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OPTIMUM STEP STRESS ACCELERATED LIFE TESTING FOR FRECHET DISTRIBUTION

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Abstract

Nowadays products become more reliable so these highly reliable products could take long time to fail under normal condition. To overcome this problem, accelerated life testing has been used to estimate the reliability of these products. In this paper, we propose Frechet step stress accelerated life test plan under cumulative exposure model assuming a log-linear relationship between Frechet scale parameter and stresses. The simulation study is used for estimation. Further, optimal plan is designed using real data set by minimizing the asymptotic variance of the maximum likelihood estimators at $100p^{th}$percentile of the design stress. Finally, sensitivity analysis is designed.

Key Words: Accelerated Life Testing, Frechet Distribution, Asymptotic Variance, Cumulative Exposure Model, Log-Linear Relationship, Simple Step Stress.

1. Introduction

Accelerated life test (ALT) is a stress testing approach used to capture lifetime characteristics of objects under some extreme conditions and then to predict the reliability under some other operating conditions. ALT is used for estimation of highly reliable product's lifetime within an acceptable testing time. It has been widely used to estimate the lifetime of products in the industry when the lifetime of products at used condition is much longer than maximum acceptable test time. There are different types of ALT, commonly used are constant stress accelerated life test, step stress accelerated life test (SSALT) and progressive stress accelerated life test. When the stress applied to a sample of units does not vary with time, then it is called a constant stress. Whereas in SSALT the stress applied to a sample of units varies with time and in progressive stress accelerated life test stress levels increase continuously.

The cumulative exposure model (CEM) is one of most useful models in analysis of step stress experiments. The life distribution of a unit at one stress level is related using CEM by Nelson (1980) and Nelson (1990) to the life distribution of that unit at the next stress level, assuming that the residual life of the unit depends only on the CEM that unit had experienced with no memory of how this exposure was accumulated. In case of simple step stress model the CEM is:

$$F(t) = \begin{cases} F_1(t), & 0 \le t < \tau, \\ F_2(t - \tau_1 + \tau_1'), & \tau \le t < \infty. \end{cases} \quad (1.1)$$

where τ is the time to change stress and τ' is the solution of $F_1(\tau') = F_2((\tau')$.
Several distributions have been used to describe the life pattern of different items. In the present study Frechet distribution is used as the lifetime distribution. French mathematician Maurice Frechet (1878-1973) introduced the Frechet distribution in 1927. It has over fifty applications ranging from ALT through to floods, horse racing, rainfall, earthquakes, wind speeds, sea currents, and track race records. The cumulative distribution function (CDF) of Frechet distribution is:

$$F_i(t) = exp\left[-\left(\frac{t}{\theta_i}\right)^{-\alpha}\right], \qquad \theta\,, t > 0, \tag{1.2}$$

where the parameter $\alpha > 0$ determines the shape of the distribution and $\theta_i > 0$ is the scale parameter. Moreover, the scale parameter θ_i of Frechet distribution is assumed to be the function of log-linear relationship of stresses. The log-linear model can be expressed as:

$$log\ \theta i = \beta_0 + \beta_1 S_i,$$

where S_i denotes the stress level. Extensive work on parameter estimation of Frechet distribution has been found; see for example Singh (1987), Singh et al. (1990) and Abbas and Tang (2013). Some of previous works related to SSALT and optimum design of ALT plans with respect to stress loading profiles have been studied by many authors. The optimum accelerated life test plans for Lognormal distribution has been designed by Kelpinski and Nelson (1975) when the data were analyzed before all test units fail. In Nelson and Kielpinski (1976) optimum plan has been presented for estimating the medians of the Lognormal and Normal distributions. Nelson and Meeker (1978) emphasized that more test units should be allocated at the lower stress level than at the higher stress level. The simple SSALT plan has been studied for exponentially distributed lifetimes by Millerand Nelson (1983). The optimum 3-SSALT plans are obtained assuming quadratic lifestress relationship see Khamisand Higgins (1996). The optimal design of SSALT to estimate a specified quantile at design stress is presented by Chung and Bai (1998) in which CEM was assumed for effect of changing stress.

The same method to find the optimal stress changing time by minimizing the asymptotic variance (AV) of MLE of a quantity of interest see Alhadeed and Yang (2002) and Srivastava and Shukla(2008). In Alhadeed and Yang (2005) the Lognormal distribution has been used for designing optimum simple step stress model. Optimum stress changing times using Log-Logistic CEM was demonstrated by Al-Haj Ebrahem and Al-Masri (2007). A new optimum ramp stress ALT plan by simultaneously determining the ramp rate and lower start level of stress based on a CEM have been designed see Hong et al. (2010). An optimum SSALT for Rayleigh distribution with log-linear life stress relationship has been presented by Saxena et al. (2012). In Hakamipour and Rezae (2015) optimal designing of SSALT for two stress levels using Gompertz lifetimes distribution was performed. They determined the optimal test plan by minimizing the AV of the MLE of reliability at time. Chandra et al. (2016) has demonstrated the optimum stress changing times for 3-SSALT under the CEM with Type I censoring. They designed optimum test plans to minimize the AV of MLEs of given pth percentile of the Weibull distribution at a design stress. Recently, Hakamipour et al. (2017) has proposed an optimization for the SSALT for the Frechet distribution under Type I censoring. By minimizing the AV of the desired life estimate and the reliability estimate, they obtained the optimal simple SSALT. Finally, they conducted the simulation study without specifying the true values of ($\beta_0\beta_1$) using large samples to illustrate the effect of the initial estimates on the optimal values.

Our development differs from that of Hakamipour et al. (2017) in two ways. First, the model parameters (α, $\beta_0\beta_1$) are estimated by the ML method considering different stress levels and then optimal plan is developed using moderate and large sample sizes also hypothesis testing is conducted. Second, optimum stress changing time $\tau$is obtained for many combinations of stress levels. Also, sensitivity analysis is used to observe the effect of initial parametric values on the optimal design.

The rest of the paper is arranged as follows. Model and assumptions are described in Section 2. MLEs are presented in Section 3. Optimal test plans are discussed in Section 4 and simulation study is given in Section 5. Optimum test design for real life data is investigated in Section 6. The sensitivity analysis of model parameters is discussed in section 7. Finally, conclusions are reported in Section 8.

2. Model and Assumptions

Under any constant stress, the life time of a test product follows a Frechet distribution with CDF of time to failure of a test unit under simple SSALT is

$$F_i(t) = \begin{cases} exp\left[-\left(\frac{t}{\theta_1}\right)^{-\alpha}\right], & 0 \le t < \tau, \\ exp\left[-\left(\frac{t}{\theta_2}\right)^{-\alpha}\right] exp\left[-\left(\left(\frac{\tau}{\theta_1}\right)^{-\alpha}\right) - \left(\left(\frac{\tau}{\theta_2}\right)^{-\alpha}\right)\right], & \tau \le t < \infty. \end{cases} \quad (2.1)$$

The shape parameter α is independent of the stresses and scale parameter θ_i is the log-linear function of stresses. *i.e.*, $\log \theta_i = \beta_0 + \beta_1 S_i$, where $i = 0,1,2$

The following assumptions are made:

1. Testing is done at two stresses S_1 and S_2, where S_1 is lower stress level and S_2 is the higher stress level where $S_1 < S_2$.
2. Under any level of stress the failure time of test product follows a Frechet distribution with unknown shape parameter α.
3. All n identical products are first placed on a lower stress S_1 and run until time τ and then those are placed at higher stress S_2 until all units fail.
4. The scale parameter θ_i at stress level i, where $i = 1,2$ is a log-linear function of stresses *i.e.*,$\log \theta_i = \beta_0 + \beta_1 S_i$, where β_0 and $\beta_1 < 0$ are unknown parameters that are estimated.
5. The lifetimes of test units are independent and identically distributed.

3. Maximum Likelihood Estimation

To obtain the MLE of the model parameters, let t_{ij}, $j = 1,2, ..., n_i$, $i = 1,2$ be the observed failure test of a unit j under the stress level i, where n_1 denotes the number of units failed at stress S_1 and $n_2$denotes the number of units failed at stress S_2respectively. The likelihood function becomes:

$$\boldsymbol{L}(\theta_1,\theta_2,\alpha)=\prod_{j=1}^{n_1}\frac{\alpha}{\theta_1}\left(\frac{t_{1j}}{\theta_1}\right)^{-(\alpha+1)}exp\left[-\left(\frac{t_{1j}}{\theta_1}\right)^{-\alpha}\right]\prod_{j=1}^{n_2}\left[\frac{\alpha}{\theta_2}\left(\frac{t_{2j}}{\theta_2}-\frac{\tau}{\theta_2}+\frac{\tau}{\theta_1}\right)^{-(\alpha+1)}\right]\prod_{j=1}^{n_2}\left[exp\left[-\left(\frac{t_{2j}}{\theta_2}\right)^{-(\alpha+1)}\right]exp\left[-\left(\left(\frac{\tau}{\theta_1}\right)^{-\alpha}\right)-\left(\left(\frac{\tau}{\theta_2}\right)^{-\alpha}\right)\right]\right], \quad (3.1)$$

The log likelihood of the likelihood function is given by:

$$logL(\theta_1,\theta_2,\alpha)=nlog(\alpha)-n_1\log(\theta_1)-n_2\log(\theta_2)-\sum_{j=1}^{n_1}\left(\frac{t_{1j}}{\theta_1}\right)^{-\alpha}-\sum_{j=1}^{n_2}\left(\frac{t_{2j}}{\theta_2}-\frac{\tau}{\theta_2}+\frac{\tau}{\theta_1}\right)^{-\alpha}-(\alpha+1)-\left[\sum_{j=1}^{n_1}log\left(\frac{t_{1j}}{\theta_1}\right)-\sum_{j=1}^{n_2}log\left(\frac{t_{2j}}{\theta_2}-\frac{\tau}{\theta_2}+\frac{\tau}{\theta_1}\right)\right], \quad (3.2)$$

where, $n = n_1 + n_2$

$\log(\theta_i)=\beta_0+\beta_1 S_i$, $i= 0,1,2$

$$logL(\theta_1,\theta_2,\alpha)=nlog(\alpha)-n_1(\beta_0+\beta_1S_1)-n_2(\beta_0+\beta_1S_2)-\sum_{j=1}^{n_1}t_{1j}{}^{-\alpha}e^{\alpha(\beta_0+\beta_1S_1)}-\sum_{j=1}^{n_2}\left(t_{2j}{}^{-\alpha}-\tau^{-\alpha}\right)e^{\alpha(\beta_0+\beta_1S_2)}+\tau^{-\alpha}e^{\alpha(\beta_0+\beta_1S_1)}-(\alpha+1)\sum_{j=1}^{n_1}\log\left(t_{1j}e^{-(\beta_0+\beta_1S_1)}\right)-(\alpha+1)\sum_{j=1}^{n_2}\log\left(\left(t_{2j}-\tau\right)e^{-(\beta_0+\beta_1S_2)}+\tau e^{-(\beta_0+\beta_1S_1)}\right), \quad (3.3)$$

$$\frac{\partial logL}{\partial\alpha}=\frac{n}{\alpha}-\sum_{j=1}^{n_1}\log\left(t_{1j}e^{-(\beta_0+\beta_1S_1)}\right)+\sum_{j=1}^{n_1}t_{1j}{}^{-\alpha}e^{\alpha(\beta_0+\beta_1S_1)}\log\left(t_{1j}\right)-\sum_{j=1}^{n_2}t_{2j}{}^{-\alpha}e^{\alpha(\beta_0+\beta_1S_2)}\log\left(t_{2j}\right)-\sum_{j=1}^{n_2}\log\left(\left(t_{2j}-\tau\right)e^{-(\beta_0+\beta_1S_2)}+\tau e^{-(\beta_0+\beta_1S_1)}\right)-\sum_{j=1}^{n_2}t_{2j}{}^{-\alpha}e^{\alpha(\beta_0+\beta_1S_2)}(\beta_0+\beta_1S_2)-\sum_{j=1}^{n_1}t_{1j}{}^{-\alpha}e^{\alpha(\beta_0+\beta_1S_1)}(\beta_0+\beta_1S_1)+n_2\tau^{-\alpha}e^{\alpha(\beta_0+\beta_1S_2)}(\beta_0+\beta_1S_2)-n_2\tau^{-\alpha}e^{\alpha(\beta_0+\beta_1S_2)}\log(\tau)-n_2\tau^{-\alpha}e^{\alpha(\beta_0+\beta_1S_1)}(\beta_0+\beta_1S_1)+n_2\tau^{-\alpha}e^{\alpha(\beta_0+\beta_1S_1)}\log(\tau)=0. \quad (3.4)$$

$$\frac{\partial logL}{\partial \beta_0} = n\alpha - \sum_{j=1}^{n_2}\left[\left(t_{2j}{}^{-\alpha} - \tau^{-\alpha}\right)\alpha e^{\alpha(\beta_0+\beta_1 S_2)} + \tau^{-\alpha}\alpha e^{\alpha(\beta_0+\beta_1 S_1)}\right] - \sum_{j=1}^{n_1} t_{1j}{}^{-\alpha}\alpha e^{\alpha(\beta_0+\beta_1 S_1)}$$
$$= 0, \tag{3.5}$$

$$\frac{\partial logL}{\partial \beta_1} = S_1 n_1 - S_1 \sum_{j=1}^{n_1} e^{\alpha(\beta_0+\beta_1 S_1)} + \alpha n_2 S_2 - \sum_{j=1}^{n_2} \alpha S_2\left(t_{2j}{}^{-\alpha} - \tau^{-\alpha}\right)e^{\alpha(\beta_0+\beta_1 S_2)} - \sum_{j=1}^{n_2} \alpha S_1 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} = 0, \tag{3.6}$$

The above equations cannot be written in closed form so the MLEs can be obtained numerically. Here, the optim function in the R software (version:2.9.2) is used for maximization of the loglikelihood function to get the MLEs. Then, estimates $\hat{\alpha}, \widehat{\beta_0}$ and $\widehat{\beta_1}$are obtained and their confidence intervals (CIs) are constructed. The second and mixed partial derivatives of log L are presented in Appendix A.

4. Optimal Test Plan

Optimal stress change time can be determined by minimizing the asymptotic variance (AV) of ML estimates at $100p^{th}$ percentile. The $100p^{th}$ percentile of the Frechet lifetime $t_p(S_0)$ at the design stress S_0 is:

$$t_p(S_0) = e^{(\beta_0+\beta_1 S_0)}(-\log(p))^{-\frac{1}{\alpha}}, \tag{4.1}$$

The AV of the percentile estimate at the design stress can be derived as:

$$AV\left(\widetilde{t_p}(S_0)\right) = \left[\frac{\partial \widehat{t_p(S_0)}}{\partial \widehat{\beta_0}} \frac{\partial \widehat{t_p(S_0)}}{\partial \widehat{\beta_1}} \frac{\partial \widehat{t_p(S_0)}}{\partial \widehat{\alpha}}\right] \Sigma \left[\frac{\partial \widehat{t_p(S_0)}}{\partial \widehat{\beta_0}} \frac{\partial \widehat{t_p(S_0)}}{\partial \widehat{\beta_1}} \frac{\partial \widehat{t_p(S_0)}}{\partial \widehat{\alpha}}\right]^T. \tag{4.2}$$

where Σ is the variance-covariance matrix and optimum stress change time τ is obtained by minimizing $AV\left(\widetilde{t_p}(S_0)\right)$.

5. Simulation Study

A simulation study is conducted to get the point and interval estimates based on the asymptotic normality of the MLEs. The simulation study is performed through following steps:

1. The values for true parameters and stress combinations are chosen to be α = 1.5, β_0 = 3, β_1=−1, τ = 14 and (S_1= 0.5, S_2= 1.5), (S_1= 0.7, S_2= 1.5),(S_1= 0.5 , S_2= 1.6) and(S_1 = 0.7 , S_2 = 1.6).

2. The random samples of size n are generated from Frechet cumulative exposure model.

3. The MLEs of model parameters $\hat{\alpha}, \widehat{\beta_0}$ and $\widehat{\beta_1}$are obtained for each sample size and all combinations of stress levels.
4. The performance of the MLEs is evaluated through variance of the estimates for all sample size and various combinations of stress levels.

5. The 90% and 95% CIs are obtained for all sample sizes and different stress levels.

The results drawn from the above simulation study are presented in Tables 1-4. From the results, conclusions are drawn regarding the behaviour of the estimators, which are summarized as follows:

- From Tables 1-4, it is observed that as the sample size increases the variance of the estimates get smaller.

- When the sample size increases, the interval of the estimates decreases. Further, the intervals of the estimates at γ = 90% is smaller than the interval of estimates at γ = 95%. It can also be observed that coverage probabilities for both γ =95% and γ =90% do not change much across the different sample sizes.

- It can also be observed from Table II that the values of estimates are closer to the true values for stress combination (S_1 = 0.5, S_2 = 1.6) as compared to other stress combinations.

n	Parameters	Estimates	Variance	95%CI	$C_{95\%}$	90%CI	$C_{90\%}$
25	α	1.8038	1.4208	0.3325, 4.1401	0.80	0.1570, 3.7647	0.88
	β_0	2.5102	0.2063	1.6191, 3.4004	0.78	1.7630, 3.2574	0.71
	β_1	-0.6727	0.0385	-1.0572, -0.2883	0.81	-0.9954, -0.3501	0.73
50	α	1.6144	0.3284	0.4914, 2.7375	0.81	0.6719, 2.5570	0.90
	β_0	2.6504	0.0420	2.2487, 3.0521	0.79	2.3133, 2.9876	0.71
	β_1	-0.7701	0.0368	-1.1462, -0.3939	0.82	-1.0858, -0.4544	0.75
75	α	1.5813	0.1317	0.8699, 2.2927	0.81	0.9842, 2.1783	0.91
	β_0	2.6799	0.0140	2.4478, 2.9120	0.80	2.4851, 2.8747	0.72
	β_1	-0.7941	0.0263	-1.1149, -0.4794	0.78	-1.0638, -0.5304	0.74
100	α	1.5315	0.0669	1.0245, 2.0385	0.81	1.1060, 1.9571	0.92
	β_0	2.7146	0.0032	2.6039, 2.8253	0.80	2.6217, 2.8075	0.72
	β_1	-0.7952	0.0212	-1.0804, -0.5091	0.81	-1.0346, -0.5558	0.74
150	α	1.5152	0.0347	1.1501, 1.8802	0.81	1.2088, 1.8216	0.93
	β_0	2.7088	0.0057	2.5602, 2.8574	0.80	2.5841, 2.8335	0.72
	β_1	-0.8059	0.0117	-1.0176, -0.5942	0.80	-0.9835, -0.6282	0.76
200	α	1.5062	0.0085	1.3250, 1.6874	0.82	1.3541, 1.6583	0.93
	β_0	2.7207	0.0014	2.6466, 2.7949	0.80	2.6585, 2.7829	0.74
	β_1	-0.8075	0.0041	-0.9336, -0.6814	0.80	-0.9134, -0.7017	0.75

Table 1: Average point and interval estimates for α = 1.5, β_0 = 3, β_1 = −1, τ = 14, S_1= 0.5 and S_2 = 1.5

n	Parameters	Estimates	Variance	95%CI	$C_{95\%}$	90%CI	$C_{90\%}$
25	α	1.7109	0.4799	0.3531, 3.0687	0.81	0.5713, 2.8504	0.83
	β_0	2.5751	0.1409	1.8394, 3.3109	0.76	1.9576, 3.1926	0.70
	β_1	-0.7649	0.0292	-1.0997, -0.4300	0.80	-1.0459, -0.4838	0.76
50	α	1.6876	0.1260	0.9920, 2.3832	0.81	1.1038, 2.2714	0.90
	β_0	2.6225	0.0563	2.3600, 2.9450	0.79	2.4070, 2.8980	0.73
	β_1	-0.7734	0.0251	-1.0640, -0.5828	0.81	-1.0254, -0.6215	0.73
75	α	1.6615	0.1174	0.3029, 2.8954	0.81	0.5112, 2.6871	0.91
	β_0	2.6255	0.0529	2.1748, 3.0763	0.79	2.2472, 3.0039	0.74
	β_1	-0.7913	0.0139	-1.0975, -0.3651	0.83	-1.0387, -0.4240	0.75
100	α	1.6014	0.2037	0.7167, 2.4861	0.85	0.8588, 2.3439	0.92
	β_0	2.6272	0.0330	2.2710, 2.9834	0.79	2.3282, 2.9261	0.73
	β_1	-0.8241	0.0150	-1.1823, -0.3059	0.83	-1.1119, -0.3763	0.79
150	α	1.5624	0.0723	1.0352, 2.0896	0.82	1.1199, 2.0049	0.93
	β_0	2.7096	0.0011	2.6448, 2.7745	0.80	2.6552, 2.7641	0.73
	β_1	-0.8568	0.0106	-1.1678, -0.4659	0.82	-1.1114, -0.5223	0.74
200	α	1.5015	0.0294	1.1626, 1.8345	0.82	1.2166, 1.7805	0.94
	β_0	2.7290	0.0022	2.6374, 2.8207	0.81	2.6521, 2.8059	0.73
	β_1	-0.9004	0.0038	-0.9719, -0.6850	0.82	-0.9488, -0.7080	0.73

Table 2: Average point and interval estimates for $\alpha = 1.5, \beta_0 = 3, \beta_1 = -1, \tau = 14$, S_1= 0.5 and S_2= 1.6

n	Parameters	Estimates	Variance	95%CI	$C_{95\%}$	90%CI	$C_{90\%}$
25	α	1.7318	0.3226	0.6185, 2.8451	0.83	0.7974, 2.6662	0.80
	β_0	2.5925	0.0387	2.2071, 2.9780	0.82	2.2690, 2.9161	0.76
	β_1	-0.7872	0.0378	-1.1680, -0.4064	0.78	-1.1068, -0.4676	0.72
50	α	1.6961	0.0839	1.1283, 2.2640	0.83	1.2195, 2.1728	0.83
	β_0	2.6531	0.0052	2.5127, 2.7953	0.79	2.5354, 2.7726	0.71
	β_1	-0.8378	0.0211	-1.1226, -0.5529	0.79	-1.0768, -0.5987	0.72
75	α	1.6594	0.0814	1.10034, 2.2184	0.84	1.1902, 2.1286	0.81
	β_0	2.6660	0.0128	2.4439, 2.8882	0.80	2.4796, 2.8525	0.74
	β_1	-0.8541	0.0107	-1.0567, -0.6515	0.87	-1.0242, -0.6841	0.74
100	α	1.6013	0.0661	1.0941, 2.1084	0.80	1.1756, 2.0269	0.89
	β_0	2.7112	0.0039	2.5895, 2.8329	0.78	2.6090, 2.8134	0.70
	β_1	-0.8598	0.0105	-1.0439, -0.6357	0.81	-1.0111, -0.6685	0.77
150	α	1.6077	0.0354	1.3057, 1.9896	0.82	1.3607, 1.9346	0.88
	β_0	2.7207	0.0027	2.5865, 2.7909	0.79	2.6030, 2.7744	0.73
	β_1	-0.8647	0.0089	-1.0298, -0.6596	0.81	-1.0000, -0.6893	0.72
200	α	1.5793	0.0349	1.2133, 1.9452	0.83	1.2721, 1.8864	0.89
	β_0	2.7209	0.0029	2.5941, 2.8041	0.80	2.6109, 2.7872	0.73
	β_1	-0.8695	0.0012	-1.0371, -0.6219	0.83	-1.004, -0.6553	0.77

Table 3: Average point and interval estimates for $\alpha = 1.5, \beta_0 = 3, \beta_1 = -1, \tau = 14$, S_1= 0.7 and S_2= 1.5

n	Parameters	Estimates	Variance	95%CI	$C_{95\%}$	90%CI	$C_{95\%}$
25	α	1.8384	0.6070	0.3113, 3.3655	0.82	0.5567, 3.1200	0.85
	β_0	2.5308	0.1617	1.7426, 3.3190	0.79	1.8693, 3.1924	0.71
	β_1	-0.8220	0.0352	-1.1895, -0.4546	0.82	-1.1304, -0.5136	0.72
50	α	1.8161	0.1650	1.02013, 2.6122	0.82	1.14806, 2.4843	0.87
	β_0	2.5892	0.0289	2.2558, 2.9226	0.79	2.3093, 2.8690	0.74
	β_1	-0.8752	0.0079	-1.0495, -0.7008	0.80	-1.0215, -0.7288	0.72
75	α	1.8090	0.5249	1.3599, 2.2580	0.82	1.4321, 2.1858	0.88
	β_0	2.6672	0.0047	2.5329, 2.8015	0.79	2.5545, 2.7799	0.74
	β_1	-0.8826	0.0071	-1.0914, -0.7606	0.79	-1.0648, -0.7872	0.74
100	α	1.7141	0.2561	0.7222, 2.7060	0.82	0.8816, 2.5466	0.88
	β_0	2.6746	0.0036	2.2080, 3.0813	0.79	2.2782, 3.0111	0.74
	β_1	-0.8850	0.0070	-1.0153, -0.6848	0.80	-0.9887, -0.7114	0.74
150	α	1.6723	0.0311	1.3268, 2.0177	0.84	1.3823, 1.9622	0.88
	β_0	2.6853	0.0027	2.5841, 2.7864	0.79	2.6004, 2.7701	0.73
	β_1	-0.8855	0.0043	-1.0147, -0.7564	0.82	-0.9939, -0.7771	0.75
200	α	1.5415	0.0172	1.2843, 1.7988	0.84	1.3256, 1.7574	0.90
	β_0	2.7204	0.0024	2.6057, 2.8350	0.79	2.6242, 2.8166	0.73
	β_1	-0.8973	0.0041	-0.9425, -0.6920	0.83	-0.9224, -0.7121	0.75

Table 4: Average point and interval estimates for $\alpha = 1.5$, $\beta 0 = 3$, $\beta 1 = -1$, $\tau = 14$, $s_1 = 0.7$ and $s_2 = 1.6$

6. Optimum Test Design for Real Data

For optimization, we considered the real-life data set for failure times in hours for the model 7835 power amplifier vacuum tube used in the Linac accelerated at Fermi National Accelerated Laboratory, used by McCrory (2006) are presented in Table 5. To determine the optimal stress changing point of the $100p^{th}$ percentile at design stress level S_0, we use the experimental data (see Table 5) with n=125, α=0.66, β_0=10, β_1= -2, $S_0 = 0.3$, S_1=1 and S_2=1.5. Moreover, the plot of $AV(\widetilde{t_p}(S_0))$ versus the stress changing point τ is presented in Figure. 1 and optimal stress changing point is $\tau^* = 16000$.

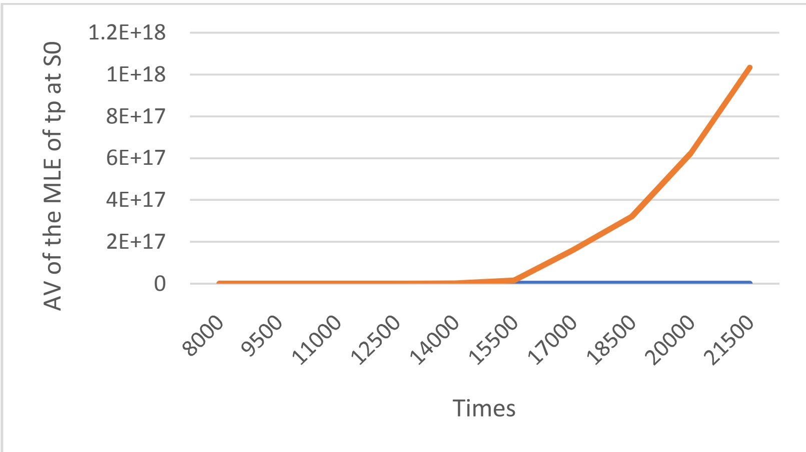

Figure 1: $AV(\widetilde{t_p}(S_0))$vs. Times

Stress levels	Failure Data
$S_1=1$	25 2664 5469 5929 6845 7190 7828 7845 7867 8050 8063 8404 8568 8589 8729 8912 8947 9022 9319 9515 9773 9896 10039 10051 10064 10216 10265 10322 10322 10378 10388 10452 10567 10615 10634 10661 10687 10830 10880 10880 10914 11184 11305 11448 11466 11674 11687 11795 11869 11914 12027 12258 12350 12484 12651 12824 12920 13084 13264 13282 13414 13444 13513 13540 13594 13736 13779 13852 13853 14004 14145 14174 14190 14438 14492 14741 14741 14791 14868 15002 15018 15299 15346 15363 15599 15812 15843 15870 15943 15981 15996
$S_2=1.5$	16214 16284 16547 16640 16752 16766 16852 16942 16987 17163 17833 17948 17973 18102 18632 19117 19203 19433 19598 20054 20340 20939 21094 21587 21983 22269 22699 22874 22910 25773 26466 31882 40859 48449

Table 5: Failure times (in hours) for power amplifier vacuum tube, model 7835

1) The MLEs are $\hat{\alpha} = 0.6580439$, $\widehat{\beta_0}$=9.4899959 and $\widehat{\beta_1}$= -1.0809954

2) The inverse of the estimated fisher information matrix is:

$$F^{-1} = \begin{bmatrix} 1.471328e-17 & -2.691183e-17 & -7.011267e-22 \\ -2.691183e-17 & 3.190937e-02 & -7.608399e-13 \\ -7.011267e-22 & -7.608399e-13 & 5.052425e-13 \end{bmatrix}$$

3) The 90% and 95% CIs for the model parameters are provided in Table 6:

Parameters	90% C.I	95% C.I
α	0.6548,0.6613	0.6542, 0.6619
β_0	8.6510,10.3290	8.4904,10.4896
β_1	-2.5928,0.4308	-2.8822, 0.7202

Table 6: Confidence intervals of the parameters

As shown from (Table 6) the interval of the estimates at γ = 90% is shorter in length than the interval of the estimates at γ = 95%. The width of CIs are too narrow which indicates that point estimates are stable and Frechet CEM is more suitable for this data set.

4. Hypotheses testing about model parameters are performed using likelihood ratio method.

An important inference problem concerning the regression coefficients is to test of hypothesis H_0: β_1= 0 against H_1: $\beta_1 \neq 0$. To test H_0against H_1, the likelihood ratio test statistic is:

$$-2log\Lambda = -2\log\left[\frac{L(\widetilde{\beta_0},0)}{L(\widehat{\beta_0},\widehat{\beta_1})}\right], \tag{6.1}$$

where $,\widehat{\beta_0}$ and $,\widehat{\beta_1}$ are unrestricted MLEs of parameters β_0 and β_1 which are obtained by solving the system of nonlinear equations. While $\widetilde{\beta_0}$ is the restricted MLE of the parameter $\beta_0$under H_0. The test with approximate size γ is to reject H_0if and only if, $\Lambda>\chi^2_{1-\gamma,1}$where $\chi^2_{1-\gamma,1}$ is the $(1-\gamma)$ quantile of the Chi-square distribution with one degrees of freedom. Similarly, we can test the hypothesis for $H_0^{'}:\beta_0 = 0$ against $H_1^{'}:\beta_0 \neq 0$. The results for hypotheses testing are provided in Table 7.

Model	Λ	d.f	$\chi^2_{0.05}$
Full model ($\boldsymbol{\beta}_0$, $\boldsymbol{\beta}_1$)	-	-	-
$\boldsymbol{\beta}_0 = 0$	1393.06	1	3.841
$\boldsymbol{\beta}_1 = 0$	1151.324	1	3.841

Table 7: Test for parameters in the model log $\boldsymbol{\theta_i = \beta_0 + \beta_1 s_i}$

Table 7 shows the likelihood ratio statistic for tests of various sub models against the full model. The tests $H_0^{'}:\beta_0 = 0$ and $H_0:\beta_1 = 0$ are rejected. As the hypotheses are rejected so we can say that there is a log-linear relationship between stresses S_i and scale parameter θ_i and conclude that parameter β_0 has an important role in model.

6.1 Optimal Times with Variation in Stress Levels

We obtained the optimum stress changing time τ for many combinations of stress levels. The results are determined from Fig. 2 and presented in Table 8. The initial values of Frechet step stress model parameters and different sets of stress levels are $\alpha = 0.66$, $\beta_0 = 10$, $\beta_1 = -2$, S_1= 0.75, 0.77 and S_2= 0.85, 0.87.

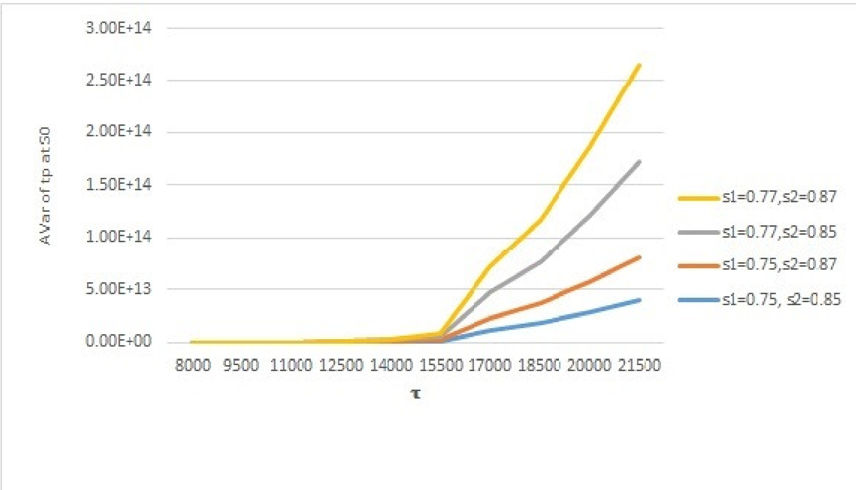

Figure 2: $\boldsymbol{AV(\widetilde{t_p}(S_0))\ vs\ \tau}$

S_1	S_2	τ^*	$AV\,\hat{t}_p(S_0)$
0.75	0.85	15700	4.11E+08
	0.87	15650	4.10E+08
0.77	0.85	15550	2.99E+08
	0.87	15500	2.96E+08

Table 8: Optimum times of changing stress

As shown in Table 8 for fixed values of S_1 the optimum times τ^* decreases as the values of S_2 increases. The AV also decreases for fixed values of S_1as the value of S_2increases. Conversely, for same values of S_2 the optimum times τ^* decreases as the values of S_1 increases. The AV also decreases for fixed values of S_2 as the value of S_1 increases.

7. Sensitivity analysis

Suppose that there are uncertainties in estimating the initial values for the model parameters (α, β_0, β_1). We are interested in the sensitivity of the optimal stress change time ($\tau*$) to the changes in the initial values for the model parameters (α, β_0, β_1). Figures 3 shows the relationship of the resultant optimal value $\tau*$with different initial values for the model parameters (α, β_0, β_1).

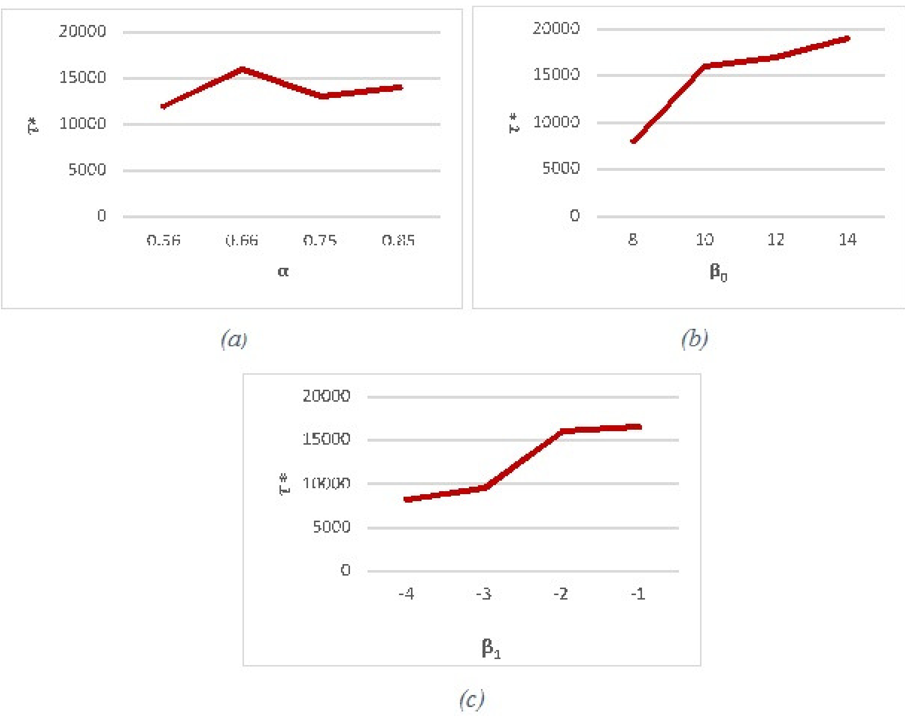

Figure 3: Optimal times $\tau*$ vs. α, β_0 and β_1

As shown in Figures 3 a small change in the initial value $\boldsymbol{\alpha}$, $\boldsymbol{\beta_0}$ and $\boldsymbol{\beta_1}$ will result in a large change in the optimal times $\boldsymbol{\tau*}$.

8. Conclusion

The Frechet distribution has widespread applications in the field of reliability. It handles complex circuits very easily and is also used for Opto-electronic device such as solar cell, photo diodes, photo transistor, light emitting devices, etc. In this paper, we propose the Frechet step stress model with two stress levels. All test units are initially put on the lower stress level and run until time τ. Then the stress is changed to high level and the test continues until all units fail.

Simulation study is used to get the point and interval estimates of model parameters. Based on the simulation results the following conclusions are drawn. For the parameters α, β_0 and β_1 the MLEs $\hat{\alpha}, \widehat{\beta_0}$ and $\widehat{\beta_1}$ has small values of variance also as the sample size increases the value of variance decreases. It is also noted that the CIs for $\hat{\alpha}, \widehat{\beta_0}$ and $\widehat{\beta_1}$ has a small interval of the estimates for bothγ = 90% and γ = 95%. The interval of the estimates decreases as the sample size increases and intervals of the estimates at γ = 95% is wider than the interval of estimates at γ = 90%. Also, their respective coverage probabilities do not change much with the different sample sizes. So, it can be concluded that the present step stress ALT plan works well and has a good choice to be considered in the field of ALT. Also, the results indicate that for given true parameter values and stress levels this combination (S_1 = 0.5, S_2 = 1.6) of stress level parameter values are closer to their true parameter values than the other sets of stress levels. From the results, it appears that for small value of S_1 and large value of S_2 parameter estimates are better than all other stress combinations. Thus, it is recommended to use a small value of S_1 and a large value of S_2.

The optimum plan is subject to practical constraints such as the minimum number of failures at the low stress level. This optimization approach is demonstrated by a real-life data set. For some selected values of the parameters and stress levels we have designed the plot between $AV\left(\widetilde{t_p}(S_0)\right)$versus Times. Thus, optimum stress change time is obtained graphically. Then we investigated point estimates and approximate interval estimates. The hypotheses testing of model parameters are designed and it is found that there is a log-linear relationship between stress levels and scale parameter of the Frechet step stress model. Researchers, also focus to obtain the optimum stress changing time $\tau$for many combinations of stress levels. Variation of optimal time is assessed and found that by increasing stress level S_2 and for fixed values of S_1 optimum stress changing times are decreasing. Conversely, optimal time decreases on increasing stress level S_1 and for the same values of S_2 while parameters are fixed. Though, stresses lie between 0.75< S_1<0.77 and 0.85< S_2<0.87, from the results, we found that stress levels have an effect on optimal stress change time which recommends that the model is suitable in the field of highly reliable products such as insulation of cables, transformers, capacitors, etc. to obtain the early failure of such products than that of the normal operating conditions. Further, sensitivity analysis shows that the initial values of α, β_0 and $\beta_1$significantly affect the resultant optimal plans. Therefore, we need to be very careful when estimating the values of parameters.

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Appendix A

Elements of FIM are as follows:

$$\frac{\partial^2 \log\, L}{\partial \beta_0^2} = n_2\tau^{-\alpha}\alpha^2 e^{\alpha(\beta_0+\beta_1 S_2)} - \sum_{j=1}^{n_2} t_{2j}{}^{-\alpha}\alpha^2 e^{\alpha(\beta_0+\beta_1 S_2)} - n_2\tau^{-\alpha}\alpha^2 e^{\alpha(\beta_0+\beta_1 S_1)}$$
$$- \sum_{j=1}^{n_1} t_{1j}{}^{-\alpha}\alpha^2 e^{\alpha(\beta_0+\beta_1 S_1)} = 0,$$

$$\frac{\partial^2 \log\, L}{\partial \beta_1^2} = n_2\tau^{-\alpha}S_2^2\alpha^2 e^{\alpha(\beta_0+\beta_1 S_2)}$$
$$- \sum_{j=1}^{n_2} t_{2j}{}^{-\alpha}\alpha^2 S_2^2 e^{\alpha(\beta_0+\beta_1 S_2)} - n_2\tau^{-\alpha}\alpha^2 S_1^2 e^{\alpha(\beta_0+\beta_1 S_1)}$$
$$- \sum_{j=1}^{n_1} t_{1j}{}^{-\alpha}\alpha^2 S_1^2 e^{\alpha(\beta_0+\beta_1 S_1)} = 0,$$

$$\frac{\partial^2 \log\, L}{\partial \beta_1 \beta_0} = \frac{\partial^2 \log\, L}{\partial \beta_0 \beta_1}$$
$$= n_2\tau^{-\alpha}S_2\alpha^2 e^{\alpha(\beta_0+\beta_1 S_2)}$$
$$- \sum_{j=1}^{n_2} t_{2j}{}^{-\alpha}\alpha^2 S_2 e^{\alpha(\beta_0+\beta_1 S_2)} - n_2\tau^{-\alpha}\alpha^2 S_1 e^{\alpha(\beta_0+\beta_1 S_1)}$$
$$- \sum_{j=1}^{n_1} t_{1j}{}^{-\alpha}\alpha^2 e S_1{}^{\alpha(\beta_0+\beta_1 S_1)} = 0,$$

$$
\begin{aligned}
\frac{\partial^2 \log\ L}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \sum_{j=1}^{n_1} {t_{1j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} \left(log t_{1j}\right)^2 \\
- \sum_{j=1}^{n_1} {t_{1j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} \left(log t_{1j}\right)^2 (\beta_0 + \beta_1 S_1)^2 \\
- \sum_{j=1}^{n_2} {t_{2j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} (\beta_0 + \beta_1 S_2)^2 \\
+ \sum_{j=1}^{n_2} {t_{2j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} (\beta_0 + \beta_1 S_2) \log\left(t_{2j}\right) \\
- n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} \log(\tau)\, (\beta_0 + \beta_1 S_1) \\
+ n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} \log(\tau)\, (\beta_0 + \beta_1 S_2) \\
- \sum_{j=1}^{n_2} {t_{2j}}^{-\alpha} \left(\log\left(t_{2j}\right)\right)^2 e^{\alpha(\beta_0+\beta_1 S_2)} \\
+ \sum_{j=1}^{n_2} {t_{2j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} \log\left(t_{2j}\right) (\beta_0 + \beta_1 S_2) \\
- n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} \log(\tau)\, (\beta_0 + \beta_1 S_2) \\
- n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} \log(\tau)\, (\beta_0 + \beta_1 S_2)^2 \\
+ \sum_{j=1}^{n_1} {t_{1j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} log t_{1j}\, (\beta_0 + \beta_1 S_1) \\
+ n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} (\log(\tau)^2) \\
+ \sum_{j=1}^{n_1} {t_{1j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} log t_{1j} (\beta_0 + \beta_1 S_1) \\
+ n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} (\log(\tau)^2) - n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} (\beta_0 + \beta_1 S_1) \\
- n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} \log(\tau) = 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \log\ L}{\partial\beta_0\partial\alpha} = \frac{\partial^2 \log\ L}{\partial\alpha\partial\beta_0} \\
&= n + \sum_{j=1}^{n_1} {t_{1j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} \alpha \, log t_{1j} - \sum_{j=1}^{n_1} {t_{1j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} \\
&- \sum_{j=1}^{n_2} {t_{2j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} + \sum_{j=1}^{n_2} {t_{2j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} \alpha \log(t_{2j}) \\
&- \sum_{j=1}^{n_1} {t_{1j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} \alpha\,(\beta_0 + \beta_1 S_1) - n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} \\
&+ n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} \alpha(\beta_0 + \beta_1 S_2) + n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} \\
&+ n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} \alpha(\beta_0 + \beta_1 S_1) - n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} \alpha \log(\tau) \\
&- \sum_{j=1}^{n_2} {t_{2j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} \alpha(\beta_0 + \beta_1 S_2) + n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} \alpha \log\,(\tau) \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \log L}{\partial\beta_1\partial\alpha} = \frac{\partial^2 \log L}{\partial\alpha\partial\beta_1} \\
&= n_1 S_1 + n_2(S_2 + S_1) + \sum_{j=1}^{n_1} {t_{1j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} \alpha \, log t_{1j} \\
&- n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} S_2 + n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} \alpha \log(\tau) \\
&- n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} \alpha(\beta_0 + \beta_1 S_1) + n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} \alpha \log(\tau) \\
&+ \sum_{j=1}^{n_2} {t_{2j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} \alpha \log(t_{2j}) + n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} \alpha(\beta_0 + \beta_1 S_2) \\
&- \sum_{j=1}^{n_1} {t_{1j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} \alpha(\beta_0 + \beta_1 S_1) - \sum_{j=1}^{n_1} {t_{1j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} S_1 \\
&- \sum_{j=1}^{n_2} {t_{2j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} \alpha(\beta_0 + \beta_1 S_2) - \sum_{j=1}^{n_2} {t_{2j}}^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_2)} S_2 \\
&- n_2 \tau^{-\alpha} e^{\alpha(\beta_0+\beta_1 S_1)} S_1 = 0.
\end{aligned}
$$

The FIM is:

$$
F_{\alpha,\beta_0,\beta_1} = \begin{bmatrix} -\frac{\partial^2 L}{\partial\alpha^2} & -\frac{\partial^2 L}{\partial\alpha\partial\beta_0} & -\frac{\partial^2 L}{\partial\alpha\partial\beta_1} \\ -\frac{\partial^2 L}{\partial\beta_0\partial\alpha} & -\frac{\partial^2 L}{\partial{\beta_0}^2} & -\frac{\partial^2 L}{\partial\beta_0\partial\beta_1} \\ -\frac{\partial^2 L}{\partial\beta_1\partial\alpha} & -\frac{\partial^2 L}{\partial\beta_1\partial\beta_0} & -\frac{\partial^2 L}{\partial{\beta_1}^2} \end{bmatrix}.
$$