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GENERAL CLASS OF EXPONENTIAL ESTIMATOR FOR ESTIMATING FINITE POPULATION VARIANCE

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Abstract

In this article, we consider the problem of estimating an unknown population variance using two auxiliary variables. A generalized exponential estimator along with a class of estimators has been proposed for estimating population variance. The bias and the mean square error of the proposed estimator are obtained to the first order of approximations. It is shown that the proposed generalized estimator is more efficient than the existing literature estimators. An empirical and a simulated study have also been carried out to demonstrate the efficiency of proposed estimator with the literature.

Key Words: Population Variance, Auxiliary Variable, Exponential Estimator, Single-phase Sampling, Mean Square Error, Bias

1. Introduction

It is well known that the auxiliary information is used for improving the efficiency of the estimator of population parameter of interest. Estimating the finite population variance has great significance in various fields such as Industry, Agriculture, Medical and Biological sciences, etc.

Let us consider a finite population $\Omega = \{\Omega 1, \Omega 2, \ldots . \Omega N\}$ with N population units. Let $S_y^2 = \frac{\sum_{i=1}^{N}(y_i - \bar{Y})^2}{N-1}$, $\bar{X}_1 = \frac{\sum_{i=1}^{N} X_1}{N}$ and $\bar{X}_2 = \frac{\sum_{i=1}^{N} X_2}{N}$ be the population variance and the population means for the first and second auxiliary variable respectively. Similarly $s_y^2 = \frac{\sum_{i=1}^{n}(\mathrm{y}_i - \bar{y})^2}{n-1}$ is the sample variance. $\bar{x}_1 = \frac{\sum_{i=1}^{n} x_1}{n}$ and $\bar{x}_2 = \frac{\sum_{i=1}^{n} x_2}{n}$ are the sample means. Let $\eta_{400} = \frac{\mu_{400}}{\mu_{200}^2}$ be the population coefficient of kurtosis of y and

$\eta_{210} = \frac{\mu_{210}}{\mu_{200}\mu_{020}^{1/2}}$ covariance between S_y^2 and $\overline{X}_1$,where $\eta_{pqr} = \frac{\mu_{pqr}}{\mu_{200}^{p/2}\mu_{020}^{q/2}\mu_{002}^{r/2}}$,

$\mu_{pqr} = \frac{1}{N}\sum_{i}^{N}(Y_i - \overline{Y})^p (X_{i1} - \overline{X}_1)^q (X_{i2} - \overline{X}_2)^r$.

For ease of mathematical computations, we ignore finite population correction (fpc). We use the following relative error terms to compute the bias and mean square error,

$$s_y^2 = S_y^2\,(1+\varepsilon_0), \bar{x}_1 = \overline{X}_1(1+e_1), \bar{x}_2 = \overline{X}_2(1+e_2)\,.\,E(\varepsilon_0) = E(e_1) = E(e_2) = 0$$

$$E(\varepsilon_0^2) = \frac{1}{n}(\eta_{400} - 1), E(e_1^2) = \frac{C_{x_1}^2}{n}, E(e_2^2) = \frac{C_{x_2}^2}{n}, E(\varepsilon_0 e_1) = \frac{\eta_{210}C_{x_1}}{n} \text{ and}$$

$$E(\varepsilon_0 e_2) = \frac{\eta_{201}C_{x_2}}{n}, E(e_1 e_2) = \frac{\rho_{x_1x_2}C_{x_1}C_{x_2}}{n}\,.\,\eta^*_{400} = (\eta_{400} - 1), \gamma = \frac{1}{n}$$

An unbiased estimator of population variance is given by

$$t_0 = s_y^2\,, \tag{1}$$

$$Var(t_0) = \gamma S_y^4\left[\eta^*_{400}\right]. \tag{2}$$

A modified form of Isaki (1983) using two auxiliary variables for estimating population variance in single phase sampling may be given as,

$$t_I = s_y^2 \frac{S_{x_1}^2}{s_{x_1}^2}\frac{S_{x_2}^2}{s_{x_2}^2}, \tag{3}$$

Up to the first order-approximation the bias and Mean Square Error (MSE) is given as,

$$Bias(t_I) \approx \gamma S_y^2\left[\eta_{040} + \eta_{004} - \eta_{220} - \eta_{202} + \eta_{022} - 1\right], \tag{4}$$

and

$$MSE(t_I) \approx \gamma S_y^4\left[\eta_{400} + \eta_{040} + \eta_{004} - 2\eta_{220} - 2\eta_{202} + 2\eta_{022} - 1\right] \tag{5}$$

A modified form of Upadhyaya& Singh (2001) using two auxiliary variables for estimating population variance in single phase sampling may be given as,

$$t_U = s_y^2 \frac{\overline{X}_1}{\bar{x}_1}\frac{\overline{X}_2}{\bar{x}_2}, \tag{6}$$

The bias and Mean square error is,

$$Bias(t_U) \approx \gamma S_y^2 \left[C_{x_1}^2 + C_{x_2}^2 - \eta_{210} C_{x_1} - \eta_{201} C_{x_2} + \rho C_{x_1} C_{x_2} \right], \quad (7)$$

and

$$MSE(t_U) \approx \gamma S_y^4 \left[\eta^*{}_{400} + C_{x_1}^2 + C_{x_2}^2 - 2\eta_{210} C_{x_1} - 2\eta_{201} C_{x_2} + 2\rho C_{x_1} C_{x_2} \right] \quad (8)$$

Many more authors including Upadhyaya and Singh (1999), Singh et al. (2011), Subramani and Kumarapandiyan (2012), Yadav and Kadilar (2013), Singh and Solanki (2013), Asghar et al. (2014) and Adichwal et al. (2015) among others have proposed estimators for the estimation of population variance in single-phase sampling design. Subramani and Kumarapandiyan (2013) used quartiles and median of auxiliary information for estimating population variance. Subramani and Kumarapandiyan (2015) proposed the variance estimation by using mean of auxiliary variable. Subramani (2015) also proposed the generalized modified ratio type estimator for variance estimator and showed the efficiency of proposed estimators numerically and graphically. Ahmed et al. (2016) used the multi-auxiliary variables for estimating the population variance using successive sampling.

In the next section we have proposed a generalized exponential type estimator for estimating the population variance. In section 3 of this paper theoretical comparison has been made where as in sections 4 and 5, numerical and simulation studies have conducted respectively.

2. Proposed Generalized Exponential-Type Estimator

Following Upadhyaya and Singh (2001) and Sanaullah et al. (2016) the ratio-type and product-type exponential estimators for population variance using two auxiliary variables are,

$$t_{r1} = s_y^2 \exp\left[\frac{\bar{X}_1 - \bar{x}_1}{\bar{X}_1 + \bar{x}_1}\right] \exp\left[\frac{\bar{X}_2 - \bar{x}_2}{\bar{X}_2 + \bar{x}_2}\right], \quad (9)$$

and

$$t_{p1} = s_y^2 \exp\left[-\left(\frac{\bar{X}_1 - \bar{x}_1}{\bar{X}_1 + \bar{x}_1}\right)\right] \exp\left[-\left(\frac{\bar{X}_2 - \bar{x}_2}{\bar{X}_2 + \bar{x}_2}\right)\right], \quad (10)$$

This leads to the generalized form as,

$$t_{FI} = s_y^2 \exp\left[\alpha\left(1 - \frac{a\bar{x}_1}{\bar{X}_1 + (a-1)\bar{x}_1}\right)\right] \exp\left[\beta\left(1 - \frac{b\bar{x}_2}{\bar{X}_2 + (b-1)\bar{x}_2}\right)\right] \quad (11)$$

When α and β takes positive values we may have exponential ratio-cum-ratio estimator, and when α and β takes negative values we may have exponential product-cum-product estimators.

Putting different values of a and b, we may get different exponential ratio-cum-ratio and product-cum-product family of estimator.

For, $\alpha = \beta = 1$ in (11) we may obtain the exponential ratio-cum ratio estimator as,

$$t_r = s_y^2 \exp\left(1 - \frac{a\bar{x}_1}{\bar{X}_1 + (a-1)\bar{x}_1}\right) \exp\left(1 - \frac{b\bar{x}_2}{\bar{X}_2 + (b-1)\bar{x}_2}\right), \tag{12}$$

Some examples of exponential ratio-cum-ratio estimator t_r may be given as follows:

For $a = 2 \,\&\, b = 2$, t_r in (12) is given as,

$$t_{r1} = s_y^2 \exp\left[\frac{\bar{X}_1 - \bar{x}_1}{\bar{X}_1 + \bar{x}_1}\right] \exp\left[\frac{\bar{X}_2 - \bar{x}_2}{\bar{X}_2 + \bar{x}_2}\right], \tag{13}$$

For, $a = 1 \,\&\, b = 1$, t_r in (12) is given as,

$$t_{r2} = s_y^2 \exp\left[\frac{\bar{X}_1 - \bar{x}_1}{\bar{X}_1}\right] \exp\left[\frac{\bar{X}_2 - \bar{x}_2}{\bar{X}_2}\right], \tag{14}$$

For, $\alpha = \beta = -1$ in (11) we may obtain the exponential product-cum product estimator as,

$$t_p = s_y^2 \exp\left[-\left(1 - \frac{a\bar{x}_1}{\bar{X}_1 + (a-1)\bar{x}_1}\right)\right] \exp\left[-\left(1 - \frac{b\bar{x}_2}{\bar{X}_2 + (b-1)\bar{x}_2}\right)\right], \tag{15}$$

Some examples of exponential Product-cum-product estimators may be given as follows:

For, $a = 2 \,\&\, b = 2$, t_p in (15) is given as,

$$t_{p1} = s_y^2 \exp\left[\frac{\bar{x}_1 - \bar{X}_1}{\bar{X}_1 + \bar{x}_1}\right] \exp\left[\frac{\bar{x}_2 - \bar{X}_2}{\bar{X}_2 + \bar{x}_2}\right], \tag{16}$$

For, $a = 1 \,\&\, b = 1$, t_p in (15) is reduced to,

$$t_{p2} = s_y^2 \exp\left[\frac{\bar{x}_1 - \bar{X}_1}{\bar{X}_1}\right] \exp\left[\frac{\bar{x}_2 - \bar{X}_2}{\bar{X}_2}\right], \tag{17}$$

The expression in (12) may be expressed in the form of e's as,

$$t_{FI} = S_y^2\,(1+\varepsilon_0)\exp\left[-\frac{\alpha\, e_1}{a}\left\{(1+\frac{a-1}{a}e_1)\right\}^{-1}\right] \exp\left[-\frac{\beta\, e_2}{b}\left\{(1+\frac{b-1}{b}e_2)\right\}^{-1}\right], \tag{18}$$

We expand the exponentials and neglect the terms in e_0, e_1 and e_2 of power higher than two as,

$$t_{FI} = S_y^2\,(1+\varepsilon_0)\exp\left[-\frac{\alpha\, e_1}{a}\left\{1-\frac{a-1}{a}e_1+\frac{(a-1)^2}{a^2}e_1^2 ----\right\}\right] \exp\left[-\frac{\beta\, e_2}{b}\left\{1-\frac{b-1}{b}e_2+\frac{(b-1)^2}{b^2}e_2^2 ---\right\}\right], \tag{19}$$

or

$$t_{FI} \approx S_y^2\left[\begin{array}{l}1+\varepsilon_0-\dfrac{\alpha\;e_1}{a}+\dfrac{\alpha(a-1)e_1^2}{a^2}+\dfrac{\alpha^2\,e_1^2}{2!a^2}-\dfrac{\beta\;e_2}{b}\\ +\dfrac{\beta(b-1)e_2^2}{b^2}+\dfrac{\beta^2e_2^2}{2!b^2}-\dfrac{\alpha\,e_0e_1}{a}-\dfrac{\beta\,e_0e_2}{b}+\dfrac{\alpha\beta e_1\,e_2}{a\,b}\end{array}\right], \tag{20}$$

Subtracting S_y^2 from both sides to obtain the expression for bias, we may get as,

$$t_{FI} - S_y^2 \approx S_y^2\left[\begin{array}{l}\varepsilon_0-\dfrac{\alpha\;e_1}{a}+\dfrac{\alpha(a-1)e_1^2}{a^2}+\dfrac{\alpha^2\,e_1^2}{2a^2}-\dfrac{\beta\;e_2}{b}\\ +\dfrac{\beta(b-1)e_2^2}{b^2}+\dfrac{\beta^2e_2^2}{2b^2}-\dfrac{\alpha\,e_0e_1}{a}-\dfrac{\beta\,e_0e_2}{b}+\dfrac{\alpha\beta e_1\,e_2}{a\,b}\end{array}\right], \tag{21}$$

Now taking expectation on both sides of (21) and we get bias as,

$$\text{Bias}(t_{FI}) \approx S_y^2 \begin{bmatrix} \left(\frac{\alpha}{a} - \frac{\alpha}{a^2} + \frac{\alpha^2}{2a^2}\right)\frac{C_{x_1}^2}{n} + \left(\frac{\beta}{b} - \frac{\beta}{b^2} + \frac{\beta^2}{2b^2}\right)\frac{C_{x_2}^2}{n} \\ -\frac{\alpha}{na}\eta_{210}C_{x_1} - \frac{\beta}{nb}\eta_{201}C_{x_2} + \frac{\alpha\beta}{nab}\rho_{x_1x_2}C_{x_1}C_{x_2} \end{bmatrix}. \tag{22}$$

Squaring of (21), neglecting the terms of power more than two and taking expectation, we may get the Mean Square error (MSE) of t_{FI} as,

$$\text{MSE}(t_{FI}) \approx \gamma S_y^4 \left[\eta_{400}^* + \frac{\alpha^2 C_{x_1}^2}{a^2} + \frac{\beta^2 C_{x_2}^2}{b^2} - \frac{2\alpha\eta_{210}C_{x_1}}{a} - \frac{2\beta\eta_{201}C_{x_2}}{b} + \frac{2\alpha\beta\rho_{x_1x_2}C_{x_1}C_{x_2}}{ab}\right]. \tag{23}$$

In order to get the optimum values of a and b which minimize the MSE(t_{FI}), we differentiate MSE (t_{FI}) w.r.t. a and b and equates to zero as,

$$\frac{\partial\,\text{MSE}(t_{FI})}{\partial a} = 0, \qquad \text{and} \qquad \frac{\partial\,\text{MSE}(t_{FI})}{\partial b} = 0. \tag{24}$$

After simplifications we may get the optimum values of a and b as,

$$a_{opt} = \frac{\alpha C_{x_1}(1-\rho_{x_1x_2}^2)}{(\eta_{210} - \rho_{x_1x_2}\eta_{201})} \;\&\; \text{b}_{opt} = \frac{\beta C_{x_2}(1-\rho_{x_1x_2}^2)}{(\eta_{201} - \rho_{x_1x_2}\eta_{210})}. \tag{25}$$

or

The $\text{MSE}(t_{FI})$ is given by,

$$\text{MSE}(t_{FI}) \approx \gamma S_y^4 \left[\eta_{400}^* + \frac{\alpha^2 C_{x_1}^2}{a_{opt}^2} + \frac{\beta^2 C_{x_2}^2}{b_{opt}^2} - \frac{2\alpha\eta_{210}C_{x_1}}{a_{opt}} - \frac{2\beta\eta_{201}C_{x_2}}{b_{opt}} + \frac{2\alpha\beta\rho_{x_1x_2}C_{x_1}C_{x_2}}{a_{opt}b_{opt}}\right]. \tag{26}$$

To obtain $\text{MSE}(t_r)$ we put $\alpha = \beta = 1$ in (26),

$$\text{MSE}(t_r) \approx \gamma S_y^4 \left[\eta_{400}^* + \frac{C_{x_1}^2}{a_{opt_r}^2} + \frac{C_{x_2}^2}{b_{opt_r}^2} - \frac{2\eta_{210}C_{x_1}}{a_{opt_r}} - \frac{2\eta_{201}C_{x_2}}{b_{opt_r}} + \frac{2\rho_{x_1x_2}C_{x_1}C_{x_2}}{a_{opt_r}b_{opt_r}}\right]. \tag{27}$$

We get the optimal values of a_{optr} and b_{optr}, by setting $\alpha = \beta = 1$ in (27).

$$a_{optr} = \frac{C_{x_1}(1-\rho^2_{x_1x_2})}{(\eta_{210}-\rho_{x_1x_2}\eta_{201})} \;\&\; \mathrm{b}_{optr} = \frac{C_{x_2}(1-\rho^2_{x_1x_2})}{(\delta_{201}-\rho_{x_1x_2}\eta_{210})}. \tag{28}$$

Or by simplifying, we may get expression for the minimum MSE of t_r as,

$$\min.\mathrm{MSE}(t_r) \approx \gamma S_y^4 \left[\eta^*_{400} + \frac{1}{(1-\rho^2_{x_1x_2})} \left\{ \begin{array}{l} (\eta_{210}-\rho_{x_1x_2}\eta_{201})^2 + (\eta_{201}-\rho_{x_1x_2}\eta_{210})^2 \\ -2\eta_{210}(\eta_{210}-\rho_{x_1x_2}\eta_{201}) - 2\eta_{201}(\eta_{201}-\rho_{xz}\eta_{210}) \\ +\dfrac{2\rho_{x_1x_2}(\eta_{210}-\rho_{x_1x_2}\eta_{201})(\eta_{201}-\rho_{x_1x_2}\eta_{210})}{(1-\rho^2_{x_1x_2})} \end{array} \right\} \right], \tag{29}$$

Now if we consider
$A=(\eta_{210}-\rho_{x_1x_2}\eta_{201})$, B$=(\eta_{201}-\rho_{x_1x_2}\eta_{210})$ & $C=(1-\rho^2_{x_1x_2})$ then,

$$\min.\mathrm{MSE}(t_r) \approx \gamma S_y^4 \left[\eta^*_{400} + \frac{1}{C}\left\{ A^2 + B^2 - 2\eta_{210}A - 2\eta_{201}B + \frac{2\rho AB}{C} \right\} \right]. \tag{30}$$

Similarly we may get $\mathrm{MSE}(t_p)$ by putting $\alpha = \beta = -1$ in optimum values of $a \,\&\, b$ in (2.18). We note that the expressions for the minimum mean square error of t_{FI} , t_r, and t_p are obtained same up to the first order of approximation, i.e.

$$\min.\mathrm{MSE}(t_r) = \min.\mathrm{MSE}(t_p) = \min.\mathrm{MSE}(t_{FI})$$

$$\approx \gamma S_y^4 \left[\eta^*_{400} + \frac{1}{C}\left\{ A^2 + B^2 - 2\eta_{210}A - 2\eta_{201}B + \frac{2\rho AB}{C} \right\} \right]. \tag{31}$$

3. Theoretical Comparison

In this section we compare our suggested estimator with unbiased estimator of sample variance, modified Isaki (1983) and Upadhyaya and Singh (2001) under single phase sampling as follows:

The proposed exponential ratio-cum-ratio estimator will be more efficient if,

$$MSE(t_r) < Var(t_0)$$

$$a^2(C_{x_2}^2 - 2b_{opt_r} C_z \eta_{201}) - 2a(b_{opt_r}^2 \eta_{210} C_{x_1} - b_{opt_r} \rho_{x_1x_2} C_{x_1} C_{x_2}) + b_{opt_r}^2 C_{x_1}^2 \le 0$$

Now consider,

$$M = C_{x_2}^2 - 2b_{opt_r} C_{x_1} \eta_{201} \quad \& \quad N = b_{opt_r}^2 C_{x_1} \eta_{210} - 2b_{opt_r} \rho_{x_1x_2} C_{x_1} C_{x_2}$$

After some simplification, we may get the range of a as,

$$\frac{N}{M} - \sqrt{\frac{\left(N/\sqrt{M}\right)^2 - b_{opt_r}^2 C_{x_1}^2}{M}} < a < \frac{N}{M} + \sqrt{\frac{\left(N/\sqrt{M}\right)^2 - b_{opt_r}^2 C_{x_1}^2}{M}}. \tag{32}$$

The proposed exponential ratio-cum-ratio estimator will be more efficient if,

$$MSE(t_r) < MSE(t_I)$$

$$a^2(C_{x_2}^2 - 2b_{opt_r} C_{x_2} \eta_{201} - b_{opt_r}^2 k) - 2a(b_{opt_r}^2 \eta_{210} C_{x_1} - b_{opt_r} \rho_{x_1x_2} C_{x_1} C_{x_2}) + b_{opt}^2 C_{x_1}^2 \le 0$$

Now if we consider

$$k = [\eta_{040} + \eta_{004} - 2\eta_{220} - 2\eta_{202} + 2\eta_{022} - 1],$$

$$M_I = (C_{x_2}^2 - 2b_{opt_r} C_{x_2} \eta_{201} - b_{opt_r}^2 k) \;\&\; N = b_{opt_r}^2 C_{x_1} \eta_{210} - b_{opt_r} \rho_{x_1x_2} C_{x_1} C_{x_2}$$

Then after simplification, we may get the range of a as,

$$\frac{N}{M_I} - \sqrt{\frac{\left(N/\sqrt{M_I}\right)^2 - b_{opt_r}^2 C_{x_1}^2}{M_I}} < a < \frac{N}{M_I} + \sqrt{\frac{\left(N/\sqrt{M_I}\right)^2 - b_{opt_r}^2 C_{x_1}^2}{M_I}}. \tag{33}$$

The proposed exponential ratio-cum-ratio estimator will be more efficient if,

$$MSE(t_r) < MSE(t_U)$$

$$a^2(C_{x_2}^2 - b_{opt_r}^2 C_{x_2}^2 + 2b_{opt_r}^2 C_{x_1} \eta_{210} - 2b_{opt_r} \eta_{201} C_{x_2} + 2b_{opt_r}^2 \eta_{201} C_{x_2} + 2\rho_{x_1x_2} C_{x_1} C_{x_2}) - 2a(b_{opt_r}^2 \eta_{210} C_{x_1} - b_{opt_r} \rho_{x_1x_2} C_{x_1} C_{x_2}) \le 0$$

Now if we consider

$$M_U = (C_{x_2}^2 - b_{opt_r}^2 C_{x_2}^2 - b_{opt}^2 C_{x_1}^2 + 2b_{opt_r}^2 C_{x_1} \eta_{210}$$
$$- 2b_{opt_r} \eta_{201} C_{x_2} + 2b_{opt_r}^2 \eta_{201} C_{x_2} + 2b_{opt_r}^2 \rho_{x_1 x_2} C_{x_1} C_{x_2}) \qquad \&$$
$$N = b_{opt}^2 C_{x_1} \eta_{210} - b_{opt_r} \rho_{x_1 x_2} C_{x_1} C_{x_2}$$

Then after some simplification, we may get the range of a as,

$$\frac{N}{M_U} - \sqrt{\frac{\left(N/\sqrt{M_U}\right)^2 - b_{opt_r}^2 C_{x_1}^2}{M_U}} < a < \frac{N}{M_U} + \sqrt{\frac{\left(N/\sqrt{M_U}\right)^2 - b_{opt_r}^2 C_{x_1}^2}{M_U}} \qquad (34)$$

The suggested estimator t_r would be more efficient than the existing estimators t_0, t_I, and t_U if the conditions in (32)-(34) are satisfied.

4. Numerical Study and Illustration

To examine the performance of our suggested estimator over the competitors we consider the three real applications\populations for positively correlated case. See Appendix for the description of the three populations. The PRE's for each estimator is calculated as,

$$\text{PRE} = \frac{MSE(t_0)}{MSE(t_i)} \text{x}100, \qquad \text{where } i = \text{I, U, r1, r2, r}$$

Estimator	I	II	III
t_0	100	100	100
t_I	98.46386	79.19756	100
t_U	111.5988	119.829	104.8395
t_{r1}	106.5255	110.3557	102.9487
t_{r2}	111.5988	119.829	104.8395
t_r	118.6027	133.3718	105.878

Table 1: Percentage Relative efficiency (PRE) of proposed estimators with t_0, t_I and t_U

Table 1 demonstrates the relative efficiency of each estimator. This empirical study shows some realistic results about the efficiency of the proposed class of ratio-cum-ratio estimators. We can observe proposed class t_{r1}, t_{r2}, and t_r are more efficient than some modified versions of Isaki's and Upadhyaya's estimators and further we note that among the proposed class t_r is the most efficient estimator.

5. Simulation Study and Discussion

However to assess the performance of our proposed estimators we have also computed the results by simulation study. Following Ahmed et al. (2016), we have under taken the simulation study by the following steps.

a. In the simulation study we have two auxiliary variables along with a study variable.

b. We generate different populations with different sample sizes for five models.

c. For model-I, the study variable y_i is generated from the linear model and auxiliary variables x_1 & x_2 are generated from normal distribution with means (20, 30) and standard deviations (2, 3).

Where model-I, model-II, model –III, model –IV and model-V are described as,

Model-I $y_i = \sum x_i + \varepsilon_i$, and $\varepsilon_i \sim N(0,1)$

Model-II $y_i = \sum \exp(\mathrm{x}_i) + \varepsilon_i$

or

$$y_i = \exp(\mathrm{x}_1) + \exp(\mathrm{x}_2) + \varepsilon_i,\ \varepsilon_i \sim N(0,1)$$

Model-III $y_i = \sum \exp\left(\frac{X_i}{\bar{X}_i + \bar{x}_i}\right)$

or

$$y_i = \exp\left(\frac{X_1}{\bar{X}_1 + \bar{x}_1}\right) + \exp\left(\frac{X_2}{\bar{X}_2 + \bar{x}_2}\right) + \varepsilon_i \text{ and } \varepsilon_i \sim N(0,1)$$

Model-IV $y_i = \prod \exp(\mathrm{x}_i) + \varepsilon_i$

or

$$y_i = \exp(\mathrm{x}_{1i} + \mathrm{x}_{2i}) + \varepsilon_i,\ \varepsilon_i \sim N(0,1)$$

Model-V $y_i = \prod \exp\left(\frac{X_i}{\bar{X} + \bar{x}_i}\right) + \varepsilon_i,$

or

$$y_i = \exp\left(\frac{X_{1i}}{\bar{X}_1 + \bar{x}_{1i}} + \frac{X_{2i}}{\bar{X}_2 + \bar{x}_{2i}}\right) + \varepsilon_i \text{ and } \varepsilon_i \sim N(0,1)$$

- We generated $k = 5000$ times, populations of size N=200 for each time a sample of size n= 30 was selected by *SRSWR* using R program.
- The MSE of t is obtained following each model separately as, $MSE(\text{t}) = \frac{1}{k}\sum_{i=1}^{k}(t_i - T)^2$, where $T = \frac{1}{k}\sum_{i=1}^{k} t_i$.
- This procedure (1 to 5) was used for four different sample sizes, i.e*n*=30,50,100,150.
- The similar procedure (1 to 7) was used for three different population sizes, i.e. *N*=200,1000,2000.
- Each time we computed the simulated results for t_j estimator, where *j*=0, I, U, r1, r2, r.

Finally results based on the above said simulation study are presented for *N*=200, *N*=1000 and *N*=2000 respectively in Table 2-Table 7.

In the above simulation study different relationships between y and x's are thought to be studied. Mode-1 is constructed on the basis if the relationship between the study variable y and auxiliary variables x_1 and x_2 is linear, where as for the construction of further four models, Model-2 to Model-5, it is assumed that the relationship between the study and auxiliary variables is exponential or exponential-type. On the basis of simulation study presented in Table 2-Table 7, it can be noted that from the proposed estimators t_{2r} and t_r are found to be more efficient than t_0, t_I, and t_U under all five models. Furthermore it is noted that efficiency of the exponential estimators are increased considerably on using model-2 and model-5.

Estimator	**Model-I**				**Model-II**				**Model-III**			
	n=30	**n=50**	**n=100**	**n=150**	**n=30**	**n=50**	**n=100**	**n=150**	**n=30**	**n=50**	**n=100**	**n=150**
t_0	100	100	100	100	100	100	100	100	100	100	100	100
t_I	41	40	61	80	64	49	52	92	80	36	52	65
t_U	102	102	100	100	108	102	102	100	100	100	102	100
t_{r1}	102	101	101	100	102	101	101	101	101	101	101	101
t_{r2}	103	103	102	101	104	102	102	102	102	102	102	102
t_r	103	104	103	101	110	103	102	102	103	104	102	102

Table 2: Percentage relative efficiency (PRE's) of proposed estimators with t_0, t_I and t_U when $k = 5000, N = 200$ & $n = 30,50,100,150$ for Model I-III

Estimator	Model-IV				Model-V			
	n=30	n=50	n=100	n=150	n=30	n=50	n=100	n=150
t_0	100	100	100	100	100	100	100	100
t_I	35	74	93	102	97	99	49	66
t_U	104	101	103	104	105	108	101	105
t_{r1}	102	102	101	102	103	102	100	102
t_{r2}	104	105	102	104	106	104	101	104
t_r	124	110	127	131	184	189	110	167

Table 3: Percentage relative efficiency (PRE's) of proposed estimators with t_0, t_I and t_U when $k = 5000, N = 200$ & $n = 30, 50, 100, 150$ for Model IV-V

Estimator	Model-I				Model-II				Model-III			
	n=30	n=50	n=100	n=150	n=30	n=50	n=100	n=150	n=30	n=50	n=100	n=150
t_0	100	100	100	100	100	100	100	100	100	100	100	100
t_I	78	95	102	87	49	25	85	98	100	58	90	98
t_U	102	106	102	101	100	103	102	103	100	103	101	102
t_{r1}	101	102	102	101	102	102	102	102	101	102	101	101
t_{r2}	102	103	103	102	105	103	104	103	103	103	101	102
t_r	109	107	106	104	115	112	115	109	101	108	102	104

Table 4: Percentage relative efficiency (PRE's) of proposed estimators with t_0, t_I and t_U when $k = 50000, N = 1000$ & $n = 30, 50, 100, 150$ for Model I-III

Estimator	Model-IV				Model-V			
	n=30	n=50	n=100	n=150	n=30	n=50	n=100	n=150
t_0	100	100	100	100	100	100	100	100
t_I	33	28	77	80	25	23	42	88
t_U	102	101	104	103	101	101	102	101

t_{r1}	101	101	102	101	102	102	100	100
t_{r2}	102	102	103	101	104	104	101	101
t_r	130	142	186	150	140	165	112	129

Table 5: Percentage relative efficiency (PRE's) of proposed estimators with t_0, t_I and t_U when $k = 50000, N = 1000$ & $n = 30, 50, 100, 150$ for Model IV-V

Esti-mator	Model-I				Model-II				Model-III			
	n=30	n=50	n=100	n=150	n=30	n=50	n=100	n=150	n=30	n=50	n=100	n=150
t_0	100	100	100	100	100	100	100	100	100	100	100	100
t_I	28	67	92	99	55	85	66	76	58	44	58	95
t_U	98	96	101	100	106	107	102	102	103	100	104	100
t_{r1}	101	100	101	101	103	104	102	102	102	101	100	100
t_{r2}	102	101	102	101	106	108	103	104	103	102	101	101
t_r	110	105	103	106	115	110	114	113	109	110	104	106

Table 6: Percentage relative efficiency (PRE's) of proposed estimators with t_0, t_I and t_U when $k = 5000, N = 2000$ & $n = 30, 50, 100, 150$ for Model I-III

Estimator	Model-IV				Model-V			
	n=30	n=50	n=100	n=150	n=30	n=50	n=100	n=150
t_0	100	100	100	100	100	100	100	100
t_I	77	53	37	100	46	103	82	80
t_U	102	106	100	104	104	100	103	102
t_{r1}	101	101	100	102	101	101	101	101
t_{r2}	102	102	101	104	103	102	101	102
t_r	132	145	131	194	144	141	140	148

Table 7: Percentage relative efficiency (PRE's) of proposed estimators with t_0, t_I and t_U when $k = 5000, N = 2000$ & $n = 30, 50, 100, 150$ for Model IV-V

6. Conclusion

From empirical and simulation study it is shown that our proposed exponential estimators perform more efficiently than the existing estimators. From simulation study it is also further concluded that the proposed exponential estimators perform better than existing estimators if relationship between the study variable and auxiliary variables is exponential. This study removes some existing gape on the issue regarding the usage of the exponential estimators.

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Appendix

S#	Source of Population
1	Cochran (1977),pg. 34
2	Gujarati (2004), pg.386
3	Douglas C. Montgomery (2012), pg. 559

Table A: Source of Population

Population	***Y***	X_1	X_2
1	Food Cost	Family Size	Weekly Income
2	Average miles per Gallon	Top seed, miles per hour	Engine horsepower
3	Concentration	Space time	Molar density

Table B: Variables used for Population 1-3

Parameters	**I**	**II**	**III**
N	33	47	28
n	17	23	12
C_{x_1}	0.40250	0.3458955	0.5924725
C_{x_2}	0.143577	0.0712406	0.0680259
ρ_{yx_1}	0.42378	0.6876045	0.2739857
ρ_{yx_2}	0.252166	0.1888663	0.2267193
$\rho_{x_1x_2}$	-0.065989	0.1214932	0.9234176
η_{400}	5.72	3.943658	20.33332
η_{210}	0.6305	0.8221768	1.016998
η_{201}	0.5506	0.3441902	0.8632808
η_{040}	2.380	3.301295	1.871055
η_{004}	2.142944	2.20406	1.228679
η_{220}	1.491798	2.487812	1.119856
η_{202}	1.432347	1.095846	1.350588
η_{022}	2.287957	1.217578	0.9852038

Table C: Parameters of populations