Journal of Reliability and Statistical Studies; ISSN (Print): 0974-8024, (Online): 2229-5666
Vol. 10, Issue 2 (2017): 117-126

ON THE VARIANCE OF $P(Y < X)$ ESTIMATOR IN BURR XII DISTRIBUTION

M. Khorashadizadeh*, S. Nayeban[1] and G.R. Mohtashami Borzadaran[2]
*Department of Statistics, University of Birjand
[1,2]Department of Statistics, Ferdowsi University of Mashhad
E Mail: m.khorashadizadeh@birjand.ac.ir*
*Corresponding Author

Received November 03, 2016
Modified October 12, 2017
Accepted October 31, 2017

Abstract

Sometimes due to complicated form of estimators, we can not compute their variances. In some cases the best way to approximate the variance is using lower bounds such as Cramer-Rao and its extension Bhattacharyya lower bounds. On the other hand, statistical inference for stress-strength measure needs the variance of unbiased estimators. In this paper, we present the maximum likelihood estimator (MLE) and uniformly minimum variance unbiased estimator (UMVUE) of stress-strength reliability measure of form $R = P(Y < X)$, when X and Y have Burr XII distribution. Also, in this distribution, we obtain the general form of Bhattacharyya matrix and then by using Bhattacharyya lower bounds, we approximate the variance of any unbiased estimator of R.

Key Words: Stress-Strength Model, Burr XII Distribution, Bhattacharyya Lower Bound, Cramer-Rao Lower Bound.

1. Introduction

When properly working of a system or a component depends on its resistance to external factors, the concept of stress-strength models can be used to determine the reliability of the system or component. So, if X represents the strength and Y be the stress imposed on the component, then the computation of $R = P(Y < X)$ could measure the reliability of that system. Birnbaum (1956) was the first researcher who considered the stress-strength model and its application in engineering fileds. Many works have been done for this model by considering different distibutions for X and Y. For a comprehensive study of these models till year 2003, see the book by Kotz et al. (2003). For some of the recent references, the readers may refer to Raqab and Kundu (2005), Kundu and Gupta (2005), Kundu and Raqab (2009), Baklizi(2008a, 2008b, 2014), Al-Mutairi et al. (2015), Hor and Seal (2017) and Jovanović (2017) .

We will concentrate our attention on considering $R = P(Y < X)$ in the Burr XII distribution (Burr, 1942), because this distribution has many properties and features that are very flexible, and has many applications in reliability and distribution modeling. Zimmer et al. (1998) have shown that this distribution represents a good model for modeling failure time data. Many researchers like Wu et al. (2007) and Silva et al. (2008) have used this distribution in different fields because of its flexibility in modeling. Also, the estimation of $P[Y < X]$ in Burr X distribution, have considered by Ahmad et al. (2011) and Surles and Padgett (2001). Panahi and Asadi (2010) by using Monte Carlo simulations, have compared different methods of estimating R when Y and X both follow Burr XII distribution.

From here assume that the Burr XII distribution has the following density and distribution functions

$$f(x) = \alpha\theta x^{\alpha-1}(1+x^{\alpha})^{-(\theta+1)}; \quad for \quad x > 0,$$

$$F(x) = 1-(1+x^{\alpha})^{-\theta}; \quad for \quad x > 0.$$

Here $\alpha > 0$ and $\theta > 0$ are the shape parameters. In special case of $\alpha = 1$, the Burr XII reduces to the log-logistic distribution.

In estimation theory, if the estimator is unbiased, calculating its variance in order to understand the accuracy of estimating is one of the most important and problematic issues, especially when the estimator has a complex form and it is not possible to find the exact variance.

In such a cases will certainly find the boundary for the variance would be useful. Many studies have been done for the lower bounds of the variance of an unbiased estimator of the parameter. The well-known lower bounds are Cramer–Rao (Cramer, 1946; Rao, 1947), Bhattacharyya (Bhattacharyya, 1946, 1947), Hammersley–Chapman–Robbins (Chapman and Robbins, 1951; Hammersley, 1950), Kshirsagar (Kshirsagar, 2000), and Koike (Koike, 2002).

Blight and Rao (1974), Tanaka and Akahira (2003) and many other researchers have used the above lower bounds to apprixomate the variance of the estimators. Cacoullos (1982) have obtined an upper and lower bounds for variance of a function of a random variable.

In stress-strength modelling for Burr XII distribution, the evaluation of variance of R's estimator have not been considered, because it is impossible or very hard to compute it. So, motiviation by this open problem and according to usefulness and wide applications of the Burr XII distribution, in this paper, we first introduce the most sharper bounds which is the Bhattacharyya bound under regularity conditions. Then, we construct the general form of the Bhattacharyya matrix which is used in its inequality. Also, we evaluate and compare different Bhattacharyya bounds for the variance of estimator of R in Burr XII distribution.

2. The Stress-Strength Reliability Measure

The reliability measure (R) represents the probability that the component do its job properly under the stress. Let X and Y are two independent Burr XII random variables with parameters θ_1, α and θ_2, α respectively. Therefore

$$R = P(Y < X) = \int_0^\infty \int_0^x f_X(x) f_Y(y) dy dx$$
$$= \int_0^\infty F_Y(x) f_X(x) dx$$
$$= \frac{\theta_2}{\theta_1 + \theta_2}.$$

Panahi and Asadi (2010) have find the MLE of R in Burr XII distribution. Consider the estimation of R when the common parameter (α) is known. Without loss of generality, assume that $\alpha = 1$ (i.e. X and Y have log-logistic distributions). Let $\underline{X} = (X_1, \ldots, X_n)$ is a random sample from Burr XII distribution with parameters $(\theta_1, 1)$ and $\underline{Y} = (Y_1, \ldots, Y_m)$ is a random sample from Burr XII distribution with parameters $(\theta_2, 1)$. Then we have:

$$L = L(\theta_1, \theta_2 \mid \underline{x}, \underline{y}) = n \ln(\theta_1) + m \ln(\theta_2) - (\theta_1 + 1) \sum_{i=1}^{n} \ln(1 + x_i) - (\theta_2 + 1) \sum_{j=1}^{m} \ln(1 + y_j).$$

Differentiating partially with respect to θ_1 and θ_2 and setting the results equal to zero we get two non-linear equations as follow:

$$\frac{\partial L}{\partial \theta_1} = \frac{n}{\theta_1} - \sum_{i=1}^{n} \ln(1 + x_i) = 0,$$
$$\frac{\partial L}{\partial \theta_2} = \frac{m}{\theta_2} - \sum_{j=1}^{m} \ln(1 + y_j) = 0,$$

and we obtain,

$$\hat{\theta}_1 = \frac{n}{\sum_{i=1}^{n} \ln(1 + x_i)},$$
$$\hat{\theta}_2 = \frac{m}{\sum_{j=1}^{m} \ln(1 + y_j)}.$$

Therefore the plug-in MLE of R is given by,

$$\hat{R} = \frac{\hat{\theta}_2}{\hat{\theta}_1 + \hat{\theta}_2}.$$

In the following we have study the UMVUE of R when the common parameter (α) is known, so $(\sum_{i=1}^{n} \ln(1 + x_i), \sum_{j=1}^{m} \ln(1 + y_j))$ is a jointly sufficient statistic for (θ_1, θ_2). Therefore using the results of Tong (1974,1977), the UMVUE of R in Burr-XII is as follow:

$$\tilde{R} = \begin{cases} 1-\sum_{i=0}^{n-1}(-1)^i \dfrac{(m-1)!(n-1)!}{(m+i-1)!(n-i-1)!}\left(\dfrac{T_2}{T_1}\right)^i & T_2 < T_1, \\ 1-\sum_{i=0}^{m-1}(-1)^i \dfrac{(m-1)!(n-1)!}{(m+i-1)!(n-i-1)!}\left(\dfrac{T_1}{T_2}\right)^i & T_1 < T_2, \end{cases}$$

where $T_1 = \sum_{j=1}^{m} \ln(1+y_j)$ and $T_2 = \sum_{i=1}^{n} \ln(1+x_i)$. The variance of $\tilde{R}$ has not closed form and has not been considered precisely yet. We show that, one can approximate the variance by lower bounds of higher orders.

3. Bhattacharyya Bound

Bhattacharyya (1946, 1947) obtained a generalized form of the Cramer-Rao inequality. These lower bounds are made by the covariance matrix of the random vector,

$$\frac{1}{f(X\mid\theta)}(f^{(1)}(X\mid\theta), f^{(2)}(X\mid\theta), \ldots, f^{(k)}(X\mid\theta)), \tag{1}$$

where $f^{(j)}(.\mid\theta)$ is the j^{th} derivative of the probability density function $f(.\mid\theta)$ w.r.t. the parameter θ. Suppose that W (called "*Bhattacharyya matrix*") be the covariance matrix of (1) with the following elements,

$$W = (W_{rs}) = \left(Cov_\theta \left\{ \frac{f^{(r)}(X\mid\theta)}{f(X\mid\theta)}, \frac{f^{(s)}(X\mid\theta)}{f(X\mid\theta)} \right\} \right),$$

such that $E_\theta\left(\dfrac{f^{(r)}(X\mid\theta)}{f(X\mid\theta)}\right) = 0$ for $r, s = 1, 2, \ldots, k$. It is clear that W_{11} is the Fisher information. Now, Let $T(X)$ be any unbiased estimator of parameter function $g(\theta)$, then under some regularity conditions, the Bhattacharyya bound stated that,

$$Var_\theta(T(X)) \geq J_\theta W^{-1} J_\theta^t := B_k(\theta), \tag{2}$$

where t refers to the transpose, $J_\theta = (g^{(1)}(\theta), g^{(2)}(\theta), \ldots, g^{(k)}(\theta))$, $g^{(j)}(\theta) = \partial^j g(\theta)/\partial\theta^j$ for $j = 1, 2, \ldots, k$ and W^{-1} is the inverse of the Bhattacharyya matrix.

It should be noted that the special case of $B_1(\theta)$ is indeed the Cramer-Rao inequality. By using the properties of the multiple correlation coefficient, it is easy to show that as the order of the Bhattacharyya matrix (k) increases, the Bhattacharyya bound becomes sharper. This means that

$$Var_\theta(T(X)) \geq \ldots \geq B_3(\theta) \geq B_2(\theta) \geq B_1(\theta) \geq 0,$$

Using the Bhattacharyya bounds, Shanbhag (1972,1979) has characterized the natural exponential family with quadratic variance function. One can see more details and information about Bhattacharyya bounds and their applications in the papers such as, Blight and Rao (1974), Tanaka and Akahira (2003), Tanaka (2003,2006), Mohtashami Borzadaran (2006), Khorashadizadeh and Mohtashami Borzadaran (2007), Mohtashami Borzadaran et al. (2010) and Nayeban et al. (2013).

In this paper, we consider θ as unknown parameter. Similar results can be obtained when α is unknown or furthermore in multiparameter case when both parameters are unknown. By some mathematical computation and simpilification, the term $\frac{f^{(r)}(X|\theta)}{f(X|\theta)}$ in Burr XII has obtained as follow,

$$\frac{f^{(r)}(X|\theta)}{f(X|\theta)} = \begin{cases} \frac{1-\ln(1+X^{\alpha})^{\theta}}{\theta}; & r=1 \\ \frac{(-1)^r}{\theta}[\ln(1+X^{\alpha})^{\theta}-r]\ln(1+X^{\alpha})^{r-1}; & r=2,3,\ldots \end{cases} \tag{3}$$

So, using the Equation (3), we obtained the general form of the 5×5 symetric Bhattacharyya matrix in Burr XII as follow,

$$W = \begin{pmatrix} \frac{1}{\theta^2} & \frac{-2}{\theta^3} & \frac{6}{\theta^4} & \frac{-24}{\theta^5} & \frac{120}{\theta^6} \\ & \frac{8}{\theta^4} & \frac{-36}{\theta^5} & \frac{192}{\theta^6} & \frac{-1200}{\theta^7} \\ & & \frac{216}{\theta^6} & \frac{-1440}{\theta^7} & \frac{10800}{\theta^8} \\ & & & \frac{11520}{\theta^8} & \frac{-100800}{\theta^9} \\ & & & & \frac{1008000}{\theta^{10}} \end{pmatrix}. \tag{4}$$

As an example for W_{11}, the $(1,1)^{th}$ element of the matrix, we have,

$$\begin{aligned} W_{11} &= E\left(\frac{f^{(1)}(X|\theta)}{f(X|\theta)}.\frac{f^{(1)}(X|\theta)}{f(X|\theta)}\right) \\ &= \int_0^{\infty}\frac{\alpha x^{\alpha-1}}{\theta(1+x^{\alpha})^{\theta+1}}(\ln(1+x^{\alpha})^{\theta}-1)^2\,dx \\ &= -\frac{[1+x^{\alpha}][1+\theta^2\ln(1+x^{\alpha})^2]}{\theta^2(1+x^{\alpha})^{\theta+1}}\Big|_0^{\infty} \\ &= \frac{1}{\theta^2}, \end{aligned}$$

which is Fisher information of Burr XII. Also for W_{24} we have,

$$\begin{aligned} W_{24} &= E\left(\frac{f^{(2)}(X|\theta)}{f(X|\theta)}.\frac{f^{(4)}(X|\theta)}{f(X|\theta)}\right) \\ &= \int_0^{\infty}\frac{\alpha x^{\alpha-1}\ln(1+x^{\alpha})^4(\ln(1+x^{\alpha})^{\theta}-2)(\ln(1+x^{\alpha})^{\theta}-4)}{\theta(1+x^{\alpha})^{\theta+1}}dx \\ &= \frac{192}{\theta^6}. \end{aligned}$$

4. Numerical Studies

In this section, using the Bhattacharyya lower bounds of different orders, we study the convergence of bounds and approximating the variance of any unbiased estimator of R in Burr XII distribution. For the remainder of this paper, we will consider the case where θ_2 is known and will, without loss of generality, take $\theta_2 = 1$. In fact we want to find lower bounds for variance of any unbiased estimator of $g(\theta) = \frac{1}{1+\theta}$ in Burr XII distribution. Using the Bhattacharyya matrix of form (4), we compute the first four orders of Bhattacharyya lower bounds as the following:

$$B_1(\theta) = \frac{\theta^2}{(1+\theta)^4}$$

$$B_2(\theta) = \frac{\theta^2(\theta^2 + 2\theta + 2)}{(1+\theta)^6}$$

$$B_3(\theta) = \frac{\theta^2(\theta^4 + 4\theta^3 + 7\theta^2 + 6\theta + 3)}{(1+\theta)^8}$$

$$B_4(\theta) = \frac{\theta^2(\theta^6 + 6\theta^5 + 16\theta^4 + 24\theta^3 + 22\theta^2 + 12\theta + 4)}{(1+\theta)^{10}}$$

The general form of Bhattacharyya lower bounds can be stated by,

$$B_i(\theta) = \frac{\theta^2}{(1+\theta)^{2(i+1)}} \sum_{r=1}^{2i-1} c_{i,r}\theta^{r-1}; \quad for \quad i = 1,2,\ldots, \tag{5}$$

where $c_{i,r}$ is a constant value depending on order of the bound (i.e i) and r. Figure 1 and Table 1 show the Bhattacharyya lower bounds for variance of any unbiased estimator of R in Burr XII distribution.

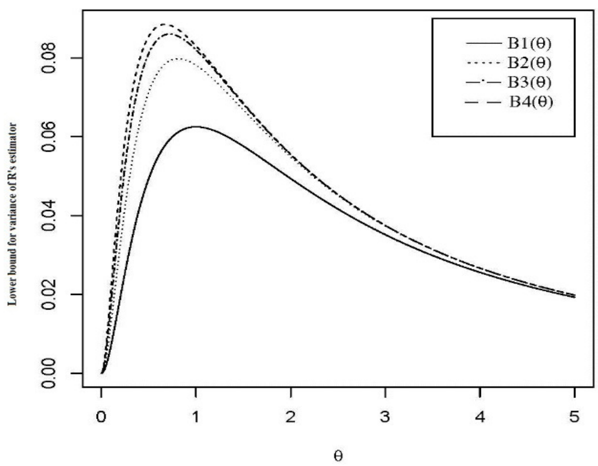

Figure 1: First four Bhattacharyya lower bounds of variance of any unbiased estimator of R in Burr XII distribution with parameters θ and $\alpha = 1$

It is seen that, for any values of θ (specially for small values) , as the order of bounds are increased, the lower bounds get sharpers and converge to a specified point and get closer to the variance of the estimator. Table 1 present the values of the lower bounds for comparisons the rate of convergence.

θ	$B_1(\theta)$ *(Cramer-Rao)*	$B_2(\theta)$	$B_3(\theta)$	$B_4(\theta)$
0.1	0.006830	0.012474	0.017139	0.020995
0.5	0.049382	0.071330	0.081085	0.085420
1	0.062500	0.078125	0.082031	0.083007
3	0.035156	0.037353	0.037490	0.037499
5	0.019290	0.019825	0.019840	0.019841
10	0.006830	0.006886	0.006887	0.006887
20	0.002056	0.002061	0.002061	0.002061

Table 1: First four Bhattacharyya lower bounds of variance of any unbiased estimator of R in Burr XII distribution for some values of θ and $\alpha = 1$.

As it is seen, for a value of θ more than about 3, the differences between the lower bounds are less than 0.0001 and can be ignored, which shows that in this subject the Cramer-Rao lower bound can be good alternative for approximation of variance of R in Burr XII distribution.

Figure 2 shows the $\frac{\partial B_i(\theta)}{\partial \theta}$ for $i = 1,2,3,4$. The results show that the lower bounds for variance of unbiased estimator of R is always decreasing with respect to $\theta \geq 1$ (since $B_i'(\theta) \leq 0$; $i = 1,2,...$) and is increasing function with respect to some $\theta < 1$ (since $B_i'(\theta) \geq 0$; $i = 1,2,...$). Furthermore, for large values of θ, the differences of lower bounds are negligible.

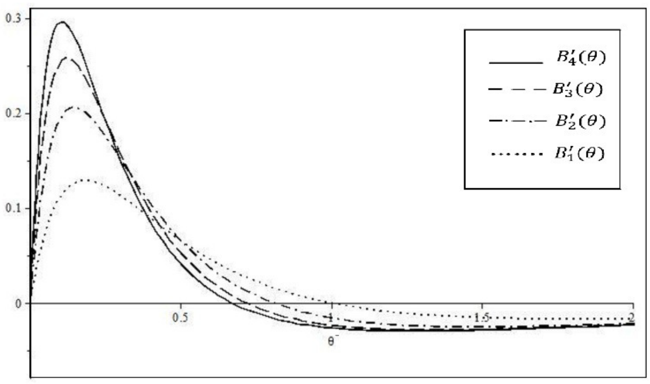

Figure 2: ***Derivatives of Bhattacharyya lower bound (*** $B_i'(\theta) = \frac{\partial B_i(\theta)}{\partial \theta}$ ***)***

Conclusion

In this paper, first we consider the MLE and UMVUE of $R = P(Y < X)$ when Y and X both follow Burr XII distribution with parameters $(\theta_1, 1)$ and $(\theta_2, 1)$, respectively. Then the general form of Bhattacharyya matrix for Burr XII distribution is obtained and in special case of $\theta_2 = 1$, first four Bhattahcaryya lower bounds for variance of any unbiased estimator of R are computed. The convergence rate of the lower bounds to the actual value of the variance is relatively good.

Acknowledgment

The authors are grateful to the referees for their useful comments and suggestions which improve the paper substantially.

References

1. Ahmad, K. E., Jaheen, Z. F., and Mohammed, H. S. (2011). Finite mixture of Burr type XII distribution and its reciprocal: properties and applications, Statistical Papers, 52(4), p. 835-845.
2. Al-Mutairi, D. K., Ghitany, M. E., and Kundu, D. (2015). Inferences on stress-strength reliability from weighted Lindley distributions, Communications in Statistics-Theory and Methods, 44(19), p. 4096-4113.
3. Baklizi, A. (2008a). Estimation of Pr(X < Y) using record values in the one and two parameter exponential distributions, Communications in Statistics-Theory and Methods, 37(5), p. 692-698.
4. Baklizi, A. (2008b). Likelihood and Bayesian estimation of Pr(X < Y) using lower record values from the generalized exponential distribution, Computational Statistics and Data Analysis, 52, p. 3468-3473.
5. Baklizi, A. (2014). Interval estimation of the stress-strength reliability in the two-parameter exponential distribution based on records, Journal of Statistical Computation and Simulation, 84, p. 2670-2679.
6. Bhattacharyya, A. (1946). On some analogues of the amount of information and their use in statistical estimation, Sankhya A, 8, p. 1-14.
7. Bhattacharyya, A. (1947). On some analogues of the amount of information and their use in statistical estimation II, Sankhya A, 8, p. 201-218.
8. Birnbaum, Z. W. (1956). On a use of Mann-Whitney statistics. Proceedings of the third Berkley Symposium in Mathematics, Statistics and Probability, 1, p. 13-17.
9. Blight, B. J. N. and Rao, R. V. (1974). The convergence of Bhattacharyya bounds, Biometrika, 61(1), p. 137-142.
10. Burr, I. W. (1942). Cumulative frequency functions, Ann. Math. Stat., 13, p. 215-232.
11. Chao, A. (1982). On comparing estimators of $P(Y < X)$ in the exponential case, IEEE Trans. Reliab., 31 (4), p. 389-392.
12. Chapman, D. G., Robbins, H. (1951). Minimum variance estimation without regularity assumptions, The Annals of Mathematical Statistics, 22, p. 581–586.
13. Church, J. D. and Harris, B. (1970). The estimation of reliability from stress strength relationships, Technometrics, 12, p. 49-54.

14. Cramer, H. (1946). Mathematical Methods of Statistics. Princeton, NJ: Princeton University Press.

15. Hammersley, J. M. (1950). On estimating restricted parameters, Journal of the Royal Statistical Society, Series B 12, p. 192-240.
16. Hor, P. and Seal, B. (2017). Minimum variance unbiased estimation of stress–strength reliability under bivariate normal and its comparisons, Communications in Statistics-Simulation and Computation, 46(3), p. 2447-2456.
17. Jovanović, M. (2017). Estimation of P {X< Y} for geometric—exponential model based on complete and censored samples, Communications in Statistics-Simulation and Computation, 46(4), p. 3050-3066.
18. Khorashadizadeh, M. and Mohtashami Borzadaran, G. R. (2007). The structure of Bhattacharyya matrix in natural exponential family and its role in approximating the variance of a statistics, J. Statistical Research of Iran (JSRI), 4(1), p. 29-46.
19. Koike, K. (2002). On the inequality of Kshirsagar. Communications in Statistics – Theory and Methods, 31, p. 1617–1627.
20. Kotz, S., Lumelski, I. P., and Pensky, M. (2003). The stress-strength Model and its Generalizations: Theory and Applications, World Scientific.
21. Kundu, D. and Gupta R. D. (2005). Estimation of $P(Y < X)$ for the generalized exponential distribution, Metrika, 61(3), p. 291-308.
22. Kundu, D. and Raqab M.Z. (2009). Estimation of R = P(Y < X) for three-parameter Weibull distribution, Statistics & Probability Letters,79, p. 1839–1846.
23. Kshirsagar, A. M. (2000). An extension of the Chapman–Robbins inequality, Journal of the Indian Statistical Association, 38, p. 355-362.
24. Mohtashami Borzadaran, G. R. (2006). A note via diagonality of the 2x2 Bhattacharyya matrices, J. Math. Sci. Inf., 1(2), p. 73-78.
25. Mohtashami Borzadaran, G. R., Rezaei Roknabadi, A. H. and Khorash-adizadeh, M. (2010). A view on Bhattacharyya bounds for inverse Gaussian distributions, Metrika, 72(2), p. 151-161.
26. Nayeban, S., Rezaei Roknabadi, A. H., and Mohtashami Borzadaran, G. R. (2013). Bhattacharyya and Kshirsagar bounds in generalized gamma distribution, Communications in Statistics-Simulation and Computation, 42(5), p. 969-980.
27. Panahi, H., and Asadi, S. (2010). Estimation of $R = P(Y < X)$ for two-parameter Burr Type XII distribution, World Academy of Science, Engineering and Technology, 72, p. 465-470.
28. Raqab M. Z. and Kundu, D. (2005). Comparison of different estimators of $P(Y < X)$ for a scaled Burr type X distribution, Communications in Statistics-Simulation and Computation, 34 (2), p. 465-483.
29. Shanbhag, D. N. (1972). Some characterizations based on the Bhattacharyya matrix, J. Appl. Probab., 9, p. 580-587.
30. Shanbhag, D. N. (1979). Diagonality of the Bhattacharyya matrix as a characterization, Theory Probab. Appl., 24(2), p. 430-433.

31. Silva, G. O., Ortega, E. M. M., Garibay, V. C. and Barreto, M. L. (2008). Log-Burr XII regression models with censored data, Computational Statistics and Data Analysis, 52, p. 3820-3842.
32. Surles, J. G. and Padgett, W. J. (2001). Inference for reliability and stress-strength for a scaled Burr-type X distribution, Lifetime Data Analy., 7, p. 187-200.
33. Tanaka, H. (2003). On a relation between a family of distributions attaining the Bhattacharyya bound and that of linear combinations of the distributions from an exponential family, Comm. Stat. Theory. Methods , 32(10), p. 1885-1896.
34. Tanaka, H. (2006). Location and scale parameter family of distributions attaining the Bhattacharyya bound, Communications in Statistics - Theory and Methods, 35(9), p. 1611-1628.
35. Tanaka, H. and Akahira, M. (2003). On a family of distributions attaining the Bhattacharyya bound, Ann. Inst. Stat. Math., 55, p. 309-317.
36. Tong, H. (1974). A Note on the Estimation of $Pr\{Y < X\}$ in the Exponential Case, Technometrics, 16(4), p. 625-625.
37. Tong, H. (1977). On The Estimation of $Pr\{Y < X\}$ for Exponential Families, IEEE Transactions on Reliability, 1, p. 54-56.
38. Wu, S. J., Chen, Y. J. and Chang. C. T. (2007). Statistical inference based on progressively censored samples with random removals from the Burr type XII distribution, Journal of Statistical Computation and Simulation, 77, p. 19-27.
39. Zimmer, W. J., Keats, J. B. and Wang, F. K. (1998). The Burr XII distribution in reliability analysis, Journal of Quality Technology, 30, p. 386-94.