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TL- MOMENTS AND L-MOMENTS ESTIMATION FOR THE TRANSMUTED WEIBULL DISTRIBUTION

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Abstract

Accurate estimation of parameters of a probability distribution is of massive importance in statistics. Biased and vague estimation of parameters can lead to misleading results. The Transmuted Weibull Distribution (TWD) has the advantage of bring capable of modeling various types of data, so the accurate estimation of the parameters of this distribution is required. The main purpose of this paper is to develop the Trimmed Linear moments (TL-moments) of the TWD and use the TL-moments to estimate unknown parameters of the TWD. An special case, linear moments (L-moments) will be obtained and used to estimate the unknown parameters of the TWD. Monte Carlo Simulation technique is used to compare the L-moments and TL-moments of TWD.

Key Words: Transmuted Weibull Distribution, Order Statistics, L-moments, TL-moments, Monte Carlo simulation.

1. Introduction

The method of L-moments was introduced Hosking (1990), as a linear combination of probability weighted moments. L-moments can also be applied in parameter estimation and hypothesis testing of parameter values of the theoretical probability distributions. L-moments can be defined for any random variable whose mean exists. The main lead of L-moments over conventional moments i.e., product moments is that they can be estimated by linear functions of sample values and are more resistant to the influence of sample variability. Furthermore, L-moments are less insightful in case of outliers in the data (Vogel and Fennessey (1993)). Sometimes L-moments bring even more competent parameter estimates for the parametric distribution as compared to those estimated by method of maximum likelihood estimation for small samples (Hosking (1990), Tomer and Kumar (2014)). The method of L-moment estimators have recently appeared. L-moment estimators for Log-Normal, Gamma and Generalized Extreme value distributions have been derived. Kundu and Raqab (2005) gave L-moment estimators for generalized Rayleigh distribution. Tomer and Kumar (2014) compared the L-moment estimators with other estimators for the Transmuted Exponentiated Lomax distribution.

A substitute version of L-moments was introduced by Elamir and Seheult (2003). This version is called “Trimmed Linear moments” and it is termed as TL-moments. In the TL-moments, a predetermined percentage of data is trimmed by assigning zero weight before estimating the moments. The main advantage of TL-moments outweighs those of usual L-moments and central moments. Even if the L–moments or central moments of a probability distribution do not exist, TL – moments may be derived. Elamir and Seheult (2003) derived TL-moment for Cauchy distribution, as its mean does not exist. Sample TL-moments are more defiant to outliers in any data. TL-moments are also used to obtain the most fitted distribution and to estimate the unknown parameters of the probability distributions. TL-moments are frequently applied in modeling of different types of data sets particularly, in hydrologic studies and especially in modeling of flood frequency. Since TL-moments are recently developed, they have not yet received significant attention. The parameters of exponential and Weibull distribution were estimated by Abdul-Monien (2007, 2009) using the methods of L-moment and TL-moment estimation. Abdul-Monien and Selim (2009) applied TL-moments and L-moments estimation for estimating the parameters of Generalized Pareto distribution. Bilkova (2014) showed that the L-moments and TL-moments are substitute methods for statistical data analysis. Parameter estimation of power function distribution with TL-moments was introduced by Neveed- Shahzad et al. (2015).

In this article we obtain the L-moments and TL-moments estimators for the TWD. The rest of the study is organized as follows: In Section 2, we introduce the population and sample L-moments and TL-moments. Section 3 is about the Transmuted Weibull distribution. The first four L-moments and TL-moments are derived in Section 4. To compare the properties of the L-moments and TL-moments for TWD, we have established a Monte Carlo simulation study in section 5. Finally, in Section 6, we have concluded our study.

2. Generalized TL-Moments

Let $X_1,\ X_2, \dots, X_n$ be a continuous random sample of size n that has a distribution with distribution function $F(x)$, density function $f(x)$ and quantile function $x(F)$. Further, let $X_{1:n},\ X_{2:n}, \dots, X_{n:n}$ denote the corresponding order statistic. The r^th^ generalized TL-moments with $t1$ smallest and $t2$ largest trimming was given by Elamir and Seheult (2003) as follows

$$L_r^{(t1,\ t2)} = \frac{1}{r}\sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \mathrm{E}(X_{r-k+t1:r+t1+t2}) \tag{1}$$

where, $t1, t2 = 0, 1, 2, \dots;\quad r = 1, 2, 3, \dots$

The expression of the expected value of the $(r-k+t1)^{\text{th}}$ order statistics of the random sample of size $(r+t1+t2)$, i.e., $\mathrm{E}(X_{r-k+t1:r+t1+t2})$ is given by

$$\mathrm{E}(X_{r-k+t1:r+t1+t2}) = \frac{(r+t1+t2)!}{(r-k+t1-1)!(t2+k)!}\int_0^1 x(F)^{r-k+t1-1}[1-F(x)]^{t2+k} dF$$

$$= \frac{(r+t1+t2)!}{(r-k+t1-1)!(t2+k)!}\int_{-\infty}^{\infty} x f(x) F(x)^{r-k+t1-1}[1-F(x)]^{t2+k} dx \tag{2}$$

where, $x(F)$ is the quantile function of random variable X with distribution function $F(x)$.

The generalized TL-moment ratios such as coefficient of variation, coefficient of skewness and coefficient of kurtosis computed from the first four generalized TL-moments, these ratios are defined as $\tau_1 = \frac{L_2^{(t1,t2)}}{L_1^{(t1,t2)}}, \tau_3 = \frac{L_3^{(t1,t2)}}{L_2^{(t1,t2)}}$ and $\tau_4 = \frac{L_4^{(t1,t2)}}{L_2^{(t1,t2)}}$, respectively.

2.1 L-moments

L-moments were introduced by Hosking (1990). He derived these moments for well-known distributions. He proved that L-moments provide superior fit, parameter estimation, hypothesis testing and empirical description of the data. He also proved many hypothetical advantages of L-moments above conservative moments. Due to the advantage of L-moments over conventional moments, many distributions are analyzed by L-moments. L-moments can be defined for any distribution variable whose mean exists. L-moments are linear functions of the expected values of certain linear combination of order statistics. Furthermore, these moments are less insightful in the case of outliers in the data (Vogel and Fennessey (1993)). Hosking (1990) defined r[th] population L-moment (L_r) as follows

$$L_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\,\mathrm{E}(X_{r-k:r}) \quad ; r = 1,2,3, \ldots. \qquad (3)$$

Substituting $r = 1$ and $r = 2$ in (3), we get first and second L-moments and these moments provide the measures of location and dispersion, respectively. The ratio of the third L-moment (L_3) to second L-moment (L_2) i.e. $\tau_{cs}^L = \frac{L_3}{L_2}$ is the measure of skewness and ratio of the fourth L-moment (L_4) to the second L-moment (L_2) i.e. $\tau_{ck}^L = \frac{L_4}{L_2}$ the measure of kurtosis. Note that the r[th] population L-moment can also be obtained from (1) by putting $t1 = t2 = 0$.

The sample L-moments can be obtained from the formula given by Elamir and Seheult (2003) as

$$l_r = \frac{1}{r}\sum_{i=1}^{n}\frac{\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k}}{\binom{n}{r}}x_{i:n} \quad ; r = 1,2,3, \ldots \qquad (4)$$

Similarly L-skewness and L-kurtosis for the sample are defined as the ratio of the third L-moment (l_3) to the second L-moment (l_2) and the ratio of the fourth L-moment (l_4) to the second L-moment (l_2) of the sample. That is, the sample $L-skewness = \frac{l_3}{l_2}$ and sample $L-kurtosis = \frac{l_4}{l_2}$, respectively. These ratios so obtained are less biased as compared to the conventional moments in estimation.

2.2 TL-Moments

An extension of L-moments called TL-moments was given by Elamir and Seheult (2003). In TL-moments a predetermined percentage of data is trimmed by assigning zero weight before estimating the moments. TL-moments for any distribution may exist regardless of the existence of the central moments of that distribution. The TL-moments for Cauchy distribution were obtained by Elamir and Seheult (2003), while the mean of this distribution did not exist. These moments are also used in obtaining most fitted distribution and estimation unknown parameters for the distribution. According to Elamir and Seheult (2003), the r[th] population TL-moment can be obtained from (1) when $t1 = t2 = t$

$$L_r^{(t)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\mathrm{E}(X_{r-k+t:r+2t}) \quad ; r = 1,2,3,\ldots \tag{5}$$

The r[th] sample TL-moment is as follows

$$l_r^{(t)} = \frac{1}{r}\sum_{i=t+1}^{n-t}\frac{\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k+t-1}\binom{n-i}{t+k}}{\binom{n}{r+2t}}x_{i:n} \quad ; r = 1,2,3,\ldots \tag{6}$$

The TL-skewness and TL-kurtosis of a sample are given as $\tau_{cs}^{(t)} = \frac{l_3^{(t)}}{l_2^{(t)}}$ and $\tau_{ck}^{(t)} = \frac{l_4^{(t)}}{l_2^{(t)}}$, respectively.

3. The Transmuted Weibull Distribution

W. Weibull, a Swedish physicist, established Weibull distribution in 1939 to analyze the breaking strength of materials. Since then, it has been widely used in reliability theory for studying the life time data. Aryal and Tsokos (2011) used quadratic rank transmutation map approach, recommended by Shaw and Buckley (2007), to present a new generalization of Weibull distribution. This new generalized distribution is termed as "Transmuted Weibull distribution". The cumulative distribution function (cdf) of random variable X having transmuted probability distribution is given by

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2 \quad ; -1 \le \lambda \le 1 \tag{7}$$

where, $G(x)$ is the cdf of the base distribution.

A random variable X follows transmuted Weibull distribution with parameters $\eta > 0, \sigma > 0$ and λ if its cdf is given by

$$F(x,\eta,\sigma,\lambda) = \left[1 - e^{-\left(\frac{x}{\sigma}\right)^\eta}\right]\left[1 + \lambda\, e^{-\left(\frac{x}{\sigma}\right)^\eta}\right] \tag{8}$$

The corresponding probability density function (pdf) of X is given by

$$f(x,\eta,\sigma,\lambda) = \frac{\eta}{\sigma}\left(\frac{x}{\sigma}\right)^{\eta-1} e^{-\left(\frac{x}{\sigma}\right)^\eta}\left[1 - \lambda + 2\lambda e^{-\left(\frac{x}{\sigma}\right)^\eta}\right] \tag{9}$$

4. Generalized TL-moments for the TWD

In this segment, the population L-moments and TL-moments for the TWD are discussed. The sample L-moments and TL-moments are also discussed. First, we obtain the generalized TL-moments for the TWD. In the subsections 4.1 and 4.2, we have derived L-moments and TL-moments respectively. Using formula given in (1) and the cdf of X given in (8) and the pdf of X given in (9), the r[th]generalized TL-moment with $t1$ smallest and $t2$ largest trimming for the TWD is given by

$$L_r^{(t1,t2)} = \frac{\sigma\,\Gamma\left(\frac{1}{\eta}+1\right)}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\frac{(r+t1+t2)!}{(r-k+t1-1)!(t2+k)!}$$
$$\times \sum_{j=0}^{r-k+t1-1}(-1)^j\sum_{i=0}^{t2+k+j}\binom{r-k+t1-1}{j}\binom{t2+k+j}{i}\lambda^i(1-\lambda)^{t2+k+j-i} \times \left[\frac{1-\lambda}{(t2+k+j+i+1)^{\frac{1}{\eta}+1}} + \frac{2\lambda}{(t2+k+j+i+2)^{\frac{1}{\eta}+1}}\right] \tag{10}$$

4.1 L-moments of TWD

The population L-moment is a special case of generalized TL-moment of order r for the TWD by putting $t1 = t2 = 0$ in (10). The r[th] population L-moment for the TWD is

$$L_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\,\mathrm{E}(X_{r-k:r})$$

$$= \frac{\sigma\,\Gamma\left(\frac{1}{\eta}+1\right)}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\frac{r!}{(r-k-1)!k!}\sum_{j=0}^{r-k-1}(-1)^j\sum_{i=0}^{k+j}\binom{r-k-1}{j}\binom{k+j}{i}$$

$$\times\ \lambda^i(1-\lambda)^{k+j-i}\left[\frac{1-\lambda}{(k+j+i+1)^{\frac{1}{\eta}+1}}+\frac{2\lambda}{(k+j+i+2)^{\frac{1}{\eta}+1}}\right] \qquad (11)$$

The first four population L-moments for the TWD are obtained by putting $r = 1, 2, 3, 4$ in (11) as

$$L_1 = \sigma\,\Gamma\left(\frac{1}{\eta}+1\right)\left[(1-\lambda)+\frac{\lambda}{2^{\frac{1}{\eta}}}\right]$$

$$L_2 = \sigma\Gamma\left(\frac{1}{\eta}+1\right)\left[(1-\lambda)-\frac{(\lambda^2-3\lambda+1)}{2^{\frac{1}{\eta}}}-\frac{2\lambda(1-\lambda)}{3^{\frac{1}{\eta}}}-\frac{\lambda^2}{4^{\frac{1}{\eta}}}\right]$$

$$L_3 = \frac{\sigma\,\Gamma\left(\frac{1}{\eta}+1\right)}{3}\Bigg[2(1-\lambda)+\frac{4}{2^{\frac{1}{\eta}+1}}(5\lambda-2-2\lambda^2)+\frac{1-\lambda}{3^{\frac{1}{\eta}+1}}(7\lambda^2-38\lambda+7)+$$
$$\frac{4\lambda}{4^{\frac{1}{\eta}+1}}(7\lambda^2-18\lambda+7)+\frac{7\lambda(1-\lambda)}{5^{\frac{1}{\eta}+1}}(4-3\lambda)+\frac{14\lambda^2}{6^{\frac{1}{\eta}+1}}\Bigg]$$

$$L_4 = \frac{\sigma\,\Gamma\left(\frac{1}{\eta}+1\right)}{4}\Bigg[(1-\lambda)-\frac{1}{2^{\frac{1}{\eta}}}(6\lambda^2-13\lambda+6)+\frac{2(1-\lambda)}{3^{\frac{1}{\eta}}}(5\lambda^2-10\lambda-5)+$$
$$\frac{4}{4^{\frac{1}{\eta}+1}}(5(1-\lambda)^4-24\lambda(1-\lambda)^2-6\lambda^2)+\frac{10\lambda(1-\lambda)}{5^{\frac{1}{\eta}}}(5\lambda-2)-$$
$$\frac{10\lambda^2}{6^{\frac{1}{\eta}}}(3\lambda^2-7\lambda+3)-\frac{20\lambda^3(1-\lambda)}{7^{\frac{1}{\eta}}}-\frac{5\lambda^4}{8^{\frac{1}{\eta}}}\Bigg]\ ,$$

respectively. First four sample L-moments can be obtained from (4).

$$l_1 = \frac{1}{n}\sum_{i=1}^{n}x_{i:n} = \bar{x}$$

$$l_2 = 2\sum_{i=1}^{n}\frac{(i-1)}{n(n-1)}x_{i:n} - \bar{x}$$

$$l_3 = \sum_{i=1}^{n}\frac{(i-1)(i-2)-4(i-1)(n-i)+(n-i)(n-i-1)}{n(n-1)(n-2)}x_{i:n}$$

$$l_4 = \sum_{i=1}^{n}\frac{(i-1)(i-2)(i-3)-9(i-1)(i-2)(n-i)+9(i-1)(n-i)(n-i-1)-9(n-i)(n-i-1)(n-i-2)}{n(n-1)(n-2)(n-3)}x_{i:n}$$

4.2 TL-Moments of TWD

TL-moments are the extension of L-moments. TL-moments are more dynamic than L-moments as they truncate the outliers present in the data (Elamir and Seheult (2003)). The population TL-moments of order r for the TWD as a special case of generalized TL-moments from (10) by taking $t1 = t2 = t$ and are given by

$$L_{(r)}^{(t)} = \frac{\sigma\Gamma\left(\frac{1}{\eta}+1\right)}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\frac{(r+2t)!}{(r-k+t-1)!(t+k)!}$$

$$\times\sum_{j=0}^{r-k+t-1}(-1)^j\sum_{i=0}^{t+k+j}\binom{r-k+t-1}{j}\binom{t+k+j}{i}\lambda^i(1-\lambda)^{t+k+j-i}$$

$$\times\left[\frac{1-\lambda}{(t+k+j+i+1)^{\frac{1}{\eta}+1}}+\frac{2\lambda}{(t+k+j+i+2)^{\frac{1}{\eta}+1}}\right] \quad ;\ t, r = 1,2,3, \dots \qquad (12)$$

Here, we take $t = 1$, then the first four TL-moments for the TWD are obtained as

$$L_{(1)}^{(1)} = 6\sigma\,\Gamma\left(\frac{1}{\eta}+1\right)\left[\frac{(1-\lambda)^2}{2^{\frac{1}{\eta}+1}} - \frac{(1-\lambda)}{3^{\frac{1}{\eta}+1}}(\lambda^2-5\lambda+1) - \frac{2\lambda}{4^{\frac{1}{\eta}+1}}(2\lambda^2-5\lambda+2) - \frac{5\lambda^2(1-\lambda)}{5^{\frac{1}{\eta}+1}} - \frac{2\lambda^3}{6^{\frac{1}{\eta}+1}}\right]$$

$$L_{(2)}^{(1)} = 6\sigma\,\Gamma\left(\frac{1}{\eta}+1\right)\left[\frac{(1-\lambda)^2}{2^{\frac{1}{\eta}+1}} - \frac{3(1-\lambda)}{3^{\frac{1}{\eta}+1}}(\lambda^2-3\lambda+1) + \frac{2}{4^{\frac{1}{\eta}+1}}\{(1-\lambda)^4 - 6\lambda(1-\lambda)^2 + \lambda^2\} + \frac{5\lambda(1-\lambda)}{5^{\frac{1}{\eta}+1}}\{2\lambda^2-7\lambda+2\} + \frac{6\lambda^2}{6^{\frac{1}{\eta}+1}}\{3\lambda^2-7\lambda+3\} + \frac{14\lambda^3(1-\lambda)}{7^{\frac{1}{\eta}+1}} + \frac{4\lambda^4}{8^{\frac{1}{\eta}+1}}\right]$$

$$L_{(3)}^{(1)} = \frac{20\sigma\,\Gamma\left(\frac{1}{\eta}+1\right)}{3}\left[\frac{(1-\lambda)^2}{2^{\frac{1}{\eta}+1}} - \frac{(1-\lambda)}{3^{\frac{1}{\eta}+1}}(5\lambda^2-13\lambda+5) + \frac{2}{4^{\frac{1}{\eta}+1}}\{4(1-\lambda^4) - 10\lambda(1-\lambda)^2 + \lambda^4\} - \frac{1}{5^{\frac{1}{\eta}+1}}\{4(1-\lambda)^5 - 40\lambda(1-\lambda)^3 + 13\lambda^2(1-\lambda) + 12\lambda^2\} - \frac{2\lambda}{6^{\frac{1}{\eta}+1}}\{2(1-\lambda^4) - 36\lambda(1-\lambda)^2 + 5\lambda^2\} - \frac{56\lambda^2(1-\lambda)}{7^{\frac{1}{\eta}+1}}(\lambda^2-3\lambda+1) - \frac{16\lambda^3}{8^{\frac{1}{\eta}+1}}\{4\lambda^2 - 9\lambda + 4\} - \frac{36\lambda^4(1-\lambda)}{9^{\frac{1}{\eta}+1}} - \frac{8\lambda^5}{10^{\frac{1}{\eta}+1}}\right]$$

$$L_{(4)}^{(1)} = \frac{15\sigma\,\Gamma\left(\frac{1}{\eta}+1\right)}{2}\left[\frac{(1-\lambda)^2}{2^{\frac{1}{\eta}+1}} - \frac{3(1-\lambda)}{3^{\frac{1}{\eta}+1}}(3\lambda^2-7\lambda+3) + \frac{2}{4^{\frac{1}{\eta}+1}}\{7(1-\lambda)^4 - 12\lambda(1-\lambda)^2 + \lambda^4\} - \frac{5\lambda^2(1-\lambda)}{5^{\frac{1}{\eta}+1}}\{3(1-\lambda)^5 - 14\lambda(1-\lambda)^3 + 6\lambda^2(1-\lambda)\} + \frac{6}{6^{\frac{1}{\eta}+1}}\{(1-\lambda)^6 - 15\lambda(1-\lambda)^4 + 21\lambda^2(1-\lambda)^2 - 2\lambda^3\} + \frac{2}{7^{\frac{1}{\eta}+1}}\{21\lambda(1-\lambda)^5 - 105\lambda^2(1-\lambda)^3 + 43\lambda^3(1-\lambda)\} + \frac{4}{8^{\frac{1}{\eta}+1}}\{25\lambda^2(1-\lambda)^4 - 60\lambda^3(1-\lambda)^2 + 7\lambda^4\} + \frac{20\lambda^3(1-\lambda)}{9^{\frac{1}{\eta}+1}}(9\lambda^2 - 25\lambda + 9) + \frac{30\lambda^4}{10^{\frac{1}{\eta}+1}}(5\lambda^2 - 11\lambda + 5) + \frac{66\lambda^5(1-\lambda)}{11^{\frac{1}{\eta}+1}} + \frac{12\lambda^6}{12^{\frac{1}{\eta}+1}}\right]$$

First four sample TL-moments for the TWD are given by

$$l_1^{(1)} = 6\sum_{i=2}^{n-1}\frac{(i-1)(n-i)}{n(n-1)(n-2)}x_{i:n}$$

$$l_2^{(1)} = 6\sum_{i=2}^{n-1}\frac{(i-1)(i-2)(n-i)-(i-1)(n-i)(n-i-1)}{n(n-1)(n-2)(n-3)}x_{i:n}$$

$$l_3^{(1)} = \frac{20}{3n(n-1)(n-2)(n-3)(n-4)}\sum_{i=2}^{n-1}\{(i-1)(i-2)(i-3)(n-i) - 3(i-1)(i-2)(n-i)(n-i-1) + (i-1)(n-i)(n-i-1)(n-i-2)\}\,x_{i:n}$$

$$l_4^{(1)} = \frac{15}{2n(n-1)(n-2)(n-3)(n-4)(n-5)}\sum_{i=2}^{n-1}\{(i-1)(i-2)(i-3)(i-4)(n-i) - 6(i-1)(i-2)(i-3)(n-i)(n-i-1) + 6(i-1)(i-2)(n-i)(n-i-1)(n-i-2) - (i-1)(n-i)(n-i-1)(n-i-2)(n-i-3)\}\,x_{i:n}$$

5. Monte Carlo Simulation Study

In this segment, a Monte Carlo simulation is performed for the comparison of estimators obtained by L-moment and TL-moment methods for TWD. This comparison is based on measures of average estimators and mean square errors (MSEs). Here, we use R i386 3.2.2 Software for simulation study.

We have executed the experiments for different sample sizes (20, 30, 40, 60) as well as different values of the parameters. The estimates are calculated from 5000 repeated samples. In the estimation of η (shape parameter) and σ (scale parameter), we equate the sample moments to corresponding population moments for each case to obtain the average estimates and MSEs of η and σ of TWD. The results of the average estimates and MSEs (in parenthesis) are listed in Table 1 and 2.

6. Results and Conclusion

The results of the simulation study are very good in terms of average estimates and MSEs even in case of small sample sizes. As the sample size n increases the average estimates of η and σ are get closer to the true values of the parameters η and σ and corresponding MSEs decrease. L-moment estimates of η and σ are better than the TL-moment estimators of η and σ in respect of average estimates and MSEs for all combinations of sample sizes and different true values of the parameters. We have observed that the MSE of η increases with the increase in true value of σ where as the MSEs of σ decreases with the increase in the true value of η.

n	η	σ	L-Moment Estimates		TL-Moment Estimates	
			$\hat{\eta}$	$\hat{\sigma}$	$\hat{\eta}$	$\hat{\sigma}$
20	1	1	1.0264 (0.0429)	1.0015 (0.0385)	1.0331 (0.0611)	0.9904 (0.0529)
		1.5	1.0301(0.0430)	1.5050 (0.0839)	1.0385 (0.0613)	1.4944 (0.1172)
		2	1.0269 (0.0432)	2.0099 (0.1521)	1.0353 (0.0624)	1.9899 (0.2091)
	1.5	1	1.5307 (0.0964)	0.8790 (0.0289)	1.5498 (0.1397)	0.9925 (0.0230)
		1.5	1.5333 (0.0983)	1.3216 (0.0650)	1.5500 (0.1399)	1.4945 (0.0526)
		2	1.5359 (0.1001)	1.7613 (0.1145)	1.5533 (0.1430)	1.9866 (0.0915)
30	1	1	1.0163 (0.0271)	1.0033 (0.0257)	1.0205 (0.0366)	0.9961 (0.0357)
		1.5	1.0152 (0.0269)	1.4974 (0.0586)	1.0193 (0.0364)	1.4872 (0.0804)
		2	1.0193 (0.0277)	2.0139 (0.1047)	1.0241 (0.0374)	2.0024 (0.1440)
	1.5	1	1.5296 (0.0611)	0.8801 (0.0239)	1.5390 (0.0817)	0.9967 (0.0151)
		1.5	1.5244 (0.0584)	1.3202 (0.0542)	1.5369 (0.0823)	1.4945 (0.0343)
		2	1.5248 (0.0607)	1.7645 (0.0952)	1.5347 (0.0836)	1.9971 (0.0626)
40	1	1	1.0119 (0.0195)	1.0002 (0.0190)	1.0148 (0.0256)	0.9952 (0.0262)
		1.5	1.0127 (0.0199)	1.4984 (0.0418)	1.0154 (0.0249)	1.4916 (0.0573)
		2	1.0123 (0.0193)	2.0029 (0.0785)	1.0150 (0.0262)	1.9939 (0.1080)

	1.5	1	1.5203 (0.0426)	0.8819 (0.0214)	1.5274 (0.0577)	0.9996 (0.0119)
		1.5	1.5201 (0.0441)	1.3209 (0.0481)	1.5271 (0.0592)	1.4960 (0.0252)
		2	1.5212 (0.0425)	1.7614 (0.0858)	1.5274 (0.0592)	1.9967 (0.0454)
60	1	1	1.0103 (0.0125)	1.0003 (0.0127)	1.0123 (0.0159)	0.9979 (0.0177)
		1.5	1.0077 (0.0125)	1.4996 (0.0300)	1.0093 (0.0163)	1.4947 (0.0413)
		2	1.0087 (0.0125)	1.9968 (0.0506)	1.0080 (0.0160)	1.9916 (0.0694)
	1.5	1	1.5076 (0.0275)	0.8795 (0.0195)	1.5118 (0.0373)	0.9973 (0.0078)
		1.5	1.5105 (0.0277)	1.3201 (0.0434)	1.5156 (0.0374)	1.4975 (0.0173)
		2	1.5123 (0.0280)	1.7625 (0.0757)	1.5166 (0.0379)	1.9988 (0.0303)

Table 1: Estimates and MSEs for η and σ when $\lambda = -0.5$

n	η	σ	L-Moment Estimates		TL-Moment Estimates	
			$\hat{\eta}$	$\hat{\sigma}$	$\hat{\eta}$	$\hat{\sigma}$
20	1	1	1.0214 (0.0422)	0.9939 (0.0501)	1.0276 (0.0594)	1.0053 (0.0626)
		1.5	1.0298 (0.0432)	1.4859 (0.1168)	1.0344 (0.0602)	1.5082 (0.1460)
		2	1.0233 (0.0435)	1.9903 (0.2088)	1.0298 (0.0611)	2.0122 (0.2523)
	1.5	1	1.5332 (0.0907)	0.9973 (0.0232)	1.5531 (0.1401)	1.0029 (0.0300)
		1.5	1.5333 (0.0925)	1.4970 (0.0517)	1.5527 (0.1402)	1.5053 (0.0665)
		2	1.5357 (0.0933)	1.9968 (0.0972)	1.5532 (0.1445)	2.0100 (0.1243)
30	1	1	1.0114 (0.0252)	1.0026 (0.0346)	1.0134 (0.0344)	1.0106 (0.0421)
		1.5	1.0132 (0.0257)	1.4955 (0.0778)	1.0173 (0.0344)	1.5053 (0.0930)
		2	1.0151 (0.0267)	1.9886 (0.1328)	1.0202 (0.0363)	2.0014 (0.1621)
	1.5	1	1.5241 (0.0559)	0.9986 (0.0156)	1.5311 (0.0779)	1.0036 (0.0198)
		1.5	1.5229 (0.0555)	1.4996 (0.0360)	1.5312 (0.0787)	1.5055 (0.0446)
		2	1.5237 (0.0559)	1.9982 (0.0643)	1.5334 (0.0801)	2.0051 (0.0793)

40	1	1	1.0123 (0.0189)	0.9939 (0.0248)	1.0150 (0.0242)	0.9987 (0.0300)
		1.5	1.0102 (0.0189)	1.4985 (0.0588)	1.0130 (0.0244)	1.5065 (0.0707)
		2	1.0130 (0.0190)	1.9949 (0.1038)	1.0146 (0.0246)	2.0077 (0.1268)
	1.5	1	1.5199 (0.0417)	0.9979 (0.0117)	1.5283 (0.0567)	0.9998 (0.0144)
		1.5	1.5174 (0.0417)	1.4959 (0.0260)	1.5227 (0.0559)	1.5008 (0.0318)
		2	1.5186 (0.0421)	2.0006 (0.0462)	1.5227 (0.0561)	2.0089 (0.0591)
60	1	1	1.0067 (0.0121)	0.9948 (0.0174)	1.0074 (0.0156)	0.9988 (0.0204)
		1.5	1.0089 (0.0127)	1.4981 (0.0399)	1.0081 (0.0159)	1.5058 (0.0467)
		2	1.0068 (0.0126)	1.9893 (0.0695)	1.0073 (0.0161)	1.9980 (0.0809)
	1.5	1	1.5117 (0.0264)	0.9989 (0.0079)	1.5156 (0.0350)	1.0014 (0.0097)
		1.5	1.5122 (0.0265)	1.4973 (0.0173)	1.5175 (0.0352)	1.4993 (0.0212)
		2	1.5132 (0.0265)	2.0000 (0.0312)	1.5184 (0.0363)	2.0033 (0.0382)

Table 2: Estimates and MSEs for η and σ when $\lambda = 1$

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