\frac{\alpha}{x^2} e^{-\frac{\alpha}{x}}[1+\beta(1-2e^{-\frac{\alpha}{x}})]}{1-e^{-\frac{\alpha}{x}}[1+\beta(1-e^{-\frac{\alpha}{x}})]} \tag{1.3}$$

Recently, [20] discussed more flexibility of TIED using two real data sets but still some issues remain unexplored. Therefore, this paper discusses the different statistical properties and estimation method to estimate unknown parameters of this model. Several lifetime distribution have been generalized using transmuted method; see Elbatal (2013), Aryal and Tsokos (2011), Khan and King (2013), Merovci (2013a), Merovci (2013b) etc. The different shapes of the CDF and PDF for the TIED are presented in Figure 1.

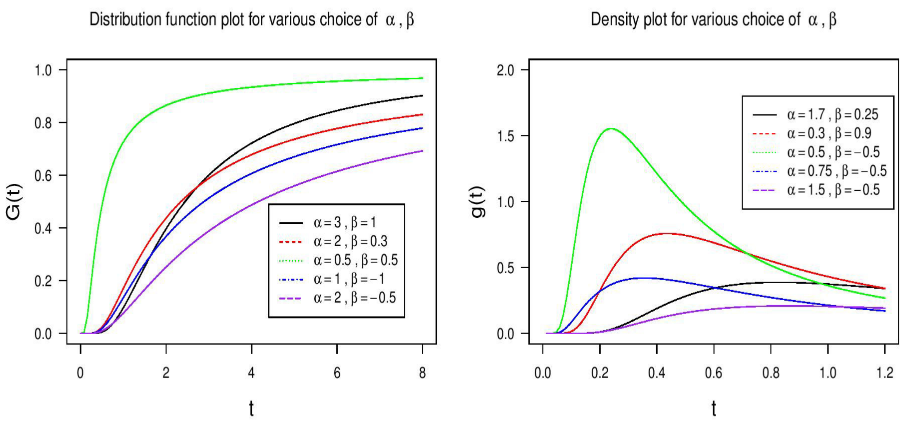


Figure 1: CDF and PDF plots for different choice of the parameters

The primary objective of this study is to discuss some statistical properties of TIED and then proposed different estimation procedures to estimate the unknown parameters of TIED. To the best of our knowledge, the proposed study has not been considered yet. Therefore, the present article is designed to fill up this gap. The organization of the paper is as follows. The introduction to the supposed problem is given in Section 1. Some statistical properties are discussed in Section 2. Order Statistics based on the considered model has been derived and presented in Section 3. Different Estimation of the parameters are considered in Section 4. The behavior of the above-said estimators is compared through simulation study in Section 5. A set of real data has been found to illustrate the applicability of the considered distribution in Section 6. The conclusions of the paper are given in Section 7.

2. Statistical Properties

This Section described the several mathematical and statistical properties of the $TIED\ (\alpha, \beta)$.

2.1 Reverse Hazard Function

The reverse hazard function for the TIED is obtained in the following equation.

$$\eta(x) = \frac{\alpha}{x^2} \tag{2.1}$$

From the above expression, it is clear that the hazard function is same as an inverted exponential distribution for $\beta = 0$. The shapes of the reliability and hazard function of transmuted inverted exponential distribution are given in following Figure 2.

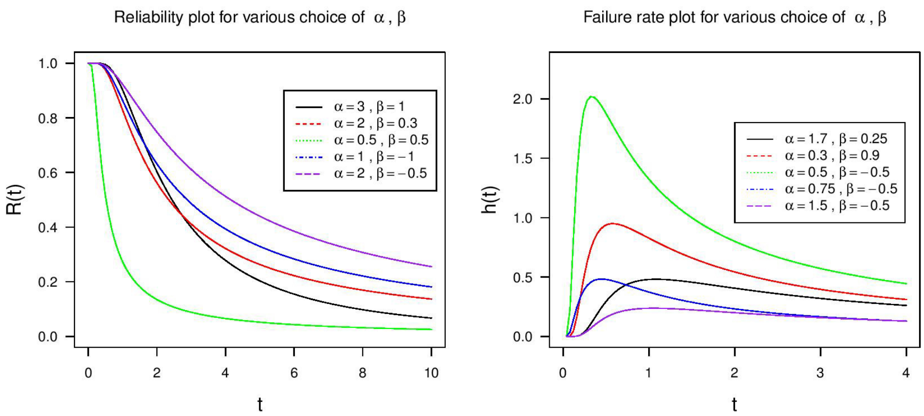


Figure 2: Reliability function and hazard function plots.

2.2 Inverse Moments

In baseline distribution, the first moment, second moment and other higher-order moments do not exist, and this is obvious in several inverted families of distributions. Therefore, here we consider inverse moment for $TIED(\alpha, \beta)$. The implementation of inverse moment theory may be useful to generate the inverse moments. The theory as well as the imporatnce of inverse moment was considered by Zakkula (1962).

Let r. v. X observed from $TIED(\alpha,\beta)$ then the inverse moment is defined as

$$\mu_r^{-1} = E\left(\frac{1}{x^r}\right) \tag{2.2}$$

Thus, after simplification of the above expression we get,

$$\mu_r^{-1} = \int_{x=0}^{\infty} \frac{\alpha e^{-\frac{\alpha}{x}}}{x^{2+r}} e^{-\frac{\alpha}{x}}[1+\beta(1-2e^{-\frac{\alpha}{x}})]dx = \frac{\Gamma(1+r)}{\alpha^r}\left[(1+\beta)-\frac{\beta}{2^r}\right] \tag{2.3}$$

where, $r = 0,1,2,3 \,\&\, 4$

2.3 Generating Functions

2.3.1 Inverse moment generating function

The inverse moment generating function (IMGF) of transmuted inverted exponential distribution is derived as;

$$\begin{aligned} IM_{X^{-1}}(t) &= E\left(e^{t/x}\right) \\ &= \int_{x=0}^{\infty} \frac{\alpha}{x^2} e^{-\frac{(\alpha-t)}{x}}[1+\beta(1-2e^{-\frac{\alpha}{x}})]dx \\ &= \frac{2\alpha^2+\alpha t(\beta-1)}{2\alpha^2-3\alpha t+t^2} \quad ;\alpha > t \,\&\, \beta \neq 1 \end{aligned} \tag{2.4}$$

From IMGF, the first four inverse moments can be simply obtained.

2.3.2 Inverse cumulants generating function

Inverse cumulants generating function $K_{X^{-1}}(t)$ is defined as;

$$K_{X^{-1}}(t) = \ln IM_{X^{-1}}(t) = \ln\left(\frac{2\alpha^2+\alpha t(\beta-1)}{2\alpha^2-3\alpha t+t^2}\right) \tag{2.5}$$

2.3.3 Inverse characteristics function

Also, the characteristics function of TIED is simply obtained by replacing t by it. Therefore, we have

$$\begin{aligned} \Phi_{X^{-1}}(t) &= E\left(e^{it/x}\right) \\ &= \frac{2\alpha^2+\alpha it(\beta-1)}{2\alpha^2-3i\alpha t-t^2} \quad ;\alpha > t\,, \beta \neq 1 \;\&\; i^2 = -1 \end{aligned} \tag{2.6}$$

2.4 Renyi Entropy

The amount of uncertainity of a r. v. X has been measured by entropy. The detailed description related to the entropy measurements are mentioned by Reniyi (1961). Rényi entropy is given by

$$\zeta(\mu)=\frac{1}{1-\mu}\ln\left\{\int_x g^{\mu}(x,\alpha,\beta)dx\right\} \tag{2.7}$$

where, $\mu>0$ and $\mu\neq 1$. For the inverted exponential distribution we obtained that

$$\zeta(\mu)=\frac{1}{1-\mu}\ln\left\{\int_x \frac{\alpha^{\mu}e^{-\frac{\alpha\mu}{x}}}{x^{2\mu}}\{1+\beta(1-2e^{-\frac{\alpha}{x}})\}^{\mu}dx\right\} \tag{2.8}$$

After simplification we get;

$$\zeta(\mu)=\frac{1}{1-\mu}\ln\left[\sum_{k=0}^{\infty}\frac{(\mu)_k}{k!}\alpha^{3\mu-1}(1+\beta)^{\mu}\left(\frac{-2\beta}{1+\beta}\right)^{k}\frac{\Gamma(2\mu-1)}{(1+\mu)^{2\mu-1}}\right] \tag{2.9}$$

where, $(\mu)_k=\mu(\mu-1)(\mu-2)\cdots(\mu-k+1)$ is called as Pochhamer symbol.

2.5 β - Entropy

β-entropy for the transmuted inverted exponential distribution is obtained by

$$\beta_{ent}=\frac{1}{\beta-1}\left[1-\int_{x=0}^{\infty}g^{\beta}(x,\alpha,\beta)dx\right] \tag{2.10}$$

Using pdf (1.2) and after simplification the expression for β-entropy is given by;

$$\beta_{ent}=\frac{1}{\beta-1}\left\{1-\left[\sum_{k=0}^{\infty}\frac{(\mu)_k}{k!}\alpha^{3\mu-1}(1+\beta)^{\mu}\left(\frac{-2\beta}{1+\beta}\right)^{k}\frac{\Gamma(2\mu-1)}{(1+\mu)^{2\mu-1}}\right]\right\} \tag{2.11}$$

2.6 Stochastic Ordering

The concept of stochastic ordering has been discussed by Mann and Whitney (1947) to compare two random variables. A random variable X is said to be stochastically greater $(Y\leq_{st}X)$ than Y if $F_X(x)\leq F_Y(x)$ for all x. Also, a random variable X is said to be stochastically greater than Y in the

- stochastic order $(X\leq_{st}Y)$ if $F_X(x)\geq F_Y(x)$ for all x.
- hazard rate order $(X\leq_{hr}Y)$ if $h_X(x)\geq h_Y(x)$ for all x.
- mean residual life order $(X\leq_{mrl}Y)$ if $m_X(x)\geq m_Y(x)$ for all x.
- likelihood ratio order $(X\leq_{lr}Y)$ if $\left(\frac{f_X(x)}{f_Y(x)}\right)$ decreases in x.

From the above relations, we analyzed that;

$$(X \le_{lr} Y) \Rightarrow (X \le_{hr} Y) \Downarrow (X \le_{st} Y) \Rightarrow (X \le_{mrl} Y)$$

The TIED is ordered with respect to the likelihood ratio test as narrate in the following lemma.

Lemma: Let $X : f_w(\alpha_1, \theta_1)$ and $Y : f_w(\alpha_2, \theta_2)$. If $\alpha_1 > \alpha_2$, then $(X \le_{lr} Y)$ and hence $(X \le_{hr} Y)$, $(X \le_{mrl} Y)$ and $(X \le_{st} Y)$.

Proof: According to the definition of the likelihood ratio order, first we obtain the ratio

$$\left[\frac{f_X(x)}{f_Y(x)}\right] \text{ i.e. } \psi = \frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1[1+\beta_1(1-2e^{-\frac{\alpha_1}{x}})]e^{-\frac{1}{x}(\alpha_1-\alpha_2)}}{\alpha_2[1+\beta_2(1-2e^{-\frac{\alpha_2}{x}})]}$$

Therefore,

$$\psi' = \frac{d\psi}{dx} = \left[\frac{(\alpha_1+\alpha_2)}{x^2} + \frac{1}{x^2}\left\{\frac{2\beta_2\alpha_2 e^{-\alpha_2/x}}{[1+\beta_2(1-2e^{-\frac{\alpha_2}{x}})]} - \frac{2\beta_1\alpha_1 e^{-\alpha_1/x}}{[1+\beta_1(1-2e^{-\frac{\alpha_1}{x}})]}\right\}\right]\frac{f_X(x)}{f_Y(x)} \tag{2.12}$$

from equation no (2.13), it is clear that, $\psi' > 0 \quad \forall \ \alpha_i, \beta_i$, hence $(X \le_{lr} Y)$. The remaining ordering behaviour can be established by proceeding same way.

2.7 System Reliability (SR)

Let $X : (\alpha_1, \beta_1)$ and $Y : (\alpha_2, \beta_2)$ are the two independent strength-stress random variables observed from TIED respectively. The SR is the measure the system performances under certain stress. Mathematcally, it is obtained as $P[Y < X]$ is calculated by;

$$SR = P[Y<X] = \int_x g(x,\alpha_1,\beta_1)G_Y(x,\alpha_2,\beta_2)dx$$

$$= \int_{x=0}^{\infty}\frac{\alpha_1}{x^2}e^{-\frac{\alpha_1}{x}}[1+\beta_1(1-2e^{-\frac{\alpha_1}{x}})]e^{-\frac{\alpha_2}{x}}[1+\beta_2(1-e^{-\frac{\alpha_2}{x}})]dx \tag{2.13}$$

After simplification, it is obtained as

$$SR = \frac{\alpha_1+\alpha_1\beta_1+\alpha_2\beta_2+2\alpha_1\beta_1\beta_2}{\alpha_1+\alpha_2} - \frac{\alpha_1\beta_2(1+\beta_1)}{\alpha_1+2\alpha_2} - \frac{2\alpha_1\beta_1(1+\beta_2)}{2\alpha_1+\alpha_2} \tag{2.14}$$

The system reliability for different choices of the parameters is obtained as follows;

- If $\beta_i = 0 \ ; i = 1,2$ then we have system reliability for the base line model i.e.

$$SR = \frac{\alpha_1}{\alpha_1+\alpha_2} \tag{2.15}$$

• If transmuted parameters $\beta_i's$ are equal say $\beta_1 = \beta_2 = \beta(say)$ but scale is different i.e.

$X : TIED \ (\alpha_1, \beta), \quad Y : TIED \ (\alpha_2, \beta)$

Then

$$SR = \frac{\alpha_1 + \alpha_1\beta + \alpha_2\beta + 2\alpha_1\beta^2}{\alpha_1 + \alpha_2} - \frac{\alpha_1\beta(1+\beta)}{\alpha_1 + 2\alpha_2} - \frac{2\alpha_1\beta(1+\beta)}{2\alpha_1 + \alpha_2} \quad (2.16)$$

• If scale parameters are same $\alpha_1 = \alpha_2 = \alpha(say)$ and transmuted parameters β_i is different i.e.

$X : TIED \ (\alpha, \beta_1), \quad Y : TIED \ (\alpha, \beta_2)$

Then

$$SR = \frac{3 - \beta_1 + 5\beta_2 + 4\beta_1\beta_2}{6} \quad (2.17)$$

3. Order Statistics

In life testing experiment numerous application of order statistics have been found in the literature. Let $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$ are the n ordered random sample observed from (1.2). The different order statistics is defined as follows;

$$x_{(1)} = min[x_{(1)}, x_{(2)}, \cdots, x_{(r)}, \cdots x_{(n)}]$$
$$x_{(n)} = max[x_{(1)}, x_{(2)}, \cdots, x_{(r)}, \cdots x_{(n)}]$$

$x_{(r)}^{th}$ is stand for rth order statistics. Then the PDF of r^{th} order statistics is given by

$$f_r(x_{(r)}, \alpha, \beta) = \frac{n!}{(r)!(n-r)!}\left[F(x_{(r)})\right]^{r-1}\left[1 - F(x_{(r)})\right]^{n-r} f(x_{(r)}, \alpha, \beta) \quad (3.1)$$

After using the expression (1.1) and (1.2) for $x_{(r)}$ in above equation we get the expression for r^{th} order statistics. i.e.

$$f_r(x_{(r)}, \alpha, \beta) = \frac{n!}{(r)!(n-r)!}\frac{\alpha}{x_{(r)}^2}\left[e^{-\frac{\alpha}{x_{(r)}}}[1+\beta(1-e^{-\frac{\alpha}{x_{(r)}}})]\right]^r\left[1-e^{-\frac{\alpha}{x_{(r)}}}[1+\beta(1-e^{-\frac{\alpha}{x_{(r)}}})]\right]^{n-r}$$

(3.2)

3.1 Distribution of Median

The PDF of minimum $(x_{(1)})$, maximum $(x_{(n)})$ order statistics are obtained by putting $r = 1$ & $r = n$ in eqn (3.1). Now, the density function for sample median for order statistics is given by $x_{(m+1:n)}$. It is computed by

$$f_{m+1}(x_{(m+1:n)},\alpha,\beta) = \frac{(2m+1)!}{(m)!(m)!}\left[F(\tilde{x}_{m+1})\right]^m\left[1-F(\tilde{x}_{m+1})\right]^m f(\tilde{x}_{m+1},\alpha,\beta)$$
$$= \frac{(2m+1)!}{(m)!(m)!}\left(\frac{\alpha}{\tilde{x}_{m+1}^2}\right)\left[e^{-\frac{\alpha}{\tilde{x}_{m+1}}}[1+\beta(1-e^{-\frac{\alpha}{\tilde{x}_{m+1}}})]\right]^{m+1}$$
$$\times\left[1-e^{-\frac{\alpha}{\tilde{x}_{m+1}}}[1+\beta(1-e^{-\frac{\alpha}{\tilde{x}_{m+1}}})]\right]^m \quad (3.3)$$

3.2 Joint Distribution of r^{th} and s^{th} Order Statistics

The joint density function of r^{th} and s^{th} $(r<s)$ order statistics is obtained by considering the following expression;

$$f_{r:s:n}(x_r,x_s,\alpha,\beta)=\xi\left[F(x_{(r)})\right]^{r-1}\left[1-F(x_{(s)})\right]^{n-s}\left[F(x_{(s)})-F(x_{(r)})\right]^{s-r-1}f(x_{(r)})f(x_{(s)}) \quad (3.4)$$

Using eqn (1.1) and (1.2) the density function for r^{th},s^{th} is given by

$$f_{r:s:n}(x_r,x_s,\alpha,\beta)=\xi\left[e^{-\frac{\alpha}{x_{(r)}}}[1+\beta(1-e^{-\frac{\alpha}{x_{(r)}}})]\right]^r\left[1-e^{-\frac{\alpha}{x_{(s)}}}[1+\beta(1-e^{-\frac{\alpha}{x_{(s)}}})]\right]^{n-s}$$
$$\times\left[e^{-\frac{\alpha}{x_{(s)}}}[1+\beta(1-e^{-\frac{\alpha}{x_{(s)}}})]-e^{-\frac{\alpha}{x_{(r)}}}[1+\beta(1-e^{-\frac{\alpha}{x_{(r)}}})]\right]^{s-r-1}$$
$$\times\left(\frac{\alpha}{x_{(r)}x_{(s)}}\right)^2 e^{-\frac{\alpha}{x_{(s)}}}[1+\beta(1-e^{-\frac{\alpha}{x_{(s)}}})]$$

In particular, when $r=1$ and $s=n$, we get the joint distribution of minimum $(x_{(1)})$, maximum $(x_{(n)})$ order statistics and is written as;

$$
\begin{aligned}
f_{1:n}(x_1, x_n, \alpha, \beta) &= n(n-1)\left[e^{-\frac{\alpha}{x_{(r)}}}[1+\beta(1-e^{-\frac{\alpha}{x_{(r)}}})]\right]\left(\frac{\alpha}{x_{(1)}x_{(n)}}\right)^2 \\
&\times\left[e^{-\frac{\alpha}{x_{(n)}}}[1+\beta(1-e^{-\frac{\alpha}{x_{(n)}}})]-e^{-\frac{\alpha}{x_{(1)}}}[1+\beta(1-e^{-\frac{\alpha}{x_{(1)}}})]\right]^{n-2} \\
&\times\left[e^{-\frac{\alpha}{x_{(s)}}}[1+\beta(1-e^{-\frac{\alpha}{x_{(n)}}})]\right]
\end{aligned}
\tag{3.5}
$$

where, $\xi = \dfrac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}$

3.3 Distribution of Range R

Let $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$ be the n ordered sample observed from density (1.1). The range is defined as;

$$R = x_{(n)} - x_{(1)}$$

Now, we derived the distribution of R using transformation method. Let us assume that $R_1 = x_{(1)}$, then $R + R_1 = x_{(n)}$. The Jacobian of the transformation is;

$$
J = \frac{\partial(x_{(1)}, x_{(n)})}{\partial(R, R_1)} = \begin{pmatrix} \frac{\partial x_{(1)}}{\partial R} & \frac{\partial x_{(n)}}{\partial R} \\ \frac{\partial x_{(1)}}{\partial R_1} & \frac{\partial x_{(n)}}{\partial R_1} \end{pmatrix} = 1 \tag{3.6}
$$

Thus, joint distribution of R and R_1 is obtained by using (3.7)

$$
\begin{aligned}
f(R, R_1) &= n(n-1)\left[e^{-\frac{\alpha}{R_1}}[1+\beta(1-e^{-\frac{\alpha}{R_1}})]\right]\left(\frac{\alpha}{(R+R_1)R_1}\right)^2 \\
&\times\left[e^{-\frac{\alpha}{R+R_1}}[1+\beta(1-e^{-\frac{\alpha}{R+R_1}})]-e^{-\frac{\alpha}{R_1}}[1+\beta(1-e^{-\frac{\alpha}{R_1}})]\right]^{n-2} \\
&\times\left[e^{-\frac{\alpha}{R+R_1}}[1+\beta(1-e^{-\frac{\alpha}{R+R_1}})]\right]
\end{aligned}
\tag{3.7}
$$

Hence, the distribution of R is obtained by solving the following integral

$$f(R) = \int_{R_1} f(R, R_1)\, dR_1 \tag{3.8}$$

4. Different Method of Estimation

This section discusses the different methods of estimation namely, maximum likelihood estimation (MLE), maximum product spacing estimation (MPSE) and least square estimation (LSE). The brief explanation of these methods is given below.

4.1 Maximum Likelihood Estimation

Let $x_1, x_2, \cdots, x_n$ are the random sample from (1.2). The likelihood function is written as

$$L(\alpha, \beta, x) = \alpha^n e^{-\sum_{i=1}^{n} \frac{\alpha}{x_i}} \prod_{i=1}^{n} \frac{[1 + \beta(1 - 2e^{-\frac{\alpha}{x_i}})]}{x_i^2} \tag{4.1}$$

Log-likelihood function is given by;

$$\ln L = n \ln \alpha - \alpha \sum_{i=1}^{n} \frac{1}{x_i} + \sum_{i=1}^{n} \ln[1 + \beta(1 - 2e^{-\frac{\alpha}{x_i}})] - 2\sum_{i=1}^{n} \ln x_i \tag{4.2}$$

The MLE's of the parameter α and β are obtained by solving following two simultaneous equations.

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \frac{1}{x_i} + \sum_{i=1}^{n} \frac{2\beta e^{-\frac{\alpha}{x_i}}}{x_i[1 + \beta(1 - e^{-\frac{\alpha}{x_i}})]} = 0 \tag{4.3}$$

and

$$\frac{\partial \ln L}{\partial \beta} = \sum_{i=1}^{n} \frac{1 - 2e^{-\frac{\alpha}{x_i}}}{[1 + \beta(1 - 2e^{-\frac{\alpha}{x_i}})]} = 0 \tag{4.4}$$

4.2 Maximum Product Spacing Estimation

Maximum product spacing estimation is a conventional and alternative method to MLE. Cheng and Amin (1983) initially introduced the MPS method, and Singh et al. (2014) have given the detailed description of the MPS method. Thus, for implementing this method, the uniform spacings based on two consecutive cdfs is defined as follows: "

$$D_i = F(x_i) - F(x_{(i-1)}) = e^{-\frac{\alpha}{x_i}}[1 + \beta(1 - e^{-\frac{\alpha}{x_i}})] - e^{-\frac{\alpha}{x_{i-1}}}[1 + \beta(1 - e^{-\frac{\alpha}{x_{i-1}}})] \tag{4.5}$$

such that $\sum D_i = 1$,

In MPS method, we choose that (α, β) which maximizes the product of spacings;

$$G = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i} \tag{4.6}$$

Taking the logarithm of G we get,

$$\begin{aligned}\ln G &= \frac{1}{n+1}\sum_{i=1}^{n+1}\ln D_i \\ &= \frac{1}{(n+1)}\left\{\ln\left[e^{-\frac{\alpha}{x_1}}[1+\beta(1-e^{-\frac{\alpha}{x_1}})]\right] + \ln e^{-\frac{\alpha}{x_n}}[1+\beta(1-e^{-\frac{\alpha}{x_n}})]\right\} \\ &+ \frac{1}{1+n}\sum_{i=2}^{n}\ln\left[e^{-\frac{\alpha}{x_i}}[1+\beta(1-e^{-\frac{\alpha}{x_i}})] - e^{-\frac{\alpha}{x_{i-1}}}[1+\beta(1-e^{-\frac{\alpha}{x_{i-1}}})]\right]\end{aligned} \tag{4.7}$$

The MPSE parameters are obtained by solving following simultenious equations;

$$\begin{aligned}\frac{\partial \ln G}{\partial \alpha} &= \frac{1}{n+1}\left[\frac{2\beta e^{-\frac{\alpha}{x_1}} - \beta - 1}{x_1[1+\beta(1-e^{-\frac{\alpha}{x_1}})]} + \frac{[2\beta e^{-\frac{\alpha}{x_n}} + \beta + 1]e^{-\frac{\alpha}{x_n}}}{x_n[1-e^{-\frac{\alpha}{x_n}}[1+\beta(1-e^{\frac{\alpha}{x_n}})]]}\right] \\ &+ \frac{1}{n+1}\sum_{i=2}^{n}\frac{[2\beta e^{-\frac{\alpha}{x_i}} - \beta - 1]\dfrac{e^{-\frac{\alpha}{x_i}}}{x_i} - [2\beta e^{-\frac{\alpha}{x_{i-1}}} - \beta - 1]\dfrac{e^{-\frac{\alpha}{x_{i-1}}}}{x_{i-1}}}{e^{-\frac{\alpha}{x_i}}[1+\beta(1-e^{\frac{\alpha}{x_i}})] - e^{-\frac{\alpha}{x_{i-1}}}[1+\beta(1-e^{\frac{\alpha}{x_{i-1}}})]} = 0\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\frac{\partial \ln G}{\partial \beta} &= \frac{1}{n+1}\left[\frac{e^{-\frac{\alpha}{x_1}}(1-e^{-\frac{\alpha}{x_1}})}{x_1[1+\beta(1-e^{-\frac{\alpha}{x_1}})]} + \frac{e^{-\frac{\alpha}{x_n}}(1-e^{-\frac{\alpha}{x_n}})}{x_n[1-e^{-\frac{\alpha}{x_n}}[1+\beta(1-e^{\frac{\alpha}{x_n}})]]}\right] \\ &+ \frac{1}{n+1}\sum_{i=2}^{n}\frac{[e^{-\frac{\alpha}{x_i}}(1-e^{-\frac{\alpha}{x_i}})] - [e^{-\frac{\alpha}{x_{i-1}}}(1-e^{-\frac{\alpha}{x_i-1}})]}{e^{-\frac{\alpha}{x_i}}[1+\beta(1-e^{\frac{\alpha}{x_i}})] - e^{-\frac{\alpha}{x_{i-1}}}[1+\beta(1-e^{\frac{\alpha}{x_{i-1}}})]} = 0\end{aligned} \tag{4.9}$$

The exlicit solution of the above normal equations are not possible due to its mathematical mess; therefore, the maximum product spacing estimates $(\hat{\alpha}_{MPS}, \hat{\beta}_{MPS})$ of the parameters (α, β) are obtained by using non-linear maximization techniques.

4.3 Least Square Estimation

Let $x_1 < x_2 < \cdots, < x_n$ be the n ordered random sample from TIED with CDF (1.1), Then, the expected empirical cumulative distribution function is defined as;

$$E(G(x_i)) = \frac{i}{(n+1)} \quad ; i = 1,2, \cdots, n \tag{4.10}$$

The least square estimates (LSE's) $\hat{\alpha}_{LSE}$ and $\hat{\beta}_{LSE}$ of α and β are obtained by minimizing

$$R^*(\alpha,\beta) = \sum_{i=1}^{n} \left(G(x_i) - \frac{i}{(n+1)} \right)^2 \tag{4.11}$$

Putting the cdf of TIED in the above equation, we get

$$R^*(\alpha,\beta) = \sum_{i=1}^{n} \left(e^{-\frac{\alpha}{x_i}} [1 + \beta(1 - e^{-\frac{\alpha}{x_i}})] - \frac{i}{(n+1)} \right)^2 \tag{4.12}$$

The least square estimates $\hat{\alpha}_{LSE}$ and $\hat{\alpha}_{LSE}$ of α and β can be obtained by minimising the above expression (4.12) w.r.t. the parameters.

5. Simulation Study

In this section, we compared the different estimation procedure for different variations of sample sizes (n), scale parameter (α) and transmuted parameter (β). To perform simulation study 5000 samples are generated from $TIED\ (\alpha,\beta)$ for different combination of $n[20,30\ \&\ 50]$, $\alpha[0.5,1.0,1\ .5]$ and $\beta[-0.65, 0.65]$. The average estimates and mean square error (MSEs) obtained using MLE, MPS and LSE methods, see Table 2-3. It has been easily seen that the MSEs of all the estimators decreases as the sample size increases for all choices of the parametric combination. However, the superiority of one method over others cannot be globally stated. After investigating the results, we observed that;

- When the transmuted parameter is positive $(\beta = 0.65)$, then the MSEs of the estimators have the following trend

$$MSE\ (\hat{\alpha}_{ML}, \hat{\beta}_{ML}) < MSE\ (\hat{\alpha}_{LS}, \hat{\beta}_{LS}) < MSE\ (\hat{\alpha}_{MP}, \hat{\beta}_{MP})$$

- When transmuted parameter is negative $(\beta = -0.65)$, then the MSEs of the estimators have following trend

$$MSE\ (\hat{\alpha}_{MP}) < MSE\ (\hat{\alpha}_{ML}) < MSE\ (\hat{\alpha}_{LS})$$

and

$$MSE\ (\hat{\beta}_{MP}) < MSE\ (\hat{\beta}_{LS}) < MSE\ (\hat{\beta}_{ML})$$

Thus from above, we see that for the positive value of transmuted parameter the MLE outperforms as compared to the rest of the considered methods while the MPS method provides efficient result when transmuted parameters are negative.

n	α	β	Estimate/MSE	MLE		MPS		LSE	
				alpha	beta	alpha	beta	alpha	beta
20	0.5	0.65	AE	0.484	0.467	0.392	0.120	0.434	0.307
			MSE	0.015	0.188	0.026	0.502	0.025	0.389
	1		AE	0.971	0.466	0.782	0.110	0.863	0.295
			MSE	0.058	0.191	0.103	0.516	0.093	0.396
	1.5		AE	1.944	0.487	1.568	0.127	1.729	0.319
			MSE	0.230	0.172	0.418	0.497	0.349	0.359
30	0.5	0.65	AE	0.490	0.537	0.418	0.260	0.455	0.427
			MSE	0.009	0.126	0.018	0.353	0.015	0.233
	1		AE	0.988	0.551	0.847	0.284	0.919	0.446
			MSE	0.039	0.124	0.068	0.327	0.060	0.238
	1.5		AE	1.967	0.540	1.689	0.278	1.825	0.432
			MSE	0.175	0.134	0.287	0.333	0.259	0.243
50	0.5	0.65	AE	0.497	0.589	0.449	0.408	0.477	0.531
			MSE	0.006	0.088	0.011	0.204	0.009	0.126
	1		AE	0.994	0.607	0.900	0.429	0.951	0.537
			MSE	0.025	0.077	0.040	0.181	0.036	0.128
	1.5		AE	1.997	0.607	1.812	0.434	1.908	0.538
			MSE	0.097	0.074	0.155	0.175	0.140	0.123

Table 1: Average estimates and MSEs of the estimators for positive values of transmuted parameter

n	α	β	Estimate/MSE	MLE		MPS		LSE	
				alpha	beta	alpha	beta	alpha	beta
20	0.5	-0.65	AE	0.603	-0.427	0.513	-0.552	0.577	-0.606
			MSE	0.037	0.110	0.018	0.027	0.044	0.066
	1		AE	1.206	-0.357	1.014	-0.515	1.114	-0.590
			MSE	0.129	0.200	0.063	0.066	0.151	0.088
	1.5		AE	2.421	-0.299	2.023	-0.472	2.296	-0.470
			MSE	0.497	0.278	0.227	0.108	0.539	0.178

30	0.5	-0.65	AE	0.588	-0.444	0.513	-0.560	0.557	-0.609
			MSE	0.028	0.109	0.013	0.032	0.028	0.067
	1		AE	1.160	-0.410	1.004	-0.546	1.068	-0.625
			MSE	0.090	0.163	0.044	0.058	0.102	0.095
	1.5		AE	2.353	-0.367	2.022	-0.518	2.265	-0.499
			MSE	0.411	0.216	0.195	0.086	0.427	0.168
50	0.5	-0.65	AE	0.569	-0.498	0.515	-0.585	0.540	-0.635
			MSE	0.022	0.090	0.010	0.031	0.020	0.070
	1		AE	1.122	-0.479	1.011	-0.577	1.048	-0.647
			MSE	0.069	0.116	0.036	0.047	0.076	0.093
	1.5		AE	2.269	-0.446	2.025	-0.561	2.174	-0.561
			MSE	0.297	0.148	0.152	0.064	0.297	0.130

Table 2: Average estimates and MSEs of the estimators for negative values of transmuted parameter

6. Application to the Cancer Data

In this section, we took a real data set to illustrate the practical utility of the TIED. The data set, which represents the remission times (in months) of bladder cancer patients reported by Lee and Wang (2003). The summary of the above data is

Minimum	First Quartile	Median	Mean	Third Quartile	Maximum
0.080	3.35	6.40	9.37	11.84	79.05

The idea about the underlying hazard rate for the considered data set TTT plot has been considered. For more detals about TTT plot, see Aarset [19]. The empirical TTT is given as

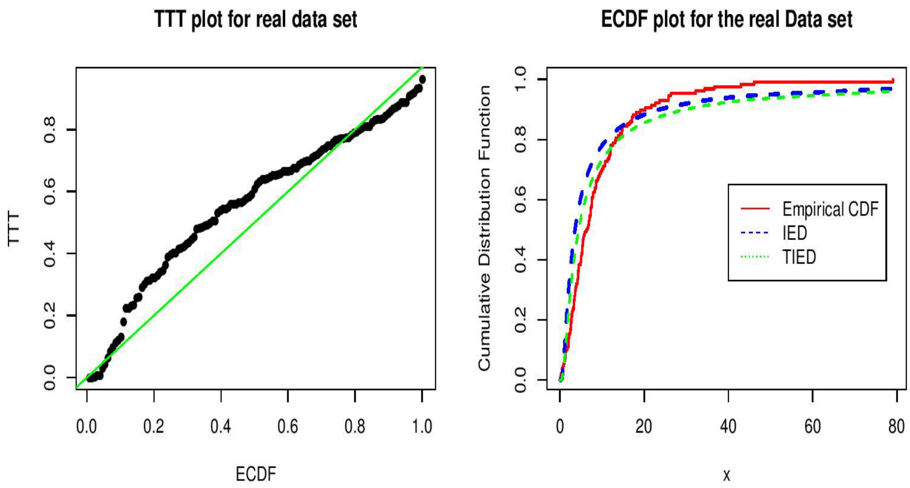
$$T_{emp} = \frac{\sum_{i=1}^{r} x_{(i)} + (n-r)x_{(r)}}{\sum_{i=1}^{n} x_{(i)}} \tag{6.1}$$

where $r = 1,2, \cdots, n$ and $x_{(r)}$ is the r^{th} order statistics. Figure 3 shows the TTT plot, which indicates that the data relates to a non-monotone hazard rate behavior. Thus, the considered data set is properly suited by TIED. Also, the validity of considered model over IED and GIED has been discussed using the relative measure such as Akaike information criterion (AIC), corrected Akaike information criterion (AICC), Bayesian information criterion (BIC), negative log-likelihood criterion (-LogL) and K-S test criterion. From Table 2, It is easy to observe that TIED has least AIC, AICC, BIC, -LogL and KS values. Thus it can be recommended to analyze such medical data which reflect the behavior of non-monotone failure rate. The empirical cumulative distribution function (ecdf) plot is also plotted and is revealed that the deviation between theoretical cdf of TIED with ecdf is less as compared to the baseline distribution. Hence, TIED can use as a good alternative model for the survival data. The real data estimates are also evaluated by using different estimation methods, see Table 1.

Parameter	MLE	MPS	LSE
α	1.677	1.61	6.125
β	-0.856	-0.854	0.650

Table 3: Real data estimates obtained by using different estimation method

Figure 3: Total time on test plot



and ecdf plot for real data

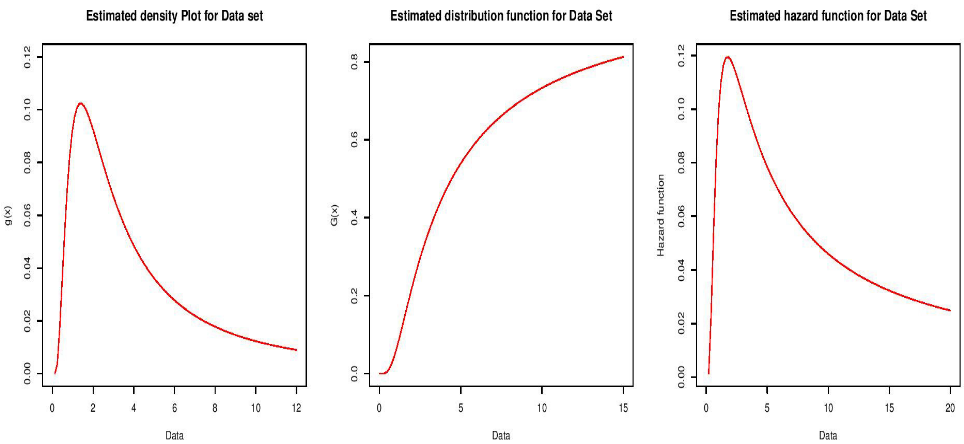


Figure 4: Estimated density, cdf and survival function plots

Model	MLE	-Log L	AIC	AICC	BIC	K-S
IED	2.485	460.382	922.765	925.617	920.796	0.224
GIED	(0.746, 1.995)	457.202	918.405	914.501	924.109	0.390
TIED	(1.678 -0.860)	442.776	889.553	885.649	895.257	0.151

Table 4: Values of different relative measures of data set-II

The estimated density, distribution and hazard function plots are also given for the considered real dataset, see Figure 4.

7. Conclusion

In this article, an extension of IED named as transmuted inverted exponential distribution has been considered. Different statistical properties such as inverse moment and inverse moment generating and characteristics function, entropy, stochastic ordering, stress-strength reliability are derived. Also, order statistics and related distributions such as the distribution of median, joint distribution of rth and sth order statistics and distribution of sample range have been computed for TIED. The model parameters are estimated by different estimation methods, namely MLE, MPSE and LSE. The Monte Carlo simulation study has been performed to compare the performances of the proposed estimation methods. The real-life application of the proposed study has been illustrated using the bladder cancer data set. The compatibility/superiority of the considered model is also shown among the inverted family of distributions using relative model selection tools. It is observed that TIED model can be used as alternative lifetime distribution to model the data with non-monotone failure rate.

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