Journal of Reliability and Statistical Studies; ISSN (Print): 0974-8024, (Online): 2229-5666 Vol. 11, Issue 2 (2018): 143-158

THE BURR XII UNIFORM DISTRIBUTION: THEORY AND APPLICATIONS

M. Arslan Nasir[1], Gamze Özel[2]* and Farrukh Jamal[3]
[1]Govt. Degree College Lodhran, Punjab, Pakistan
[2]Department of Statistics, Hacettepe University, 06800, Ankara, Turkey
[3]Department of Statistics, Govt. S. A. Postgraduate College, Pakistan
E Mail: arslannasir147@gmail.com; gamzeozl@hacettepe.edu.tr; farrukhjamalmphil@gmail.com
*Corresponding Author

Received January 05, 2018
Modified October 20, 2018
Accepted December 12, 2018

Abstract

In this study, we obtain a new flexible distribution named Burr XII Uniform distribution with several hazard rate shapes including decreasing, increasing and bathtub. Most mathematical properties of the Burr XII Uniform distribution are studied including infinite mixture representation, moments, moment generating function, quantile function, stochastic ordering, order statistics and entropy. Parameter estimations are given using the maximum likelihood approach. Applications are given to present the usage of the maximum likelihood estimates and proposed model.

Key Words: Burr XII Distribution, Quantile Function, Entropy,Estimation.

1. Introduction

The well-known statistical distributions generally do not fit the skewed and asymmetric data sets. Hence, as an improvement, many new families of distributions have been created and extended using the classical distributions. These new families have been used and yielded more flexibility in applications. The beta-generated (beta-G) family of distributions was studied by Eugene et al. (2002). Zografos and Balakrishnan (2009) derived a gamma-G family. In addition to the beta-G and gamma-G families of distributions, several other classes or families of generalized distributions have been introduced. These are the Kummer beta-G family by Pescim et al. (2012), exponentiated generalized-G family by Cordeiro et al. (2013), Weibull-G family by Bourguignon et al. (2014), Lomax-G family by Cordeiro et al. (2014), the type I half-logistic-G family by Cordeiro et al. (2016). Garhy-G family was introduced by Elgarhy et al. (2016). Then, Kumaraswamy Weibull-G family was given by Hassan and Elgarhy (2016a) and exponentiated Weibull-G family was defined by Hassan and Elgarhy (2016b). Type I Topp-Leone-G family was defined by Al-Shomrani et al. (2016). Cakmakyapan (2017) introduced the Lindley family of distributions and exponentiated extended-G family defined by Elgarhy et al. (2017). Then, Hassan et al. (2017)

introduced type II half logistic-G family. The odd Frèchet-G family and transmuted Weibull power function distribution introduced by Haq et al. (2018a) and (2018b), respectively.

Although there are many families of distributions in the literature, there is a clear need for flexible family of distributions. The uniform distribution is not flexible enough to analyzedata set related with reliability. So, new alternative distributions should beuseful. Therefore, we introduce a flexible lifetime distribution where the uniform distribution is the baseline.

Type I Topp-Leone-G family defined by Al-Shomrani et al. (2016). Cakmakyapan (2017) introduced the Lindley family of distributions and the exponentiated extended-G family defined by Elgarhy et al. (2017). Then, Hassan et al. (2017) introduced the type II half logistic-G family. Recently, the odd Frèchet-G family and transmuted Weibull power function distribution have been introduced by Haq et al. (2018 a) and (2018 b), respectively. Although there are many distributions in the literature, there is still need more flexible distributions in applied areas including lifetime analysis, finance and insurance.

The Burr type XII (BXII) distribution has been used in different fields due to its great flexibility for fitting data. These applications have typically involved data showing heavy-tailed behaviors. Further, an interesting feature of this distribution is its relation without well-known distributions as Lomax, compound Weibull, Weibull-exponential, logistic, log-logistic, Weibull distributions and the Kappa family of distributions which shows its wide flexibility for fitting different kinds of data. With respect to the applications, this distribution has been used in a variety of fields thanks to its coverage of a wide range of skewness and kurtosis values, for instance survival analysis, hydrology, environment and reliability. The cumulative distribution function (cdf) of generalized BXII distribution is defined as

$$F(x) = 1 - (1 + x^c)^{-k}, \qquad 0 < x < \infty \tag{1}$$

and the corresponding probability density function (pdf) is obtained as

$$f(x) = c\,k\,x^{c-1}(1 + x^c)^{-k-1}, c, k > 0 \tag{2}$$

Recently, the cdf of the generalized BXII-G family has been proposed by Nasir et al. (2017) as

$$\begin{aligned} F(x) &= \int_0^{-\log \bar{G}(x;\xi)} c\,k\,x^{c-1}(1 + x^c)^{-k-1}\,dx \\ &= 1 - [1 + \{-\log \bar{G}(x;\xi)\}^c]^{-k}, \end{aligned} \tag{3}$$

where the shape parameters are *c, k>0* and the baseline cdf is G(x; ξ).

Motivated by Burr XII-G family of Nasir et al. (2017), a new distribution is proposed called as the Burr XII-Uniform (for short BXIIU) distribution using the uniform distribution as a baseline distribution. Hence, the cdf for BXIIU distribution is obtained as

$$F(x) = \int_0^{-\log\left(1-\frac{x}{\theta}\right)} c\,k\,x^{c-1}(1+x^c)^{-k-1}\,dx = 1-\left[1+\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^c\right]^{-k} \qquad (4)$$

The pdf is derived as

$$f(x) = \frac{c\,k}{\theta}\frac{\left(1-\frac{x}{\theta}\right)^{-1}\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{c-1}}{\left[1+\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^c\right]^{k+1}},\ c,\ k>0 \qquad (5)$$

The first aim to study the BXIIU distribution is due to its ability of analyzing hydrologic, environmental, survival and reliability data. The second aim is to provide empirical evidence on the great flexibility of the BXIIU distribution to fit practical data from different domains which is studied in the application section. The BXIIU distribution is very flexible with regard to the uniform distribution and provides a great flexibility.

In Section 2, the characteristics of the new distribution, including moments, extreme values, order statistics are provided. Inference and estimation with the maximum likelihood method are presented in Section 3. Order statistics are given in Section 4 and entropy function is obtained in Section 5. Applications are considered to show performance for the proposed model in Section 6. In Section 7, conclusion is given.

2. Statistical Characteristics

2.1 Survival and Hazard Functions

In reliability theory, survival function and hazard rate function (hrf) are very important to analyze lifetime data. The survival function of the BXIIU distribution is given by

$$S(x) = \left[1+\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^c\right]^k \qquad (6)$$

and the hrf of the BXIIU distribution is given by

$$\tau\,(x) = \frac{c\,k}{\theta}\frac{\left(1-\frac{x}{\theta}\right)^{-1}\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{c-1}}{1+\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^c}. \qquad (7)$$

The limiting behaviors for the pdf and hrf are obtained as if x→ 0, then the limiting behavior of $f(x)$ and $\tau\,(x)$ are obtained as$f(x) \to ck\,\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{c-1}$ and $\tau\,(x) \to$

$ck\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{c-1}$, respectively. If x→ ∞, then the limiting behaviours of $f(x)$and $\tau(x)$ are obtained as$f(x) \to 0$ and $(x) \to 0$, respectively.

The pdf and hrf of the BXIIU distribution are shown in Figures 1 and 2, respectively. As seen from Figure 1, the BXIIU distribution has different shapes depending on the parameter values. Figure 1 present that unimodal, decreasing, increasing, symmetric, left-skewed and right-skewed shapes are possible which proves the flexibility of the BXIIU distribution according to the uniform distribution. Furthermore, the hrfs has non-monotone shapes and hence the BXIIU is a suitable distribution in real life situations.

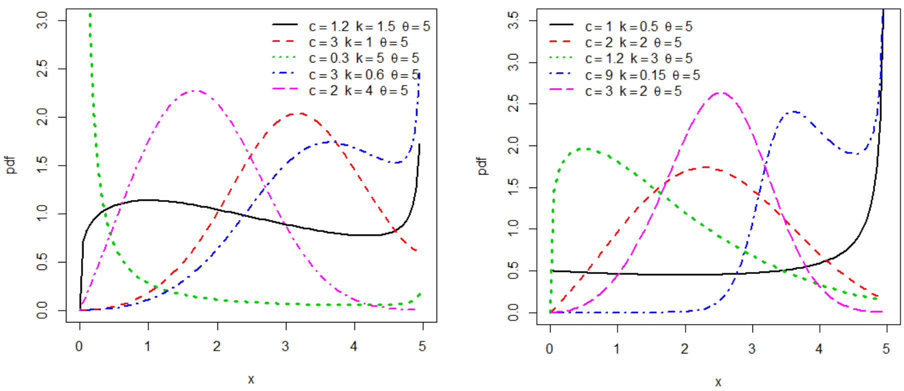

Figure 1: Plots of density function of the BXIIU distribution with several parameter values.

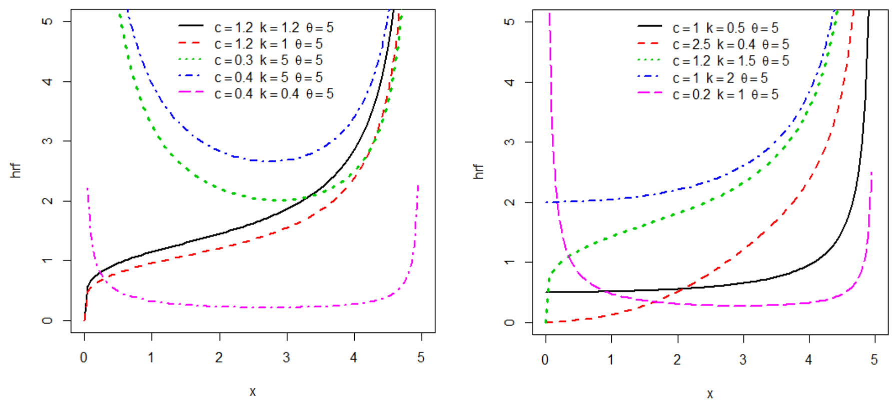

Figure 2: Plots for BXIIU distribution with several parameter values

2.2. Infinite mixture representations

Now, we give the mixture representations for the cdf and pdf of the BXIIU distribution using baseline distribution. Several important expansions for (4) and (5) is obtained from the concept of power series. Consider following series expansion

$$(1-z)^{-b} = \sum_{j=0}^{\infty} \binom{b+j-1}{j} z^j \tag{8}$$

The cdf in (1.4) is written as

$$F(x) = 1 - \sum_{j=0}^{\infty} \binom{k+j-1}{j} \left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{c\,j} \tag{9}$$

Using log-power expansion, we obtain

$$\left\{\log\left(1-\frac{x}{\theta}\right)\right\}^{c\,j} = cj \sum_{i=0}^{\infty} \sum_{q=0}^{i} \binom{i-cj}{i} \binom{i}{q} \frac{(-1)^{q+i}}{c\,j-q} P_{q,i} \left(\frac{x}{\theta}\right)^i,$$

where$P_{q,i} = i^{-1} \sum_{n=1}^{\infty} (n(q+1)-i) c_n P_{q,i-n}$ and $P_{q,0} = 1$, $c_i = \frac{(-1)^i}{i+1}$. The cdf for BXIIU distribution is obtained as

$$F(x) = 1 - \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{q=0}^{i} \binom{k+j-1}{j} \binom{i-cj}{i} \binom{i}{q} cj \frac{(-1)^{q+i+c\,j}}{c\,j-q} P_{q,i} \left(\frac{x}{\theta}\right)^i$$

$$F(x) = 1 - \sum_{i=0}^{i} a_i \left(\frac{x}{\theta}\right)^i \tag{10}$$

Rewriting Eq. (10), we obtain

$$F(x) = \sum_{i=0}^{i} b_i \left(\frac{x}{\theta}\right)^i, \tag{11}$$

where$b_i = -a_i$ and $b_0 = 1 - a_0$.Here, $a_i = \sum_{j=0}^{\infty} \sum_{q=0}^{i} \binom{k+j-1}{j} \binom{i-cj}{i} \binom{i}{q} cj \frac{(-1)^{q+i+c\,j}}{c\,j-q} P_{q,i}$.

Now, the pdf of BXIIU distribution is given as

$$f(x) = \sum_{i=0}^{i} b_i \frac{i}{\theta} \left(\frac{x}{\theta}\right)^{i-1} \tag{12}$$

Many properties of the BXII-U distribution is derived from the expansions of the pdf and cdf given in (11) and (12), respectively.

2.3. Moments

r^{th} moment of BXIIU distribution is derived by the infinite mixture representation in (12) as

$$\mu_r' = \int_0^{\theta} x^r \sum_{i=0}^{i} b_i \frac{i}{\theta}\left(\frac{x}{\theta}\right)^{i-1} dx, \tag{13}$$

where the coefficients are already defined in Section 2.3. Then, we have

$$\mu_r' = \sum_{i=0}^{i} b_i \frac{i\,\theta^r}{r+i} \tag{14}$$

Similarly, s^{th}incomplete moment for BXIIU distribution is obtained as

$$T_s'(x) = \int_0^x x^s \sum_{i=0}^{i} b_i \frac{i}{\theta}\left(\frac{x}{\theta}\right)^{i-1} dx$$

Then, we have

$$T_s'(x) = \sum_{i=0}^{i} b_i \frac{i}{\theta^i}\frac{x^{s+i}}{s+i}$$

The mgf for the BXII-U is obtained as

$$M_0(x) = \int_0^x e^{t\,x} \sum_{i=0}^{i} b_i \frac{i}{\theta}\left(\frac{x}{\theta}\right)^{i-1} dx \tag{15}$$

The mgf is also written as

$$M_0(x) = \sum_{i=0}^{i} b_i \gamma\left(i, \frac{-\theta}{t}\right)\left(\frac{-1}{t}\right)^i \tag{16}$$

The mean deviations of the BXIIU distribution about median and mean are defined as

$$D_{\mu} = 2\,\mu\, F(\mu) - 2\,T_s'(\mu) \tag{17}$$

$$D_M = \mu - 2\,T_s'(Median), \tag{18}$$

whereμ is the mean, $Q_x(0.5)$ is the median and $T_1'(c)$ is the first incomplete moment at $x = c$.

2.4. Quantile Function

The quantile function for BXII-U distribution is obtained by

$$Q_x(u) = \theta\left[1 - \exp\left\{(1-u)^{-\frac{1}{k}} - 1\right\}^{\frac{1}{c}}\right] \tag{19}$$

For the BXII-U distribution, Galton's skewness is obtained using Eq. (19) as

$$S = \frac{Q(3/4) - 2Q(1/2) + Q(1/4).}{Q(3/4) - Q(1/4)}$$

Similarly, kurtosis of Moors is given by

$$K = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)},$$

where Q(.) is quantile function for BXIIU distribution. For a symmetric distribution, S equals to 0 and for the right-skewed (left-skewed) distribution S is lower (greater) than 0. The tail of the distribution becomes heavier when K increases.

Figure 3 shows the skewness and kurtosis measures of BXIIU distribution when θ=0.5. Based on the Figure 3, we conclude that the parameter ***k*** controls the both skewness and kurtosis. As seen in Figure 3, it is clear that the parameter ***k*** has more effect than the parameter ***c*** on these measures.

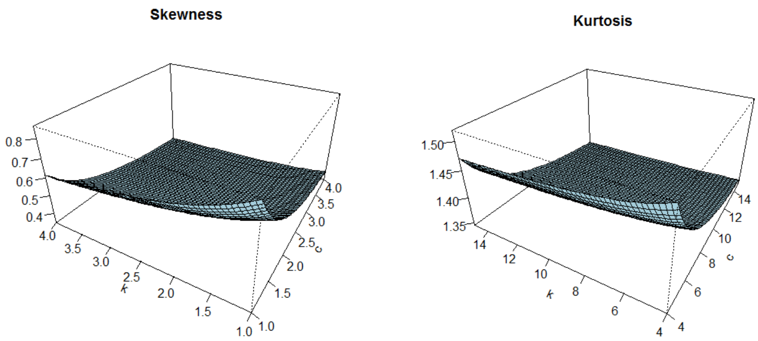

Figure 3: Plots for skewness and kurtosis for BXIIU distribution.

2.5 Stochastic ordering

Let $X_1 \sim BXIIU(c, k_1, \theta)$ and $X_2 \sim BXIIU(c, k_2, \theta)$ with density functions under

$$f(x) = \frac{c_1 k_1}{\theta_1} \frac{\left(1 - \frac{x}{\theta_1}\right)^{-1} \left\{-\log\left(1 - \frac{x}{\theta_1}\right)\right\}^{c_1 - 1}}{\left[1 + \left\{-\log\left(1 - \frac{x}{\theta_1}\right)\right\}^{c_1}\right]^{k_1 + 1}}$$

and

$$g(x) = \frac{c_2 k_2}{\theta_2} \frac{\left(1 - \frac{x}{\theta_2}\right)^{-1} \left\{-\log\left(1 - \frac{x}{\theta_2}\right)\right\}^{c_2 - 1}}{\left[1 + \left\{-\log\left(1 - \frac{x}{\theta_2}\right)\right\}^{c_2}\right]^{k_2 + 1}}$$

Then, the ratio $\frac{f(x)}{g(x)}$ is defined as

$$\frac{f(x)}{g(x)}=\frac{c_1k_1\theta_2}{c_2k_2\theta_1}\frac{\left(1-\frac{x}{\theta_2}\right)}{\left(1-\frac{x}{\theta_1}\right)}\frac{\left\{-\log\left(1-\frac{x}{\theta_1}\right)\right\}^{c_1-1}}{\left\{-\log\left(1-\frac{x}{\theta_2}\right)\right\}^{c_2-1}}\frac{\left[1+\left\{-\log\left(1-\frac{x}{\theta_2}\right)\right\}^{c_2}\right]^{k_2+1}}{\left[1+\left\{-\log\left(1-\frac{x}{\theta_1}\right)\right\}^{c_1}\right]^{k_1+1}}$$

Taking derivative of $log\ \frac{f(x)}{g(x)}$, one canobtain

$$\frac{d}{d\,x}\log\frac{f(x)}{g(x)}=\left(\frac{1}{\theta_1-x}\right)-\left(\frac{1}{\theta_2-x}\right)+\frac{(c_1-1)\left(1-\frac{x}{\theta_1}\right)^{-1}}{\theta_1\left\{-\log\left(1-\frac{x}{\theta_1}\right)\right\}}-\frac{(c_2-1)\left(1-\frac{x}{\theta_2}\right)^{-1}}{\theta_2\left\{-\log\left(1-\frac{x}{\theta_2}\right)\right\}}$$
$$+\frac{c_2(k_2+1)}{\theta_2}\frac{\left\{-\log\left(1-\frac{x}{\theta_2}\right)\right\}^{c_2-1}}{\left[1+\left\{-\log\left(1-\frac{x}{\theta_2}\right)\right\}^{c_2}\right]\left(1-\frac{x}{\theta_2}\right)}$$
$$-\frac{c_1(k_1+1)}{\theta_1}\frac{\left\{-\log\left(1-\frac{x}{\theta_1}\right)\right\}^{c_1-1}}{\left[1+\left\{-\log\left(1-\frac{x}{\theta_1}\right)\right\}^{c_1}\right]\left(1-\frac{x}{\theta_1}\right)}$$

If $\theta_1=\theta_2$ and $c_1=c_2$, then the equation given above becomes

$$\frac{d}{d\,x}\log\frac{f(x)}{g(x)}=(k_2-k_1)\frac{c}{\theta}\frac{\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{c-1}}{\left[1+\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{c}\right]\left(1-\frac{x}{\theta}\right)}.$$

If $k_2<k_1$, then $\frac{d}{d\,x}\log\frac{f(x)}{g(x)}<0$, this implies that the likelihood ratio ordering exists.

2.6 Stress-Strength Model

In reliability theory, the stress-strength model shows a component life where a random strength X_1which subjects to a random stress X_2. Therefore, $R=P(X_2<X_1)$ shows the component reliability. We present the reliability function R if the independent, random X_1 and X_2 variables are BXIIU distributed.

Let f_i and F_i, are pdf and cdf of X_i for i=1,2, then the reliability function for the BXIIU distribution is defined as

$$R=\int_0^\infty f_1(x)\,F_2(x)dx, \qquad (20)$$

where

$$f(x)=\frac{c_1k_1}{\theta_1}\frac{\left(1-\frac{x}{\theta_1}\right)^{-1}\left\{-\log\left(1-\frac{x}{\theta_1}\right)\right\}^{c_1-1}}{\left[1+\left\{-\log\left(1-\frac{x}{\theta_1}\right)\right\}^{c_1}\right]^{k_1+1}}$$

and

$$F_2(x) = 1 - \left[1 + \left\{-\log\left(1 - \frac{x}{\theta_2}\right)\right\}^{c_2}\right]^{-k_2}.$$

Now equation (20) becomes

$$R = \int_0^\infty = \frac{c_1 k_1}{\theta_1} \frac{\left(1 - \frac{x}{\theta_1}\right)^{-1} \left\{-\log\left(1 - \frac{x}{\theta_1}\right)\right\}^{c_1 - 1}}{\left[1 + \left\{-\log\left(1 - \frac{x}{\theta_1}\right)\right\}^{c_1}\right]^{k_1+1}} \left\{1 - \left[1 + \left\{-\log\left(1 - \frac{x}{\theta_2}\right)\right\}^{c_2}\right]^{-k_2}\right\} dx$$

After some algebra, we have

$$R = 1 - \int_0^\infty \left(1 - \frac{x}{\theta_1}\right)^{-1} \left\{-\log\left(1 - \frac{x}{\theta_1}\right)\right\}^{c_1 - 1} \left\{1 - \left[1 + \left\{-\log\left(1 - \frac{x}{\theta_2}\right)\right\}^{c_2}\right]^{-k_2 - k_1 - 1}\right\} dx$$

The above expression reduces to $R = \frac{k_2}{k_1 + k_2}$.

3. Estimation

Now, we obtain the maximum likelihood estimates (MLEs) for model parameters about the BXIIU distribution. Let $x_1, x_2, \ldots, x_n$ be a random sample from BXIIU distribution. First, we express log-likelihood function with complete samples as follows:

$$\ell(\Theta) = n\, log\left(\frac{c\,k}{\theta}\right) - \sum_{i=1}^{n} \log\left(1 - \frac{x_i}{\theta}\right) + (c-1) \sum_{i=1}^{n} \log\left\{-\log\left(1 - \frac{x_i}{\theta}\right)\right\} - (k+1) \sum_{i=1}^{n} \log\left[1 + \left\{-\log\left(1 - \frac{x}{\theta}\right)\right\}^{c}\right]$$

The components for the score vector $\Theta = (c, k, \theta)^T$ are given by

$$U_k = \frac{n}{k} - \sum_{i=1}^{n} \log\left[1 + \left\{-\log\left(1 - \frac{x}{\theta}\right)\right\}^{c}\right]$$

$$U_c = \frac{n}{c} + \sum_{i=1}^{n} \log\left\{-\log\left(1 - \frac{x_i}{\theta}\right)\right\} - (k+1) \sum_{i=1}^{n} \left[\frac{\left\{-\log\left(1 - \frac{x}{\theta}\right)\right\}^{c} \log\left[-\log\left(1 - \frac{x}{\theta}\right)\right]}{1 + \left\{-\log\left(1 - \frac{x}{\theta}\right)\right\}^{c}}\right]$$

$$U_\theta = -n\log\theta - \sum_{i=1}^{n}\left[\frac{x_i}{\theta^2\left(1-\frac{x_i}{\theta}\right)}\right] - (c-1)\sum_{i=1}^{n}\left[\frac{x_i}{\theta^2\left(1-\frac{x_i}{\theta}\right)\left\{-\log\left(1-\frac{x_i}{\theta}\right)\right\}}\right]$$
$$-(k+1)\sum_{i=1}^{n}\left[\frac{c\,x_i\left\{-\log\left(1-\frac{x_i}{\theta}\right)\right\}^{c-1}}{\theta^2\left(1-\frac{x_i}{\theta}\right)\left[1+\left\{-\log\left(1-\frac{x_i}{\theta}\right)\right\}^{c}\right]}\right]$$

These equations set to zero and simultaneous solution gives the MLEs for BXIIU distribution.

4. Order statistics

Let $X_1, X_2, \ldots, X_n$be the random sample from a population with pdf and cdf given in (4) and (5), then the pdf of i[th] order statistics is obtained as

$$f_{i:n}(x) = \frac{n!}{(n-i)!\,(i-1)!} f(x)F^{i-1}(x)[1-F(x)]^{n-i}$$

Using the generalized binomial expansion, this function is given by

$$f_{i:n}(x) = \frac{n!}{(n-i)!(i-1)!}\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j f(x)\bar{F}^{n+j-i}(x) \tag{21}$$

Now, consider $F^{i-1}(x)[1-F(x)]^{n-i}$**.** Using the cdf given in (4), we have

$$f(x)\bar{F}^{n+j-i}(x) = \left[\frac{c\,k}{\theta}\frac{\left(1-\frac{x}{\theta}\right)^{-1}\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{c-1}}{\left[1+\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{c}\right]^{k+1}}\right]$$
$$\times\left[\left[1+\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{c}\right]^{-k}\right]^{n+j-i}$$

Then, we obtain

$$f(x)\bar{F}^{n+j-i}(x) = \frac{c\,k}{\theta}\left(1-\frac{x}{\theta}\right)^{-1}\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{c-1} \times \left[\left[1+\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{c}\right]^{-k}\right]^{-\Delta_1},$$

where$\Delta_1 = \mathrm{k}(\mathrm{n}+\mathrm{j}+1-\mathrm{i})+1$. From generalized binomial expansion, the following expansion can be written as

$$f(x)\bar{F}^{n+j-i}(x) = \frac{c\,k}{\theta}\left(1-\frac{x}{\theta}\right)^{-1}\sum_{l=0}^{\infty}\binom{\Delta_1+\mathrm{l}-1}{l}(-1)^l\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{\Delta_2} \tag{22}$$

where $\Delta_2 = \mathrm{c}(\mathrm{l}+1)-1$. From the log-power expansion, we have

$$\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{\Delta_2} = (-1)^{\Delta_2}\sum_{p=0}^{\infty}\sum_{q=0}^{p}\binom{p-\Delta_2}{p}\binom{p}{q}\frac{(-1)^{2q}}{\Delta_2-\mathrm{q}}M_{p,q}\left(\frac{x}{\theta}\right)^{q}$$

and

$$\left(1-\frac{x}{\theta}\right)^{-1} = \sum_{m=0}^{\infty}(-1)^{m}\left(\frac{x}{\theta}\right)^{m}$$

Then, equation (21) can be written as

$$f_{i:n}(x) = \frac{n!}{(n-i)!\,(i-1)!}\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^{j}\,(c\,k)\sum_{m=0}^{\infty}(-1)^{m}\left(\frac{x}{\theta}\right)^{m}\sum_{l=0}^{\infty}\binom{\Delta_1+\mathrm{l}-1}{l}(-1)^{l}\,(-1)^{\Delta_2}\sum_{p=0}^{\infty}\sum_{q=0}^{p}\binom{p-\Delta_2}{p}\binom{p}{q}\frac{(-1)^{2q}}{\Delta_2-\mathrm{q}}M_{p,q}\left[\frac{1}{\theta}\left(\frac{x}{\theta}\right)^{q}\right]$$

Rewriting the above expression, we have

$$f_{i:n}(x) = \frac{n!(c\,k)}{(n-i)!(i-1)!}\sum_{j=0}^{i-1}\sum_{m,l,q=0}^{\infty}\sum_{l=0}^{\infty}\sum_{q=0}^{p}\binom{i-1}{j}\binom{\Delta_1+\mathrm{l}-1}{l}\binom{p-\Delta_2}{p}\binom{p}{q}\frac{(-1)^{2q+i+m+\Delta_2}}{\Delta_2-\mathrm{q}}M_{p,q}\left[\frac{1}{\theta}\left(\frac{x}{\theta}\right)^{q}\right]$$

,

where $\left[\frac{1}{\theta}\left(\frac{x}{\theta}\right)^{q}\right]$ equals to $f(x)F^{q}(x)$. Hence, the density of i*th* order statistics is the infinite mixture representation of the uniform distribution.

5. Entropy

Now, we will consider measure of Rényi entropy. According to Rényi (1961), it is defined by

$$I_R = \frac{1}{1-R}\log\int_0^{\theta} f^{R}(x)dx$$

Using the pdf given in equation (5), we have

$$I_R = \frac{1}{1-R}\log\int_0^{\theta}\left[\frac{c\,k}{\theta}\frac{\left(1-\frac{x}{\theta}\right)^{-1}\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{c-1}}{\left[1+\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{c}\right]^{k+1}}\right]^{R}dx \tag{23}$$

Considering $f^{R}(x)$, after some algebra, we have

$$f^{R}(x) = \left[\frac{c\,k}{\theta}\right]^{R}\left(1-\frac{x}{\theta}\right)^{-R}\sum_{j=0}^{\infty}\binom{R(k+1)+j-1}{j}(-1)^{j}\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{c(R+j)-R} \tag{24}$$

If $\Delta_3 = c(R+j) - R$, using the log-power expansion, we have

$$\left\{-\log\left(1-\frac{x}{\theta}\right)\right\}^{\Delta_3} = (-1)^{\Delta_3} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p-\Delta_3}{p} \binom{p}{q} \frac{(-1)^{2q}}{\Delta_3 - \mathrm{q}} M_{p,q} \left(\frac{x}{\theta}\right)^{q}$$

and

$$\left(1-\frac{x}{\theta}\right)^{-R} = \sum_{m=0}^{\infty} \binom{R+m-1}{m} (-1)^m \left(\frac{x}{\theta}\right)^m,$$

where the coefficients are given in (5.2). Now, equation (5.2) becomes

$$f^R(x) = \left[\frac{c\,k}{\theta}\right]^R \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{R(k+1)+j-1}{j} \binom{R+m-1}{m} \binom{p-\Delta_3}{p} \binom{p}{q} \frac{(-1)^{2q+m+j+\Delta_3}}{\Delta_3 - \mathrm{q}} x\, M_{p,q} \left(\frac{x}{\theta}\right)^{q+m}$$

Now consider the integration

$$\int_0^\theta \left(\frac{x}{\theta}\right)^{q+m} dx = \left(\frac{1}{\theta}\right)^{q+m} \int_0^\theta x^{q+m} dx = \frac{\theta}{q+m+1}$$

and equation (23) becomes

$$I_R = \frac{1}{1-R} \log\log\left[\left[\frac{c\,k}{\theta}\right]^R \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{R(k+1)+j-1}{j} \binom{R+m-1}{m} \binom{p-\Delta_3}{p} \cdot \binom{p}{q} \frac{(-1)^{2q+m+j+\Delta_3}}{\Delta_3 - \mathrm{q}} M_{p,q} \left\{\frac{\theta}{q+m+1}\right\}\right]$$

Note that if R→1, then the Rényi entropy tends to the Shannon entropy.

6. Application

Now, two data sets are used to show performance for the BXIIU distribution and to compare some sub-models: namely, Logistic uniform (LU) and gamma uniform (GU) distributions. The MLEs are obtained for parameters. To present the performance of different models, we give criteria including -log-likelihood function (*L),* Anderson-Darling (*A**), Cramér–von Mises (*W**) Kolmogorov-Smirnov (*K-S*) statistic, Akaike information criterion (*AIC*), Bayesian information criterion (*BIC*) and p-value. Note that small *A**, *W**, *AIC*, *BIC*, *K-S* criteria values indicate big p-value. Then, the histograms of data sets and the estimated pdfs for BXIIU, LU and GU models are presented. Finally, the plots of empirical cdfs and estimated pdfs of the BXIIU, LU and GU models are presented in Figures 4 and 5, respectively.

The first data about failure times are reported in Murthy *et al.* (2004). The second data set consists of the natural increase rate in Ristic et al. (2012). In Table 1, we give MLEs and their standard errors (in parentheses).

Model	$\hat{c}$	$\hat{k}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$
BXIIU	2.2706	66.7068	19.7679	-	-
	(0.1446)	(12.5342)	(3.1313)	-	-
GU	2.0290	2.5997	-	0.0001	6.8052
	(1.0581)	(2.3054)	-	(0.3819)	(1.5611)
LU	-2.1231	0.0809	-	0.0001	77.2479
	(1.3222)	(0.0503)	-	(8.4865)	(21.3233)

Table 1: Parameter estimation for first data

Some statistics and P-value are presented in Table 2.

Model	L	AIC	BIC	A*	W*	K-S	P-value
BXIIU	130.6554	267.3107	274.6387	0.5504	0.0562	0.0561	0.9518
GU	131.3072	269.8143	276.5849	0.6477	0.0983	0.0843	0.5810
LU	131.9609	270.1218	279.8924	0.5546	0.0753	0.0766	0.6996

Table 2: Some statistics of models fitted to first data

As seen in Table 2, the BXII-U model gives the lowest values. Therefore, the BXIIU distribution yields the best fit and is more flexible model to explain data.

Table 3 provides the MLEs for model parameters of second data set.

Model	$\hat{c}$	$\hat{k}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$
BXIIU	1.9188	19.3998	70.9816	-	-
	(0.5880)	(20.7429)	(33.3698)	-	-
GU	1.4201	6.7145	-	2.1337	63.9932
	(0.3238)	(0.3275)	-	(0.5153)	(24.1212)
LU	-1.5126	0.4237	-	0.0001	63.0409
	(0.2621)	(0.1070)	-	(3.0110)	(20.3233)

Table 3: The MLEs and their standard errors of the second data

The values L, AIC, BIC, A*, W* and K-S in Table 3 indicates that BXIIU distribution is much more better fit than the GU and LU distributions.

Model	L	AIC	BIC	A*	W*	K-S	P-value
BXII-U	190.1662	388.3323	395.6154	0.4992	0.0614	0.0828	0.8049
GU	191.6134	388.6268	397.0042	0.5561	0.0807	0.1001	0.5843
LU	194.7689	397.5377	405.9151	0.7888	0.1117	0.0897	0.8028

Table 4: Some statistics of models fitted to second data

As seen in Table 4, the BXIIU model gives the lowest values which proves that the BXII-U distribution is adequate model for explaining data.

Histograms, estimated pdfs, cdfs of BXIIU model with other competitive models are shown in Figures 4 and 5, respectively. Figures 4 and 5 present that the BXIIU model fit to the both data sets very well.

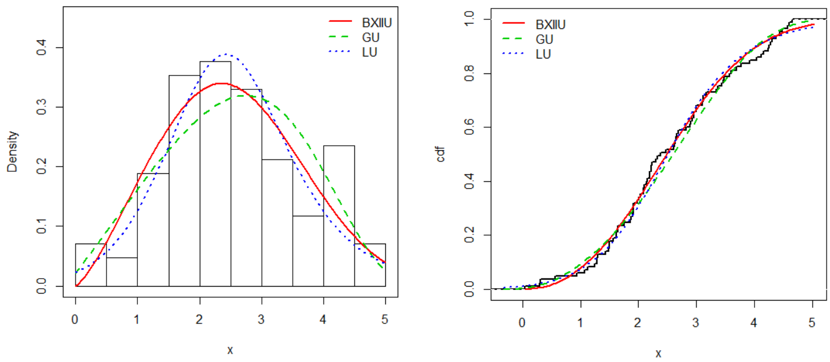

Figure 4: Plots for estimated pdfs in(a) and cdfsin (b) for BXIIU and other models for first data

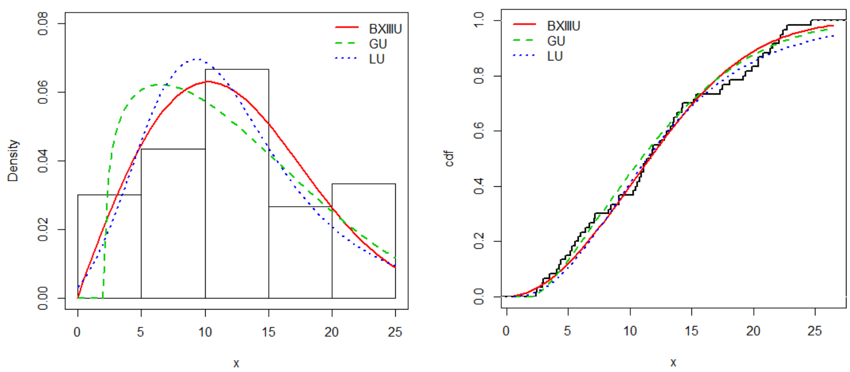

Figure 5: Plots for estimated pdfs in (a) and cdfs in (b) for BXII-U and other models for second data

Summary statistics and the BXIIU distribution for fitted data sets are presented in Tables 5 and 6.

Data	Mean	S.D	Median	MD (mean)	MD (median)	R	S	K	Entropy
I	2.56	1.11	2.38	0.917	0.954	0.09	-0.69	0.12	4.410
II	12.1	6.26	11.55	5.166	9.290	0.25	-1.09	0.81	4.025

Table 5: Summary statistics of the data sets

Data	Mean	S.D	Median	MD (mean)	MD (median)	S	K	Entropy
I	2.421	1.101	2.1238	0.8975	0.9495	-0.700	0.1942	5.1100
II	12.022	6.212	11.162	5.416	9.180	-1.002	0.798	4.054

Table 7: Summary statistics for BXIIU distribution for fitted data sets

7. Conclusion

We have introduced a new distribution, named BXII-U by extending uniform distribution. We have generalized the uniform distribution to provide a larger flexibility to model real data sets. We have derived various properties of new distribution including the moments, mean deviations. Entropy, orders statistics etc. The maximum likelihood approach is used to estimate model parameters. Applications show that fit for new model is superior to other models.

References

1. Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S. and Shahbaz, M.Q. (2016).Topp Leone family of distributions: Some properties and application, Pakistan Journal of Statistics and Operation Research, 12(3), p. 443-451.
2. Nasir, M.A., Tahir, M.H., Jamal, F. and Ozel, G. (2017). A new generalized Burr family of distributions for the lifetime data, Journal of Statistics Applications and Probability, 6, p. 1–17.
3. Bourguignon, M., Silva, R.B., and Cordeiro, G.M. (2014).The Weibull-G family of probability distributions, Journal of Data Science, 12, p. 53-68.
4. Cakmakyapan, S. and Ozel G. (2017). The Lindley family of distributions: properties and applications, Hacettepe Journal of Mathematics and Statistics, 46, p. 1-27.
5. Cordeiro, G.M., Ortega E.M.M. and da Cunha D.C.C. (2013). The exponentiated generalized class of distribution, Journal of Data Science, 11, p. 1-27.
6. Cordeiro, G.M., Ortega, E.M.M., Bozidar, P.V and, Pescim, R.R. (2014). The Lomax generator of distributions: Properties, minification process and regression model, Applied Mathematics and Computation, 247, p. 465-486.

7. Cordeiro, G.M., Alizadeh, M. and Marinho, P.R.D. (2016). The type I half-logistic family of distributions, Journal of Statistical Computation and Simulation, 86(4), p. 707-728.
8. Eugene, N., Lee C., and Famoye, F. (2002). Beta-normal distribution and its applications, Communications in Statistics-Theory Methods, 6(31), p. 497-512.
9. Elgarhy, M., Hassan, A.S. and Rashed, M. (2016).Garhy-generated family of distributions with application, Mathematical Theory and Modeling, 6, p. 1-15.
10. Elgarhy, M., Haq, M.A., Ozel, G. and Nasir, A. (2017). A new exponentiatedextended family of distributions with applications, Gazi University Journal of Science, 30(3), p. 101-115.
11. Haq, M. and Elgarhy, M. (2018a). The odd Frèchet-G family of probability distribution, Journal Statistics Applications and Probability, 7, p.185-201.
12. Haq, M. A., Elgarhy, M., Hashmi, S., Ozel, G. and Ul, Ain, Q. (2018b). Transmuted Weibull power function distribution: its properties and applications, Journal of Data Science, 16(2), p. 397-418.
13. Hassan, A.S. and Elgarhy, M. (2016a). Kumaraswamy Weibull-generated family of distributions with applications, Advances and Applications in Statistics, 48, p. 205-239.
14. Hassan, A.S. and Elgarhy, M. (2016b). A new family of exponentiated Weibull-generated distributions, International Journal of Mathematics and its Applications, 4, p. 135-148.
15. Hassan, A.S., Elgarhy, M., and Shakil, M. (2017). Type II half logistic family of distributions with applications, Pakistan Journal of Statistics and Operation Research, 13(2), p. 245-264.
16. Murthy, D.N.P., Xie, M. and Jiang, R. (2004). Weibull Models, John Wiley and Sons, New Jersey.
17. Nasir, A., Tahir, M.A., Jamal, F. andOzel, G., (2017). A new generalized Burr family of distributions for the lifetime data, Journal of Statistics Applications and Probability, 6, p. 401-417.
18. Pescim, R.R., Cordeiro, G.M., Demetrio, C.G.B.,Ortega, E.M.M. and Nadarajah, S. (2012).The new class of Kummer beta generalized distributions, SORT, 36(2), p. 153-180.
19. Rényi, A. (1961). On measures of entropy and information, In: Proceedings of the 4th Fourth Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley, p. 547- 561.
20. Ristic, M.M., and Balakrishnan, N. (2012). The gamma-exponentiated exponential distribution, Journal of Statistical Computation and Simulation, 82, p. 1191-1206.
21. Zografos, K. and Balakrishnan, N. (2009). On families of beta and generalized gamma-generated distributions and associated inference, Statistical Methodology, 6, p. 344–36.