Journal of Reliability and Statistical Studies; ISSN (Print): 0974-8024, (Online): 2229-5666
Vol. 11, Issue 2 (2018): 193-217

ON BAYESIAN ANALYSIS OF RIGHT CENSORED WEIBULL DISTRIBUTION USING APPROXIMATE METHODS

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Received March 22, 2018
Modified November 03, 2018
Accepted December 10, 2018

Abstract

In this paper, we have discussed the estimation for parameters of Weibull model under right censored samples. We have assumed two priors and two loss functions for the posterior estimation. As the Bayes estimators from the concerned posterior distributions do not exist in the explicit form, we have considered Quadrature method (QM), Gibbs sampler (GS), importance sampling (IS), Lindley's approximation (LA) and Tierney and Kadane's approximation (TKA) to obtain the numerical solutions for the Bayes estimators. The performance of the different estimators has been compared using simulated results along with real example. The findings of the study suggest that estimators based on IS and TKA are superior in performance with certain conditions.

Key Words: Quadrature Method, Gibbs Sampler, Importance Sampling (IS), Lindley's Approximation, Tierney and Kadane's Approximation (TKA), Posterior Distributions, Loss Functions.

1. Introduction

The Weibull model has many advantages over the other competitive failure-time models due to its flexibility, various shapes of its hazard rate and closed form distribution function. Few valuable contributions based on Weibull distributions have been reported as follows. Choudhury (2005) derived the moments for the exponentiated-Weibull distribution in a simple and convenient way. Nadarajah and Gupta (2005) discussed the estimation of the moments for the exponentiated-Weibull model. Bebbington et al. (2007) proposed an extension for the Weibull model which is more flexible than the standard Weibull distribution. Lee et al. (2007) discussed the applications of the proposed beta Weibull model to the censored data. Cordeiro, et al. (2010) discussed various properties of the Kumaraswamy-Weibull model which contains special sub-models. Some real life applications of the model to investigate failure time data have also been discussed. Cordeiro and de Castro (2011) introduced a family of generalized models including generalizations for the Weibull model. Almalki and Yuan (2013) developed a modification of the Weibull model and confirmed the applications of the said distribution to model failure time data. Cordeiro et al. (2013) derived the general results for the beta-Weibull model. Hanook et al. (2013) discussed the properties and estimation of beta inverse Weibull distribution. Mahmoudi and Sepahdar (2013) proposed an exponentiated Weibull-Poisson distribution and illustrated its applications. Alizadeh et al. (2015) proposed the estimation for different statistical properties of the exponentiated-Weibull model. Castellares and Lemonte (2015) proposed a simple representation for the gamma-G family of models and called it gamma-exponentiated-Weibull model and studied its details. El-Gohary et al. (2015) developed an inverse Weibull extension distribution and discussed its estimation. Ortega et al. (2015) predicted breast carcinoma by introducing a power series beta Weibull regression model. Bagheri et al. (2016) proposed a better estimation for the properties of Weibull extension model. Kopal et al. (2016) discussed the applications of the Weibull distribution for modeling data regarding temperature dependence of polyurethane storage modulus. Liu et al. (2017) proposed an accrual failure detector using Weibull distribution and named it Weibull distribution failure detector. The results show that Weibull model has superior performance especially in cloud computing.

According to Kundu and Joarder (2006), if some prior information is available, then the Bayesian estimation out performs the MLE in terms of efficiency. As the informative priors have been considered under the study, so the proposed results will be better than MLE, therefore the comparison of MLE and Bayesian estimation has not been presented here.

The paper estimates the weibull distribution using Bayesian inference under right censoring. The choice of Weibull distribution made due to its superiority over the standard lifetime distributions. However, the expressions for the Bayes estimators do not exist in the closed form; therefore the five different approximation techniques have been proposed to obtain the approximate solution for the posterior estimates. We have considered these five approximate methods to develop a comprehensive comparison among these methods to estimate the right censored Weibull model.

The remaining part of the contribution is arranged as follows. The section-2 includes the derivation of likelihood function for the proposed model. The construction of the posterior distribution has been presented in the section-3. The section-4 defines the proposed loss functions. The Bayesian estimation using QM, GS, IS, LA and TKA has been presented in the section-5. The simulated results have been reported in the section-6. The real application has been reported in the section-7. The conclusion of the study has been given in the section-8.

2. The Model and Likelihood Function

The probability density function (pdf) of the weibull distribution is

$$f(x)=\alpha\beta x^{\alpha-1}e^{-\beta x^{\alpha}} \qquad x>0,\ \alpha,\beta>0 \tag{1}$$

where α and β are the parameters of the model.

The cumulative distribution function (CDF) of the distribution is

$$F(x) = 1 - e^{-\beta x^{\alpha}} \qquad x > 0,\ \alpha, \beta > 0 \tag{2}$$

Let us consider a random sample of size 'n' from a Weibull distribution, and let $x_1, \ldots, x_r$ be the ordered observations remaining when the 'n – r' largest observations have been censored. The likelihood function for the Type II right censored sample x = ($x_1, \ldots, x_r$), is

$$L(\mathrm{x}|\alpha,\beta) \propto \left[1 - F(x_r|\alpha,\beta)\right]^{n-r} \prod_{i=1}^{r} f(x_i|\alpha,\beta) \tag{3}$$

Putting results in (3), we have

$$L(\mathrm{x}|\alpha,\beta) \propto \left[e^{-\beta x_r^{\alpha}}\right]^{n-r} \prod_{i=1}^{r} \alpha\beta x_i^{\alpha-1} e^{-\beta x_i^{\alpha}}$$

After simplifications it becomes

$$L(\mathrm{x}|\alpha,\beta) \propto \alpha^r \beta^r e^{(\alpha-1)\sum_{i=1}^{r}\log x_i} e^{-\beta\left(\sum_{i=1}^{r} x_i^{\alpha} + (n-r)x_r^{\alpha}\right)} \tag{4}$$

3. Prior and Posterior Distribution

Here, the non-informative and informative priors have been assumed for the parameters α and β to obtain the combined posterior distribution for the said parameters. Suppose the non-informative prior for the parameters α is $\pi_1(\alpha) \propto 1, \alpha > 0$ and that for the β is $\pi_2(\beta) \propto 1, \beta > 0$. Therefore, the combined prior distribution for the model parameters is

$$\pi_1(\alpha,\beta) \propto 1\ ,\ \alpha,\beta > 0 \tag{5}$$

Considering (4) and (5) the combined posterior distribution for the model parameters is

$$g_1(\alpha,\beta|\mathrm{x}) = \frac{L(\mathrm{x}|\alpha,\beta)\pi_1(\alpha,\beta)}{\int_0^{\infty}\int_0^{\infty} L(\mathrm{x}|\alpha,\beta)\pi_1(\alpha,\beta)\,d\alpha\,d\beta} \tag{6}$$

Further, suppose the informative priors for the model parameters are $\pi_3(\alpha) \propto \alpha^{a-1}e^{-b\alpha}, \alpha > 0$

and $\pi_4(\beta) \propto \beta^{c-1}e^{-d\beta}, \beta > 0$ respectively. Therefore, the combined informative prior for the said parameters is

$$\pi_2(\alpha,\beta) \propto \alpha^{a-1}\beta^{c-1}\alpha^{a-1}e^{-d\beta}, \alpha,\beta > 0 \tag{7}$$

Considering (4) and (7) the combined posterior distribution of the model parameters is

$$g_2(\alpha,\beta|\mathrm{x}) = \frac{L(\mathrm{x}|\alpha,\beta)\pi_2(\alpha,\beta)}{\int_0^{\infty}\int_0^{\infty} L(\mathrm{x}|\alpha,\beta)\pi_2(\alpha,\beta)\,d\alpha\,d\beta} \tag{8}$$

4. Loss Functions

In this section, two loss functions have been proposed for the Bayesian estimation of the parameters of the model given in (1). The introduction of these loss functions are as follows.

Squared Error Loss Function (SELF): Legendre (1805) and Gauss (1810) proposed the SELF that can be defined as $L(\theta,\theta_S) = (\theta-\theta_S)^2$, where θ is a parameter. Based on SELF, the Bayes estimator for the parameter θ is $\theta_S = \mathrm{E}(\theta)$.

Precautionary Loss Function (PLF): The PLF has been introduced by Norstrom (1996), which can be formulated as $L(\theta_P,\theta) = \theta_P^{-1}(\theta_P-\theta)^2$. Using PLF the Bayes estimator is $\theta_P = \left[\mathrm{E}(\theta^2)\right]^{\frac{1}{2}}$.

5. Bayesian Estimation

This section reports the Bayesian inference of the Weibull model under right censored samples. From (6) and (8) it can be seen that the Bayes estimators under the proposed loss functions do not provide explicit expressions, therefore we have considered Quadrature method, Gibbs sampler, importance sampling, Tierney and Kadane's approximation and Lindley's approximation for the numerical estimation.

5.1 Quadrature Method

The posterior estimators for the model parameters under SELF from the posterior distribution using non-informative prior distribution are

$$\alpha_{NS} = \int_0^\infty\int_0^\infty \alpha g_1(\alpha,\beta|\mathrm{x})\,d\alpha d\beta \tag{9}$$

$$\beta_{NS} = \int_0^\infty\int_0^\infty \beta g_1(\alpha,\beta|\mathrm{x})\,d\alpha d\beta \tag{10}$$

The posterior estimators for the model parameters under SELF from the posterior distribution using informative prior distribution are

$$\alpha_{IS} = \int_0^\infty\int_0^\infty \alpha g_2(\alpha,\beta|\mathrm{x})\,d\alpha d\beta \tag{11}$$

$$\beta_{IS} = \int_0^\infty\int_0^\infty \beta g_2(\alpha,\beta|\mathrm{x})\,d\alpha d\beta \tag{12}$$

Note that from (09)-(12), the explicit expressions for the Bayes estimators are impossible, therefore we have used the Quadrature method for the numerical solution of the estimates. Consider the posterior density $g(\alpha,\beta)$, where α and β are the parameters. The Quadrature method gives following solution to the equations (09)-(12).

$$\int_0^\infty \int_0^\infty g(\alpha,\beta)\, d\alpha\, d\beta = \sum_{i=0}^{r}\sum_{i=0}^{r} w_i g(\alpha_i,\beta_i) \tag{13}$$

where w_i, w_j, ..., w_k are the increments and α_i and β_i are the quadrature points. We have developed a program in the software mathematica to obtain Bayes estimates for the model parameters using proposed loss functions considering informative and non-informative priors.

5.2 Gibbs Sampler

Consider a posterior distribution with two parameters α and β, where 'x' denote the data. Further suppose that the full conditional densities $g(\alpha|\beta,\mathrm{x})$ and $g(\beta|\alpha,\mathrm{x})$ are tractable, and we aim to obtain $g(\alpha|\mathrm{x})$ and $g(\beta|\mathrm{x})$. In Gibbs sampler, first we choose some initial values for the parameters α and β and denote them by α_0,β_0. These can be any reasonable values of α,β and then we take draws from the two conditional distributions in the following sequence

$$\begin{matrix}\alpha_1 \sim g(\alpha|\beta_0,\mathrm{x}) \\ \beta_1 \sim g(\beta|\alpha_1,\mathrm{x})\end{matrix}, \quad \begin{matrix}\alpha_2 \sim g(\alpha|\beta_1,\mathrm{x}) \\ \beta_2 \sim g(\beta|\alpha_2,\mathrm{x})\end{matrix}, \ldots, \quad \begin{matrix}\alpha_m \sim g(\alpha|\beta_{m-1},\mathrm{x}) \\ \beta_m \sim g(\beta|\alpha_m,\mathrm{x})\end{matrix}$$

As values at the step (m) depend on the step (m-1), therefore this sequence will construct a Markov chain. For implementation of Gibbs sampler for the posterior distributions (6) and (8), we need to extract the conditional distributions from these posterior distributions for each unknown parameter.

The conditional distribution of the parameter β given α under non-informative prior is

$$g_1(\beta|\alpha,\mathrm{x}) \propto \beta^m e^{-\beta\left(\sum_{i=1}^{r} x_i^\alpha + (n-r)x_r^\alpha\right)}$$

The density becomes $\beta \sim Gamma\left(m+1, \sum_{i=1}^{r} x_i^\alpha + (n-r)x_r^\alpha\right)$ (14)

The conditional distribution of the parameter α given β under non-informative prior is

$$g_1(\alpha|\beta,\mathrm{x}) \propto \alpha^m e^{(\alpha-1)\sum_{i=1}^{r}\log x_i - \beta\left(\sum_{i=1}^{r} x_i^\alpha + (n-r)x_r^\alpha\right)} \tag{15}$$

which is a log-concave as:

$$\frac{\partial^2 \ln g(\alpha|\beta,\mathrm{x})}{\partial\alpha^2} = -\frac{m}{\alpha^2} - \beta\left\{(n-r)x_r^\alpha(\log x_r)^2 + \sum_{i=1}^{r} x_i^\alpha(\log x_i)^2\right\} < 0 \tag{16}$$

Algorithm

Step 1: Generate α from the log-concave density function (12) using the method proposed by Devroye (1984).

Step 2: Generate β from $\beta \sim Gamma\left(m+1, \sum_{i=1}^{r} x_i^{\alpha} + (n-r) x_r^{\alpha} \right)$

Step 3: Obtain the posterior samples $(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)$ by repeating the Steps 1 and 2, m times.

Now, the Bayes estimate and posterior risks under SELF can be obtained by using the formulae $\theta_S = \frac{\sum_{i=1}^{r} \theta_i}{r}$ and $\rho(\theta_S) = \sum_{i=1}^{r} (\theta_i - \theta_S)^2$. Similarly the Bayes estimate and posterior risk under PLF can be calculated by using the formulae $\theta_P = \sqrt{\frac{\sum_{i=1}^{r} \theta_i^2}{r}}$ and $\rho(\theta_P) = 2\left\{ \sqrt{\frac{\sum_{i=1}^{r} \theta_i^2}{r}} - \frac{\sum_{i=1}^{r} \theta_i}{r} \right\}$. Similar process can be followed to implement Gibbs sampler for the posterior distribution under informative prior.

The graphs regarding the history for generation of Gibbs samples, marginal densities and percentile points for the parameters α = 0.5 and β = 0.5 for n = 30 have been presented in the figures 1-6 in the followings.

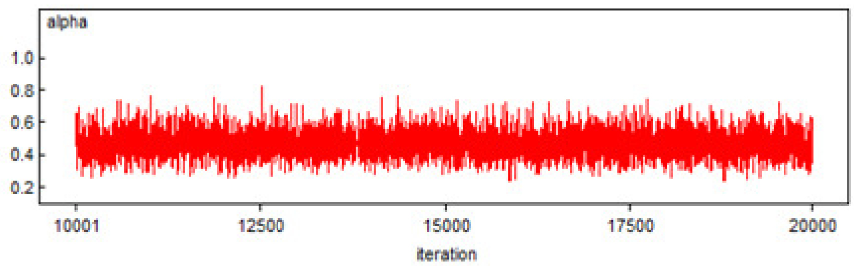

Figure 01: The graph showing the history for generation of Gibbs samples for the parameter α

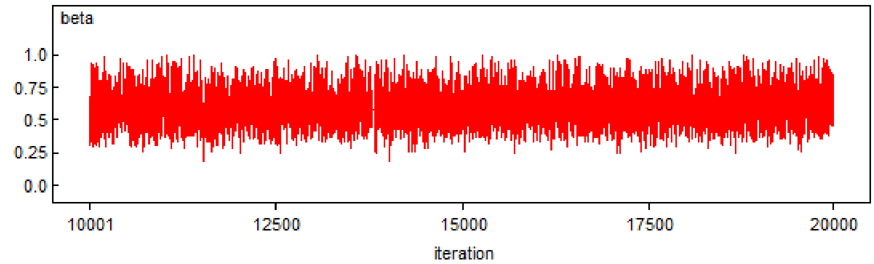

Figure 02: The graph showing the history for generation of Gibbs samples for the parameter β

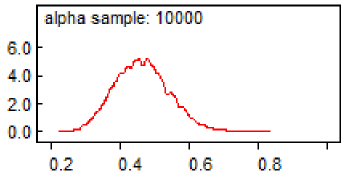

Figure 03: The marginal density for the parameter α

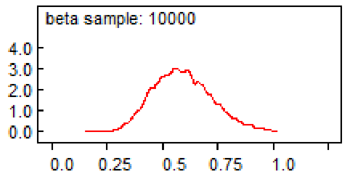

Figure 04: The marginal density for the parameter β

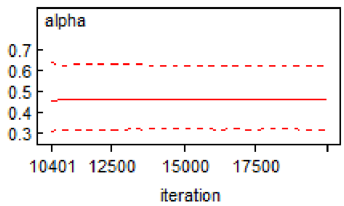

Figure 05: Percentile points graph for the parameter α

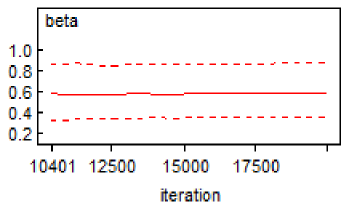

Figure 06: Percentile points graph for the parameter β

5.3 Importance Sampling

Here, the importance sampling has been used to derive the approximate Bayesian estimators of the model parameters under the proposed loss functions. Using importance sampling, the Bayes estimator for the parameter θ under SELF is presented as $\theta_S = \dfrac{E'\left[\theta h(\theta)\right]}{E'\left[h(\theta)\right]}$ and $\theta_P = \sqrt{\dfrac{E'\left\{\theta^2 h(\theta)\right\}}{E'\left\{h(\theta)\right\}}}$,

where $h(\theta)$ is any function of the parameter θ and E' is the expectation with respect to distribution of the parameter θ.

For simplicity, the likelihood function in (6) can be written as

$$L\left(\alpha,\beta\left|\mathrm{x}\right.\right)\propto\alpha^{r}\beta^{r}e^{-\alpha\left(-\sum_{i=1}^{r}\log x_{i}\right)}e^{-\beta\left(\sum_{i=1}^{r}x_{i}^{\alpha}+(n-r)x_{r}^{\alpha}\right)} \quad (17)$$

Now the posterior distribution under non-informative prior can be written as

$$g_{1}\left(\alpha,\beta\left|\mathrm{x}\right.\right)\propto\alpha^{r}\beta^{r}\left\{\psi\left(x\right)\right\}^{-r}e^{-\alpha\xi(x)}e^{-\beta\psi(x)} \quad (18)$$

where $\xi\left(x\right)=-\sum_{i=1}^{r}\log x_{i}$ and $\psi\left(x\right)=\sum_{i=1}^{r}x_{i}^{\alpha}+\left(n-r\right)x_{r}^{\alpha}$. Now α follows gamma distribution with parameters r and $\xi\left(x\right)$ and the conditional distribution of β given α is $h_{1}\left(\beta\left|\alpha,\mathrm{x}\right.\right)\propto\left\{\psi\left(x\right)\right\}^{-r}\beta^{r}e^{-\beta\psi(x)}$. Hence (18) can be decomposed to have following form

$$g_{1}\left(\alpha,\beta\left|\mathrm{x}\right.\right)\propto h_{11}\left(\alpha\left|\mathrm{x}\right.\right)h_{12}\left(\beta\left|\alpha,\mathrm{x}\right.\right)h_{13}\left(\alpha,\beta\left|\mathrm{x}\right.\right) \quad (19)$$

where, $h_{11}\left(\alpha\left|\mathrm{x}\right.\right)\sim\alpha^{r}e^{-\alpha\xi(x)}$, $h_{12}\left(\beta\left|\alpha,\mathrm{x}\right.\right)\sim\beta^{r}e^{-\beta\psi(x)}$ and $h_{13}\left(\alpha,\beta\left|\mathrm{x}\right.\right)\sim\left\{\psi\left(x\right)\right\}^{-r}$

Hence, under importance sampling the Bayes estimators for the parameter α under non-informative prior using SELF and PLF are respectively

$$\alpha_{NS}=\frac{\mathrm{E}'\left[\alpha h_{13}\left(\alpha,\beta\left|\mathrm{x}\right.\right)\right]}{\mathrm{E}'\left[h_{13}\left(\alpha,\beta\left|\mathrm{x}\right.\right)\right]}\text{ and }\alpha_{NP}=\sqrt{\frac{\mathrm{E}'\left[\alpha^{2}h_{13}\left(\alpha,\beta\left|\mathrm{x}\right.\right)\right]}{\mathrm{E}'\left[h_{13}\left(\alpha,\beta\left|\mathrm{x}\right.\right)\right]}},$$

where E' is the expectation with respect to $gamma\left(r+1,\xi\left(x\right)\right)$.

Therefore, using importance sampling the Bayes estimators for the parameter β under non-informative prior using SELF and PLF are respectively

$$\beta_{NS}=\frac{\mathrm{E}'\left[\beta h_{13}\left(\alpha,\beta\left|\mathrm{x}\right.\right)\right]}{\mathrm{E}'\left[h_{13}\left(\alpha,\beta\left|\mathrm{x}\right.\right)\right]}\text{ and }\beta_{NP}=\sqrt{\frac{\mathrm{E}'\left[\beta^{2}h_{13}\left(\alpha,\beta\left|\mathrm{x}\right.\right)\right]}{\mathrm{E}'\left[h_{13}\left(\alpha,\beta\left|\mathrm{x}\right.\right)\right]}},$$

where E' is the expectation with respect to $gamma\left(r+1,\psi\left(x\right)\right)$.

Now the posterior distribution under informative prior can be written as

$$g_{2}\left(\alpha,\beta\left|\mathrm{x}\right.\right)\propto\alpha^{\nu-1}\beta^{\tau-1}\left\{\mu\left(x\right)\right\}^{-\tau+1}e^{-\alpha\eta(x)}e^{-\beta\mu(x)} \quad (20)$$

where $\nu=r+a$, $\tau=r+c$, $\eta\left(x\right)=\xi\left(x\right)+b$ and $\mu\left(x\right)=\psi\left(x\right)+d$. Now α follows gamma distribution with parameters ν and $\eta\left(x\right)$ and the conditional distribution of β given α is $h_{2}\left(\beta\left|\alpha,\mathrm{x}\right.\right)\propto\left\{\mu\left(x\right)\right\}^{-\tau+1}\beta^{\tau-1}e^{-\beta\mu(x)}$. Therefore the posterior distribution can be decomposed to have following form

$$g_{2}\left(\alpha,\beta\left|\mathrm{x}\right.\right)\propto h_{21}\left(\alpha\left|\mathrm{x}\right.\right)h_{22}\left(\beta\left|\alpha,\mathrm{x}\right.\right)h_{23}\left(\alpha,\beta\left|\mathrm{x}\right.\right) \quad (21)$$

where, $h_{21}\left(\alpha|\mathrm{x}\right)\sim\alpha^{\nu-1}e^{-\alpha\eta(x)}$, $h_{22}\left(\beta|\alpha,\mathrm{x}\right)\sim\beta^{\tau-1}e^{-\beta\mu(x)}$ and $h_{23}\left(\alpha,\beta|\mathrm{x}\right)\sim\left\{\mu\left(x\right)\right\}^{-\tau+1}$

Again, using importance sampling the Bayes estimators for the parameter α under informative prior using SELF and PLF are respectively

$$\alpha_{IS}=\frac{\mathrm{E}'\left[\alpha h_{23}\left(\alpha,\beta|\mathrm{x}\right)\right]}{\mathrm{E}'\left[h_{23}\left(\alpha,\beta|\mathrm{x}\right)\right]}\text{ and }\alpha_{IP}=\sqrt{\frac{\mathrm{E}'\left[\alpha^2 h_{23}\left(\alpha,\beta|\mathrm{x}\right)\right]}{\mathrm{E}'\left[h_{23}\left(\alpha,\beta|\mathrm{x}\right)\right]}},$$

where E' is the expectation with respect to $gamma\left(\nu,\eta\left(x\right)\right)$.

Similarly, using importance sampling the Bayes estimators for the parameter β under informative prior using SELF and PLF are respectively

$$\beta_{IS}=\frac{\mathrm{E}'\left[\beta h_{23}\left(\alpha,\beta|\mathrm{x}\right)\right]}{\mathrm{E}'\left[h_{23}\left(\alpha,\beta|\mathrm{x}\right)\right]}\text{ and }\beta_{IP}=\sqrt{\frac{\mathrm{E}'\left[\beta^2 h_{23}\left(\alpha,\beta|\mathrm{x}\right)\right]}{\mathrm{E}'\left[h_{23}\left(\alpha,\beta|\mathrm{x}\right)\right]}},$$

where E' is the expectation with respect to $gamma\left(\tau,\mu\left(x\right)\right)$.

5.4 Lindley's Approximation

In this section, the Lindley's approximation has been proposed to have the approximate Bayes estimators for the parameters of the said model. Based on the Lindley's approximation, due to Lindley (1980), the expression of the form

$$I\left(\theta\right)=\mathrm{E}\left[g\left(\alpha,\beta\right)\right]=\frac{\int\limits_{(\alpha,\beta)} g\left(\alpha,\beta\right)e^{l\left(\alpha,\beta|\mathrm{x}\right)+G\left(\alpha,\beta\right)}d\left(\alpha,\beta\right)}{\int\limits_{(\alpha,\beta)} e^{l\left(\alpha,\beta|\mathrm{x}\right)+G\left(\alpha,\beta\right)}d\left(\alpha,\beta\right)} \tag{22}$$

where $g\left(\alpha,\beta\right)$ is any function of α or β, $l\left(\alpha,\beta|\mathrm{x}\right)$ is the log-likelihood function and $G\left(\alpha,\beta\right)$ is the logarithmic of joint prior for the parameters α and β, can be evaluated as

$$I\left(\theta\right)=g\left(\hat{\alpha},\hat{\beta}\right)+\left(g_1d_1+g_2d_2+d_3+d_4\right)+\frac{1}{2}\left(A_1B_1+A_2B_2\right) \tag{23}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are MLEs of the parameters α and β respectively,

$B_i=g_1S_{i1}+g_2S_{i2}$, $A_i=S_{11}L_{11i}+S_{22}L_{22i}+2S_{12}L_{12i}$, $d_i=P_1S_{i1}+P_2S_{i2}$, $i=1,2$,

$d_3=g_{12}S_{12}$, $d_4=\frac{1}{2}\left(g_{11}S_{11}+g_{22}S_{22}\right)$, $P_i=\frac{\partial G\left(\theta\right)}{\partial\theta_i}$, $i=1,2$, $\theta=\left(\alpha,\beta\right)$,

$g_{ij}=\frac{\partial^2 g\left(\theta\right)}{\partial\theta_i\partial\theta_j}$, $L_{ij}=\frac{\partial^2 L\left(\theta\right)}{\partial\theta_i\partial\theta_j}$, $i,j=1,2$, $L_{ijk}=\frac{\partial^3 L\left(\theta\right)}{\partial\theta_i\partial\theta_j\partial\theta_k}$, $i,j,k=1,2$

and S_{ij} is the $(i, j)^{th}$ element of the inverse of the matrix $\{L_{ij}\}$, all evaluated at the MLEs of the parameters.

Now, the log-likelihood function from (4) can be obtained as

$$L(\alpha, \beta | \mathrm{x}) \propto r \log \alpha + r \log \beta + \alpha \sum_{i=1}^{r} \log x_i - \beta \left\{ \sum_{i=1}^{r} x_i^{\alpha} + (n-r) x_r^{\alpha} \right\} \tag{24}$$

The maximum likelihood estimates (MLEs) of the parameters α and β can be obtained by differentiating (24) with respect to α and β and equating to zero respectively as

$$\frac{r}{\alpha} + \sum_{i=}^{r} \log x_i - \beta \left\{ (n-r) x_r^{\alpha} \log x_r + \sum_{i=1}^{r} x_i^{\alpha} \log x_i \right\} = 0 \tag{25}$$

$$\frac{r}{\beta} - (n-r) x_r^{\alpha} - \sum_{i=1}^{r} x_i^{\alpha} = 0 \tag{26}$$

As the explicit expressions for the parameters using the equations (25)-(26) is not possible, the iterative procedures have been used to obtain the MLEs for the model parameters.

The second and third order derivatives of the log-likelihood function are presented in the followings

$$L_{11} = -\frac{r}{\hat{\alpha}^2} - \hat{\beta} \left\{ (n-r) x_r^{\hat{\alpha}} (\log x_r)^2 + \sum_{i=1}^{r} x_i^{\hat{\alpha}} (\log x_i)^2 \right\} \tag{27}$$

$$L_{12} = -(n-r) x_r^{\hat{\alpha}} \log x_r - \sum_{i=1}^{r} x_i^{\hat{\alpha}} \log x_i \tag{28}$$

$$L_{22} = -\frac{r}{\hat{\beta}^2} \tag{29}$$

$$L_{111} = \frac{2r}{\hat{\alpha}^3} + \hat{\beta} \left\{ (n-r) x_r^{\hat{\alpha}} (\log x_r)^3 + \sum_{i=1}^{r} x_i^{\hat{\alpha}} (\log x_i)^3 \right\} \tag{30}$$

$$L_{212} = L_{221} = L_{122} = 0 \tag{31}$$

$$L_{121} = L_{112} = L_{211} = -(n-r) x_r^{\hat{\alpha}} (\log x_r)^2 - \sum_{i=1}^{r} x_i^{\hat{\alpha}} (\log x_i)^2 \tag{32}$$

$$L_{222} = \frac{2r}{\hat{\beta}^3} \tag{33}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the MLEs of the parameters α and β respectively.

Based on the second order derivates, the matrix $\{L_{ij}\}$ is

$$\left\{L_{ij}\right\} = -\begin{bmatrix} L_{11} & L_{21} \\ L_{12} & L_{22} \end{bmatrix} \text{and its inverse is} \left\{L_{ij}\right\}^{-1} = \begin{bmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{bmatrix} \quad (34)$$

Therefore, the Bayes estimators considering the Lindley's approximation for the model parameters assuming non-informative prior using SELF are respectively presented as

$$\alpha_{NS} = \hat{\alpha} + \frac{1}{2}\left(S_{11}A_1 + S_{21}A_2\right) \quad (35)$$

$$\beta_{NS} = \hat{\beta} + \frac{1}{2}\left(S_{12}A_1 + S_{22}A_2\right) \quad (36)$$

Further, the Bayes estimators considering the Lindley's approximation for the model parameters under non-informative prior using PLF are respectively presented as

$$\alpha_{NP} = \sqrt{\hat{\alpha}^2 + \frac{1}{2}\left(2\hat{\alpha}S_{11} + S_{11}A_1 + S_{21}A_2\right)} \quad (37)$$

$$\beta_{NP} = \sqrt{\hat{\beta}^2 + \frac{1}{2}\left(2\hat{\beta}S_{22} + S_{12}A_1 + S_{22}A_2\right)} \quad (38)$$

Now, using Lindley's approximation the Bayes estimators for the parameters α and β under informative prior using SELF are respectively presented as

$$\alpha_{IS} = \hat{\alpha} + \frac{1}{2}\left(S_{11}A_1 + S_{21}A_2\right) + \left(\frac{a-1}{\hat{\alpha}} - b\right)S_{11} + \left(\frac{c-1}{\hat{\beta}} - d\right)S_{12} \quad (39)$$

$$\beta_{IS} = \hat{\beta} + \frac{1}{2}\left(S_{12}A_1 + S_{22}A_2\right) + \left(\frac{a-1}{\hat{\alpha}} - b\right)S_{21} + \left(\frac{c-1}{\hat{\beta}} - d\right)S_{22} \quad (40)$$

Again, the Bayes estimators considering the Lindley's approximation for the model parameters under informative prior using PLF are respectively presented as

$$\alpha_{IP} = \sqrt{\hat{\alpha}^2 + \frac{1}{2}\left(2\hat{\alpha}S_{11} + S_{11}A_1 + S_{21}A_2\right) + \left(\frac{a-1}{\hat{\alpha}} - b\right)S_{11} + \left(\frac{c-1}{\hat{\beta}} - d\right)S_{12}} \quad (41)$$

$$\beta_{IP} = \sqrt{\hat{\beta}^2 + \frac{1}{2}\left(2\hat{\beta}S_{22} + S_{12}A_1 + S_{22}A_2\right) + \left(\frac{a-1}{\hat{\alpha}} - b\right)S_{21} + \left(\frac{c-1}{\hat{\beta}} - d\right)S_{22}} \quad (42)$$

5.5 Tierney and Kadane's Approximation

This section considers the Tierney and Kadane's approximation for the approximate Bayesian analysis of the model parameters of the right censored Weibull distribution. In case of non-informative prior, consider $Q(\alpha,\beta) = \log \pi_1(\alpha,\beta) + \log L(\mathrm{x}|\alpha,\beta)$ where $\log \pi_1(\alpha,\beta)$ is the logarithmic of the joint non-informative prior for the parameters (α,β) and $\log L(\mathrm{x}|\alpha,\beta)$ is the logarithmic of likelihood function given in (4). Further consider

$H(\alpha,\beta)=Q(\alpha,\beta)/n$ and $H^*(\alpha,\beta)=\left[\log h(\alpha,\beta)+Q(\alpha,\beta)\right]/n$, where $\log h(\alpha,\beta)$ is the logarithmic of the function of the parameter(s) α or β. Then according to Tierney and Kadane (1986) the expression $\mathrm{E}\left\{h(\alpha,\beta|\mathrm{x})\right\}$ using (6) can be re-expressed as

$$\mathrm{E}\left\{h(\alpha,\beta|\mathrm{x})\right\}=\frac{\int_0^\infty\int_0^\infty e^{nH^*(\alpha,\beta)}d\alpha d\beta}{\int_0^\infty\int_0^\infty e^{nH(\alpha,\beta)}d\alpha d\beta} \tag{43}$$

Now using the Laplace's method, the approximation for $\mathrm{E}\left\{h(\alpha,\beta|\mathrm{x})\right\}$ can be given as

$$\hat{h}(\alpha,\beta)=\left[\frac{\det\Sigma^*}{\det\Sigma}\right]^{1/2}\exp\left[n\left\{H^*\left(\hat{\alpha}^*,\hat{\beta}^*\right)-H\left(\hat{\alpha},\hat{\beta}\right)\right\}\right] \tag{44}$$

where $\left(\hat{\alpha}^*,\hat{\beta}^*\right)$ and $\left(\hat{\alpha},\hat{\beta}\right)$ maximize $H^*(\alpha,\beta)$ and $H(\alpha,\beta)$ respectively, and Σ^* and Σ are the negatives of the inverse Hessians of $H^*(\alpha,\beta)$ and $H(\alpha,\beta)$ evaluated at $\left(\hat{\alpha}^*,\hat{\beta}^*\right)$ and $\left(\hat{\alpha},\hat{\beta}\right)$ respectively.

Here we have

$$H(\alpha,\beta)=\frac{1}{n}\left[k+r\log\alpha+r\log\beta+\alpha\sum_{i=1}^{r}\log x_i-\beta\left\{\sum_{i=1}^{r}x_i^\alpha+(n-r)x_r^\alpha\right\}\right] \tag{45}$$

$$H^*(\alpha,\beta)=\frac{1}{n}\left[k+\log h(\alpha,\beta)+r\log\alpha+r\log\beta+\alpha\sum_{i=1}^{r}\log x_i-\beta\left\{\sum_{i=1}^{r}x_i^\alpha+(n-r)x_r^\alpha\right\}\right] \tag{46}$$

where k is any constant independent of the parameters α and β.

$$\frac{\partial H(\alpha,\beta)}{\partial\alpha}=\frac{1}{n}\left[\frac{r}{\alpha}+\sum_{i=1}^{r}\log x_i-\beta\left\{\sum_{i=1}^{r}x_i^\alpha\log x_i+(n-r)x_r^\alpha\log x_r\right\}\right]=0 \tag{47}$$

$$\frac{\partial H(\alpha,\beta)}{\partial\beta}=\frac{1}{n}\left[\frac{r}{\beta}-\left\{\sum_{i=1}^{r}x_i^\alpha+(n-r)x_r^\alpha\right\}\right]=0 \tag{48}$$

Now, $\left(\hat{\alpha},\hat{\beta}\right)$ can be obtained by solving (45) and (46). The second order derivatives from $H(\alpha,\beta)$ have been presented in (49)-(51).

$$\frac{\partial^2H(\alpha,\beta)}{\partial\alpha^2}=-\frac{1}{n}\left[\frac{r}{\alpha^2}+\beta\left\{\sum_{i=1}^{r}x_i^\alpha(\log x_i)^2+(n-r)x_r^\alpha(\log x_r)^2\right\}\right] \tag{49}$$

$$\frac{\partial^2 H(\alpha,\beta)}{\partial\beta^2} = -\frac{1}{n}\left(\frac{r}{\beta^2}\right) \tag{50}$$

$$\frac{\partial^2 H(\alpha,\beta)}{\partial\alpha\partial\beta} = -\frac{1}{n}\left\{\sum_{i=1}^{r} x_i^{\alpha}\log x_i + (n-r)x_r^{\alpha}\log x_r\right\} \tag{51}$$

The determinant for the negative of the inverse Hessian of $H(\alpha,\beta)$ evaluated at $(\hat{\alpha},\hat{\beta})$ is $\det\Sigma = \left(H_{11}H_{22} - H_{12}^2\right)^{-1}$

where

$$H_{11} = \left.\frac{\partial^2 H(\alpha,\beta)}{\partial\alpha^2}\right|_{\hat{\alpha},\hat{\beta}} = \frac{1}{n}\left[\frac{r}{\hat{\alpha}^2} + \hat{\beta}\left\{\sum_{i=1}^{r} x_i^{\hat{\alpha}}\left(\log x_i\right)^2 + (n-r)x_r^{\hat{\alpha}}\left(\log x_r\right)^2\right\}\right] \tag{52}$$

$$H_{22} = \left.\frac{\partial^2 H(\alpha,\beta)}{\partial\beta^2}\right|_{\hat{\alpha},\hat{\beta}} = \frac{1}{n}\left(\frac{r}{\hat{\beta}^2}\right) \tag{53}$$

$$H_{12} = \left.\frac{\partial^2 H(\alpha,\beta)}{\partial\alpha\partial\beta}\right|_{\hat{\alpha},\hat{\beta}} = \frac{1}{n}\left\{\sum_{i=1}^{r} x_i^{\hat{\alpha}}\log x_i + (n-r)x_r^{\hat{\alpha}}\log x_r\right\} \tag{54}$$

Similarly MLEs and the second order derivatives for the $H^*(\alpha,\beta)$ can be obtained. Further the Bayes estimators and the posterior risks under informative prior can be obtained in a similar manner.

6. Simulation Study

In this section, the simulation study has been performed to assess the features of various posterior estimators. The performance of loss functions, prior distribution and approximation methods have been compared under different parametric values and the sample sizes based on the magnitudes of the posterior risks. The parametric space used is (α , β) = {(0.5, 0.5), (1, 1), (1.5, 1.5), (2, 2)}. Different sample sizes have been considered. The inverse transformation technique has been used to generate the samples from the Weibull model. The hyper-parameters, in the prior distributions, have been assumed to be so that the prior mean is equal to the true parametric values. For example, for α = β = 0.50, we have assumed a = 0.50, b = 1.00, c = 0.50 and d = 1.00 since E(α) = a/b = 0.50 and E(β) = c/d = 0.50. The test termination point is considered to be so that the various percentages (5%, 10% and 205) of the observations in the respective samples have been censored. Following abbreviations have been used in the tables below: AM = Approximation Methods, LF = Loss Functions, QM = Quadrature Method, GS = Gibbs Sampler, IM = Importance Sampling, LA = Lindley's Approximation, TKA = Tierney and Kadane's Approximation, IP = Informative prior.

The numerical results for the simulation study have been reported in the tables 1-6 presented in the appendix-A and in the figures 7-8. From the tables 1-7 and figures 7-8, the findings of the simulation study can be summarized in the following comments

1. The posterior estimates (B.Es) tend to come closer and closer to the true figures and the corresponding magnitudes of posterior risks (P.Rs) becomes smaller and smaller for increased the sample sizes. Hence the said estimates are consistent. However the bigger choice of the original parametric values has a negative impact on the convergence of the estimates and results in greater amounts of the posterior risks.
2. Based on the amounts of the posterior risks the performance of the SELF is superior to PLF whenever the true parametric values are less than one and vice versa. This property is true for the estimation under both priors and for all approximation methods.
3. The performance of the informative prior is better than non-informative prior throughout.
4. Whenever the true parametric values are less than one, the performance of the importance sampling seems better than all its counter parts. However, when the true parametric values are greater than or equals to one then the estimation under TKA is found superior among all approximation methods. The other approximation techniques have mixed behavior in terms of performance.
5. In conclusion, when the true parametric value is less than one, the estimation under the combination of SELF, informative prior and importance sampling is superior to its counterparts. In case when true parametric values are greater than or equals to one, then estimation under the combination of PLF, informative prior and TKA is superior to other competitors.

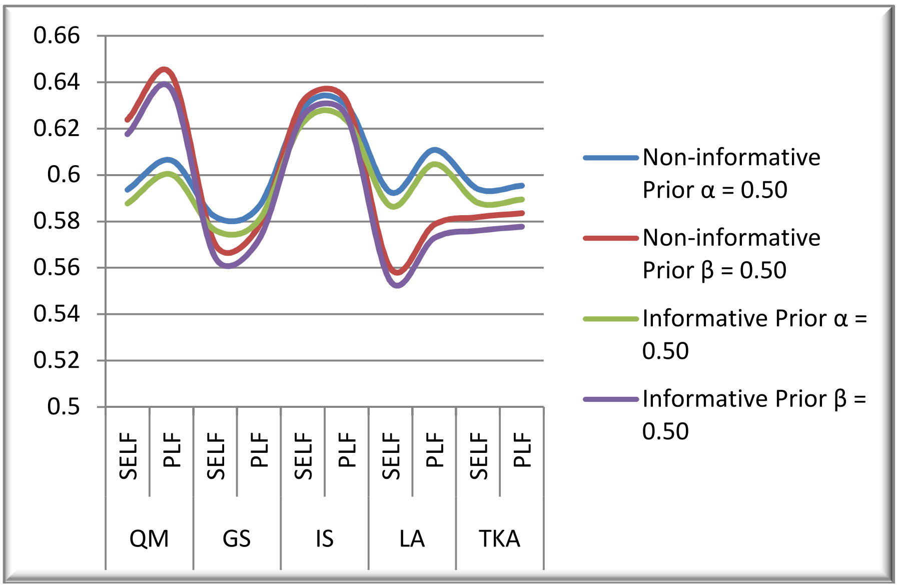

Figure 07: Bayes estimates for the parameters under different priors and loss functions using n = 30

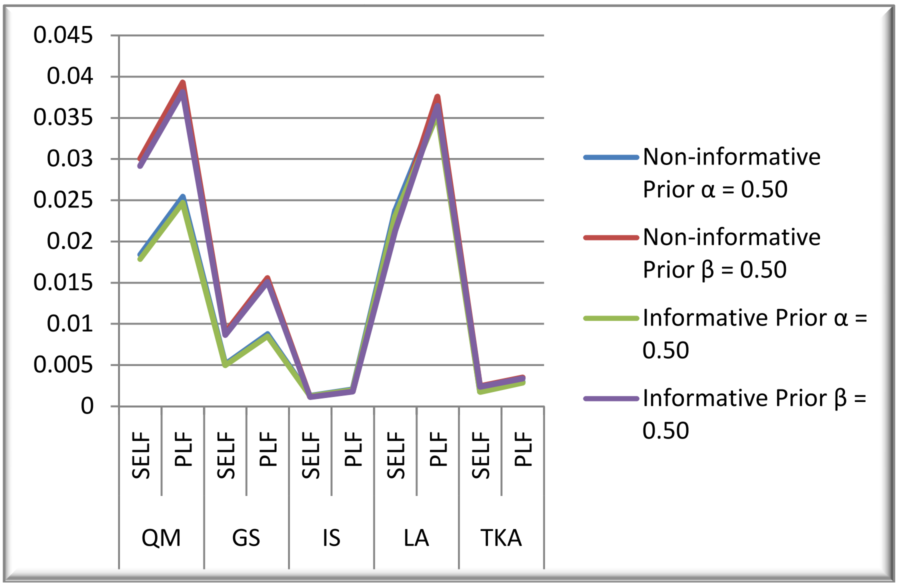

Figure 08: Posterior risks for the parametric estimates under different priors and loss functions using n = 30

7. Real Life Example

In this section we have analyzed the real dataset reported by Fleming and Harrington (1991). This dataset is regarding Group IV of the primary biliary cirrhosis (PBC) liver study group conducted by Mayo Clinic. The failure time has been considered as time to expiry of the PBC patients. The survival periods (in days) for the 36 patients having highest grad of bilirubinare: 77, 400, 71, 859, 334, 1037, 1427, 733, 549, 51, 1413, 1170, 41, 890, 853, 1882+, 216, 1067+, 223, 131, 1827, 1297, 2540, 264, 930, 797, 1329+, 1350, 264, 1191, 943, 130, 974, 1765+, 790, 1320+.

The '+' with the observations indicates the censored observations. The Chi-square and the Kolomogorov–Smirnov tests have been used to verify that the dataset come from the Weibull model. The resultant Bayesian estimates have been reported in the table-8 below. For simplicity, in case of the informative prior, the values of the hyper-parameters have been assumed to be $a = b = c = d = 1$ for the estimation.

AM	Non-informative Prior			
	SELF		PLF	
	α	β	α	β
QM	0.95796 (0.18479)	0.00229 (0.00330)	0.97803 (0.25776)	0.00236 (0.00499)
GS	0.93930 (0.05147)	0.00210 (0.00098)	0.94594 (0.08895)	0.00212 (0.00197)
IS	0.97478	0.00232	0.98598	0.00232

	(0.01307)	(0.00013)	(0.02099)	(0.00023)
LA	0.91142 (0.17642)	0.00206 (0.00228)	0.93039 (0.29256)	0.00211 (0.00433)
TKA	0.95847 (0.01810)	0.00214 (0.00027)	0.96043 (0.03016)	0.00214 (0.00044)
	Informative Prior			
QM	0.94838 (0.17925)	0.00227 (0.00320)	0.96825 (0.25003)	0.00234 (0.00484)
GS	0.92991 (0.04993)	0.00207 (0.00095)	0.93648 (0.08628)	0.00210 (0.00191)
IS	0.96463 (0.01268)	0.00230 (0.00012)	0.96582 (0.02036)	0.00230 (0.00023)
LA	0.90230 (0.17113)	0.00204 (0.00222)	0.92109 (0.28378)	0.00209 (0.00420)
TKA	0.94888 (0.01755)	0.00212 (0.00026)	0.95082 (0.02926)	0.00212 (0.00043)

Table 8: B. E.s and P.R.s (in parenthesis) under non-informative and informative priors using real data set

From the Table-8, it can be seen that estimation under the combination of SELF, informative prior and importance sampling is better than its counterparts. This in accordance with the results derived from the simulations. Hence, the results from the analysis of the real dataset further strengthened the findings of the simulation study.

8. Conclusion

In this paper, the posterior inference of the right-censored samples from the Weibull model has been reported. Unfortunately, the Bayes estimators for the parameters of the Weibull model are not possible in the closed form. Therefore, we have considered five approximation methods to calculate the numerical estimates of the model parameters. The findings from the analysis of the simulated and real dataset suggest that (i) when the true parametric value is less than one, the estimation under the combination of SELF, informative prior and importance sampling is superior to its counterparts, (ii) and in case when true parametric values are greater than or equals to one, then estimation under the combination of PLF, informative prior and TKA is superior to its competitors.

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Appendix-A

AM	LF	α = 0.50	β = 0.50	α = 1.00	β = 1.00
QM	SELF	0.59359 (0.01841)	0.62376 (0.03006)	1.19305 (0.04101)	1.15497 (0.06885)
	PLF	0.60631 (0.02544)	0.64341 (0.03929)	1.21356 (0.03588)	1.18095 (0.05196)
GS	SELF	0.58203 (0.00513)	0.57013 (0.00892)	1.15211 (0.02781)	1.09342 (0.05091)
	PLF	0.58642 (0.00878)	0.57790 (0.01555)	1.16411 (0.02401)	1.11646 (0.04608)
IS	SELF	0.62880 (0.00130)	0.63162 (0.00116)	1.17399 (0.02873)	1.14726 (0.03224)
	PLF	0.62983 (0.00207)	0.63172 (0.00183)	1.19957 (0.02117)	1.15198 (0.02645)
LA	SELF	0.59253 (0.02385)	0.55976 (0.02198)	1.13589 (0.05518)	1.07694 (0.05973)
	PLF	0.61080 (0.03654)	0.57855 (0.03759)	1.15790 (0.04401)	1.10339 (0.05290)
TKA	SELF	0.59391 (0.00180)	0.58177 (0.00245)	1.17562 (0.01267)	1.11573 (0.01367)
	PLF	0.59540 (0.00298)	0.58352 (0.00350)	1.18081 (0.01037)	1.12110 (0.01075)

Table 1(a): B. Es and P.Rs (in parenthesis) under non-informative prior for n = 30

AM	LF	α = 1.50	β = 1.50	α = 2.00	β = 2.00
QM	SELF	1.69644 (0.05780)	1.65015 (0.07969)	2.36370 (0.06140)	2.28493 (0.08436)
	PLF	1.71985 (0.04682)	1.66894 (0.06758)	2.38599 (0.04457)	2.29588 (0.06909)
GS	SELF	1.68014 (0.06729)	1.59996 (0.12331)	2.16867 (0.12408)	2.15438 (0.23319)
	PLF	1.70005 (0.03981)	1.63804 (0.07616)	2.19709 (0.05684)	2.20784 (0.10691)
IS	SELF	1.64335 (0.04317)	1.62425 (0.04998)	2.25629 (0.07792)	2.30234 (0.09107)
	PLF	1.64848 (0.02996)	1.66093 (0.03698)	2.26804 (0.05351)	2.32668 (0.07869)
LA	SELF	1.68509 (0.09201)	1.57162 (0.11335)	2.19487 (0.11956)	2.18342 (0.23694)
	PLF	1.71064 (0.05110)	1.59618 (0.06911)	2.22146 (0.05319)	2.23324 (0.09964)
TKA	SELF	1.71443 (0.03198)	1.63261 (0.03864)	2.21293 (0.05658)	2.19835 (0.07834)
	PLF	1.72362 (0.01837)	1.64230 (0.01939)	2.22562 (0.02538)	2.21269 (0.02868)

Table 1(b): B. Es and P.Rs (in parenthesis) under non-informative prior for n = 30

AM	LF	α = 0.50	β = 0.50	α = 1.00	β = 1.00
QM	SELF	0.57955 (0.01450)	0.59327 (0.01792)	1.11631 (0.02493)	1.12849 (0.04189)
	PLF	0.59169 (0.02023)	0.61145 (0.02706)	1.13400 (0.02356)	1.15206 (0.03397)
GS	SELF	0.56826 (0.00404)	0.54226 (0.00532)	1.07800 (0.01691)	1.06835 (0.03098)
	PLF	0.57228 (0.00698)	0.54920 (0.01071)	1.08780 (0.01577)	1.08914 (0.03013)
IS	SELF	0.61392 (0.00103)	0.60075 (0.00069)	1.09847 (0.01747)	1.12096 (0.01961)
	PLF	0.61465 (0.00165)	0.60034 (0.00126)	1.12093 (0.01390)	1.12379 (0.01729)
LA	SELF	0.55139 (0.01384)	0.53429 (0.01239)	1.05123 (0.02833)	1.03369 (0.03410)
	PLF	0.56287 (0.02296)	0.54603 (0.02347)	1.06404 (0.02563)	1.04977 (0.03216)
TKA	SELF	0.57986 (0.00142)	0.55333 (0.00146)	1.10000 (0.00770)	1.09015 (0.00832)
	PLF	0.58104 (0.00237)	0.55453 (0.00241)	1.10340 (0.00681)	1.09367 (0.00703)

Table 2(a): B. Es and P.Rs (in parenthesis) under non-informative prior for n = 50

AM	LF	α = 1.50	β = 1.50	α = 2.00	β = 2.00
QM	SELF	1.60629 (0.03367)	1.61344 (0.04154)	2.24508 (0.03729)	2.15130 (0.04102)
	PLF	1.62543 (0.02891)	1.62811 (0.04068)	2.26202 (0.02846)	2.15632 (0.04044)
GS	SELF	1.59085 (0.03920)	1.56436 (0.06428)	2.05984 (0.07536)	2.04839 (0.11339)
	PLF	1.60671 (0.02458)	1.59796 (0.04585)	2.08294 (0.03630)	2.08364 (0.06257)
IS	SELF	1.55602 (0.02515)	1.58811 (0.02605)	2.14306 (0.04732)	2.16770 (0.04429)
	PLF	1.36896 (0.01850)	1.32763 (0.02226)	2.15020 (0.03417)	2.18525 (0.04606)
LA	SELF	1.56730 (0.04646)	1.55162 (0.06945)	2.11614 (0.07286)	2.06888 (0.11816)
	PLF	1.58162 (0.02865)	1.57314 (0.04305)	2.13272 (0.03315)	2.09652 (0.05527)
TKA	SELF	1.62332 (0.01863)	1.59629 (0.02014)	2.10188 (0.03436)	2.06979 (0.03809)
	PLF	1.62899 (0.01134)	1.60212 (0.01167)	2.10998 (0.01620)	2.07819 (0.01679)

Table 2(b): B. Es and P.Rs (in parenthesis) under non-informative prior for n = 50

AM	LF	α = 0.50	β = 0.50	α = 1.00	β = 1.00
QM	SELF	0.54067 (0.00798)	0.55289 (0.00906)	1.06593 (0.01170)	1.07083 (0.01828)
	PLF	0.55159 (0.01201)	0.56934 (0.01521)	1.08124 (0.01078)	1.09147 (0.01645)
GS	SELF	0.53014 (0.00222)	0.50535 (0.00269)	1.02934 (0.00793)	1.01376 (0.01352)
	PLF	0.53349 (0.00414)	0.51138 (0.00602)	1.03718 (0.00789)	1.03186 (0.01259)
IS	SELF	0.57274 (0.00056)	0.55986 (0.00035)	1.04889 (0.00820)	1.06368 (0.00856)
	PLF	0.57299 (0.00098)	0.55899 (0.00071)	1.06878 (0.00695)	1.06469 (0.00837)
LA	SELF	0.53993 (0.00645)	0.52875 (0.00665)	1.04945 (0.01432)	1.02224 (0.01692)
	PLF	0.54565 (0.01144)	0.53494 (0.01238)	1.05604 (0.01319)	1.03041 (0.01634)
TKA	SELF	0.54096 (0.00078)	0.51567 (0.00074)	1.05035 (0.00361)	1.03445 (0.00363)
	PLF	0.54166 (0.00141)	0.51634 (0.00135)	1.05206 (0.00341)	1.03615 (0.00340)

Table 3(a): B. Es and P.Rs (in parenthesis) under non-informative prior for n = 100

AM	LF	α = 1.50	β = 1.50	α = 2.00	β = 2.00
QM	SELF	1.53597 (0.01634)	1.56124 (0.01987)	2.21056 (0.01853)	2.11466 (0.02298)
	PLF	1.55175 (0.01472)	1.57269 (0.01758)	2.22310 (0.01446)	2.11533 (0.02003)
GS	SELF	1.52121 (0.01902)	1.51375 (0.03075)	2.02817 (0.03745)	2.02384 (0.05247)
	PLF	1.53388 (0.01252)	1.54358 (0.02319)	2.04710 (0.01844)	2.03421 (0.03099)
IS	SELF	1.48790 (0.01220)	1.53673 (0.01247)	2.11011 (0.02352)	2.13077 (0.02349)
	PLF	1.30690 (0.00942)	1.28245 (0.01126)	2.11320 (0.01736)	2.14371 (0.02281)
LA	SELF	1.52232 (0.02270)	1.51670 (0.03283)	2.09146 (0.03589)	2.00138 (0.05497)
	PLF	1.52959 (0.01453)	1.52737 (0.02134)	2.09986 (0.01681)	2.01493 (0.02710)
TKA	SELF	1.55226 (0.00904)	1.54464 (0.00964)	2.06956 (0.01708)	2.03453 (0.01763)
	PLF	1.55515 (0.00578)	1.54759 (0.00590)	2.07368 (0.00823)	2.03868 (0.00831)

Table 3(b): B. Es and P.Rs (in parenthesis) under non-informative prior for n = 100

AM	LF	α = 0.50	β = 0.50	α = 1.00	β = 1.00
QM	SELF	0.58765 (0.01786)	0.61752 (0.02916)	1.18112 (0.03978)	1.14342 (0.06679)
	PLF	0.60024 (0.02468)	0.63697 (0.03811)	1.20143 (0.03480)	1.16914 (0.05040)
GS	SELF	0.57621 (0.00497)	0.56443 (0.00866)	1.14059 (0.02697)	1.08248 (0.04939)
	PLF	0.58055 (0.00851)	0.57212 (0.01508)	1.15247 (0.02329)	1.10529 (0.04470)
IS	SELF	0.62251 (0.00126)	0.62531 (0.00113)	1.16225 (0.02787)	1.13579 (0.03127)
	PLF	0.62353 (0.00201)	0.62540 (0.00178)	1.18758 (0.02053)	1.14046 (0.02566)
LA	SELF	0.58660 (0.02314)	0.55416 (0.02132)	1.12453 (0.05352)	1.06617 (0.05794)
	PLF	0.60469 (0.03544)	0.57276 (0.03646)	1.14632 (0.04269)	1.09236 (0.05132)
TKA	SELF	0.58797 (0.00175)	0.57595 (0.00237)	1.16386 (0.01229)	1.10457 (0.01326)
	PLF	0.58944 (0.00289)	0.57768 (0.00339)	1.16900 (0.01006)	1.10989 (0.01043)

Table 4(a): B. Es and P.Rs (in parenthesis) under informative prior for n = 30

AM	LF	α = 1.50	β = 1.50	α = 2.00	β = 2.00
QM	SELF	1.67948 (0.05606)	1.63365 (0.07730)	2.34006 (0.05955)	2.26208 (0.08183)
	PLF	1.70265 (0.04542)	1.65225 (0.06555)	2.36213 (0.04323)	2.27292 (0.06702)
GS	SELF	1.66334 (0.06527)	1.58396 (0.11961)	2.14698 (0.12036)	2.13284 (0.22619)
	PLF	1.68305 (0.03862)	1.62166 (0.07388)	2.17512 (0.05514)	2.18576 (0.10371)
IS	SELF	1.62692 (0.04187)	1.60800 (0.04848)	2.23373 (0.07558)	2.27932 (0.08834)
	PLF	1.43400 (0.02906)	1.34732 (0.03587)	2.24536 (0.05190)	2.30341 (0.07633)
LA	SELF	1.66824 (0.08925)	1.55590 (0.10995)	2.17292 (0.11597)	2.16159 (0.22983)
	PLF	1.69353 (0.04957)	1.58022 (0.06704)	2.19925 (0.05160)	2.21091 (0.09665)
TKA	SELF	1.69729 (0.03102)	1.61628 (0.03748)	2.19080 (0.05488)	2.17637 (0.07599)
	PLF	1.70638 (0.01782)	1.62588 (0.01880)	2.20336 (0.02462)	2.19056 (0.02782)

Table 4(b): B. Es and P.Rs (in parenthesis) under informative prior for n = 30

AM	LF	α = 0.50	β = 0.50	α = 1.00	β = 1.00
QM	SELF	0.57375 (0.01407)	0.58734 (0.01738)	1.10515 (0.02419)	1.11721 (0.04063)
	PLF	0.58577 (0.01962)	0.60533 (0.02625)	1.12266 (0.02286)	1.14054 (0.03295)
GS	SELF	0.56258 (0.00392)	0.53684 (0.00516)	1.06722 (0.01640)	1.05766 (0.03005)
	PLF	0.56655 (0.00677)	0.54371 (0.01038)	1.07692 (0.01530)	1.07825 (0.02922)
IS	SELF	0.60778 (0.00100)	0.59474 (0.00067)	1.08749 (0.01694)	1.10975 (0.01902)
	PLF	0.60850 (0.00160)	0.59433 (0.00122)	1.10972 (0.01349)	1.11256 (0.01678)
LA	SELF	0.54588 (0.01343)	0.52895 (0.01202)	1.04072 (0.02748)	1.02335 (0.03307)
	PLF	0.55724 (0.02227)	0.54057 (0.02277)	1.05340 (0.02486)	1.03927 (0.03120)
TKA	SELF	0.57406 (0.00138)	0.54779 (0.00141)	1.08900 (0.00747)	1.07925 (0.00807)
	PLF	0.57523 (0.00230)	0.54898 (0.00234)	1.09237 (0.00661)	1.08273 (0.00682)

Table 5(a): B. Es and P.Rs (in parenthesis) under informative prior for n = 50

AM	LF	α = 1.50	β = 1.50	α = 2.00	β = 2.00
QM	SELF	1.59022 (0.03266)	1.59731 (0.04029)	2.22263 (0.03617)	2.12979 (0.03979)
	PLF	1.60917 (0.02804)	1.61183 (0.03946)	2.23940 (0.02760)	2.13476 (0.03922)
GS	SELF	1.57494 (0.03803)	1.54872 (0.06235)	2.03924 (0.07310)	2.02791 (0.10999)
	PLF	1.59064 (0.02384)	1.58198 (0.04447)	2.06211 (0.03521)	2.06280 (0.06069)
IS	SELF	1.54046 (0.02439)	1.57223 (0.02527)	2.12163 (0.04590)	2.14602 (0.04296)
	PLF	1.35527 (0.01794)	1.31436 (0.02159)	2.12869 (0.03314)	2.16340 (0.04467)
LA	SELF	1.55163 (0.04507)	1.53610 (0.06736)	2.09498 (0.07068)	2.04819 (0.11462)
	PLF	1.56580 (0.02779)	1.55741 (0.04176)	2.11139 (0.03216)	2.07555 (0.05361)
TKA	SELF	1.60709 (0.01807)	1.58033 (0.01954)	2.08086 (0.03333)	2.04909 (0.03695)
	PLF	1.61270 (0.01100)	1.58610 (0.01132)	2.08888 (0.01572)	2.05741 (0.01628)

Table 5(b): B. Es and P.Rs (in parenthesis) under informative prior for n = 50

AM	LF	α = 0.50	β = 0.50	α = 1.00	β = 1.00
QM	SELF	0.53526 (0.00774)	0.54736 (0.00879)	1.05527 (0.01135)	1.06012 (0.01773)
	PLF	0.54607 (0.01165)	0.56365 (0.01475)	1.07043 (0.01043)	1.08055 (0.01595)
GS	SELF	0.52484 (0.00216)	0.50030 (0.00261)	1.01905 (0.00769)	1.00362 (0.01312)
	PLF	0.52816 (0.00402)	0.50626 (0.00584)	1.02681 (0.00765)	1.02154 (0.01115)
IS	SELF	0.56701 (0.00055)	0.55426 (0.00034)	1.03840 (0.00795)	1.05305 (0.00830)
	PLF	0.56726 (0.00095)	0.55340 (0.00069)	1.05809 (0.00674)	1.05404 (0.00812)
LA	SELF	0.53453 (0.00626)	0.52346 (0.00645)	1.03896 (0.01389)	1.01202 (0.01642)
	PLF	0.54019 (0.01110)	0.52959 (0.01201)	1.04548 (0.01279)	1.02011 (0.01585)
TKA	SELF	0.53555 (0.00076)	0.51051 (0.00071)	1.03985 (0.00350)	1.02411 (0.00352)
	PLF	0.53624 (0.00136)	0.51118 (0.00131)	1.04154 (0.00330)	1.02579 (0.00330)

Table 6(a): B. Es and P.Rs (in parenthesis) under informative prior for n = 100

AM	LF	α = 1.50	β = 1.50	α = 2.00	β = 2.00
QM	SELF	1.52061 (0.01585)	1.54562 (0.01928)	2.18846 (0.01798)	2.09351 (0.02141)
	PLF	1.53623 (0.01428)	1.55697 (0.01996)	2.20087 (0.01403)	2.09418 (0.01942)
GS	SELF	1.50600 (0.01845)	1.49861 (0.02983)	2.00789 (0.03633)	2.00360 (0.05089)
	PLF	1.51854 (0.01214)	1.52814 (0.02250)	2.02663 (0.01789)	2.01387 (0.03006)
IS	SELF	1.47302 (0.01184)	1.52136 (0.01209)	2.08901 (0.02281)	2.10946 (0.02288)
	PLF	1.29383 (0.00914)	1.26962 (0.01092)	2.09207 (0.01684)	2.12227 (0.02212)
LA	SELF	1.50710 (0.02202)	1.50153 (0.03184)	2.07055 (0.03481)	2.028137 (0.05332)
	PLF	1.51429 (0.01410)	1.51210 (0.02070)	2.07886 (0.01630)	2.039478 (0.02628)
TKA	SELF	1.53674 (0.00877)	1.52919 (0.00935)	2.04886 (0.01656)	2.01418 (0.01710)
	PLF	1.53960 (0.00560)	1.53211 (0.00573)	2.05294 (0.00799)	2.01829 (0.00806)

Table 6(b): B. Es and P.Rs (in parenthesis) under informative prior for n = 100

AM	LF	Complete Samples		5% Censored Samples	
QM	SELF	0.54884 (0.01263)	0.56140 (0.01558)	0.54884 (0.01263)	0.56140 (0.01558)
	PLF	0.56063 (0.01762)	0.57934 (0.02357)	0.56063 (0.01762)	0.57934 (0.02357)
GS	SELF	0.53880 (0.00352)	0.51394 (0.00463)	0.53880 (0.00352)	0.51394 (0.00463)
	PLF	0.54455 (0.00610)	0.52163 (0.00934)	0.54455 (0.00610)	0.52163 (0.00934)
IS	SELF	0.58521 (0.00090)	0.57456 (0.00061)	0.58521 (0.00090)	0.57456 (0.00061)
	PLF	0.58606 (0.00145)	0.57475 (0.00111)	0.58606 (0.00145)	0.57475 (0.00111)
LA	SELF	0.52639 (0.01215)	0.51220 (0.01092)	0.52639 (0.01215)	0.51220 (0.01092)
	PLF	0.53812 (0.02018)	0.52356 (0.02069)	0.53812 (0.02018)	0.52356 (0.02069)
TKA	SELF	0.54943 (0.00124)	0.52427 (0.00127)	0.54943 (0.00124)	0.52427 (0.00127)
	PLF	0.55092 (0.00207)	0.52556 (0.00210)	0.55092 (0.00207)	0.52556 (0.00210)

Table 7(a): B. Es and P.Rs (in parenthesis) under informative prior for α = 0.50, β = 0.50 and n = 50 using different censoring schemes.

M	LF	10% Censored Samples		20% Censored Samples	
QM	SELF	0.56016 (0.01337)	0.57297 (0.01647)	0.57375 (0.01407)	0.58734 (0.01738)
	PLF	0.57219 (0.01863)	0.59129 (0.02487)	0.58577 (0.01962)	0.60533 (0.02625)
GS	SELF	0.54991 (0.00371)	0.52453 (0.00488)	0.56258 (0.00392)	0.53684 (0.00516)
	PLF	0.55578 (0.00640)	0.53239 (0.00982)	0.56655 (0.00677)	0.54371 (0.01038)
IS	SELF	0.59728 (0.00094)	0.58641 (0.00063)	0.60778 (0.00100)	0.59474 (0.00067)
	PLF	0.59815 (0.00151)	0.58661 (0.00115)	0.60850 (0.00160)	0.59433 (0.00122)
LA	SELF	0.53725 (0.01264)	0.52276 (0.01135)	0.54588 (0.01343)	0.52895 (0.01202)
	PLF	0.54922 (0.02096)	0.53435 (0.02146)	0.55724 (0.02227)	0.54057 (0.02277)
TKA	SELF	0.56076 (0.00131)	0.53508 (0.00134)	0.57406 (0.00138)	0.54779 (0.00141)
	PLF	0.56228 (0.00218)	0.53639 (0.00221)	0.57523 (0.00230)	0.54898 (0.00234)

Table 7(b): B. Es and P.Rs (in parenthesis) under informative prior for $\alpha = 0.50$, $\beta = 0.50$ and $n = 50$ using different censoring schemes.